Symbolic Methods for the Equivalence Problem for Systems of Implicit Ordinary Differential Equations

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1 Introduction

This contribution deals with the equivalence problem for dynamic systems described by a set of n_e nonlinear implicit ordinary differential equations of the type

$$f^{i_e}\left(t, z, \tfrac{\mathrm{d}}{\mathrm{d}t} z\right) = 0 \ , \quad i_e = 1, \ldots, n_e \ , \tag{1}$$

with the independent variable t and the dependent variables z^{α_z}, $\alpha_z = 1, \ldots, q$. We assume that the coordinates (t, z) of (1) are local coordinates of a smooth manifold $\mathcal{E}$ with $\dim \mathcal{E} = q + 1$. The equivalence problem under consideration is the following one: Given the system (1) and a prescribed normal form, does there exist transformations such that these transformations map each solution of (1) to a solution of the normal form and vice versa? The class of systems (1) covers implicit systems, DAE-systems (differential algebraic equation)

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} x^{\alpha_x} &= f^{\alpha_x}(t, x, u) \ , \ \alpha_x = 1, \ldots, n_x \\ 0 &= f^{\alpha_s}(t, x, u) \ , \ \alpha_s = n_x + 1, \ldots, n_e \end{aligned} \tag{2}$$

and descriptor systems like

$$n^{\alpha_e}_{\alpha_z}(t, z) \tfrac{\mathrm{d}}{\mathrm{d}t} z^{\alpha_z} = m^{\alpha_e}(t, z) \ , \quad \alpha_e = 1, \ldots, n_e \tag{3}$$

in the unknowns z^{α_z}, $\alpha_z = 1, \ldots, n_z$. Of course, (3) contains (2) as a subclass, if one puts the input u, the set of functions that can be chosen freely, and the state x on an equal footing by setting $z = (x, u)$.

The investigation of nonlinear systems like (1), (2), (3) requires new mathematical tools. A modern approach considers differential equations as submanifolds of new spaces, called jet-bundles. One may state that the theory of jet-bundles has been invented for dealing with calculations involving any arbitrary order of differentiation, a prerequisite to solve problems concerning linear or nonlinear partial differential equations. This paper is organized as follows: In Section 2 we present some mathematical preliminaries which include manifolds and submanifolds, as well as certain rings over smooth functions defined on the manifolds under consideration and ideals of these rings. The goal is to combine methods from differential and algebraic geometry. The Section 3 deals with jet bundles and their application to systems of ordinary or partial differential equations. The main results of this contribution are presented in Section 4. Finally, this contribution ends with some conclusions. Let us finish the introduction with some technical remarks. Throughout this contribution, we consider only generic problems, because we are interested in methods based on computer algebra to derive algorithms for their investigation. Therefore, we assume that all functions are smooth and that several rank conditions are fulfilled. Roughly speaking, these conditions guarantee that the dynamic systems under consideration describe regular submanifolds of certain jet-spaces. Furthermore, we use the tensor-notation to keep the formulas short and readable.

2 Some Preliminaries

Before we start to repeat some mathematical preliminaries, we introduce some useful notations. Let $\mathcal{M}$ be a smooth finite-dimensional manifold, then $\mathcal{T}\mathcal{M}$ denotes the tangent bundle of $\mathcal{M}$, $\wedge_k^*\mathcal{M}$ the exterior k bundle and $\wedge^*\mathcal{M}$ the exterior algebra over $\mathcal{M}$. $\mathrm{d} : \wedge_k^*\mathcal{M} \to \wedge_{k+1}^*\mathcal{M}$ is the exterior derivative and $\mathrm{i} : \mathcal{T}\mathcal{M} \times \wedge_{k+1}^*\mathcal{M} \to \wedge_k^*\mathcal{M}$ is the interior product written as $\mathrm{i}_X(\omega)$ with $X \in \mathcal{T}\mathcal{M}$ and $\omega \in \wedge_{k+1}^*\mathcal{M}$. $\wedge$ denotes the exterior product of the exterior algebra $\wedge^*\mathcal{M}$. Furthermore, the Lie derivative of $\omega \in \wedge^*\mathcal{M}$ along the tangent vector field $f \in \mathcal{T}\mathcal{M}$ is written as $f(\omega)$.

Although the methods being presented are independent of the special choice of the coordinates for a manifold $\mathcal{M}$, any computer algebra implementation relies on coordinates. The following theorem shows that the choice of the coordinates influences the representation of certain maps significantly.

Theorem 1. *Let $\mathcal{M}$, $\mathcal{N}$ be two m- and n-dimensional smooth manifolds and $\varphi : \mathcal{M} \to \mathcal{N}$ a smooth map of maximal rank at $p \in \mathcal{M}$. Then there exist neighborhoods $\mathcal{U}_p \subset \mathcal{M}$, $\mathcal{V}_{\varphi(p)} \subset \mathcal{N}$ and local coordinates $(x) = \left(x^1, \ldots, x^m\right)$ for $\mathcal{U}_p$ and $(y) = \left(y^1, \ldots, y^n\right)$ for $\mathcal{V}_{\varphi(p)}$ such that φ takes the simple form*

$$\begin{array}{ll} (y) = \left(x^1, \ldots, x^m, 0, \ldots 0\right) & \textit{if } n > m \\ (y) = \left(x^1, \ldots, x^n\right) & \textit{if } n \leq m \end{array} \tag{4}$$

in these coordinates.

This theorem is an easy consequence of the implicit function theorem [1]. Later on we identify dynamic systems with submanifolds of certain manifolds. To avoid subtleties, we will assume that these submanifolds are at least locally regular.

The following definition tells us that these manifolds have a very simple representation in special coordinates [1], [3].

