Andrea Lauß
Peter Zinterhof
Monika Feldbacher

University of Salzburg
Research Institute of Software Technology
RIST ++

Parallel Implementation of Fast Algorithms for Good Lattice Points

Abstract

For $2 \leq s \leq 10$ we determine optimal coefficients for the s-dimensional numerical integration of a special class of functions. We use the algorithms of Korobov, where the calculations enlarge for higher dimensions s and larger modules N because the algorithms have the complexities $O(N^2)$, $O(N^{\frac{4}{3}})$, $O(N)$ and $O(N \log N)$, with N a prime number or with $N = 2^n$. This leads to problems regarding the computing time needed. By parallelizing the algorithms an enormous reduction of this time is reached. The implementations are done in FORTRAN-77 under PVM (= Parallel Virtual Machine) version 3.1 by using a coarse-grained parallelism. The calculations were performed on a heterogeneous cluster of workstations consisting of DECstations 5000/33, DECs 3000 AXP/400 and a Silicon Graphics Iris INDIGO.
This report is part 3 of the NEWTON-(New Technology of Numerics-) Project, which is devoted to the development and practical applications of NTN (Number Theoretical Numerics). NEWTON encompasses theoretical investigations and implementations onto parallel computers. (See also part 1([7]) and part 2([9])).

1. Introduction

Owing to the application of computer science, number theory became of great interest for numerical analysis. In this work we compute low-discrepancy point sets for the numerical integration of periodic functions, which belong to the class $E_s^{\alpha}(C)$. We use number theoretic methods to construct the lattice-points in the s-dimensional unit cube $G_s \quad s \geq 2$,

$$(\{\frac{a_1 k}{N}\}, \{\frac{a_2 k}{N}\}, ..., \{\frac{a_s k}{N}\}),$$

with $1 \leq k \leq N$ and suitable $a = (a_1, a_2, ..., a_s) \in \mathbf{Z}^s$. These lattice-points lead to

$$\int_{G_s} f(x_1,\ x_2, \ldots, x_s) dx_1 dx_2 \ldots dx_s = \frac{1}{N} \sum_{k=1}^{N} f(\{\frac{a_1 k}{N}\},\ \{\frac{a_2 k}{N}\}, \ldots,\ \{\frac{a_s k}{N}\}) - R, \qquad (1)$$

the quadrature with error R. The theoretically best results are obtained by using for a the *optimal coefficients* or equivalently the *good lattice points* in the sense of Hlawka and Korobov. The integers $a_1, a_2, ..., a_s$ are *optimal coefficients* modulo N, iff for each nontrivial solution $(m_1, m_2, ..., m_s) \in \mathbf{Z}^s$ of the congruence

$$a_1 m_1 + a_2 m_2 + \cdots + a_s m_s \equiv 0 \quad mod\ N$$

holds :

$$R((m_1, m_2, ..., m_s)) = \prod_{i=1}^{s} max(1, |m_i|) \quad \geq \quad C(s) \frac{N}{ln^{\beta} N}$$

with $C = C(s) > 0$ and $\beta = \beta(s) > 0$ constants only depending on the dimension s.

2. Korobov's Algorithms

A mathematically detailed description of the algorithms and proofs respectively can be found in Korobov [5] and [6]. These works contain although all the information about the complexities. The number theoretical background of the algorithms is beyond the paper, for surveys of this theory see Hua and Wang [4] and Korobov [5].
Let $\{x\}$ be the fractional part of x, $[x]$ the next integer to x, $\|x\|$ the distance to the next integer to x and let $\sum^*$ denote the summation of odd integers.

Algorithm K1: [[5] , theorem 23]
Let $p > s$ be prime. For $z \in \mathbf{Z}$ the function $H(z)$ is defined as

$$H(z) := \frac{3^s}{p} \sum_{k=1}^{p} (1 - 2\{\frac{k}{p}\})^2 (1 - 2\{\frac{kz}{p}\})^2 \cdots (1 - 2\{\frac{kz^{s-1}}{p}\})^2.$$

If $H(a)$ is the minimal value of H on $\{1,\ 2, \ldots, p-1\}$, then the numbers

$$a_1 := 1,\ a_2 := a, \ldots,\ a_s := a^{s-1}$$

are optimal coefficients modulo p. One needs $O(p^2)$ operations for the computation of a and this leads for great p to a considerable computing time.

Algorithm K2: [[5] , theorem 24]
Let $p_1, p_2 > s$ be prime, let $p := p_1 p_2$ and a be defined as in algorithm K1 for p_1. For $z \in \mathbf{Z}$ we define $\tilde{H}(z)$ as

$$\tilde{H}(z) := \frac{3^s}{p} \sum_{k=1}^{p} (1 - 2\{\frac{(p_1 + p_2)}{p}\})^2 (1 - 2\{\frac{(p_1 z + p_2 a)}{p}\})^2 \cdots (1 - 2\{\frac{(p_1 z^{s-1} + p_2 a^{s-1})}{p}\})^2.$$

If $\tilde{H}(b)$ is the minimal value of $\tilde{H}$ on $\{1,\ 2, \ldots,\ p_2 - 1\}$, then the numbers

$$a_1 := p_1 + p_2,\ a_2 := p_1 b + p_2 a, \ldots,\ a_s := p_1 b^{s-1} + p_2 a^{s-1}$$

are optimal coefficients modulo p. This algorithm needs $O(p^{\frac{4}{3}})$ operations in the case p_2 is of order $O(\sqrt{p_1})$.

Algorithm K3: [[6] , theorem 1]
Let be $n, s \in \mathbf{N}$ and $x_1, x_2, \ldots, x_s$ odd integers. For $1 \le \nu \le n$ we define

$$h_\nu(x_1, x_2, \ldots, x_s) := \frac{1}{2^\nu} \sum_{m=1}^{2^\nu}{}^{*}(2n - 2\nu + \frac{1}{\|\frac{mx_1}{2^\nu}\|})(2n - 2\nu + \frac{1}{\|\frac{mx_2}{2^\nu}\|}) \cdots (2n - 2\nu + \frac{1}{\|\frac{mx_s}{2^\nu}\|}).$$

Let be $a_{11} = a_{21} = \ldots = a_{s1} := 1$. Suppose the odd integers $a_{1\nu-1},\ a_{2\nu-1}, \ldots,\ a_{s\nu-1}$ are already computed, then we define $a_{1\nu},\ a_{2\nu}, \ldots,\ a_{s\nu}$, with $2 \le \nu \le n$ as

$$a_{1\nu} := a_{1\nu-1} + 2^{\nu-1} z'_1,\ a_{2\nu} := a_{2\nu-1} + 2^{\nu-1} z'_2, \ldots, a_{s\nu} := a_{s\nu-1} + 2^{\nu-1} z'_s,$$

where $h_\nu(a_{1\nu-1} + 2^{\nu-1} z'_1, a_{2\nu-1} + 2^{\nu-1} z'_2, \ldots, a_{s\nu-1} + 2^{\nu-1} z'_s)$ is the minimum of h_ν for $(z'_1, \ldots,\ z'_s) \in \{0,1\}^s$. For $n \in \mathbf{N}$

$$a_1 := a_{1n},\ a_2 := a_{2n}, \ldots,\ a_s := a_{sn}$$

are optimal coefficients modulo $p = 2^n$. The number of elementary operations is of order $O(p)$ and doesn't reach $Cs2^s p$, where C is an absolute constant.

