THE PROJECTION OF QUASI VARIETIES AND IT'S APPLICATION ON GEOMETRY THEOREM PROVING

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Abstract

Wu's method[11], as well as others based on computing of Grobner basis[1], has been fruitfully applied to automated theorem proving in elementary geometry. However often a geometric statement is true only in a "generic" sense, that is, certain non-degenerate conditions must be ruled out. In many cases, these conditions are too strong.

In this paper, we give an algorithm about computing the projection of quasi variety, and then apply it for geometry theorem proving and non-degenerate conditions deduction. Through this method the non-degenerate conditions we obtain are proved the weakest, i.e. the geometry theorem is true iff these conditions are not satisfied. At last, we also apply the algorithm of projection on geometry formula deduction. The formula about predetermined variables can be deduced.

The algorithm in this paper has be translated to program using Maple 7. Every example illustrated has also be tested in computer.

Keywords: Quasi Variety, projection, non-degenerate conditions, characteristic set

1 Introduction

Automated geometry theorem proving is the most successful application of the characteristic set method of Wu. Using Wu's method, more than 600 theorems have been proved by computer[2]. However, in most cases, non-degenerate conditions must be needed for these theorem to be true. For example, triangles collapse to a line segment or circles to a point. In Wu's method of characteristic set, the initials product of characteristic set are given as non-degenerate conditions. More often, these conditions are too strong, i.e. the geometry theorem is still true when such conditions are not satisfied.

As for other methods based on Grobner bases, various way to find non-degenerate conditions are reported. Kupar[**?**] describes a complete method of Grobner bases to prove geometry theorems, including how to obtain non-degenerate conditions. As Kapur claims that "conditions found using this approach are often simple and weaker than the ones reported using Wu's method or reported by an early version of Kutzler & Stifter's[5] paper as well as chou & schelter[3] based on the Grobner basis method.". So how to construct the simplest and weakest non-degenerate conditions is nowadays object.

F.Winkler[8] has given a method whereby to compute the simplest non-degenerate conditions. His criteria for *simplest* is "of as low a degree as possible" or "involving only certain variables."

Now we briefly describe Winkler's method. All polynomials here are in $K[x_1, \cdots x_n]$, K is a field of character 0. Assuming hypothesis consists of a polynomial set $p_1, \cdots, p_s$, the conclusion is a polynomial c, then a non-degenerate condition g described by Wu must satisfy two conditions:

$$(1)(\forall x \in K^n)(p_1(x) = \cdots = p_s(x) = 0 \wedge g(x) \neq 0 \Rightarrow c(x) = 0)$$
$$(2)(\exists x \in K^n)(p_1(x) = \cdots = p_s(x) = 0 \wedge g(x) \neq 0)$$

Winkler has proved that all polynomials satisfying (1) constitute an ideal, and among the polynomials in the Grobner basis of the ideal which satisfies (2), the one with the least leading power product is the simplest non-degenerate condition.

D.Wang [9] gives a decision method using projection and apply it to geometric theorem proving. He also get some non-degeneracy conditions for the theorem to be true. In this paper, we provide an algorithm of the projection of quasi variety which based on Wu's method and his theorems about the projection of quasi variety[10]. Then we apply it on geometry theorem proving and formula deduction and give new methods. During proving geometry theorem, the weakest non-degenerate conditions can be obtained through this method. Moreover the variables occurring in such conditions are determined in advance. As for geometry formula deduction, we can derive all conclusion about certain variables. At last, two examples reported show our method of practical value.

2 Preliminary

Let K be a computable field of characteristic 0, E be the algebraic closed field containing K. Consider a polynomial set PS in $K[X]$, where $X = (x_1, \cdots, x_n)$. The set $Zero(PS)$ defines then an algebraic variety in the $n-$dimensional affine space $A_n(X)$ over E in coordinates $(x_1, \cdots, x_n)$. Unless stated otherwise, all zero set this paper defines over E. Let $A_m(Z)$ be the $m-$dimensional affine space over E in coordinates $Z = (x_1, \cdots, x_m)$ alone, here $0 \leq m \leq n$. the projection of $A_n(X)$ to $A_m(Y)$ defined by