Definition 1. *Let $\mathcal{M}$ be an m-dimensional smooth manifold. An n-dimensional submanifold $\mathcal{S} \subset \mathcal{M}$ is* regular *iff for each $p \in \mathcal{S}$ there exist local coordinates $(x) = \left(x^1, \ldots, x^m\right)$ defined on a neighborhood $\mathcal{U}_p \subset \mathcal{M}$ of p such that*

$$\mathcal{S} \cap \mathcal{U}_p = \left\{x : x^{n+1} = \cdots = x^m = 0\right\} \tag{5}$$

is met. This coordinate chart is called a flat coordinate chart.

Often, we can derive only local results, which are valid in a sufficiently small neighborhood $\mathcal{U}_p \subset \mathcal{M}$ of $p \in \mathcal{S}$ only. To shorten the notation, we will write $\mathcal{S}_p$ for $\mathcal{S} \cap \mathcal{U}_p$. Regular submanifolds follow often from systems of implicit equations. The general concept is presented by the following theorem, which follows from Theorem 1.

Theorem 2. *Let $\mathcal{M}$ be an m-dimensional smooth manifold with local coordinates (x) and let the $f^i : \mathcal{M} \to \mathbb{R}$, $i = 1, \ldots, m - s$ denote smooth functions, which meet*

$$\bigwedge_{i=1}^{m-s} \mathrm{d}f^i \neq 0 \tag{6}$$

on the subset $\mathcal{S} = f^{-1}(\{0\})$, then $\mathcal{S}$ is a closed regular s-dimensional submanifold of $\mathcal{M}$.

A crucial observation for the problems discussed later on is that the functions f^i of Theorem 2 have no intrinsic meaning in contrast to $\mathcal{S}$. Let $\mathcal{F}_p(\mathcal{M})$ denote the set of smooth functions $f : \mathcal{M} \cap \mathcal{U}_p \to \mathbb{R}$ then it is easy to see that $\mathcal{F}_p$ has the structure of a unitary commutative local ring denoted by $R(\mathcal{F}_p(\mathcal{M}))$. Let $\mathcal{S}_p \subset \mathcal{M}$ be a regular submanifold, then the subset of all functions of $\mathcal{F}_p(\mathcal{M})$, $p \in \mathcal{S}$ vanishing on $\mathcal{S}_p$ forms an ideal denoted by $I(\mathcal{S}_p) \subset R(\mathcal{F}_p(\mathcal{M}))$. On the other hand, we can start with a set of functions $B = \{f\} = \left\{f^1, \ldots, f^{m-s}\right\}$, $f^i(\mathcal{S}_p) = \{0\}$ and study the ideal $\langle f \rangle = \left\langle f^1, \ldots, f^{m-s}\right\rangle$ generated by the functions f. The following theorem tells us, whether the ideals $\langle f \rangle$ and $I(\mathcal{S}_p)$ coincide.

Theorem 3. *Let $\mathcal{M}$ be a smooth m-dimensional manifold and $\mathcal{S}_p$ be a regular s-dimensional submanifold. Let the functions $f^i \in \mathcal{F}_p(\mathcal{M})$, $i = 1, \ldots, m - s$, $p \in \mathcal{S}$ meet $f^i(\mathcal{S}_p) = \{0\}$ and (6) on $\mathcal{S}_p$, then*

$$\langle f \rangle = I(\mathcal{S}_p) \tag{7}$$

is fulfilled. Furthermore, the set $B = \{f\} = \left\{f^1, \ldots, f^{m-s}\right\}$ forms a minimal basis of $I(\mathcal{S}_p)$, or given any function $g \in I(\mathcal{S}_p)$, then one can find functions $\lambda_i \in R(\mathcal{F}_p(\mathcal{M}))$ such that the relation

$$g = \lambda_i f^i \tag{8}$$

is met.

To prove this theorem, we show first that B is a basis of $I(\mathcal{S}_p)$. Therefore, we choose an arbitrary $g \in I(\mathcal{S}_p)$. From

$$\mathrm{d}g \overset{\mathcal{S}_p}{=} \lambda_i \mathrm{d}f^i \ ,$$

where the symbol $\overset{\mathcal{S}}{=}$ indicates that equality has to hold on the submanifold $\mathcal{S}$ only, it follows

$$\mathrm{d}g = \mathrm{d}\left(\lambda_i f^i\right) = \lambda_i \mathrm{d}f^i + f^i \mathrm{d}\lambda_i \overset{\mathcal{S}_p}{=} \lambda_i \mathrm{d}f^i \ .$$

Therefore, $g - \lambda_i f^i$ is constant on $\mathcal{M}$, or $g = \lambda_i f^i$ because of $\lambda_i f^i(\mathcal{S}_p) = g(\mathcal{S}_p)$. The relation (7), as well as that B is a minimal basis, follows directly from Theorem 2.

Let the two manifolds $\mathcal{M}$, $\mathcal{N}$ and the map φ be like in Theorem 1 with $n < m$. The pullback $g \circ \varphi \in \mathcal{F}_p(\mathcal{M})$ of $g \in \mathcal{F}_{\varphi(p)}(\mathcal{N})$ by φ is denoted by $\varphi^*(g)$. Since $\varphi^* : \mathcal{F}_{\varphi(p)}(\mathcal{N}) \to \mathcal{F}_p(\mathcal{M})$ is a ring homomorphism, we obtain $\varphi^*\left(R\left(\mathcal{F}_{\varphi(p)}(\mathcal{N})\right)\right) \subset R(\mathcal{F}_p(\mathcal{M}))$. The following theorem characterizes the intersection of this subring with an ideal of $R(\mathcal{F}_p(\mathcal{M}))$.

