Distance Coordinates Used in Geometric Constraint Solving*

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In this paper, an invariant method based on distance geometry is proposed to construct the constraint equations for spatial constraint solving.

1 Distance Geometry

Given an n-tuple of points and oriented hyperplanes in an Euclidean space, $P = (p_1, p_2, \cdots, p_n)$, we define a mapping $g : P \times P \to \mathbf{R}$ by letting

- $g(p_i, p_j)$ be the square of the distance between p_i and p_j if both are points,
- $g(p_i, p_j)$ be the signed distance from p_i to p_j, if one is a point and the other an oriented hyperplane,
- $g(p_i, p_j)$ be $-\frac{1}{2}\cos(p_i, p_j)$, if both are oriented hyperplanes.

By g_{ij} denote $g(p_i, p_j)$ and G denote the matrix $(g_{ij})_{n\times n}$, and let

$$\delta = (\delta_1, \delta_2, \cdots, \delta_n)$$

where

$$\delta_i = \begin{cases} 1, & \text{if } p_i \text{ is a point} \\ 0, & \text{if } p_i \text{ is a hyperplane} \end{cases}$$

where $i = 1, \ldots, n$. Then, let

$$M(p_1, p_2, \cdots, p_n) = \begin{pmatrix} G & \delta^T \\ \delta & 0 \end{pmatrix}$$

which is called the Cayley-Menger matrix of P, and

$$D(p_1, p_2, \cdots, p_n) = \begin{vmatrix} G & \delta^T \\ \delta & 0 \end{vmatrix}$$

which is called the Cayley-Menger determinant of P. The following theorem is an extension of the classical result on Cayley-Menger determinant.

Theorem 1. Let $D(p_1, p_2, \cdots, p_n)$ be an n-tuple of points and oriented hyperplanes in d-dimensional space. If $n \geq d + 2$, then

$$D(p_1, p_2, \cdots, p_n) = 0. \tag{1}$$

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This theorem is actually the Theorem 1.1 of article [9], and can also be found in earlier articles [7, 8].

By $A[i,j]$ denote the cofactor of a determinant A by removing the i-th row and the j-th collum from A. As a corollary of Theorem 1, we have

Corollary 1. Let D be the Cayley-Menger determinant of an n-tuple of points and oriented hyperplanes in d-dimensional space. If $n \geq d+3$, then

$$D[i,j] = 0 \qquad (\text{for } i,j = 1,\ldots,n.) \tag{2}$$

Proof: This depends on a lemma of determinants: For $i < j$,

$$A[i,i] \cdot A[j,j] - A[i,j] \cdot A[j,i] = A \cdot (A[j,j])[i,i]. \tag{3}$$

The identity (3) was used in Blumenthal's book [1] several times without proof, maybe he thought it well-known, see the pages 100, 102, etc. A proof can be found in [7] or [8]. Applying (3) to D,

$$D[i,i] \cdot D[j,j] - D[i,j] \cdot D[j,i] = D \cdot (D[j,j])[i,i],$$

where clearly $D[i,i]$ is the Cayley-Menger determinant of an $(n-1)$-tuple, by Theorem 1, $D[i,i] = 0, D = 0$ because $n \geq d+3$. Noting D is symmetric, we have $D[i,j] = 0$. Corollary 1 thus holds.

For practical convenience sometimes we need also consider an n-tuple of points, oriented hyperplanes and oriented hyperspheres in Euclidean space,

$$P^* = \{p_1, p_2, \cdots, p_n\},$$

with a mapping $g^* : P^* \times P^* \to \mathbf{R}$ as follows.

- $g^*(p_i, p_j)$ be $\dfrac{r_i^2 + r_j^2 - \overline{o_i o_j}^2}{2\, r_i r_j}$ if both are oriented hyperspheres with radii r_i, r_j and centers o_i, o_j, so $g^*(p_i, p_j) = \cos(p_i, p_j)$ when p_i intersects p_j. Note that r_i and r_j are signed since p_i and p_j oriented.
- $g^*(p_i, p_j)$ be $\cos(p_i, p_j)$ if both are oriented hyperplanes, or one is oriented hyperplane and the other oriented hypersphere that p_i intersects p_j.
- $g^*(p_i, p_j)$ be $\dfrac{h_{ij}}{r_i}$ if p_i is an oriented hypersphere with radius r_i and center o_i, and p_j an oriented hyperplane whereof the signed distance from o_i is h_{ij}.
- $g^*(p_i, p_j)$ be $-\dfrac{1}{2}\overline{p_i p_j}^2$ if both are points.
- $g(p_i, p_j)$ be the signed distance from p_i to p_j, if one is a point and the other an oriented hyperplane.
- $g(p_i, p_j)$ be $\dfrac{1}{2r_i}(r_i^2 - \overline{o_i p_j}^2)$ if p_i is an oriented hypersphere with radius r_i and center o_i, and p_j a point.