Algorithm K4: [[6] , theorem 2]
Let be $n, s \in \mathbf{N}$ and $x_1, x_2, \ldots, x_s$ odd integers. For $\nu = 1$ and $r = s$, and for $2 \le \nu \le n$ and $1 \le r \le s$ the function $h_{r\nu}(x_1, x_2, \ldots, x_s)$ is defined as

$$h_{r\nu}(x_1, x_2, \ldots, x_s) := \frac{1}{2^\nu} \sum_{m=1}^{2^\nu}{}^{*} \prod_{j=1}^{r} (2n - 2\nu + \frac{1}{\|\frac{mx_j}{2^\nu}\|}) \prod_{j=r+1}^{s} (2n - 2\nu + 2 + \frac{1}{\|\frac{mx_j}{2^{\nu-1}}\|}).$$

Let be $a_{11} = a_{21} = \ldots = a_{s1} := 1$. Suppose for $r \ge 1$ and $\nu \ge 2$ we have already computed the odd integers $a_{1\nu-1},\ a_{2\nu-1}, \ldots,\ a_{s\nu-1}$ and $a_{1\nu},\ a_{2\nu}, \ldots,\ a_{r-1\nu}$, then we get $a_{r\nu} := a_{r\nu-1} + 2^{\nu-1} z'$ with $2 \le \nu \le n$, where

$$h_{r\nu}(a_{1\nu},\ a_{2\nu}, \ldots, a_{r-1\nu}, a_{r\nu-1} + 2^{\nu-1} z', a_{r+1\nu-1}, \ldots, a_{s\nu-1})$$

is minimal for $z' \in \{0,1\}$. For any $n \in \mathbf{N}$ the integers

$$a_1 := a_{1n},\ a_2 := a_{2n}, \ldots,\ a_s := a_{sn}$$

are optimal coefficients modulo $p = 2^n$. The number of elementary operations doesn't reach $Cs^2 p$.

Algorithm K5: [[6] , theorem 3]
Let be $n, s \in \mathbf{N}$ and x an odd integer. For $1 \le \nu \le n$ and $1 \le r \le s$ the function $h_{r\nu}$ is defined as

$$h_{r\nu}(x) := \sum_{k=\nu}^{n} \frac{1}{2^{k-\nu}} \sum_{m=1}^{2^k}{}^{*} ln(\frac{1}{sin^2 \pi(\frac{ma_{1n}}{2^k})}) \cdots ln(\frac{1}{sin^2 \pi(\frac{ma_{r-1n}}{2^k})}) ln(\frac{1}{sin^2 \pi(\frac{mx}{2^\nu})}).$$

Let be $a_{1n} = a_{r1} := 1$. Suppose we already knew the values $a_{2n}, \ldots, a_{r-1n}$ and $a_{r2}, \ldots, a_{r\nu-1}$ with $r \geq 2$ and $\nu \geq 2$, then we put $a_{r\nu} := a_{r\nu-1} + 2^{\nu-1} z'$ with $2 \leq \nu \leq n$, where $h_{r\nu}(z')$ is a minimum of $h_{r\nu}$ on $\{0, 1\}$. For any $n \in \mathbf{N}$

$$a_1 := a_{1n},\ a_2 := a_{2n}, \ldots,\ a_s := a_{sn}$$

are optimal coefficients modulo $p = 2^n$. The number of elementary operations is of order $O(p \log p)$.

The following 'useful coefficients' modulo N computed with a method of Zinterhof can be good compared with the optimal coefficients of K1, K2, ..., K5.

Algorithm ZG: [[8] , theorem]
Let be $N \in \mathbf{N}$, $(\theta_1,\ \theta_2, \ldots, \theta_s) \in \mathbf{R^s}$ and $(a_1,\ a_2, \ldots,\ a_s) := ([\theta_1 N],\ [\theta_2 N], \ldots, [\theta_s N])$. Then for almost all $(\theta_1,\ \theta_2, \ldots, \theta_s) \in \mathbf{R^s}$ and for $\varepsilon > 0$ exists $C = C(\varepsilon,\ \theta_1,\ \theta_2, \ldots, \theta_s)$ such that for every nontrivial solution $(m_1,\ m_2, \ldots,\ m_s) \in \mathbf{Z}^s$ the congruence

$$a_1 m_1 + a_2 m_2 + \cdots + a_s m_s \equiv 0 \ mod\ N$$

holds

$$\prod_{i=1}^{s} max(1, |m_i|) > \frac{C}{N^\delta}, \qquad \delta = \frac{1}{2+\varepsilon}.$$

The theorem holds especially for $\theta_i := e^{r_i}$ with $r_i \in \mathbf{Q},\ r_i \neq 0,\ i = 1,\ 2, \ldots,\ s$, and $r_i \neq r_k,\ i \neq k$.
Because of their practically not requiring computing time (only s elementary operations are needed) they were called *gratis lattice points.* The gratis lattice points provide theoretically worse estimations. However the subsequent calculations indicate much better quality than can be proven by theory. It should have been remarked that in [8] were used results from Allan Baker for which he received the fields-medal.

3. Numerical Error

We define the integration error of the quadrature (1) :

$$R(n,\ h_1,\ h_2, \ldots,\ h_s,\ f) :=$$

$$\int\limits_{G_s} f(x_1,\ x_2, \ldots, x_s) dx_1 dx_2 \ldots dx_s - \frac{1}{n} \sum_{k=1}^{n} f(\{\frac{h_1 k}{n}\},\ \{\frac{h_2 k}{n}\}, \ldots,\ \{\frac{h_s k}{n}\}), \tag{2}$$

where

$$(\{\frac{h_1 k}{n}\},\ \{\frac{h_2 k}{n}\},\ldots,\ \{\frac{h_s k}{n}\}) \qquad 1 \leq k \leq n$$

are good lattice-points.

