Testing the normal crossing property of hypersurfaces

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Abstract

Given finitely many hypersurfaces, in some nonsingular ambient variety W. They are normal crossing if each of their irreducible components are nonsingular, and when r such irreducible components meet at a point $a \in W$, their local defining equations form a part of a regular system of parameters at a. We give a straightforward algorithm to test this property. The test relies on partial differentiation over W and Gröbner basis computation.

1 Introduction

The normal crossing property has a prominent role in the problem of resolution of singularities (see e.g. [5] for the foundations of the classical theory, or [4] for a more recent survey on the field). The definition of embedded resolution of singularities requires that a desingularization morphism $\pi : W' \to W$ of an embedded variety $X \subset W$ has the property that the irreducible components of $\pi^{-1}(S)$ (also called as the set of exceptional divisors) are normal crossing, where $S \subset X$ is the set of singularities of X.

In the resolution algorithm of Villamayor (for the theoretical presentation see [3, 6], for the implementation see [1, 2]) it is essential to maintain the normal crossing property in the intermediate steps, when the desingularization map π is constructed as the composition of well chosen elementary transformations (called blowing ups). The simplification of this subproblem motivated the implementation of the normal crossing test, which allows the algorithm to reduce the number of blowing ups significantly. The integration of the test into the theory of Villamayor is beyond the scope of this presentation, and is planned to be described in an upcoming paper.

The normal crossing test itself is a straightforward application of the available computer algebra machinery on the problem. I am aware of the fact that the difficulty of the solution falls into the category of the textbook exercises; still, it might be worthwhile to be documented, since the normal crossing property is ubiquitous in resolution of singularities and in singularity theory, and the solution provides a first constructive approach to the problem.

2 The normal crossing test

Let k be an algebraically closed field of characteristic zero, let W be a nonsingular affine variety of dimension n with coordinate ring $k[W] = R/J$, where $R = k[x_1, \ldots, x_m]$, $0 \neq J \subset R$. Let $\partial_{p_1}, \ldots \partial_{p_n} \in \mathrm{Der}(W)$ be derivations on W, coming from a basis $dp_1, \ldots, dp_n$ of the module differential forms $\Omega(W)$.

Let $X_1, \ldots, X_s \subset W$ be defined by $f_1, \ldots, f_s \in k[W]$ respectively. The set of singularities of the hypersurface $X_i \subset W$ is

$$\mathrm{Sing}(X_i) = \{a \in X_i \mid \partial_{p_j} f_i(a) = 0,\ j = 1, \ldots, n\}.$$

Definition 1. The hypersurfaces $X_1, \ldots, X_s \subset W$ are *normal crossing* if for all i: $\mathrm{Sing}(X_i) = \emptyset$ and for all $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, s\}$: if $a \in X_{i_1} \cap \cdots \cap X_{i_l}$, the elements of $\{f_{i_1}\mathcal{O}_{W,a}, \ldots, f_{i_l}\mathcal{O}_{W,a}\}$ are linearly independent $\bmod \mathfrak{m}_a^2$.

Let us note that the second part of the conjunction in the above definition can be expressed via "Jacobian conditions". We can formulate it as a linear algebra problem with the help of the differential basis $dp_1, \ldots dp_n$.

Proposition 2. *In the context of Definition 1,* $\{f_{i_1}\mathcal{O}_{W,a}, \ldots, f_{i_l}\mathcal{O}_{W,a}\}$ *are linearly independent* $\bmod \mathfrak{m}_a^2$ *iff the* $l \times n$ *matrix*

$$\mathrm{Jacobian}_W(f_{i_1}, \ldots, f_{i_l}) = \begin{pmatrix} \partial_{p_1} f_{i_1} & \ldots & \partial_{p_n} f_{i_1} \\ \vdots & & \vdots \\ \partial_{p_1} f_{i_l} & \ldots & \partial_{p_n} f_{i_l} \end{pmatrix}$$

has maximal rank at a.

