Using computer algebra tools to classify serial manipulators (Extended abstract)

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Industrial robotic 3-DOF manipulators are currently designed with very simple geometric rules. In order to enlarge the the possibilities of such manipulators, it is interesting to relax some constraints.

The behavior of the manipulators when changing posture depends strongly on the design parameters and it can be very different from the one of manipulators commonly used in Industry. P. Wenger and J. El Omri [6], [10] have shown that for some choices of the parameters, 3-DOF manipulators may be able to change posture without meeting a singularity in the joint space. This kind of manipulators is called **cuspidal**.

It is worth noting that in case of obstructed environment, this property would yield more flexibility which can be very useful in practice for industrial purpose.

They succeed in characterizing 3-revolute jointed manipulators using a homotopy based classification scheme [9], but they needed general conditions on the design parameters, more precisely they wanted to find answers to the following issues:

- **Problem 1** : For given parameters, is the manipulator cuspidal?
- **Problem 2** : For which values of the parameters is a manipulator cuspidal?

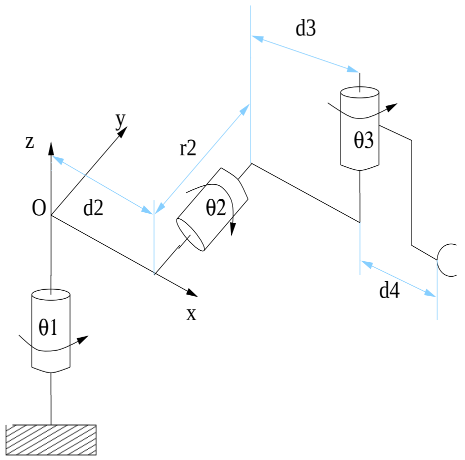

Figure 1: The manipulator under our hypothesis

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We restrict the study to 3-DOF manipulators as described in figure . As recalled in the first section, testing if such a manipulator is cuspidal or not is equivalent to deciding if an algebraic set has real roots or not (the first problem, equivalent to decide if a zero-dimensional algebraic system has real roots or not, is a particular case of the second one).

We manage to answer both questions under a few hypothesis which are not restricting ones according to roboticians : for example it is impossible to construct, in practice, a manipulator whose parameters lies on a strict hypersurface of the space of the parameters.

Precisely, we compute a partition of the space of parameters, such that in cell of maximal dimension, the behavior of the manipulator is known (cuspidal or not) while the other cells are embedded inside strict algebraic subsets of the parameters' space.

1 The kinematic map

The 4 design parameters are d_2, d_3, d_4 and r_2. In order to normalize (d_2, d_3, d_4, r_2) and to reduce the number of parameters, we assume that d_2 is equal to 1.

Along our study, we will work into two different spaces:

- the **joint space** described by the joint variables $(\theta_1, \theta_2, \theta_3)$, this space is isomorphic to $]-\pi, \pi]^3$ as the joint limits will be ignored.
- the **task space** representing the position of the end-effector in the Cartesian coordinates (x, y, z), which is isomorphic to $\mathbb{R}^3$ (no obstacles).

The kinematic map f maps the joint space on the task space :

$$\begin{array}{cccc} f : &]-\pi,\pi]^3 & \longrightarrow & \mathbb{R}^3 \\ & (\theta_1,\theta_2,\theta_3) & \longmapsto & (x,y,z) \end{array}$$

The image of this map in the task space is called the **workspace**.

Under our hypothesis, the expression of f is the following one:

$$\left\{ \begin{array}{lcl} x & = & (d_3 + \cos\theta_3 d_4)(\cos\theta_1 \cos\theta_2) + (r_2 + d_4 \sin\theta_3)\sin\theta_1 + \cos\theta_1 \\ y & = & (d_3 + \cos\theta_3 d_4)(\sin\theta_1 \cos\theta_2) - (r_2 + d_4 \sin\theta_3)\cos\theta_1 + \sin\theta_1 \\ z & = & (d_3 + \cos\theta_3 d_4)\sin\theta_2 \end{array} \right. \tag{1}$$

2 Algebraic characterization of cuspidal manipulators

The cuspidal manipulators are the ones able to change posture without meeting a singularity in the joint space. It was shown in [6] that a 3-DOF manipulator can execute a non-singular change of posture if and only if there exist at least one point in its workspace with exactly three coincident inverse kinematic solutions (corresponding in a cross section of the workspace to cusp points, hence the word **cuspidal**).

Expressing such a condition using directly the kinematic map lead to solve a huge system of equations.

By taking $s_i = sin(\theta_i)$ and $c_i = cos(\theta_i), i = 1 \ldots 3$ and $t = tan(\theta_3/2)$ and adding the algebraic relations $s_i^2 + c_i^2 = 1$ $i = 1 \ldots 3$ and $s_3 = 2t/1 - t^2, c_3 = (1+t^2)/(1-t^2)$, we have to study an algebraic system of equations. On can then show that under the conditions $x^2 + y^2 \neq 0$ and $z \neq 0$, deciding if a robot is cuspidal is equivalent to deciding if a polynomial P of degree 4 in t (whose coefficients are polynomial with respect to $x, y, z, d_4, d_3, r2$) admits real triple roots. Moreover, to a triple root of P corresponds one unique cuspidal configuration.