By g^*_{ij} denote $g^*(p_i, p_j)$ and G^* denote the matrix $(g^*_{ij})_{n\times n}$, by

$$D^*(p_1, p_2, \cdots, p_n)$$

denote the determinant $|G^*|$ and call it D^*-determinant of P^*.

Analogously, we have

Theorem 2. Let $D^*(p_1, p_2, \cdots, p_n)$ be the D^*-determinant of an n-tuple of oriented hyperplanes in d-dimensional space. If $n \geq d + 3$, then

$$D^*(p_1, p_2, \cdots, p_n) = 0. \tag{4}$$

This theorem is actually the Theorem 1.3 of article [9], and can also be found in earlier article [8]. And consequently,

Corollary 2. Let D^* be the D^*-determinant of an n-tuple of oriented hyperplanes and oriented hyperspheres in d-dimensional space. If $n \geq d + 4$, then

$$D^*[i, j] = 0 \qquad (\text{for } i, j = 1, \ldots, n.) \tag{5}$$

As for a systematic and complete investigation on metric relations of points, hyperplanes and hyperspheres, see [10].

2 Distance Coordinates and Constraint Equations

There are different ways to introduce a distance coordinate system for solving geometric constraint problems, see below for example.

Triple coordinate system in $\mathbf{E}^3$

Choose 3 points or oriented planes as a *reference triangle*, say, $\{p_{i_1}, p_{i_2}, p_{i_3}\}$. For any point or oriented plane p_j, take

$$(\, g(p_j, p_{i_1}),\ g(p_j, p_{i_2}),\ g(p_j, p_{i_3})\,)$$

as the coordinates of p_j, where function g was well-defined in last section.

To solve a geometric constraint problem on $p_1, \cdots, p_n$, we need only find the distance coordinates of every p_i. For any couple (p_i, p_j) with $i < j$, if $g(p_i, p_j)$ is given as a constraint and $\{p_i, p_j\} \cap \{p_{i_1}, p_{i_2}, p_{i_3}\} = \emptyset$, then we take

$$D(p_{i_1}, p_{i_2}, p_{i_3}, p_i,\ p_j) = 0$$

as a constraint equation. A complete set of constraint equations is established in this way. Let k be the number of these equations. On the other hand, let

$$S_1 = \{(p_i, p_j) \,|\, 1 \leq i < j \leq n, \{p_i, p_j\} \cap \{p_{i_1}, p_{i_2}, p_{i_3}\} \neq \emptyset\}, \tag{6}$$

then the cardinal number of set S_1 is $3n - 6$.

Assume this geometric constraint problem is well-constrained. Then, the number of independent constraints should be $3n-6$. It was known from above argument that exactly k constraints do not concern $p_{i_1}, p_{i_2}, p_{i_3}$, so exactly $3n-6-k$ constraints concern $p_{i_1}, p_{i_2}, p_{i_3}$. Therefore, among $3n-6$ distances (or angles) on S_1, there are exactly $3n-6-k$ items are given as constraints, so that k items are unknown. Thus, the number of unknowns equals that of constraint equations.

Quartuple coordinate system in $\mathbf{E}^3$

Choose 4 points or oriented planes as a *reference tetrahedron*, say, $\{p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}\}$. For any point or oriented plane p_j, take

$$(\, g(p_j, p_{i_1}),\; g(p_j, p_{i_2}),\; g(p_j, p_{i_3}),\; g(p_j, p_{i_4})\,)$$

as the coordinates of p_j.

To solve a geometric constraint problem on $p_1, \cdots, p_n$, we need only find the distance coordinates of every p_i. For any couple (p_i, p_j) with $i < j$, if $g(p_i, p_j)$ is given as a constraint and $\{p_i, p_j\} \cap \{p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}\} = \emptyset$, then we take

$$D(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_i,\, p_j)[5, 6] = 0 \tag{7}$$

as a constraint equation. Let l be the number of these equations. Furthermore, for every $p_j \notin \{p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}\}$, take

$$D(p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}, p_j) = 0 \tag{8}$$

as a constraint equation, we obtain $n-4$ equations. A complete set of constraint equations with $l + n - 4$ members is established in the two ways.