Proof. Since $p_1 - p_1(a), \ldots, p_n - p_n(a)$ form a regular system of parameters of $\mathcal{O}_{W,a}$, they also form a vector space basis of $\mathfrak{m}_a\mathcal{O}_{W,a}/\mathfrak{m}_a^2$. With respect to this basis, $\partial_{p_t} f_{i_j}(a)$ is just the t-th coordinate of $f_{i_j}\mathcal{O}_{W,a}/\mathfrak{m}_a^2$. □

Thus we attempt to find points within $X_{i_1} \cap \cdots \cap X_{i_l}$, where the test of Proposition 2 fails. The *Jacobian ideal* of $f_{i_1}, \ldots, f_{i_l}$ is generated by all the minors of maximal size of $\mathrm{Jacobian}_W(f_{i_1}, \ldots, f_{i_l})$. Considering that all the $k[W]$ elements and ideals are given by representatives from R, if

$$\mathtt{gbasis}(\langle f_{i_1}, \ldots, f_{i_l}\rangle + \mathrm{Jacobian\_ideal}_W(f_{i_1}, \ldots, f_{i_l}) + J,\ \mathtt{tdeg}(x_1, \ldots, x_m))$$

contains a nonzero constant, $X_{i_1}, \ldots, X_{i_l}$ are normal crossing *within their intersection*, otherwise they are not (and the result is a Gröbner basis, defining the points where they fail to have the property).

Remark 3. (i) In the above test, we cannot replace $\mathrm{Jacobian\_ideal}_W(f_{i_1}, \ldots, f_{i_l})$ by $\mathrm{Jacobian\_ideal}_W(f_1, \ldots, f_s)$. The geometric reason for this is that the Jacobian ideal defines the "linearly dependent locus" of the vector field given by its input. This could falsify the test when the selected hypersurfaces are normal crossing along their intersection, which happens to have a nonempty intersection with the "linearly dependent locus" defined by all the hypersurfaces.

(ii) We can speed up the computations when $n < l$. In this case, it is enough to check whether the intersection of the hypersurfaces are empty, since at most n intersecting hypersurfaces can have normal crossing in W.

(iii) When $l = 1$, the above computation just checks whether the selected hypersurface is nonsingular.

Because the method checks the normal crossing property for a selected subset of hypersurfaces along their intersection, the complete algorithm has to go through all the nonempty subsets of $X_1, \ldots, X_s$. A reasonable strategy is do the easier tests first, i.e. to check subsets by increasing cardinality.

Input: W a nonsingular affine variety, $(f_1, \ldots, f_s)$ a list of defining equations of hypersurfaces in W.
Output: a boolean indicating the normal crossing property of the input hypersurfaces in W.
isNormalCrossing(W,$(f_1, \ldots, f_s)$)
$n := \dim(W)$; $R/J := k[W]$; $O := \texttt{tdeg}(\mathrm{Generators}(R))$;
for $l = 1$ **to** s **do**
  for each $\{i_1, \ldots, i_l\} \subseteq \{1, \ldots, s\}$ **do**
    if $l \leq n$ **then**
      $G := \texttt{gbasis}(\langle f_{i_1}, \ldots, f_{i_l} \rangle + \mathrm{Jacobian\_ideal}_W(f_{i_1}, \ldots, f_{i_l}) + J, O)$;
    else $G := \texttt{gbasis}(\langle f_{i_1}, \ldots, f_{i_l} \rangle + J, O)$;
    if $\mathrm{NormalForm}(1, G, O) \neq 0$ **then** **return** false;
return true;

Remark 4. (i) The termination of the algorithm is obvious. (ii) The correctness follows from Proposition 2, which implies the correctness of the elementary test for a selected subset of the input hypersurfaces, and the fact that the algorithm performs an exhaustive check. (iii) The algorithm is of exponential complexity if Gröbner basis computation is taken as a primitive step; the worst case bit complexity is double exponential.

References

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