Straightening in The Whitney Algebra of a Matroid

by Henry Crapo and William Schmitt

crapo@ehess.fr — wschmitt@gwu.edu

The concept of a *matroid*, as introduced by Hassler Whitney, captures the abstract combinatorial properties of linear (or even algebraic) dependence of finite configurations of points in projective space. The information carried by the matroid structure is often 'all you need to know' about a configuration of projective points, that is, all its *intrinsic* properties: which sets of points are collinear, coplanar, cospatial, etc. The matroid (that is, the matroid of linear dependence of points) does not, however, carry information about how the configuration has been embedded in projective space. The matroid does not determine whether a given set of points lies together on a non-linear algebraic surface, nor does it predict whether three pairs of points generate lines that are concurrent, unless a point of concurrence happens to be itself a point of the configuration. Such properties could be termed *extrinsic* to the configuration, requiring information about the environment of the figure.

The deepest problems in matroid theory involve the search for combinatorial obstacles to the representability of matroids in projective spaces over specified fields. This is another sign that the gap between intrinsic and extrinsic properties of matroids is highly significant.

The best-known earlier attempts to bridge this gap are the Orlik-Solomon algebra, extensively used to study hyperplane configurations, and Neil White's bracket ring. Both of these constructions start by taking anticommutative products of elements of the matroid.

We propose a further step in this direction, associating with each matroid an appropriately weakened form of Hopf algebra, called a *lax Hopf algebra*, constructed from the free exterior algebra generated by the elements of the matroid, forming tensor products, then taking the quotient by the ideal generated by *homogeneous components of coproducts of dependent words*. In this manner we effectively impose those algebraic relations that necessarily would hold in any representation of the matroid, and over any field. We call this quotient the *Whitney algebra* of the matroid.

That this is a natural construction follows from a simple observation. Consider the Hopf algebra structure of the exterior algebra $\Lambda = \bigoplus \Lambda^k$ generated by a finite set of points in a projective space. Recall that the coproduct $\delta : \Lambda \to \Lambda \otimes \Lambda$ is the multiplicative map determined by $\delta(a) = a \otimes 1 + 1 \otimes a$, for all vectors $a \in \Lambda^1$; for example,

$$\begin{aligned} \delta(abc) &= \delta(a)\ \delta(b)\ \delta(c) \\ &= abc \otimes 1 + ab \otimes c - ac \otimes b + bc \otimes a \\ &\qquad + c \otimes ab - b \otimes ac + a \otimes bc + 1 \otimes abc, \end{aligned}$$

for vectors a, b, c (where the signs are determined by anticommutativity). Now if the set

$\{a, b, c\}$ is dependent, then the wedge product abc is equal to zero in Λ, and hence the coproduct $\delta(abc)$ is also zero. Since Λ is graded by the nonnegative integers $\mathbf{N}$, the tensor product $\Lambda \otimes \Lambda$ is thus graded by $\mathbf{N} \times \mathbf{N}$, and an element of $\Lambda \otimes \Lambda$ is equal to zero if and only if all its $(\mathbf{N} \times \mathbf{N})$-homogeneous components are zero. Hence, in particular, if $\{a, b, c\}$ is linearly dependent, then the homogeneous component $a \otimes bc - b \otimes ac + c \otimes ab$ of *shape* $(1, 2)$ in the coproduct $\delta(abc)$ is equal to zero. We obtain similar relations in each component $T^k(\Lambda) = \Lambda \otimes \cdots \otimes \Lambda$ of the tensor algebra $T(\Lambda) = \bigoplus T^k(\Lambda)$ from the fact that the iterated coproduct $\delta^k(a_1 \cdots a_r)$ is zero for any dependent set of vectors $\{a_1, \ldots, a_r\}$.

In this talk, we exhibit our main technical result, the *Zipper lemma*, a cancellation theorem that points to the basic *exchange properties* of the Whitney algebra of a matroid, and proves the (skew) commutativity of a geometric product. This geometric product is essentially the product defined by Hermann Grassmann in his 1844 *Ausdehnungslehre*, roughly speaking, the tensor product of join and meet operators.

In the spirit of the present assembly, we demonstrate the straightening algorithm for the Whitney algebra of uniform matroids, based on the Rutherford method of interpolants, here applied to tableaux with *holes*. Such Whitney algebras are torsion-free, so this method yields a basis for each homogeneous component of the algebra. An analogous method for Whitney algebras (not necessarily torsion free) of arbitrary matroids is not yet known, but we will indicate a possible approach. On the other hand, a simple, but highly exponential, algorithm yields a normal form for the integer matrix of coefficients of the defining relations of each homogeneous component of the algebra. The torsion properties of this algebra reveal coordinatizability properties of the underlying matroid.