On the other hand, let

$$S_2 = \{(p_i, p_j) \,|\, 1 \le i < j \le n, \{p_i, p_j\} \cap \{p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}\} \ne \emptyset\}, \tag{9}$$

then the cardinal number of set S_2 is $4n - 10$.

Assume this geometric constraint problem is well-constrained. Then, the number of independent constraints should be $3n - 6$. It was known from above argument that exactly l constraints do not concern $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}$, so exactly $3n-6-l$ constraints concern $p_{i_1}, p_{i_2}, p_{i_3}, p_{i_4}$. Therefore, among $4n-10$ distances (or angles) on S_2, there are exactly $3n - 6 - l$ items are given as constraints, so that $l + n - 4$ items are unknown. Thus, the number of unknowns equals that of constraint equations.

As one of the advantages, the quartuple coordinates can uniquely determine the Cartesian coordinates when the reference tetrahedron is fixed, while the triple coordinates usually give alternate Cartesian coordinate representations.

3 Examples

`The 6p Octahedral Problem.` This was discussed in many articles [2, 5, 4]. Hoffmann mentioned a paper by Michelucci [6] where Cayley-Menger determinant was used to solve this problem, but it did not follow a systematic algorithm.

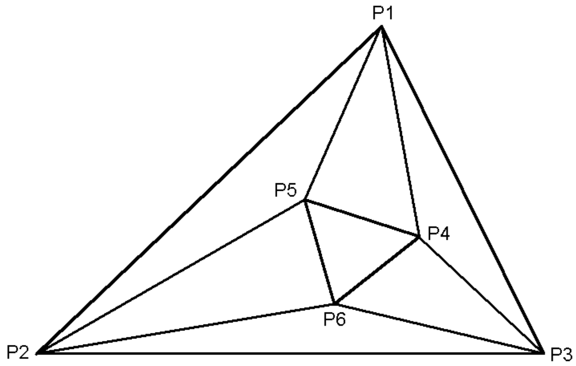

Fig. 1

Referring to Fig. 1, where each p_i may be a point or an oriented plane, the quartuple coordinate system with reference tetrahedron $\{p_2, p_3, p_4, p_5\}$ produces a complete set of constraint equations consisting of 2 equations,

$$\begin{cases} D(p_2, p_3, p_4, p_5, p_1) = 0, \\ D(p_2, p_3, p_4, p_5, p_6) = 0, \end{cases}$$

with 2 unknowns, $g(p_2, p_4)$ and $g(p_3, p_5)$. This coincides with the result of [6]. Alternatively, the triple coordinate system with reference triangle $\{p_1, p_2, p_3\}$ produces a complete set of constraint equations consisting of 3 equations,

$$\begin{cases} D(p_1, p_2, p_3, p_4, p_5) = 0, \\ D(p_1, p_2, p_3, p_4, p_6) = 0, \\ D(p_1, p_2, p_3, p_5, p_6) = 0, \end{cases}$$

with 3 unknowns, $g(p_2, p_4)$, $g(p_3, p_5)$ and $g(p_1, p_6)$.

`The 12p Icosahedral Problem.` Referring to Fig. 2, the triple coordinate system with reference triangle $\{p_1, p_2, p_3\}$ produces a complete set of constraint equations consisting of 18 equations, while the routine Cartesian coordinate approach would produce a set of 30 constraint equations.

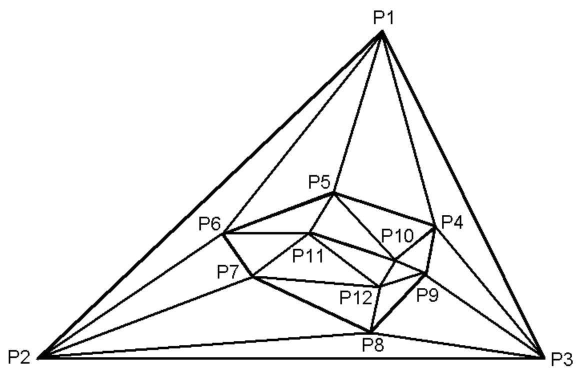

Fig. 2

Alternatively, the coordinate system with reference tetrahedron $\{p_1, p_2, p_3, p_4\}$ produces a complete set of constraint equations consisting of 23 equations, and that with reference tetrahedron $\{p_1, p_2, p_9, p_{11}\}$ produces a complete set of constraint equations consisting of 19 equations.