$$Proj : (x_1, \cdots, x_n) \to (x_1, \cdots, x_m).$$

DEFINITION 2.1 For polynomial sets PS and DS in $K[X]$, where $X = (x_1, \cdots, x_n)$, we define

$$Zero(PS/DS) = Zero(PS) - \bigcup_{d \in DS} Zero(d)$$

DEFINITION 2.2 For polynomial sets PS and DS as shown above, we define the projection with the $x_{m+1}, \cdots, x_n$ as follows

$$Proj_{x_{m+1},\cdots,x_n} Zero(PS/DS) = \{e \in E^m \mid \exists a \in E^{(n-m)} s.t. (e, a) \in Zero(PS/DS)\}$$

where E is an algebraic closed field containing K, $0 \leq m \leq n$.

when $m = n$, we define $Proj_{x_1,\cdots,x_m} Zero(PS/DS) = Truth$, if $Zero(PS/DS) \neq \emptyset$; and $False$ otherwise.

DEFINITION 2.3 For any finite number of polynomial sets PS_i and polynomials G_i in $K[X]$, where $X = (x_1, \cdots, x_n)$. the set

$$SUM_i\ Zero(PS_i/G_i)$$

is called a QUASI VARIETY in the affine space $A_n(X)$.

THEOREM 2.4 The projection of a quasi variety in $A_n(X)$ to $A_m(Z)$ is a quasi variety in $A_m(Z)$.

Proof. see[10].

THEOREM 2.5 For two finite polynomial sets PS and DS in $K[X]$, we may either detect the emptiness of $Zero(PS/DS)$ or find ascending chains $CS_i, i = 1, \cdots, l$, such that

$$Zero(PS/DS) = U_{i=1}^{l} Zero(CS_i/\{J_i\} \cup DS)$$

where J_i is the production of the initials of the polynomials in CS_i.

Among the theorem, polynomial sets of CS_i is characteristic set gained by Wu's zero-decomposition theorem. see [11].

Now we introduce three important lemmas about the projection of quasi variety. The algorithm is based on them. Firstly let variable order as $(x_1 < \cdots < x_n)$.

LEMMA 2.6 Let $B = \sum_{i=0}^{t} B_i x_n^i \in K[x_1, \cdots, x_n]$ and B_i contains only $x_1, \cdots, x_{n-1}$, then

$$Proj_{x_n} Zero(\emptyset/B) = \cup_{i=0}^{t} Zero(\emptyset/B_i)$$

Proof. For we define the variety over algebraic closed field E, so it is easy to check. Furthermore if x_n here is replaced by the power product of $x_{m+1}, \cdots, x_n$, and B_i contains only $x_1, \cdots, x_m$, the lemma is obviously also true.

LEMMA 2.7 Let P, Q be polynomials in $K[x_1, \cdots, x_n]$, such that $d = degree(P, x_n) > 0$ and $degree(Q, x_n) > 0$, then

$$Proj_{x_n} Zero(P/QI) = Proj_{x_n} Zero(\emptyset/RI)$$

where I is the initial of P and $R = prem(Q^d, P)$. here $prem(\ )$ is the function to computer the pseudo remainder.

Proof. See[10].

LEMMA 2.8 Let P be polynomial in $K[x_1, \cdots, x_n]$, and Q be polynomial in $K[x_1, \cdot, x_{n-1}]$ such that $d = degree(P, x_n) > 0$, then $Proj_{x_1} Zero(P/Q) = Proj_{x_1} Zero(\emptyset/Q)$.

Proof. It is obvious.

By THEOREM2.5, $Zero(PS/DS) = U_{i=1}^{l} Zero(CS_i/\{J_i\} \cup DS)$. According to Wu's method, Every polynomial in CS_i has a leading variable. In fact LEMMA2.62.72.8 give a method to project with the leading variable from high to low, meanwhile to eliminate polynomials containing the main variable. Successively we can gain the result of $Proj_{x_{m+1},\cdots,x_n} Zero(CS_i/\{J_i\} \cup DS)$, for any $0 \leq m \leq n$. So the complete result is easy to get.

Here is an algorithm of projection to compute $Proj_{x_{m+1},\cdots,x_n} Zero(PS/DS)$, here $0 \leq m \leq n$.