For fixed n and f we say, that $(h_1,\ h_2,\ldots,\ h_s)$ is better then $(\tilde{h_1},\ \tilde{h_2},\ldots,\ \tilde{h_s})$, if for the integration error (2) the inequality

$$|R(n,\ h_1,\ h_2,\ldots,\ h_s,\ f)| \leq |R(n,\ \tilde{h_1},\ \tilde{h_2},\ldots,\ \tilde{h_s},\ f)|$$

holds.

Hua Loo Keng and Wang Yuan gave in [4] estimators for the numerical error.

Theorem: [[4], theorem 8.2]

$$\sup_{f\in E_s^2(C)} |R(n,h_1,\ h_2,\ldots,\ h_s,f)| \leq CW_2(n,\ h_1,\ h_2,\ldots,\ h_s),$$

where

$$W_2(n,\ h_1,\ h_2,\ldots,\ h_s) := \begin{cases} \frac{1}{n}(1+\frac{\pi^2}{3})^s + \frac{2}{n}\sum\limits_{k=1}^{\frac{n-1}{2}}\prod\limits_{\nu=1}^{s}(1+2\pi^2 B_2(\{\frac{kh_\nu}{n}\})) - 1, \text{ if } 2 \nmid n, \\ \frac{1}{n}(1+\frac{\pi^2}{3})^s + \frac{1}{n}(1-\frac{\pi^2}{6})^\mu \frac{1}{n}(1+\frac{\pi^2}{3})^{s-\mu} + \\ \qquad \frac{2}{n}\sum\limits_{k=1}^{\frac{n}{2}-1}\prod\limits_{\nu=1}^{s}(1+2\pi^2 B_2(\{\frac{kh_\nu}{n}\})) - 1, \text{ if } 2 \mid n, \end{cases}$$

with $B_2(x) := x^2 - x + \frac{1}{6}$, the second Bernoulli polynomial, and μ the number of odd integers of h_ν $(1 \leq \nu \leq s)$.

Theorem: [[4], theorem 8.3]

$$\sup_{f\in E_s^4(C)} |R(n,h_1,\ h_2,\ldots,\ h_s,f)| \leq CW_4(n,\ h_1,\ h_2,\ldots,\ h_s),$$

where

$$W_4(n,\ h_1,\ h_2,\ldots,\ h_s) := \begin{cases} \frac{1}{n}(1+\frac{\pi^4}{45})^s + \frac{2}{n}\sum\limits_{k=1}^{\frac{n-1}{2}}\prod\limits_{\nu=1}^{s}(1-\frac{2\pi^4}{3} B_4(\{\frac{kh_\nu}{n}\})) - 1, \text{ if } 2 \nmid n, \\ \frac{1}{n}(1+\frac{\pi^4}{45})^s + \frac{1}{n}(1-\frac{7\pi^4}{360})^\mu \frac{1}{n}(1+\frac{\pi^4}{45})^{s-\mu} + \\ \qquad \frac{2}{n}\sum\limits_{k=1}^{\frac{n}{2}-1}\prod\limits_{\nu=1}^{s}(1-\frac{2\pi^4}{3} B_4(\{\frac{kh_\nu}{n}\})) - 1, \text{ if } 2 \mid n, \end{cases}$$

with $B_4(x) := x^4 - 2x^3 + x^2 - \frac{1}{30}$, the fourth Bernoulli polynomial, and μ the number of odd integers of h_ν $(1 \leq \nu \leq s)$.

4. Implementation of algorithms

PVM version 3.1 is a widely excepted software system, which is a byproduct of the heterogeneous network research project going on at Oak Ridge National Laboratory and the University of Tennessee. For more information see PVM 3.0 User's Guide and Reference Manual [2], the versions 3.0 and 3.1 don't differ for the user.

In 1986 Monika Feldbacher has programmed the algorithms in [1] in FORTRAN-77 and although the codes would run faster nowadays, the needed computing time for high dimensions and large nodes isn't very reasonable. The codes consisting of do-loops, which represent large sums and minimal-searching over large intervals, are predestined for parallelization. By parting up the computation it is necessary to make sure that the needed data in the isolated parts are independent. So the codes were worked over and parallelism was installed at the outmost possible loop. That means, the computation of that do-loop is distributed in equal parts onto the available machines and with their local results the global solution is formed. In the algorithms K1 and K2 the outmost possible loop is the do-loop which contains the searching process for the minimum of $H(z)$ and $\tilde{H}(z)$ on $\{1, 2, \ldots, p-1\}$. Similar in K3 it is the do-loop which includes the minimum search of h_ν for fixed ν on $\{0,1\}^s$. By the last two methods we parted each time the computation of $h_{r\nu}$. The calculations for the functions W_2 and W_4 were parallelized at the great sums.

5. Computational Results

The following tables show the optimal coefficients computed with the algorithms K1, K2, ..., K5 and the gratis lattice-points. The coefficients $(a_1,\ a_2, \ldots,\ a_s)$ for the dimensions $s = 2,\ 3, \ldots,\ 9$ and the qualifiers $W_2(n\ , a_1, \ldots,\ a_s)$ and $W_4(n\ , a_1, \ldots,\ a_s)$ are ordered in groups of similar nodes n, in the groups the qualifier $W_2(n\ , a_1, \ldots,\ a_s)$ is determining the succession. Algorithm K1 was only carried out up to dimension 5 because of the enormous time used. Some results of method K4 in dimension 10 are appended.

The algorithms K3, K4, K5 and ZG produce good lattice points modulo $n = 2^\nu$. We choose the primes p and $p_1,\ p_2$ for the methods K1 and K2 in such a way that p and p_1p_2 are near to 2^ν:

ν	$n = 2^{\nu}$	p	p_1	p_2	$p_1 \cdot p_2$
10	1024	1021	97	11	1067
11	2048	2053	157	13	2041
12	4096	4093	241	17	4097
13	8192	8191	431	19	8191
14	16384	16381	563	29	16327
15	32768	32771	1129	29	32741
16	65536	65537	1597	41	65477
17	131072	131071	2789	47	131083
18	262144	262139	3917	67	262439
18	262144	262139	4297	61	262177
19	524288	524287	6637	79	52432
20	1048576	1048573	10181	103	1048643
21	2097152		16007	131	2096917
22	4194304		25733	163	4194479
23	8388608		37619	223	8389037
24	16777216		66841	251	16777091

The computing time of the gratis lattice-points is so small, that we never mention. For the computation we take the rational numbers $r_i := \frac{p_i}{p_{s+1}}$ $1 \le i \le s$ with p_i the i-th prime and we take the next odd integer to $\theta_i N$ modulo N to get better qualifiers W_2 and W_4.