`The 5p1L Problem.` Consider a spatial configuration of points $p_1, \cdots, p_5$ and a straight line L, given the lengths of some segments,

$$p_1p_2 = a_1,\ p_2p_3 = a_2,\ p_3p_4 = a_3,\ p_4p_1 = a_4,$$
$$p_1p_5 = b_1,\ p_2p_5 = b_2,\ p_3p_5 = b_3,\ p_4p_5 = b_4,$$

and the distances from every p_i to L, $\text{distance}(p_i, L) = d_i$, for $i = 1, \ldots, 5$.

Since $(p_1, \cdots, p_5,\ L)$ is not a point-plane configuration, we consider a new one instead. Through point p_5 draw a plane p_6 perpendicular to L, and assume p_6 intersects L at point p_7. Now we have a point-plane configuration, $p_1, \cdots, p_7$. By x_1, x_2, x_3, x_4 denote the unknown signed distances from p_1, p_2, p_3, p_4 to plane p_6, respectively, i.e.

$$x_1 = g(p_1, p_6),\ x_2 = g(p_2, p_6),\ x_3 = g(p_3, p_6),\ x_4 = g(p_4, p_6).$$

It follows by Pythagorean Theorem that

$$g(p_1, p_7) = x_1^2 + d_1^2, \qquad g(p_2, p_7) = x_2^2 + d_2^2,$$
$$g(p_3, p_7) = x_3^2 + d_3^2, \qquad g(p_4, p_7) = x_4^2 + d_4^2.$$

With $\{p_5, p_6, p_7\}$ as reference triangle, construct a triple coordinate system, and obtain a set of 4 constraint equations,

$$\begin{cases} D(p_5, p_6, p_7, p_1, p_2) = 0, \\ D(p_5, p_6, p_7, p_2, p_3) = 0, \\ D(p_5, p_6, p_7, p_3, p_4) = 0, \\ D(p_5, p_6, p_7, p_4, p_1) = 0, \end{cases}$$

with just 4 unknowns, x_1, x_2, x_3, x_4. Write these equations in detail:

$$\begin{vmatrix} 0 & 0 & d_5^2 & b_1^2 & b_2^2 & 1 \\ 0 & -\frac{1}{2} & 0 & x_1 & x_2 & 0 \\ d_5^2 & 0 & 0 & x_1^2 + d_1^2 & x_2^2 + d_2^2 & 1 \\ b_1^2 & x_1 & x_1^2 + d_1^2 & 0 & a_1^2 & 1 \\ b_2^2 & x_2 & x_2^2 + d_2^2 & a_1^2 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{vmatrix} = 0, \tag{10}$$

$$\begin{vmatrix} 0 & 0 & d_5^2 & b_2^2 & b_3^2 & 1 \\ 0 & -\frac{1}{2} & 0 & x_2 & x_3 & 0 \\ d_5^2 & 0 & 0 & x_2^2 + d_2^2 & x_3^2 + d_3^2 & 1 \\ b_2^2 & x_2 & x_2^2 + d_2^2 & 0 & a_2^2 & 1 \\ b_3^2 & x_3 & x_3^2 + d_3^2 & a_2^2 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{vmatrix} = 0, \tag{11}$$

$$\begin{vmatrix} 0 & 0 & d_5^2 & b_3^2 & b_4^2 & 1 \\ 0 & -\frac{1}{2} & 0 & x_3 & x_4 & 0 \\ d_5^2 & 0 & 0 & x_3^2+d_3^2 & x_4^2+d_4^2 & 1 \\ b_3^2 & x_3 & x_3^2+d_3^2 & 0 & a_3^2 & 1 \\ b_4^2 & x_4 & x_4^2+d_4^2 & a_3^2 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{vmatrix} = 0, \tag{12}$$

$$\begin{vmatrix} 0 & 0 & d_5^2 & b_4^2 & b_1^2 & 1 \\ 0 & -\frac{1}{2} & 0 & x_4 & x_1 & 0 \\ d_5^2 & 0 & 0 & x_4^2+d_4^2 & x_1^2+d_1^2 & 1 \\ b_4^2 & x_4 & x_4^2+d_4^2 & 0 & a_4^2 & 1 \\ b_1^2 & x_1 & x_1^2+d_1^2 & a_4^2 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 & 0 \end{vmatrix} = 0. \tag{13}$$

See [5] for different approaches to the 5p1L problem.

4 More Distance Coordinate Systems

When we investigate the planar configurations formed by circles, (straight) lines and points, it is sometimes more convenient to use a coordinate system with circles and lines as reference triangle/tetrahedron.

`The Malfatti Problem.` Draw 3 circles into a triangle such that each circle contacts other two and contacts two sides of the triangle, as shown in Fig. 3.