2.1 Algorithm of Projection

COMMENT: *All polynomials in this algorithm are in $K[X]$, $X = [x_1, \cdots, x_n]$. The variables order is $(x_1 < x_2 < \cdots < x_n)$, $Y = [x_{m+1}, \cdots, x_n]$ is the variables list to be eliminated during the procedure. We denote a quasi variety as $[[PS], [DS]]$; or $[[PS], DS]$, if DS has single polynomial.*

Project*(PS, DS, X, Y)*
Input: *PS =a polynomial set, DS=a polynomial set*
    X=a variable list, Y=a variable list
Output: *QVS=a set of Quasi Varieties*

  QVS:=$\emptyset$
  CSS:=the set of characteristic sets of PS
  for *every $CS \in CSS$* loop

    J:=the product of the initials of CS
    $DS := DS \bigcup J$
    $D := \prod DS$
    QVS' :=ProjectCS(CS, D, X, Y)
    $QVS := QVS \bigcup QVS'$

  endloop
  return QVS

endProject

ProjectCS*(CS, D, X, Y)*
Input*: CS=a characteristic set or a part, D=a polynomial,*
X=a variable list, Y=a variable list
Output*: QVS=a set of Quasi Varieties*

if $\forall y \in Y$, *Degree*$(D, y) = 0$ then

CS:=the part of CS without Y, return $QVS := [[CS], D]$

endif
if CS *has no* Y then

return QVS :=Decompose(CS, D, X, Y)

endif
factorize D to $\Pi_{i=1}^{k} D_i$, $QVS := \bigcap_{i=1}^{m} QV_i = \bigcap_{i=1}^{m} [[CS], D_i]$
for *every* $QV_i \in QVS$ loop*:*

HP:=the highest polynomial of CS
LV:=the leading variable of HP
if *Degree*$(D_i, LV) = 0$ then
$CS := CS - HP$
QV_i :=ProjectCS(CS, D, X, Y)

else

$d := degree(HP, LV)$, $D_i := prem(D_i^d, HP)$
$CS := CS - HP$, $QV_i := [[CS], D_i]$
if *Degree*$(D_i, LV) > 0$ then
QV_i :=Decompose$(CS, D_i, X, [LV]) =$
$\bigcup_{j=1}^{n} QV_{ij} = \bigcup_{j=1}^{n} [[CS], D_{ij}]$
for *every* $QV_{ij} \in QV_i$ loop*:*
QV_{ij} :=ProjectCS(CS, D_{ij}, X, Y)
endloop
else QV_i :=ProjectCS(CS, D_i, X, Y)
endif

endif

endloop
return QVS

endProjectCS

Decompose(CS, D, X, Y)
Input: *CS =a characteristic set or a part, D=a polynomial*
X=a variable list, Y=a variable list
Output: *QVS = a union of QuasiVarietes*

$$QVS = \bigcup_{i=1}^{n} [[CS], D_i]$$

here D_i is coefficients of D w.r.t Y.
endDecompose

After above computation, for each CS we obtain a tree structure with the nodes $\bigcap$ *or* $\bigcup$ and leafs $[[CS], D]$. Then we tidy up them to the formation of $\cup QVS_i = \cup[[CS_i], [DS_i]]$. $[DS_i]$ contains polynomials not equal to zero simultaneous. Due to its unimportant rule in this paper, we will not present the detailed program here.

3 Application on geometry theorem proving and formula deduction

Let K be a computable field with characteristic zero (in practice, $K = Q$), E be algebraic closed field. All polynomials mentioned in this section are over K. Suppose for a geometric theorem T, after adopting an appropriate coordinate system $(x_1, \cdots, x_n)$, the corresponding geometric configuration of hypothesis can be expressed by a set of polynomial equations as below

$$PS = \{p_1(x_1, \cdots, x_n) = 0 \bigwedge \cdots \bigwedge p_s(x_1, \cdots, x_n) = 0\}$$

and set of polynomial inequations

$$DS = \{d_1(x_1, \cdots, x_n) \neq 0 \bigwedge \cdots \bigwedge d_s(x_1, \cdots, x_n) \neq 0\}$$

Finally, translate the conclusion to one polynomial equation $C = 0$.

As before, let $X = (x_1, \cdots, x_n), Z = (x_1, \cdots, x_m), Y = (x_{m+1}, \cdots, x_n)$, variables order is $(x_1 < \cdots < x_n)$.