Theorem 4. *Let the two manifolds $\mathcal{M}$, $\mathcal{N}$ and the map φ be like in Theorem 1 with $n < m$. Let $\mathcal{S}_p \subset \mathcal{M}$ and $\varphi(\mathcal{S}_p) \subset \mathcal{N}$ be regular submanifolds and let $I(\mathcal{S}_p)$, $I(\varphi(\mathcal{S}_p))$ denote the corresponding ideals, then $\pi_\varphi(I(\mathcal{S}_p))$*

$$\pi_\varphi(I(\mathcal{S}_p)) = I(\mathcal{S}_p) \cap \varphi^*\left(R\left(\mathcal{F}_{\varphi(p)}(\mathcal{N})\right)\right) = \varphi^*(I(\varphi(\mathcal{S}_p))) \tag{9}$$

is an ideal such that its elements are functions of φ only. The corresponding manifold will be denoted by $\pi_\varphi(\mathcal{S}_p)$.

To prove this theorem, we choose any function $g \in \mathcal{F}_{\varphi(p)}(\mathcal{N})$, and get $\varphi^*(g) \in \varphi^*\left(R\left(\mathcal{F}_{\varphi(p)}(\mathcal{N})\right)\right)$. Now, $g \in I(\varphi(\mathcal{S}_p))$ implies $\varphi^*(g)(\mathcal{S}_p) = \{0\}$, or $\varphi^*(g) \in \pi_\varphi(\mathcal{S}_p)$ is met. Now, we choose any function $f \in \pi_\varphi(I(\mathcal{S}_p))$. From $f \in \varphi^*\left(R\left(\mathcal{F}_{\varphi(p)}(\mathcal{N})\right)\right)$ it follows that there exists a function $g \in \mathcal{F}_{\varphi(p)}(\mathcal{N})$ such that $f = \varphi^*(g)$ is met, and $f = g \circ \varphi \in I(\mathcal{S}_p)$ implies $g \in I(\varphi(\mathcal{S}_p))$, since φ is onto.

To give a geometric interpretation of Theorem 4, we choose a map $\psi : \mathcal{N} \to \mathcal{M}$ of maximal rank such that $\psi \circ \varphi(p) = p$ is met, and that $\mathcal{P}_p = \psi(\mathcal{N}) \cap \mathcal{U}_p \subset \mathcal{M}$ is a regular submanifold. Since the function $\pi_{\mathcal{P}}^{\mathcal{M}} = \psi \circ \varphi : \mathcal{M}_p \to \psi(\mathcal{N}_p)$ maps all points of the set $\left(\pi_{\mathcal{P}}^{\mathcal{M}}\right)^{-1}(\{q\})$ to $q \in \mathcal{P}_p$, we may consider $\pi_{\mathcal{P}}^{\mathcal{M}}$ as a projection of $\mathcal{U}_p$ on $\mathcal{P}_p$. In particular, the map $\pi_{\mathcal{P}}^{\mathcal{M}}$ projects $\mathcal{S}_p$ on $\mathcal{P}_p$. Therefore, the ideal $\pi_\varphi(I(\mathcal{S}_p))$ may be identified with the ideal $I\left(\pi_{\mathcal{P}}^{\mathcal{M}}(I(\mathcal{S}_p))\right)$. Obviously, this interpretation depends on the construction of $\mathcal{P}_p$. The situation becomes simpler, if we choose the special coordinates of Theorem 1. Since the ring $\varphi^*\left(R\left(\mathcal{F}_{\varphi(p)}(\mathcal{N})\right)\right)$ contains the functions, which depend only on $x^1, \ldots, x^n$, the ideal $\pi_\varphi(I(\mathcal{S}_p))$ contains all functions of $I(\mathcal{S}_p)$, which are independent of the coordinates $x^{n+1}, \ldots, x^m$. Additionally, let the point p have the coordinates $x = 0$, then $\mathcal{P}_p$ is the plane $x^{n+1} = \cdots = x^m = 0$ and $\pi_{\mathcal{P}}^{\mathcal{M}}$ takes the form $\left(x^1, \ldots, x^m\right) \to \left(x^1, \ldots, x^n, 0, \ldots, 0\right)$.

Given a minimal basis f^i, $i = 1, \ldots, m-s$ for $I(\mathcal{S}_p)$ one can construct a minimal basis for $\pi_\varphi(I(\mathcal{S}_p)) \neq \{0\}$. We form a new basis

$$\hat{f}^l = \lambda_i^l f^i \,, \quad l = 1, \ldots, r \tag{10a}$$
$$\check{f}^{\bar{l}} = \lambda_i^{\bar{l}} f^i \,, \quad \bar{l} = r+1, \ldots, m-s \tag{10b}$$

with a regular matrix $[\lambda_i^{\bar{\imath}}]$, $\bar{\imath} = 1, \ldots, m-s$ such that the equations

$$\lambda_i^l v_k\left(f^i\right) = \mathrm{i}_{v_k}\left(\mathrm{d}f^i\right) = 0 \,, \quad k = 1, \ldots, n-m \tag{11a}$$
$$v_k\left(\varphi^j\right) = \mathrm{i}_{v_k}\left(\mathrm{d}\varphi^j\right) = 0 \,, \quad j = 1, \ldots, n \tag{11b}$$

are met, for all linearly independent solutions of (11b). The set of functions $\hat{f}^l \mathrm{mod}\left\{\check{f}^{\bar{l}} = 0\right\}$ form a basis of $\pi_\varphi(I(\mathcal{S}_p))$. To show our claim, we choose a function $\hat{f}^l$. From $\hat{f}^l \in \varphi^*\left(R\left(\mathcal{F}_p\left(\mathcal{N}\right)\right)\right)$ and $\hat{f}^l \in I(\mathcal{S}_p)$ it follows

$$\mathrm{d}\hat{f}^{\bar{l}} = \mu_j^l \mathrm{d}\varphi^j = \lambda_i^l \mathrm{d}f^i .$$