For the computations 10 DECstations 5000/33, 7 DECs 3000 AXP/400 and a Silicon Graphics Iris INDIGO were used, but they weren't always all in action. The last table contains some of the execution times for the dimensions $s = 4, ..., 9$ and should only gave a hint, because the elapsed time is dependent on the CPU-time other users needed and could therefore vary. For dimension $s = 2, 3$ the computing time was always lower than 30 seconds. The elapsed time is denoted in hours: minutes: seconds.

The subsequent tables of optimal coefficients contain some hitherto unknown coefficients for some N and dimension s.

Dimension $s = 2$ $\quad a = (a_1, a_2)$

Alg.	n	a_1	a_2	$W_2(n,a)$	$W_4(n,a)$
K1	1021	1	374	0.194356×10^{-3}	0.115233×10^{-8}
K2	1067	108	482	0.196192×10^{-3}	0.161260×10^{-8}
ZG	1024	503	841	0.254934×10^{-3}	0.389121×10^{-8}
K3	1024	485	959	0.327062×10^{-3}	0.840364×10^{-8}
K4	1024	1	165	0.327062×10^{-3}	0.840364×10^{-8}
K5	1024	1	165	0.327062×10^{-3}	0.840364×10^{-8}
K1	2053	1	468	0.536956×10^{-4}	0.924480×10^{-10}
K2	2041	108	482	0.100870×10^{-3}	0.952935×10^{-9}
K3	2048	1551	1443	0.126762×10^{-3}	0.351612×10^{-8}
K4	2048	1	1189	0.126762×10^{-3}	0.351612×10^{-8}
ZG	2048	1007	1683	0.290705×10^{-3}	0.269507×10^{-7}
K5	2048	1	1957	0.134353×10^{-2}	0.519008×10^{-6}
K1	4093	1	1715	0.139255×10^{-4}	0.465628×10^{-11}
ZG	4096	2015	3367	0.194784×10^{-4}	0.213256×10^{-10}
K3	4096	2085	1409	0.195680×10^{-4}	0.224807×10^{-10}
K4	4096	1	3237	0.195680×10^{-4}	0.224814×10^{-10}
K5	4096	1	165	0.923135×10^{-4}	0.272159×10^{-8}
K2	4097	108	482	0.113909×10^{-1}	0.259174×10^{-4}
K1	8191	1	2431	0.383950×10^{-5}	0.353051×10^{-12}
K3	8192	373	1903	0.606922×10^{-5}	0.306999×10^{-11}
K4	8192	1	3237	0.606922×10^{-5}	0.307243×10^{-11}
K2	8189	108	482	0.101988×10^{-4}	0.125904×10^{-10}
ZG	8192	4029	6735	0.153432×10^{-4}	0.446518×10^{-10}
K5	8192	1	8101	0.329033×10^{-3}	0.311971×10^{-7}
K1	16381	1	6789	0.105940×10^{-5}	0.290878×10^{-13}
K4	16384	1	11429	0.126114×10^{-5}	0.630607×10^{-13}
K3	16384	4605	5137	0.126114×10^{-5}	0.632827×10^{-13}
K2	16327	108	482	0.174320×10^{-5}	0.240918×10^{-12}
K5	16384	1	14501	0.216837×10^{-5}	0.456746×10^{-12}
ZG	16384	8059	13469	0.296378×10^{-5}	0.167910×10^{-11}
K1	32771	1	12714	0.263451×10^{-6}	0.155431×10^{-14}
K3	32768	12645	21953	0.354180×10^{-6}	0.532907×10^{-14}
K4	32768	1	27813	0.354180×10^{-6}	0.888178×10^{-14}
K5	32768	1	12453	0.571484×10^{-6}	0.370814×10^{-13}
K2	32741	108	482	0.629383×10^{-6}	0.426326×10^{-13}
ZG	32768	16117	26939	0.911803×10^{-6}	0.103029×10^{-12}

Dimension $s = 3$ $\qquad a = (a_1, a_2, a_3)$

Alg.	n	a_1	a_2	a_3	$W_2(n, a)$	$W_4(n, a)$
K1	1021	1	315	188	0.610325×10^{-2}	0.731485×10^{-6}
K2	1067	108	482	576	0.742573×10^{-2}	0.244598×10^{-5}
K3	1024	869	63	103	0.119105×10^{-1}	0.113576×10^{-4}
ZG	1024	339	547	43	0.156033×10^{-1}	0.319418×10^{-4}
K4	1024	1	421	729	0.398449×10^{-1}	0.494181×10^{-3}
K5	1024	1	165	793	0.754527×10^{-1}	0.157446×10^{-2}
K1	2053	1	430	130	0.195723×10^{-2}	0.717199×10^{-7}
K4	2048	1	1701	473	0.369931×10^{-2}	0.100883×10^{-5}
K3	2048	1829	1919	1447	0.396360×10^{-2}	0.145510×10^{-5}
K2	2041	108	482	542	0.666349×10^{-2}	0.501228×10^{-5}
ZG	2048	677	1095	87	0.948541×10^{-2}	0.142899×10^{-4}
K5	2048	1	1957	277	0.171903×10^{-1}	0.970763×10^{-4}
K1	4093	1	1884	825	0.582591×10^{-3}	0.729991×10^{-8}
K3	4096	2529	123	3443	0.592909×10^{-3}	0.633927×10^{-8}
ZG	4096	1355	2191	175	0.715942×10^{-3}	0.130445×10^{-7}
K4	4096	1	3749	3033	0.116360×10^{-2}	0.126849×10^{-6}
K5	4096	1	165	2565	0.688063×10^{-2}	0.190669×10^{-4}
K2	4097	108	482	4074	0.118056×10^{-1}	0.259227×10^{-4}
K1	8191	1	739	5515	0.166141×10^{-3}	0.394274×10^{-9}
K3	8192	7049	2783	4019	0.235955×10^{-3}	0.209458×10^{-8}
ZG	8192	2709	4383	351	0.393451×10^{-3}	0.107410×10^{-7}
K4	8192	1	3237	5593	0.440828×10^{-3}	0.318571×10^{-7}
K5	8192	1	8101	6537	0.179294×10^{-2}	0.814169×10^{-6}
K2	8189	108	482	5518	0.487685×10^{-2}	0.861296×10^{-5}
K1	16381	1	6943	12347	0.515469×10^{-4}	0.350573×10^{-10}
K3	16384	89	4735	731	0.929320×10^{-4}	0.662460×10^{-9}
ZG	16384	5419	8767	701	0.102845×10^{-3}	0.672523×10^{-9}
K4	16384	1	11429	13785	0.116047×10^{-3}	0.110965×10^{-8}
K5	16384	1	14501	5485	0.126560×10^{-3}	0.118335×10^{-8}
K2	16327	108	482	15795	0.318244×10^{-1}	0.488401×10^{-3}
K1	32771	1	3586	13164	0.152954×10^{-4}	0.310219×10^{-11}
K3	32768	20133	16127	30759	0.213328×10^{-4}	0.924927×10^{-11}
ZG	32768	10837	17533	1401	0.397836×10^{-4}	0.876357×10^{-10}
K2	32741	108	482	27055	0.587101×10^{-4}	0.547748×10^{-9}
K4	32768	1025	165	6617	0.662303×10^{-4}	0.684568×10^{-9}
K5	32768	1	12453	8181	0.112345×10^{-3}	0.225675×10^{-8}