THEOREM 3.1 For polynomial sets PS, DS and polynomial C as shown above, then

(1) if $Proj_Y Zero(PS/DS * C) = \emptyset$ and $Proj_Y Zero(PS/DS) \neq \emptyset$, then the theorem T is true without any other conditions.

(2) if $Proj_Y Zero(PS/DS * C) \neq \emptyset$, Z_0 is possible value about variables Z in T, then the theorem T is not true iff $Z_0 \in Proj_Y Zero(PS/DS * C)$ or T is true iff $Z_0 \notin Proj_Y Zero(PS/DS * C)$

Proof.(1) If $Proj_Y Zero(PS/DS*C) = \emptyset$, then $Zero(PS/DS*C) = \emptyset$, moreover $Proj_Y Zero(PS/DS) \neq \emptyset$, i.e $Zero(PS/DS) \neq \emptyset$. So $Zero(PS/DS) \subseteq Zero(C)$

(2) If $Proj_Y Zero(PS/DS*C) \neq \emptyset$, then $\forall Z_0 \in Proj_Y Zero(PS/DS*C)$, $\exists Y_0$ s.t $(Z_0, Y_0) \in Zero(PS/DS)$, but $(Z_0, Y_0) \notin Zero(C)$.

On the other hand, in $Proj_Y Zero(PS/DS)$, $\forall Z_0 \notin Proj_Y Zero(PS/DS * C)$, $\forall Y_0$ s.t $(Z_0, Y_0) \in Zero(PS/DS)$, then by the definition of project, $(Z_0, Y_0) \notin Zero(C)$.

By THEOREM3.1, we have had a new method based on computing the projection of quasi variety. It's obvious to see that we can predetermine the variables occurring in non-degenerate conditions. So it's convenient for us to observe the range of possible value for any variables.

Moreover, we can be certain that conditions found through this method are the weakest in as far as using Wu's method, Kapur's and Winkler's approach based on Grobner basis. Let's firstly show the relation of $Zero(g)$, $Zero(C)$, $Zero(PS/DS)$ as following Figure1, here polynomial g represents non-degenerate condition.

As for other method, the non-degenerate conditions are all defined as mentioned in Introduction, corresponding to $Zero(g')$ in the figure. However the non-degenerate conditions in THEOREM3.1 just correspond to the black area. So they are the weakest.

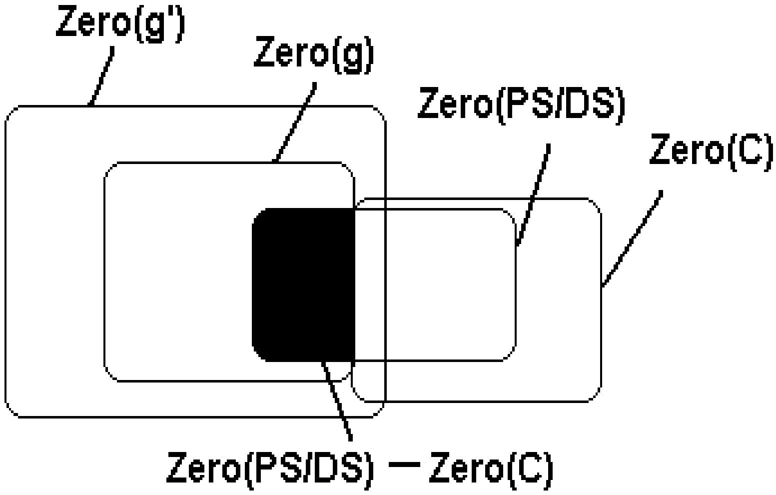

Figure 1: zero set relation

Now we consider the application of projection method on formula deduction.

THEOREM 3.2 For polynomial sets PS, DS are shown above, but there is no polynomial C, then $Proj_{x_k,\cdots,x_n} Zero(PS/DS * C)$ gives a conclusion about variables $(x_1, \cdots, x_{k-1})$.

This conclusion is expressed by a Quasi Variety. By the definition of the projection of quasi variety, it contains all relations under the hypothesis.

4 Example

Now let us give some theorems to illustrate how the method works.

Example 1. If P is any point in the plane of the parallelogram $ABCD$, find the relation among the areas of triangles PAB,PCD,and ABC.