This equation is solvable, iff (11a) is met for all linearly independent solutions of (11b). Let us denote the inverse of $[\lambda_i^{\bar{\imath}}]$ by $\left[\bar{\lambda}_i^{\bar{\imath}}\right]$. Then the relations

$$\begin{aligned} v_k\left(\lambda_i^l f^i\right) &= v_k\left(\lambda_i^l\right) f^i + \lambda_i^l v_k\left(f^i\right) \\ &= v_k\left(\lambda_i^l\right) \bar{\lambda}_l^i \hat{f}^l + \underbrace{v_k\left(\lambda_i^l\right) \bar{\lambda}_{\bar{l}}^i \check{f}^{\bar{l}} + \lambda_i^l v_k\left(f^i\right)} \end{aligned}$$

show that $v_k\left(\lambda_i^l f^i\right) \in \pi_\varphi(I(\mathcal{S}_p))$ since the underbraced term vanishes on the solution set $\check{f}^{\bar{l}} = 0$ because of (11a). Now $v_k\left(\lambda_i^l f^i\right) \in \pi_\varphi(I(\mathcal{S}_p))$ implies that $v_k\left(\hat{f}^l\right) \overset{\pi_\varphi(\mathcal{S}_p)}{=} 0$ or $\left\langle \hat{f} \right\rangle = \pi_\varphi(I(\mathcal{S}_p))\, mod \left\{\check{f}^{\bar{l}} = 0\right\}$ is met. It is worth mentioning that in general the computation of the functions needs the implicit function theorem, but for polynomial systems we can use Groebner basis for the determination of the corresponding elimination ideal [2].

3 Jet Manifolds and Differential Equations

Throughout this contribution we use the concept of fibered manifolds and jet-bundles. A fibered manifold is a triple $(\mathcal{E}, \pi, \mathcal{B})$ with the total manifold $\mathcal{E}$, the base manifold $\mathcal{B}$ and the surjective submersion $\pi : \mathcal{E} \to \mathcal{B}$ ([4], [5]). For each point $x \in \mathcal{B}$, the subset $\pi^{-1}(x) = \mathcal{E}_x$ is called the fiber over x. If all fibers $\mathcal{E}_x$ are diffeomorphic to a manifold $\mathcal{F}$, then we call $(\mathcal{E}, \pi, \mathcal{B})$ a bundle. In the finite dimensional case with $\dim \mathcal{E} = p + q$, $\dim \mathcal{B} = p$ we can introduce adapted coordinates $\left(x^i, u^\alpha\right)$ at least locally, with the independent coordinates x^i, $i = 1, \ldots, p$ and the dependent ones u^α, $\alpha = 1, \ldots, q$. We will write $\mathcal{E}$ instead of $(\mathcal{E}, \pi, \mathcal{B})$, whenever the projection π and the base manifold $\mathcal{B}$ follow from the context. A section σ of $\mathcal{E}$ is a map $\sigma : \mathcal{B} \to \mathcal{E}$ such that $\pi \circ \sigma = \mathrm{id}_{\mathcal{B}}$ is met, where $\mathrm{id}_{\mathcal{B}}$ denotes the identity map on $\mathcal{B}$. From now on we use Latin indices for the independent and Greek indices for the dependent variables.

Although jet-bundles and systems of differential equations admit a coordinate free description, we will us adapted coordinates from now on to simplify the following consideration. Let f be a smooth section of $(\mathcal{E},\pi,\mathcal{B})$. The k^{th} order partial derivatives of f will be denoted by

$$\frac{\partial^k}{\partial_1^{j_1}\cdots\partial_p^{j_p}}f=\partial_J f=f_J\ ,\quad \partial_i=\frac{\partial}{\partial x^i}$$

with $J=j_1,\ldots,j_p$, and $k=\#J=\sum_{i=1}^{p}j_i$. J is nothing else than an ordered multi-index ([4], [5]). The special index $J=j_1,\ldots,j_p$, $j_i=\delta_{ik}$ will be denoted by 1_k and $J+1_k$ is a shortcut for $j_i+\delta_{ik}$ with the Kronecker symbol δ_{ik}. Let $(\mathcal{E},\pi,\mathcal{B})$ be a bundle with adapted coordinates $\left(x^i,u^\alpha\right)$ and let f be a smooth section of $\mathcal{E}$. We can prolong f to a map $j^1f:x\to(x,f(x),\partial_i f(x))$, then the first jet $j_x^1 f$ of f at x is the equivalence class of sections f, g with $j^1f(x)=j^1g(x)$. Roughly speaking, the first jet manifold $J^1(\mathcal{E})$ may be considered as a container for all first order jets of smooth sections of $\mathcal{E}$. An adapted coordinate system of $\mathcal{E}$ induces an adapted system on $J^1(\mathcal{E})$, which is denoted by $\left(x^i,u^\alpha,u^\alpha_{1_i}\right)$ with the pq new coordinates $u^\alpha_{1_i}$. The manifold $J^1(\mathcal{E})$ has two natural projections, $\pi:J^1(\mathcal{E})\to\mathcal{B}$ and $\pi_0^1:J^1(\mathcal{E})\to\mathcal{E}$ with $\pi\left(j^1f(x)\right)=x$ and $\pi_0^1\left(j^1f(x)\right)=f(x)$. Furthermore, $\left(J^1(\mathcal{E}),\pi_0^1,\mathcal{E}\right)$ is an affine bundle and $\left(J^1(\mathcal{E}),\pi,\mathcal{B}\right)$ is bundle, if $\mathcal{E}$ is one. Analogously to the first jet of a section f, we define the n^{th}-jet j^nf of f by $j^nf=(x,f(x),\partial_J f(x))$, $\#J=1,\ldots,n$. The n^{th}-jet manifold $J^n(\mathcal{E})$ of $\mathcal{E}$ may be considered as a container for n^{th}-jets of sections of $\mathcal{E}$ ([3], [4], [5]). Furthermore, an adapted coordinate system of $\mathcal{E}$ induces an adapted system on $J^n(\mathcal{E})$ with $\left(x^i,u^{(n)}\right)$ and $u^{(n)}=u^\alpha_J$, $\alpha=1,\ldots,q$, $\#J=0,\ldots,n$. The natural projections are given by $\pi:J^n(\mathcal{E})\to\mathcal{B}$ and $\pi_m^n:J^n(\mathcal{E})\to J^m(\mathcal{E})$, $m=1,\ldots,n-1$ with $\pi(j^nf(x))=x$ and $\pi_m^n(j^nf(x))=j^mf(x)$. To simplify certain formulas later on, we set $J^0(\mathcal{E})=\mathcal{E}$.