Dimension $s = 4$ $a = (a_1, a_2, a_3, a_4)$

Alg.	n	a_1	a_2	a_3	a_4	$W_2(n,a)$	$W_4(n,a)$
K1	2053	1	960	1856	1809	0.302546×10^{-1}	0.937361×10^{-5}
K2	2041	170	1948	303	1383	0.378095×10^{-1}	0.359291×10^{-4}
K3	2048	1901	1321	703	1371	0.387300×10^{-1}	0.253972×10^{-4}
K4	2048	417	133	1657	397	0.450393×10^{-1}	0.613866×10^{-4}
ZG	2048	490	643	1179	1821	0.515407×10^{-1}	0.108098×10^{-3}
K5	2048	1	1957	277	273	0.425478	0.327601×10^{-1}
K1	4093	1	1238	1862	797	0.101771×10^{-1}	0.103172×10^{-5}
K3	4096	3373	2661	1535	23	0.136748×10^{-1}	0.385732×10^{-5}
K2	4097	258	2128	751	3590	0.154592×10^{-1}	0.535198×10^{-5}
K4	4096	545	2821	2041	525	0.166306×10^{-1}	0.116085×10^{-4}
ZG	4096	817	1285	2357	3643	0.179370×10^{-1}	0.214503×10^{-4}
K5	4096	1	165	2565	281	0.199139×10^{-1}	0.236231×10^{-4}
K1	8191	1	2488	5939	7859	0.375607×10^{-2}	0.166362×10^{-6}
K2	8189	450	6773	3473	5668	0.467299×10^{-2}	0.417007×10^{-6}
K4	8192	2369	4773	6681	2221	0.570976×10^{-2}	0.875112×10^{-6}
K5	8192	1	8101	6537	7129	0.599422×10^{-2}	0.129672×10^{-5}
K3	8192	7789	3557	1983	3239	0.687588×10^{-2}	0.214116×10^{-5}
ZG	8192	1633	2569	4715	7287	0.980895×10^{-2}	0.869180×10^{-5}
K1	16381	1	2994	3629	4623	0.135002×10^{-2}	0.285701×10^{-7}
K2	16327	592	8755	10361	11221	0.228213×10^{-2}	0.260240×10^{-6}
K4	16384	2593	11013	6137	2061	0.234854×10^{-2}	0.262061×10^{-6}
K3	16384	1005	13237	7039	4183	0.345139×10^{-2}	0.137009×10^{-5}
K5	16384	1	14501	5485	15373	0.388405×10^{-2}	0.135706×10^{-5}
ZG	16384	3267	5137	9429	14575	0.611714×10^{-2}	0.551483×10^{-5}
K1	32771	1	3015	12658	18426	0.377521×10^{-3}	0.118885×10^{-8}
K2	32741	1158	3403	2734	25451	0.755661×10^{-3}	0.375441×10^{-7}
K5	32768	1	12453	8181	12733	0.105829×10^{-2}	0.814363×10^{-7}
K4	32768	25633	9037	649	10501	0.127930×10^{-2}	0.378605×10^{-6}
K3	32768	14957	18917	7103	16551	0.212116×10^{-2}	0.119873×10^{-5}
ZG	32768	6533	10275	18857	29151	0.319720×10^{-2}	0.326263×10^{-5}
K3	65536	18137	19393	41115	50195	0.253811×10^{-3}	0.192751×10^{-8}
K2	65477	1638	40359	12618	26684	0.303393×10^{-3}	0.356208×10^{-8}
K5	65536	1	53413	28761	23313	0.312315×10^{-3}	0.467638×10^{-8}
K4	65536	23153	1109	57801	35805	0.356241×10^{-3}	0.112559×10^{-7}
ZG	65536	13067	20549	37713	58301	0.857340×10^{-3}	0.593646×10^{-7}

Dimension $s = 5$ $a = (a_1, a_2, a_3, a_4, a_5)$

Alg.	n	a_1	a_2	a_3	a_4	a_5	$W_2(n,a)$	$W_4(n,a)$
K1	8191	1	1386	4302	7715	3735	0.425885×10^{-1}	0.995767×10^{-5}
ZG	8192	1363	2127	3843	5843	2709	0.542835×10^{-1}	0.230092×10^{-3}
K4	8192	3409	7413	5801	3965	961	0.550679×10^{-1}	0.317486×10^{-4}
K2	8189	450	8081	5922	2837	6853	0.579493×10^{-1}	0.652691×10^{-4}
K3	8192	5077	5773	7177	2481	4129	0.655358×10^{-1}	0.128712×10^{-3}
K5	8192	1	6557	1701	733	4669	0.108243	0.176681×10^{-2}
K1	16381	1	7449	5154	11463	10115	0.196388×10^{-1}	0.329361×10^{-5}
K2	16327	592	10818	4077	10352	8248	0.236913×10^{-1}	0.102074×10^{-4}
K3	16384	4829	4205	15897	4197	15681	0.262677×10^{-1}	0.194079×10^{-4}
ZG	16384	2725	4253	7685	11687	5417	0.300346×10^{-1}	0.122440×10^{-3}
K5	16384	1	3209	3613	4669	6365	0.301998×10^{-1}	0.118445×10^{-3}
K4	16384	8817	12949	11977	7709	14049	0.324480×10^{-1}	0.286322×10^{-4}
K2	32741	1158	21517	28492	413	24610	0.786604×10^{-2}	0.100258×10^{-5}
K4	32768	23041	11877	26585	5741	32113	0.998355×10^{-2}	0.222924×10^{-5}
K5	32768	1	3209	13289	9469	13565	0.137982×10^{-1}	0.584361×10^{-4}
K3	32768	4301	19733	4617	12257	2225	0.150898×10^{-1}	0.161194×10^{-4}
K1	32771	1	14887	25267	4291	9440	0.158324×10^{-1}	0.931575×10^{-5}
ZG	32768	5449	8505	15369	23375	10835	0.277742×10^{-1}	0.114417×10^{-3}
K2	65477	1638	37126	10583	16655	51627	0.268882×10^{-2}	0.486649×10^{-7}
K5	65536	1	22165	38665	47157	753	0.335128×10^{-2}	0.156231×10^{-6}
K1	65537	1	21178	37993	18005	15616	0.446423×10^{-2}	0.131361×10^{-5}
ZG	65536	10899	17011	30739	46751	21671	0.781832×10^{-2}	0.268970×10^{-4}
K4	65536	1569	3845	11257	57869	44177	0.856665×10^{-2}	0.105015×10^{-4}
K3	65536	45357	23157	28261	52329	59137	0.114064×10^{-1}	0.158005×10^{-4}
K2	131083	2836	78404	78917	95318	44745	0.138564×10^{-2}	0.126961×10^{-6}
K1	131071	1	17518	43113	22432	12918	0.182845×10^{-2}	0.139725×10^{-6}
ZG	131072	21799	34023	61479	93503	43341	0.254071×10^{-2}	0.131978×10^{-4}
K5	131072	1	63125	91569	51081	76661	0.279375×10^{-2}	0.374621×10^{-6}
K3	131072	101757	15097	119057	129269	130985	0.611459×10^{-2}	0.125612×10^{-4}
K4	131072	73985	76133	111577	37741	100721	0.723521×10^{-2}	0.104696×10^{-4}
K2	262439	3984	183671	49095	177114	178734	0.360927×10^{-3}	0.168888×10^{-8}
K3	262144	72285	181221	174785	258029	61337	0.630115×10^{-3}	0.171929×10^{-7}
ZG	262144	43597	68045	122959	187005	86681	0.820409×10^{-3}	0.605639×10^{-5}
K5	262144	1	106405	114077	130653	163997	0.985996×10^{-3}	0.705109×10^{-7}
K4	262144	3337	254669	43681	206981	142897	0.149599×10^{-2}	0.248633×10^{-6}
K1	262139	1	111802	113267	66322	65536	0.594903×10^{-2}	0.125567×10^{-4}