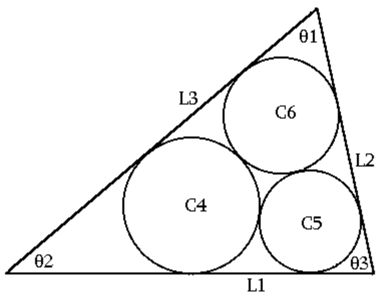

Fig. 3

By L_1, L_2, L_3 and $\theta_1, \theta_2, \theta_3$ denote the sides and interior angles, and C_4, C_5, C_6 the circles, respectively. Let all the lines and circles has the counter-clockwise orientation. Referring to Theorem 2, construct a triple coordinate system with reference triangle $\{L_1, L_2, L_3\}$. Take

$$(\, g^*(C_j, L_1),\ g^*(C_j, L_2),\ g^*(C_j, L_3)\,)$$

as the coordinates of C_j for $j = 4, 5, 6$, where function g^* was well-defined before. This triple coordinate system produces a complete set of constraint equations consisting of 3 equations,

$$\begin{cases} D^*(L_1, L_2, L_3, C_4, C_5) = 0, \\ D^*(L_1, L_2, L_3, C_4, C_6) = 0, \\ D^*(L_1, L_2, L_3, C_5, C_6) = 0, \end{cases} \tag{14}$$

with 3 unknowns, $g^*(L_2, C_4)$, $g^*(L_3, C_5)$, $g^*(L_1, C_6)$, which we denote by x, y, z. Note that $\angle(L_2, L_3) = \pi - \theta_1$, $\angle(L_3, L_1) = \pi - \theta_2$, $\angle(L_1, L_2) = \pi - \theta_3$, hence

$$\cos(L_2, L_3) = -\cos\theta_1, \ \cos(L_3, L_1) = -\cos\theta_2, \ \cos(L_1, L_2) = -\cos\theta_3.$$

Write the equations of (14) in detail:

$$\begin{vmatrix} 1 & -\cos\theta_3 & -\cos\theta_2 & 1 & 1 \\ -\cos\theta_3 & 1 & -\cos\theta_1 & x & 1 \\ -\cos\theta_2 & -\cos\theta_1 & 1 & 1 & y \\ 1 & x & 1 & 1 & -1 \\ 1 & 1 & y & -1 & 1 \end{vmatrix} = 0, \tag{15}$$

$$\begin{vmatrix} 1 & -\cos\theta_3 & -\cos\theta_2 & 1 & z \\ -\cos\theta_3 & 1 & -\cos\theta_1 & x & 1 \\ -\cos\theta_2 & -\cos\theta_1 & 1 & 1 & 1 \\ 1 & x & 1 & 1 & -1 \\ z & 1 & 1 & -1 & 1 \end{vmatrix} = 0, \tag{16}$$

$$\begin{vmatrix} 1 & -\cos\theta_3 & -\cos\theta_2 & 1 & z \\ -\cos\theta_3 & 1 & -\cos\theta_1 & 1 & 1 \\ -\cos\theta_2 & -\cos\theta_1 & 1 & y & 1 \\ 1 & 1 & y & 1 & -1 \\ z & 1 & 1 & -1 & 1 \end{vmatrix} = 0. \tag{17}$$

Alternatively, the coordinate system with reference tetrahedron $\{L_2, L_3, C_4, C_5\}$ produces a complete set of constraint equations consisting of 2 equations,

$$\begin{cases} D^*(L_2, L_3, C_4, C_5, L_1) = 0, \\ D^*(L_2, L_3, C_4, C_5, C_6) = 0, \end{cases} \tag{18}$$

with 2 unknowns, $x = g^*(L_2, C_4)$ and $y = g^*(L_3, C_5)$. Write the equations of (18) in detail:

$$\begin{vmatrix} 1 & -\cos\theta_1 & x & 1 & -\cos\theta_3 \\ -\cos\theta_1 & 1 & 1 & y & -\cos\theta_2 \\ x & 1 & 1 & -1 & 1 \\ 1 & y & -1 & 1 & 1 \\ -\cos\theta_3 & -\cos\theta_2 & 1 & 1 & 1 \end{vmatrix} = 0, \tag{19}$$

$$\begin{vmatrix} 1 & -\cos\theta_1 & x & 1 & 1 \\ -\cos\theta_1 & 1 & 1 & y & 1 \\ x & 1 & 1 & -1 & -1 \\ 1 & y & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 \end{vmatrix} = 0. \tag{20}$$

Contrastively, it was reported in [3] that, for the same problem, a complete set of constraint equations consisting of 30 equations with 30 unknowns was obtained by using Cartesian coordinate system.

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