Let us consider a bundle $(\mathcal{E},\pi,\mathcal{B})$ with adapted coordinates $\left(x^i,u^\alpha\right)$, $i=1,\ldots,p$, $\alpha=1,\ldots,q$ and its n^{th}-order jet bundle $J^n(\mathcal{E})$. Because of definition 1, a regular submanifold $\mathcal{S}_p^n\subset J^n(\mathcal{E})$, $p\in\mathcal{S}^n$ corresponds locally to a system of equations

$$f^i\left(x,u^{(n)}\right)=0\ ,\quad i=1,\ldots,n_e \tag{12}$$

that involves jet coordinates $u^{(n)}$ of u up to the order n. If we substitute the n^{th}-jet $j^n\sigma$ of a smooth section σ into (12), then we get a system of ordinary $(p=1)$ or partial differential equations $(p>1)$ of the type

$$f^i\left(j^n\sigma(x)\right)=0\ ,\quad i=1,\ldots,n_e\ . \tag{13}$$

¿From now on, we will deal exclusively with the submanifold $\mathcal{S}_p^n$, and the ideal $I\left(\mathcal{S}_p^n\right)$ instead of the differential equations (12). There exist two natural operations for $\mathcal{S}_p^n$, $I\left(\mathcal{S}_p^n\right)$, their projection and their prolongation [4]. The projection of $I\left(\mathcal{S}_p^n\right)$ is simply given by $I\left(\mathcal{S}_p^n\right)\cap R\left(J^m(\mathcal{E})\right)$, see Theorem 4 with $\varphi=\pi_m^n$ and will be denoted by $\pi_m^n\left(I\left(\mathcal{S}_p^n\right)\right)$. We assume that the ideal $\pi_m^n\left(I\left(\mathcal{S}_p^n\right)\right)$ describes a regular submanifold of $J^m(\mathcal{E})$ denoted by $\pi^n_m\left(\mathcal{S}^n_p\right)$. To define the prolongation of $\mathcal{S}^n_p$, we introduce the total derivative d_i,

$$d_i = \partial_i + u^\alpha_{J+1_i}\partial^J_\alpha \ , \quad \partial^J_\alpha = \frac{\partial}{\partial u^\alpha_J} \tag{14}$$

with respect to the independent variables x^i [3], [4], [5]. It is straightforward to see that for any section σ of $\mathcal{E}$ and $f \in \mathcal{F}(J^n(\mathcal{E}))$ the following relations

$$d_i f \circ j^{k+1}\sigma = \partial_i f\left(j^k\sigma\right) \tag{15}$$

are met. Therefore, d_i is a map $d_i : \mathcal{F}(J^n(\mathcal{E})) = \mathcal{F}\left(J^{n+1}(\mathcal{E})\right)$. The ideal $J\left(I\left(\mathcal{S}^n_p\right)\right) = I\left(\mathcal{S}^n_p\right) + \left\langle d_i f,\ i = 1,\ldots,q,\ f \in I\left(\mathcal{S}^n_p\right)\right\rangle$ is the first prolongation of $I\left(\mathcal{S}^n_p\right)$. Obviously, the ideal $J^1\left(I\left(\mathcal{S}^n_p\right)\right)$ contains all functions of $I\left(\mathcal{S}^n_p\right)$, as well as their total derivatives with respect to the independent variables. Again, we assume that $J\left(I\left(\mathcal{S}^n_p\right)\right)$ defines a regular submanifold, which we denote by $J\left(\mathcal{S}^n_p\right)$ by some abuse of notation. The n^{th} prolongation of $I\left(\mathcal{S}^n_p\right)$ is, as usual defined by $J^n\left(I\left(\mathcal{S}^n_p\right)\right) = J\left(J^{n-1}\left(I\left(\mathcal{S}^n_p\right)\right)\right)$ and $J^0\left(I\left(\mathcal{S}^n_p\right)\right) = I\left(\mathcal{S}^n_p\right)$. Given a minimal basis $B\left(\mathcal{S}^n_p\right) = \left\{f^1,\ldots,f^k\right\}$ of $I\left(\mathcal{S}^n_p\right)$ one can construct a minimal basis of $J\left(I\left(\mathcal{S}^n_p\right)\right)$ in a straightforward manner. We determine the set $B \cup \left\{d_i f^j : j = 1,\ldots,k,\ i = 1,\ldots,q\right\}$ and take out a minimal number of functions such that the members of the remaining set $B\left(J\left(\mathcal{S}^n_p\right)\right)$ are functionally independent.