Firstly let us convert the statement into algebraic form. We let $A = (0,0), B = (x_1, 0), C = (x_2, x_3), D = (x_4, x_3), P = (x_5, x_6)$, where AB is X-axis. The hypothesis of the theorem can be translated into the following equations:

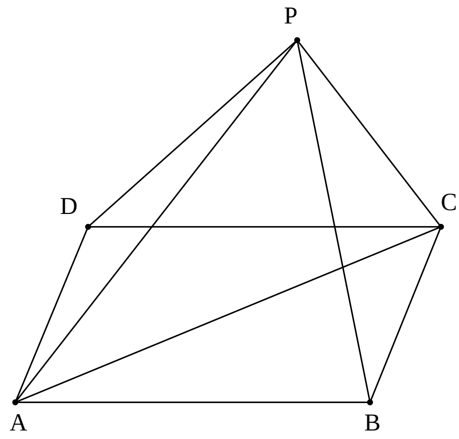

Figure 2: area theorem

$H_1 = x3 * x4 - x3 * (-x1 + x2) = 0$
$H_2 = 2 * u1 - x6 * x1 = 0$
$H_3 = 2 * u2 - (x2 - x5) * x3 - (-x3 + x6) * x4 - x5 * x3 + x6 * x2 = 0$
$H_4 = 2 * u3 - x1 * x3u_4(x_3 - x_1) - (u_5 - u_3)x_2 = 0$

where H_1 means AD is parallel to BC), H_2, H_3, H_4 assaign areas of $S_{PAB}, S_{PCD}, S_{ABC}$ to $u1, u2, u3$ respectively. The final conclusion is: $C = u3 - u2 - u1 = 0$

Before carrying out the projection method, we first use Wu's method to deal with this theorem. According to Wu's method, we let the variables order:($u1 < u2 < u3 < x1 < x2 < x3 < x4 < x5 < x6$), so the characteristic set of $[H_1, H_2, H_3]$ as follows:

$$\begin{aligned}
A_1 &= u3 * u1 * (-u2 - u1 + u3),\\
A_2 &= -2 * u3 + x1 * x3,\\
A_3 &= u3 * (-x4 + x2 - x1)\\
A_4 &= -2 * u1 + x6 * x1
\end{aligned}$$

where the initials are $I_1 = u1, I_2 = x1, I_3 = -u3, I_4 = x1$. It is easy to check that polynomial C can be reduced to zero by A_1, A_2, A_3. So the conclusion $C = 0$ is true under the non-degenerate conditions: $u1 * x1 * u3 \neq 0$. But this conditions are too strong.Obviously when $u1 = u2 = u3 = 0$, $c = 0$.

Now we try to use the projection method. According to the example and the program, we let $ord = [x1, x2, x3, x4, x5, x6, u1, u2, u3]$,$pord = [x5, x6, u1, u2, u3]$.This means we want to get the non-degenerate conditions in variables $x1, x2, x3, x4$. Then we input $PS = [H_1, H_2, H_3], DS = [C], ord, pord$. The output of $Proj_{u3,u2,u1,x_6,x_5}Zero(PS/DS * C)$ is as follows:

$$[[x3], [-x4 + x2 - x1]]$$

Let us consider the result above. $x3 = 0$ means A, B, C, D are collinear. But it also satisfy that opposite sides are parallel. And $-x4 + x2 - x1 \neq 0$ means $CD \neq BA$. This is the exact condition, containing $x1, x2, x3, x4$ only, makes the conclusion false. It is equal to that when $x3 \neq 0$ or $-x4+x2-x1 = 0$, there is conclusion $C = 0$.

If we don't know the conclusion and want to deduce the relation of $u1, u2, u3$, we can try to let $ord = [u1, u2, u3, x1, x2, x3, x4, x5, x6]$, $pord = [x1, x2, x3, x4, x5, x6]$, and input $PS = [H_1, H_2, H_3], DS = [\,], ord, pord$. The output of $Proj_{x6,x5,x4,x3,x2,x1}Zero(PS/DS)$ is:

$$[[[u2 + u1 - u3], [1]], [[u3], [1]]]$$

this show that either the area of triangle ABC is 0, or the areas of triangles ABC, PCD,PAB has the relation: $S_{PCD} + S_{PAB} - S_{ABC} = 0$

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