Dimension $s = 6$ $\quad a = (a_1, a_2, a_3, a_4, a_5, a_6)$

Alg.	n	a_1	a_2	a_3	a_4	a_5	a_6	$W_2(n,a)$	$W_4(n,a)$
K2	16327	592	13592	13325	8298	3419	1938	0.171669	0.246730×10^{-3}
ZG	16384	2045	3163	5603	8347	14909	2431	0.211254	0.818063×10^{-3}
K3	16384	5581	7445	837	585	6641	10849	0.226539	0.188821×10^{-2}
K4	16384	11777	4709	985	2973	6513	4205	0.246549	0.335430×10^{-2}
K5	16384	1	14501	5045	117	8085	6181	0.700457	0.125873
K3	32768	17485	27477	24133	27081	5361	16801	0.777457×10^{-1}	0.563173×10^{-4}
K2	32741	1158	13502	29972	27089	20956	20640	0.817558×10^{-1}	0.129925×10^{-3}
ZG	32768	4091	6325	11205	16695	29817	4861	0.845109×10^{-1}	0.228635×10^{-3}
K4	32768	10849	9413	14969	12045	3729	16117	0.130552	0.823676×10^{-3}
K5	32768	1	12453	21613	24837	24641	257	0.135095	0.404274×10^{-3}
K3	65536	4297	33777	5965	161	34389	10565	0.333643×10^{-1}	0.172426×10^{-4}
K2	65477	1638	5053	60958	6646	2194	57295	0.380756×10^{-1}	0.404476×10^{-4}
ZG	65536	8181	12649	22409	33389	59633	9723	0.510175×10^{-1}	0.812690×10^{-4}
K5	65536	1	22165	38665	47157	753	8221	0.683296×10^{-1}	0.357846×10^{-3}
K4	65536	19849	53077	56113	36877	19873	55005	0.806827×10^{-1}	0.435587×10^{-3}
K3	131072	89677	84937	58709	22597	79601	50081	0.115842×10^{-1}	0.238096×10^{-5}
K4	131072	44049	71477	5289	51965	49473	16997	0.162517×10^{-1}	0.606151×10^{-5}
ZG	131072	16363	25297	44819	66779	119265	19447	0.290463×10^{-1}	0.104137×10^{-3}
K5	131072	1	63125	91569	51081	76661	48757	0.298437×10^{-1}	0.476054×10^{-4}
K2	131083	2836	2970	97956	77625	57720	20880	0.704613×10^{-1}	0.155056×10^{-2}
K5	262144	1	106405	114077	130653	163997	179117	0.625370×10^{-2}	0.530818×10^{-6}
K3	262144	245881	117649	39421	70241	255109	173581	0.694143×10^{-2}	0.271719×10^{-5}
ZG	262144	32727	50593	89637	133557	238529	38893	0.711313×10^{-2}	0.293182×10^{-5}
K4	262144	87521	137925	135353	172541	110929	40909	0.734041×10^{-2}	0.278959×10^{-5}
K2	262439	3984	157684	57372	193366	127408	193400	0.198553×10^{-1}	0.185365×10^{-4}
K2	524323	6716	497120	511859	383691	160536	270675	0.258677×10^{-2}	0.173181×10^{-6}
K5	524288	1	368549	343057	30813	147053	93	0.292662×10^{-2}	0.188932×10^{-6}
K4	524288	393249	92741	346617	219149	444233	453333	0.591851×10^{-2}	0.416045×10^{-5}
K3	524288	444357	32249	475061	109373	251041	236113	0.742653×10^{-2}	0.739480×10^{-5}
ZG	524288	65455	101187	179277	267113	477057	77785	0.163522×10^{-1}	0.975857×10^{-4}