Let σ be a section $\mathcal{E}$ such that $j^n_x\sigma \in \mathcal{S}^n_p$ is met. By construction any section of (12) is a solution of the prolonged and projected systems. Unfortunately, this is not true for points in general, since a point $j^{n+r}_x\sigma$ with $\pi^{n+r}_n\left(j^{n+r}_x\sigma\right) \in J^n(\mathcal{E})$ may fail to meet $j^{n+r}_x\sigma \in J^r\left(\mathcal{S}^n_p\right)$. This is a result of the fact that in general $I\left(\mathcal{S}^n_p\right) \subset \pi^{n+r}_n\left(J^r\left(I\left(\mathcal{S}^n_p\right)\right)\right)$ or $\pi^{n+r}_n\left(\mathcal{S}^n_p\right) \subset \mathcal{S}^n_p$ is met only. Formally integrable systems do not show this unpleasant behavior.

Definition 2. *Let $\mathcal{S}^n_p$ be a locally regular submanifold of the n^{th} jet manifold $J^n(\mathcal{E})$ of the bundle $\mathcal{E}$. We call the system $\mathcal{S}^n_p$* formally integrable*, iff its r^{th} prolongations $J^r\left(\mathcal{S}^n_p\right)$ are regular submanifolds of $J^{n+r}(\mathcal{E})$ and*

$$\pi^{n+r+s}_{n+r}\left(J^{r+s}\left(\mathcal{S}^n_p\right)\right) = J^r\left(\mathcal{S}^n_p\right)$$

is met for all $r, s \geq 0$.

Roughly speaking, formally integrable systems do not contain hidden constraints. These constraints are often called integrability conditions. It is an extremely hard job to check a general system for formal integrability. Fortunately, this is not true for systems of ordinary differential equations and systems of linear partial differential equations of the Frobenius-type.

4 Main Results

Let $(\mathcal{E}, \pi, \mathcal{B})$ be a smooth manifold with adapted coordinates (t, z^{α_z}) in one independent variable t and q dependent variables z^{α_z}, $\alpha_z = 1,\ldots,q$ with $n_e \leq$

q. In the case $\dim \mathcal{B} = 1$, we denote the single independent coordinate by t and choose x or z for the dependent coordinates, as customary. We study the following set of n_e nonlinear ordinary differential equations

$$f^{i_e}(t, z, z_1) = 0 \ , \quad i_e = 1, \dots, n_e \tag{16}$$

with smooth functions $f^{i_e} : J^1(\mathcal{E}) \to \mathbb{R}$. To simplify the following considerations we assume that

$$\mathrm{d}t \wedge \bigwedge_{i_e=1}^{n_e} \mathrm{d}f^{i_e}(p) \neq 0 \tag{17}$$

is met for a point $p \in J^1(\mathcal{E})$ with $f^{i_e}(p) = 0$. This condition ensures that the system (16) defines a regular submanifold $\mathcal{S}_p^1 \subset J^1(\mathcal{E})$ of $J^1(\mathcal{E})$ near p, which contains an open interval of the independent variable t.

Let us choose a point $p \in \mathcal{S}^1$ and a neighborhood $\mathcal{U}_p \subset J^1(\mathcal{E})$. Then we are ready to pose the following problem. Is it possible to construct a transformation near p that converts the implicit system (16) to an explicit system of the form

$$x_1^{\alpha_x} = f^{\alpha_x}\left(t, x, u^{(n)}\right) \ , \quad \alpha_x = 1, \dots, n_x \ . \tag{18}$$

Here, x denotes the state and $u^{(n)}$ the n^{th} jet of the input u^{α_u}, $\alpha_u = 1, \dots, n_u$, the functions, which we can choose freely.

A crucial observation for the problem above is that the system (16) has no intrinsic meaning in contrast to $\mathcal{S}_p^1$ or the ideal $I\left(\mathcal{S}_p^1\right)$. Taking into account the considerations of Section 2, this problem takes the form of the ideal membership problem, whether the relations

$$(x_1^{\alpha_x} - f^{\alpha_x}) \circ j^n \varphi \in I\left(\mathcal{S}_p^1\right)$$

can be fulfilled for suitable functions f^{α_x} or not. Obviously, we have to construct a suitable basis to perform the ideal membership test.

Let us assume that the system (16) is rewritten in the following form

$$\begin{aligned} f^{\alpha_x}(t, z, z_1) &= 0 \ , \quad \alpha_x = 1, \dots, n_x \\ f^{\alpha_s}(t, z, z_1) &= 0 \ , \quad \alpha_s = n_x + 1, \dots, n_e \ , \end{aligned}$$

where the functions f^{α_s} meet the relations

$$\partial_{z^{\alpha_z}}^1 f^{\alpha_s} = 0$$

on the solution set $f^{\alpha_x}(t, z, z_1) = 0$. This implies that we can eliminate the variable z_1 from the equations f^{α_s}. Therefore, we will write $f^{\alpha_s} = f^{\alpha_s}(t, z)$ and consider the special system

$$f^{\alpha_x}(t, z, z_1) = 0 \ , \quad \alpha_x = 1, \dots, n_x \tag{19a}$$

$$\begin{aligned} d_1 f^{\alpha_s}(t, z) &= 0 \\ f^{\alpha_s}(t, z) &= 0 \end{aligned} \ , \quad \alpha_s = n_x + 1, \dots, n_e \ . \tag{19b}$$

Additionally, we assume that the system (19) is formally integrable or

$$\mathrm{d}t \wedge \bigwedge_{\alpha_z} \mathrm{d}z^{\alpha_z} \wedge \bigwedge_{\alpha_x} \mathrm{d}f^{\alpha_x} \wedge \bigwedge_{\alpha_s} \mathrm{d}d_1 f^{\alpha_s} \overset{\mathcal{S}_p^1}{\neq} 0 \ , \tag{20}$$

is fulfilled.

Now, we claim that the formally integrable system (19) with $q \geq n_e$ is equivalent to the explicit form (18)

$$x_1^{\alpha_x} - \bar{f}^{\alpha_x}\left(t, x, s, u^{(1)}\right) = 0 \ , \quad \alpha_x = 1, \ldots, n_x \tag{21a}$$

$$\begin{aligned} s_1^{\alpha_s} &= 0 \\ s^{\alpha_s} &= 0 \end{aligned} \ , \quad \alpha_s = n_x + 1, \ldots, n_e \tag{21b}$$

with $n = 1$. To prove this claim, we introduce the new bundle $\bar{\mathcal{E}}$ with adapted coordinates (t, x, s, u) and the bundle diffeomorphism $\varphi : \mathcal{E} \to \bar{\mathcal{E}}$,

$$\begin{aligned} x^{\alpha_x} &= \varphi^{\alpha_x}(t, z) \ , & \alpha_x &= 1, \ldots, n_x \\ s^{\alpha_s} &= \varphi^{\alpha_s}(t, z) = f^{\alpha_s} \ , & \alpha_s &= n_x + 1, \ldots, n_e \\ u^{\alpha_u} &= \varphi^{\alpha_u}(t, z) \ , & \alpha_u &= n_e + 1, \ldots, q \end{aligned} \tag{22a}$$

$$z^{\alpha_z} = \psi^{\alpha_z}(t, x, s, u) \tag{22b}$$

that does not change the independent variable t. Following the considerations above, we have to check, whether the functions of (21) are contained in the ideal $I\left(\mathcal{S}_p^1\right)$, or

$$\left(x_1^{\alpha_x} - \bar{f}^{\alpha_x}\right) \circ j\varphi = \lambda_{\beta_x}^{\alpha_x} f^{\beta_x} + \lambda_{\beta_s}^{\alpha_x,1} d_1 f^{\beta_s} + \lambda_{\beta_s}^{\alpha_x} f^{\beta_s} \tag{23a}$$

$$s_1^{\alpha_s} \circ j\varphi = \lambda_{\beta_x}^{\alpha_s} f^{\beta_x} + \lambda_{\beta_s}^{\alpha_s,1} d_1 f^{\beta_s} + \lambda_{\beta_s}^{\alpha_s} f^{\beta_s} \tag{23b}$$

is met. It is straightforward to solve (23b) and the solution is

$$\varphi^{\alpha_s} = f^{\alpha_s} \ , \quad \lambda_{\beta_x}^{\alpha_s} = 0 \ , \quad \lambda_{\beta_s}^{\alpha_s,1} = \delta_{\beta_s}^{\alpha_s,1} \ , \quad \lambda_{\beta_s}^{\alpha_s} = 0 \ . \tag{24}$$

To solve (23a) we consider its total derivative with respect to t,

$$\partial_{\alpha_z} \varphi^{\alpha_x} z_2^{\alpha_z} - \partial_{\alpha_u}^1 \bar{f}^{\alpha_x} \partial_{\alpha_z} \varphi^{\alpha_u} z_2^{\alpha_z} + \ldots \overset{\mathcal{S}_p^1}{=} \lambda_{\beta_x}^{\alpha_x} \partial_{\alpha_z}^1 f^{\beta_x} z_2^{\alpha_z} + \lambda_{\beta_s}^{\alpha_x,1} \partial_{\alpha_z} \varphi^{\beta_s} z_2^{\alpha_z} + \ldots \ ,$$

where the terms not involving z_2 are omitted. Comparing the coefficients of $z_2^{\alpha_z}$, we derive

$$\partial_{\alpha_z} \varphi^{\alpha_x} - \partial_{\alpha_u}^1 \bar{f}^{\alpha_x} \partial_{\alpha_z} \varphi^{\alpha_u} = \lambda_{\beta_x}^{\alpha_x} \partial_{\alpha_z}^1 f^{\beta_x} + \lambda_{\beta_s}^{\alpha_x,1} \partial_{\alpha_z} \varphi^{\beta_s}$$

and rewrite these relations as

$$\left(\mathrm{d}\varphi^{\alpha_x} - \partial_{\alpha_u}^1 \bar{f}^{\alpha_x} \circ j\varphi \mathrm{d}\varphi^{\alpha_u}\right) \wedge \mathrm{d}t \overset{\mathcal{S}_p^1}{=} \left(\lambda_{\beta_x}^{\alpha_x} \partial_{\alpha_z}^1 f^{\beta_x} \mathrm{d}z^{\alpha_z} + \lambda_{\beta_s}^{\alpha_x,1} \mathrm{d}\varphi^{\beta_s}\right) \wedge \mathrm{d}t \ . \tag{25}$$

Let us choose functions φ^{α_x}, φ^{α_u} such that φ is a diffeomorphism and

$$\bigwedge_{\alpha_x} \partial_{\alpha_z}^1 f^{\alpha_x} \mathrm{d}z^{\alpha_z} \wedge \bigwedge_{\alpha_u} \mathrm{d}\varphi^{\alpha_u} \wedge \bigwedge_{\alpha_s} \mathrm{d}\varphi^{\alpha_s} \overset{\mathcal{S}_p^1}{\neq} 0$$

is met, this is always possible, then it is easy to see that one derives the remaining functions $\lambda_{\beta_x}^{\alpha_x}$, $\lambda_{\beta_s}^{\alpha_x,1}$ immediately from (25). Finally, the functions $\bar{f}^{\alpha_x}$ follow from (23a). This proves our claim and leads to the following theorem.