Dimension $s = 7$ $a = (a_1, a_2, a_3, a_4, a_5, a_6, a_7)$

Alg.	n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	$W_2(n,a)$	$W_4(n,a)$
ZG	262144	29099	44837	78913	116773	205563	257479	117097	0.422081×10^{-1}	0.114964×10^{-4}
K3	262144	184301	139573	7973	52825	207529	57937	120705	0.436471×10^{-1}	0.132460×10^{-4}
K5	262144	1	106405	98973	98141	245661	82365	31269	0.464183×10^{-1}	0.171497×10^{-4}
K4	262144	16897	235877	83737	24685	35441	219541	79913	0.471440×10^{-1}	0.466863×10^{-4}
K2	262439	3984	141891	170492	186993	201780	16078	229056	0.652322×10^{-1}	0.123077×10^{-3}
ZG	524288	58197	89675	157827	233545	411125	514959	234195	0.179331×10^{-1}	0.722009×10^{-5}
K3	524288	71005	519153	91341	483285	240581	138529	270665	0.213909×10^{-1}	0.800510×10^{-5}
K4	524288	190497	197637	8249	384013	319121	387637	168777	0.237077×10^{-1}	0.119522×10^{-4}
K2	524323	6716	30495	395554	11219	341755	17934	272400	0.296976×10^{-1}	0.262876×10^{-4}
K5	524288	1	499621	522749	61437	278433	106605	454609	0.299673×10^{-1}	0.392768×10^{-4}
ZG	1048576	116395	179351	315655	467089	822251	1029919	468391	0.855206×10^{-2}	0.297270×10^{-5}
K3	1048576	856029	784237	315839	931147	768345	993957	939433	0.926367×10^{-2}	0.322789×10^{-5}
K4	1048576	465537	446309	10137	749101	879153	358421	148969	0.107884×10^{-1}	0.326862×10^{-5}
K2	1048643	10284	276818	256724	315421	116039	729621	40671	0.163663×10^{-1}	0.319788×10^{-4}
K5	1048576	1	1032805	210981	6609	16785	1030237	735277	0.194293×10^{-1}	0.328967×10^{-4}
K4	2097152	1647809	1145509	2025433	1710957	655473	1224533	424489	0.270039×10^{-2}	0.752354×10^{-6}
K2	2096917	16138	506559	370591	1517122	797038	1683321	1620108	0.385723×10^{-2}	0.341807×10^{-6}
K3	2097152	870067	693315	14857	836513	1214075	2027413	303175	0.454926×10^{-2}	0.194883×10^{-5}
K5	2097152	1	155557	1251081	1285257	163961	15365	1245189	0.904703×10^{-2}	0.619380×10^{-5}
ZG	2097152	232791	358703	631311	934177	1644501	2059837	936781	0.346080×10^{-1}	0.489100×10^{-3}
K3	4194304	479079	3405803	2808241	298721	2142451	2047389	683191	0.154815×10^{-2}	0.634054×10^{-7}
K5	4194304	1	3039141	3653513	1164301	2957193	1805581	1613821	0.155106×10^{-2}	0.438380×10^{-6}
ZG	4194304	465581	717407	1262621	1868355	3289003	4119673	1873561	0.166429×10^{-2}	0.404069×10^{-6}
K4	4194304	676945	3831797	920233	3741613	401081	1448701	20033	0.180359×10^{-2}	0.449093×10^{-6}
K2	4194479	25896	1198382	1271701	2788681	3117842	2118998	3671507	0.294090×10^{-2}	0.470375×10^{-6}
K3	8388608	7877401	7900617	4618289	5104993	4701299	7742947	2641771	0.500959×10^{-3}	0.176190×10^{-6}
K4	8388608	6966817	6093189	5793337	7742797	416721	3601205	2499721	0.632900×10^{-3}	0.179488×10^{-6}
ZG	8388608	931161	1434813	2525241	3736711	6578007	8239345	3747123	0.212816×10^{-2}	0.740405×10^{-6}
K5	8388608	1	6409373	58549	4364093	4374789	1706237	4122397	0.218390×10^{-2}	0.752244×10^{-6}
K2	8389037	37842	1456777	580357	7166534	2103873	2477945	5007662	0.105913×10^{-1}	0.219093×10^{-4}

Dimension $s = 8$ $a = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8)$

Alg.	n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	$W_2(n,a)$	$W_4(n,a)$
K4	262144	212993	116581	141593	250477	34161	113949	175209	170709	0.296154	0.812395×10^{-3}
K5	262144	1	147365	141109	112885	17029	100413	73773	104453	0.307372	0.731622×10^{-3}
ZG	262144	23815	36523	63657	93255	160765	199185	24671	74543	0.339754	0.219127×10^{-2}
K2	262439	3984	84230	165321	162964	34216	48853	2460	177842	0.465667	0.918849×10^{-2}
K3	262144	49267	247747	43157	218373	60169	184135	216849	73441	0.844951	0.127148
ZG	524288	47631	73045	127313	186511	321529	398369	49343	149087	0.126096	0.854235×10^{-4}
K3	524288	128293	455169	417241	259863	9827	486611	395467	126033	0.157319	0.437420×10^{-3}
K5	524288	1	425893	397821	223677	40317	458845	170365	398717	0.180193	0.766416×10^{-3}
K4	524288	116017	40341	506953	102413	408993	118533	334649	201725	0.203632	0.183378×10^{-2}
K2	524323	6716	127519	102000	353747	432725	104984	325147	57880	0.211757	0.197375×10^{-2}
K2	1048643	10284	322427	333422	302183	138006	815700	958330	811064	0.536047×10^{-1}	0.150620×10^{-4}
K3	1048576	942969	663081	791485	679471	1022907	257611	634515	741121	0.620820×10^{-1}	0.377066×10^{-4}
K5	1048576	1	16997	390453	65081	540677	487489	555469	137717	0.655352×10^{-1}	0.454880×10^{-4}
K4	1048576	322305	673381	581657	533869	76657	256661	444713	156893	0.689618×10^{-1}	0.117688×10^{-3}
ZG	1048576	95263	146091	254627	373021	643057	796739	98685	298175	0.831835×10^{-1}	0.109027×10^{-3}
ZG	2097152	190525	292183	509253	746043	1286113	1593479	197371	596351	0.274647×10^{-1}	0.101622×10^{-4}
K4	2097152	1046561	1266693	1034297	883725	2094737	1437237	2007881	481661	0.279350×10^{-1}	0.108671×10^{-4}
K3	2097152	175121	1338209	629109	1519239	1483411	1692311	1368067	1674373	0.287148×10^{-1}	0.134181×10^{-4}
K5	2097152	1	155557	1251081	1285257	163961	15365	1245189	990269	0.287987×10^{-1}	0.937527×10^{-5}
K2	2096917	16138	1008453	1535329	327715	1333185	1512254	776232	1473139	0.618323×10^{-1}	0.498295×10^{-3}
ZG	4194304	381049	584365	1018505	1492085	2572225	3186959	394743	1192703	0.108987×10^{-1}	0.291031×10^{-5}
K5	4194304	1	3039141	3653513	1164301	2957193	1805581	1613821	11281	0.110366×10^{-1}	0.167401×10^{-5}
K3	4194304	2977721	3046633	840203	2782737	2604739	2926107	528557	3652737	0.132963×10^{-1}	0.489746×10^{-5}
K4	4194304	3851337	2235853	2201057	2884805	1187697	4131733	460473	2641693	0.193462×10^{-1}	0.320696×10^{-4}
K2	4194479	25896	2099037	2918613	112449	1187868	1572768	1226681	2807264	0.326233×10^{-1}	0.128216×10^{-3}
K2	8389037	37842	4948823	994576	4006707	3440945	1101328	1769666	3598508	0.474585×10^{-2}	0.230669×10^{-6}
ZG	8388608	762099	1168731	2037009	2984169	5144451	6373919	789485	2385405	0.717370×10^{-2}	0.403383×10^{-5}
K4	8388608	5384457	3368149	5114737	6604573	8114913	925125	4450073	4644237	0.755876×10^{-2}	0.330635×10^{-5}
K3	8388608	5769049	4731497	3441105	3536731	1699147	6468947	7952099	4870017	0.763772×10^{-2}	0.369525×10^{-5}
K5	8388608	1	5143453	6152093	167689	7353253	2099193	4589321	2162441	0.108243×10^{-1}	0.754505×10^{-5}