Theorem 5. *Let (19) describe a formally integrable system, then there exists locally a diffeomorphism (22) such that the equivalent system takes the form (21).*

An interesting problem is, whether we can choose $\bar{f}^{\alpha_x}(t,x,s,u)$ in (22a) instead of $\bar{f}^{\alpha_x}\left(t,x,s,u^{(1)}\right)$. It is straightforward to see that this restriction has no impact on (25), but the terms involving $\bar{f}^{\alpha_x}$ in (25) vanish. This implies that we cannot use the simple strategy from above to determine the functions φ^{α_x}, φ^{α_u}. Now we have to find functions φ^{α_x} such that

$$\mathrm{d}\varphi^{\alpha_x} \wedge \mathrm{d}t \overset{\mathcal{S}_p^1}{=} \left(\lambda_{\beta_x}^{\alpha_x} \partial_{\alpha_z}^1 f^{\beta_x} \mathrm{d}z^{\alpha_z} + \lambda_{\beta_s}^{\alpha_x,1} \mathrm{d}\varphi^{\alpha_s}\right) \wedge \mathrm{d}t \tag{26}$$

is met. The compatibility conditions for this problem are

$$\mathrm{d}\left(\partial_{\alpha_z}^1 f^{\beta_x} \mathrm{d}z^{\alpha_z}\right) \wedge \mathrm{d}t \overset{\mathcal{S}_p^1}{=} \left(\theta_{\beta_x}^{\alpha_x} \wedge \partial_{\alpha_z}^1 f^{\beta_x} \mathrm{d}z^{\alpha_z} + \theta_{\alpha_s}^{\alpha_x} \wedge \mathrm{d}\varphi^{\alpha_s}\right) \wedge \mathrm{d}t \tag{27}$$

for suitable 1-forms $\theta_{\beta_x}^{\alpha_x}$, $\theta_{\alpha_s}^{\alpha_x}$. Now, the theorem of Frobenius [1], [3] tells us that the relation (27) implies locally the existence of the functions φ^{α_x}. Unfortunately, here we have to solve a system of partial differential equations. The next theorem summarizes this result.

Theorem 6. *Let (19) describe a formally integrable system that meets (27) additionally. Then there exists locally a diffeomorphism (22) such that the equivalent system takes the form (21), where the functions $\bar{f}^{\alpha_x}$ are independent of u_1.*

It is worth mentioning that the compatibility conditions are always met by DAE-systems like (2), see [6]. The Theorems 5, 6 deal with the two cases where derivatives of all inputs or no derivatives of the inputs are admitted. An open problem, at least according to the knowledge of the authors, is to minimize the number of inputs, whose derivatives appear in the normal form (21a). Unfortunately, this problem is beyond the scope of this contribution, since it leads to a system of nonlinear partial differential equations. Of course, the case of one input is solved, which follows from a trivial argument.

Finally, we have to show that we can transform the system (16) to the formally integrable one (19), if some regularity conditions are met. We choose $\varphi = \pi_0^1 : J(\mathcal{E}) \to \mathcal{E}$ or in coordinates $\varphi : (t,z,z_1) \to (t,z)$ and apply Theorem 4 and the relations (10), (11) to rewrite (16) as

$$I\left(\mathcal{S}_p^1\right) = \left\langle f^{\alpha_x^0}, f^{\alpha_s^0} \right\rangle \ , \quad f^{\alpha_s^0} \in \pi_0^1\left(I\left(\mathcal{S}_p^1\right)\right)$$

with $\alpha_x^0 = 1,\ldots,n_x^0$, $\alpha_s^0 = n_x^0+1,\ldots,n_e$. We set $r=0$ and run the following algorithm.

1. Using Theorem 4 and the relations (10), (11), we calculate N^r,
$$N^r = \left\langle f^{\alpha_x^r}, d_1 f^{\alpha_s^r} \right\rangle \cap R\left(\mathcal{F}_p\left(\mathcal{E}\right)\right) \ .$$
If $N^r = \langle 0 \rangle$ or equivalently (20) is met then we are done.
2. Eliminate a minimal number of functions from the set $\left\{f^{\alpha_x^r}\right\}$, $\alpha_x^{r+1} = 1, \ldots, n_x^{r+1}$ such that the set $\left\{f^{\alpha_x^{r+1}}\right\}$, $\alpha_x^{r+1} = 1, \ldots, n_x^{r+1}$ meets (20).
3. Determine a minimal number of functions $f^{\alpha_s^{r+1}}$, $\alpha_s^{r+1} = n_x^{r+1} + 1, \ldots, n_e^{r+1}$ such that $\left\langle f^{\alpha_s^{r+1}} \right\rangle = N^r + \left\langle f^{\alpha_s^r} \right\rangle$ is fulfilled. Set $r = r + 1$. If the solution set of $f^{\alpha_x^r} = d_1 f^{\alpha_s^r} = f^{\alpha_s^r} = 0$, $\alpha_x^r = 1, \ldots, n_x^r$, $\alpha_s^r = n_x^r + 1, \ldots, n_e^r$ describes a regular manifold, then go to 1, otherwise stop.

Two facts are worth mentioning. If all manifolds are regular, then this algorithm stops reliably after a finite number of steps. Carrying out the elimination step 1, one has to deal with nonlinear algebraic equations. The general case is a hard problem for any computer algebra system, but two cases admit a significant simplification. Linear algebra is enough to solve this problem for descriptor systems (3), see [6], and one can use Groebner basis [2], if the equations are polynomial ones.

5 Conclusions

This contribution is devoted to the equivalence problem for systems of nonlinear implicit ordinary differential equations, where equivalence means that the solution set of the original system and a given normal form coincide. Since regular submanifolds in certain jet spaces were chosen for the mathematical models of these dynamic systems, one can identify geometric objects like submanifolds with algebraic objects like ideals over certain rings. This combination of methods from differential and algebraic geometry allows us to reduce the equivalence problem to an ideal membership problem. This problem has been solved for the two cases where the derivatives of all inputs or no derivatives of the inputs are admitted, but there is still the open problem to minimize the number of inputs, whose derivatives appear in the normal form. Unfortunately, the latter problem leads to the test, whether a system of nonlinear partial differential equations admits a solution or not. To the best of the authors´ knowledge a test for this problem exists only for the analytic scenario. Therefore, the future research will be devoted to this problem.

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