Dimension $s = 9$ $a = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9)$

Alg.	n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	$W_2(n,a)$	$W_4(n,a)$
K3	2^{20}	837533	301325	907977	895829	358897	868677	922777	41593	875617	0.373147	0.588061×10^{-3}
K5	2^{20}	1	630693	954397	853233	262849	773569	621889	58 9837	760829	0.415121	0.229779×10^{-2}
ZG	2^{20}	74867	114283	197309	286265	483683	593085	835881	970431	220459	1.02696	0.128231
K4	2^{20}	90209	627077	625209	953165	350345	697277	666929	368021	448465	1.13605	0.153801
K5	2^{21}	1	368549	892189	1416413	457981	1415677	1568253	1842173	1309789	0.172167	0.226485×10^{-3}
K4	2^{21}	2007585	693317	950393	1316365	399505	1581653	664777	1833853	1983073	0.192896	0.606365×10^{-3}
K3	2^{21}	654461	1222701	279925	2008421	2087173	109881	408553	29585	433345	0.193591	0.375805×10^{-3}
ZG	2^{21}	149735	228565	394619	572531	967367	1186171	1671763	1940861	440917	0.236148	0.544426×10^{-3}
K4	2^{22}	1394961	198133	1542761	1827645	14561	423813	4108473	1535629	2060033	0.688968×10^{-1}	0.212918×10^{-4}
K5	2^{22}	1	499621	810621	417917	532373	1721253	486917	235 0333	3211277	0.738525×10^{-1}	0.285162×10^{-4}
K3	2^{22}	3206717	2689325	2173673	1005301	152145	2753689	3017629	2445093	3847553	0.825607×10^{-1}	0.821621×10^{-4}
ZG	2^{22}	299471	457131	789239	1145061	1934733	2372343	3343527	3881721	881833	0.163789	0.179258×10^{-2}
K4	2^{23}	3684033	7307109	7534617	7256621	3477297	6769525	6989929	19869	7884033	0.404072×10^{-1}	0.250335×10^{-4}
K5	2^{23}	1	4144029	2850301	4195969	4718213	458677	2114557	160 9861	5930493	0.426102×10^{-1}	0.201738×10^{-4}
K3	2^{23}	1015755	6330119	7714633	1768917	4143923	766801	4639045	4660355	8159009	0.487510×10^{-1}	0.105496×10^{-3}
ZG	2^{23}	598941	914261	1578479	2290123	3869467	4744685	6687053	7763443	1763665	0.735603×10^{-1}	0.239920×10^{-3}
K4	2^{24}	4177921	1258917	12873561	1485421	1942641	16487861	7865257	2165213	13411905	0.167404×10^{-1}	0.284848×10^{-5}
K3	2^{24}	3916535	7811043	5576625	11311841	15350875	14597005	2314521	15567211	6754817	0.198090×10^{-1}	0.621430×10^{-5}
ZG	2^{24}	1197881	1828523	3156959	4580245	7738935	9489371	13374105	15526885	3527329	0.308881×10^{-1}	0.100441×10^{-3}

K4 Dimension $s = 10$ $a = (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9, a_{10})$

n	a_1	a_2	a_3	a_4	a_5	a_6	a_7	a_8	a_9	a_{10}
2^{23}	7631113	4892053	4713585	3181533	119297	7682981	5530329	580941	2295457	3080005
2^{24}	5522065	9274805	2619689	12986621	11172257	2449221	11957113	10964077	16217537	14225765
2^{25}	30878929	26902781	21617209	31923829	6215777	11900101	9702697	30296877	4177217	7704229
2^{26}	29488641	6610853	16019289	63964269	58775409	8766389	3606953	5559261	56869697	9630949
2^{27}	125327393	46479941	40424953	37655053	91023761	108873301	77615817	120876413	50717633	85339109

n	$W_2(n,a)$	$W_4(n,a)$
2^{23}	0.169763	0.701971×10^{-4}
2^{24}	0.842269×10^{-1}	0.323656×10^{-4}
2^{25}	0.690556×10^{-1}	0.149504×10^{-3}
2^{26}	0.719413×10^{-1}	0.154757×10^{-2}
2^{27}	0.863297×10^{-2}	0.103303×10^{-5}

dim. and n	alg.	elapsed time	number of machines
	K1	2:06:18	11
$s=4$	K2	22	8
$n \approx 2^{16}$	K3	24	6
	K4	7	11
	K5	1:51	9
	K1	16:14:08	15
$s=5$	K2	2:21	10
$n \approx 2^{18}$	K3	1:01	10
	K4	12	10
	K5	10 :01	10
	K2	2:47	9
$s=6$	K3	2:17	18
$n \approx 2^{18}$	K4	13	6
	K5	13:51	10
	K2	4:42:05	15
$s=7$	K3	2:47:50	11
$n \approx 2^{23}$	K4	7:08	11
	K5	9:45:34	15
	K2	13:23:50	15
$s=8$	K3	9:02:24	15
$n \approx 2^{24}$	K4	9:27	15
	K5	25:56:51	15
$s=9$	K3	20:22:12	7
$n=2^{24}$	K4	18:48	7
$s=10$			
$n=2^{27}$	K4	2:09:42	16

6. Conclusion

The resulting optimal coefficients for large dimensions and nodes confirm the parallelization of the Korobov's algorithms with the software package PVM, which is a defacto standard in distributed computing. PVM delivers the inexpensive virtual machine, which appears as large memory computer, by exploiting the existing infrastructure of computer networks. The speed-up is quite linear, although the CPU-time other users needed causes some troubles. PVM, which is a byproduct of the heterogeneous network research project going on at Oak Ridge National Laboratory and the University of Tennesse, continues to evolve to meet the needs of users and to keep up with the changing technology. For example, a sophisticated scheduling software is one focus of future research [3].

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