The full expression of P is the following : $P(t) = (at^4 + bt^3 + ct^2 + dt + e)$, with
$\left\{\begin{array}{lcl} a & = & m_5 - m_2 + m_0 \\ b & = & -2m_3 + 2m_1 \\ c & = & -2m_5 + 4m_4 + 2m_0 \\ d & = & 2m_3 + 2m_1 \\ e & = & m_5 + m_2 + m_0 \end{array}\right.$, $\left\{\begin{array}{lcl} m_0 & = & -R + z^2 + r_2^2 + \frac{(R+1-L)^2}{4} \\ m_1 & = & 2r_2d_4 + (L-R-1)d_4r_2 \\ m_2 & = & (L-R-1)d_4d_3 \\ m_3 & = & 2r_2d_3d_4^2 \\ m_4 & = & d_4^2(r_2^2+1) \\ m_5 & = & d_4^2d_3^2 \end{array}\right.$, with $R = x^2 + y^2 + z^2$ and
$L = d_4^2 + d_3^2 + r_2^2$.

We will show later that the design parameters d_4, d_3, r_2 of cuspidal manipulators such that $z(x^2 + y^2) = 0$ are in a strict hypersurface of $\mathbb{R}^3$.

So, finding parameters' values defining cuspidal manipulators remains to find the values of d_4, d_3, r_2 such that $P(t)$ has triple points by solving the following system (as the boundaries of the workspace form a revolution surface around the axis Oz, it is generically zero-dimensional once the parameters are fixed) :

$$\left\{\begin{array}{lcl} P & = & 0 \\ \frac{\partial P}{\partial t} & = & 0 \\ \frac{\partial^2 P}{\partial t^2} & = & 0 \end{array}\right. \qquad (2)$$

3 Partition's boundaries of the parameter's space

Solving system 2 consists in finding values of the parameters d_4, d_3, r_2 such that the induced robots are cuspidal or equivalently such that the system 2 admits real roots. Precisely, our goal is now to compute a partition of the parameter's space such that the number of real solutions of system 2 in each cell is constant. Due to practical constrains, we are only interested computing one sample point or a bowl in the cells of highest dimension : the other possible cells will be embedded inside strict algebraic subsets of the parameter's space.

3.1 Second elimination step

Due to she shape of the system to be solved and to the properties of the solutions (the boundaries of the workspace form a revolution surface around the axis Oz), our first step consists in eliminating two of the variables t, R and $Z = z^2$ in the system 2. The hope is to obtain an eliminating polynomial depending on the parameters d_4, d_3, r_2 and on one of the three variables t, R and Z, with uniquely defined solutions with respect to the two remaining variables in the fibers. Theoretically, the possibility of getting such an equivalent system without loosing the algebraic structure (ideal) depends strongly on the shape of the solutions and is in general not possible.

So, we choose to represent the solutions of system 2 as *regular* zeroes of triangular sets (with respect to the terminology of [1]), with the hope that the non regular solutions correspond to values of the parameters contained in strict hypersurfaces of the space of the parameters.

Thanks to an clever intermediate change of variables (after several attempts), we manage to define the solutions of the problem as *regular* (with respect to the terminology of [1]) roots of a triangular system with the following shape :

$$\left\{\begin{array}{l} surf(R, d_4, d_3, r_2) = 0 \\ lc_Z(d_4, d_3, r_2)Z + tr_Z(R, d_4, d_3, r_2) \\ lc_t(d_4, d_3, r_2)t + tr_t(R, Z, d_4, d_3, r_2) \end{array}\right. \qquad (3)$$

3.2 *Regular* solutions conditions

The solutions of system 3 which are regular in the triangular sets terminology lies on strict algebraic varieties in the parameter's space defined by the equations : $lc_Z(d_4, d_3, r_2) = 0$ and $lc_t(d_4, d_3, r_2) = 0$.

In other words, system 3 describes all the solutions of the problem for values of the parameters taken outside the two algebraic varieties $lc_Z(d_4, d_3, r_2) = 0$ and $lc_t(d_4, d_3, r_2) = 0$, which are closed subsets of strict smaller dimension of the parameter's space and so can be excluded for practical issues.

3.3 Real Roots existence

Under the conditions given previously, the initial system has real solutions if and only if the polynomial $surf$ has real positive roots with respect to variable R. The number of real roots of $surf$ varies if and only if its discriminant or its leading coefficient with respect to R vanishes.

So, the last set of equations to be computed for defining our partition of the parameters' space in cells where the number of real solutions to system 3 is constant is defined by these two conditions.

Let denote by $dis_R(d_4, d_3, r_2)$ and $lc_R(d_4, d_3, r_2)$ the two polynomials defining these two varieties.

The real roots of $surf(R) = 0$ must verify $Z = z^2 > 0$ and $R - Z = x^2 + y^2 > 0$ to be admissible. Adding the condition $Z = 0$ (resp. $R - Z = 0$) to the system, give us (after an elimination process) two polynomials in the parameters d_3, d_4 and r_2.

Let note $Hyp_{Z=0}(d_4, d_3, r_2)$ and $Hyp_{R-Z=0}(d_4, d_3, r_2)$ those two polynomials.

4 Partition's cells computation

As established before, in each connected subset of the parameter's space where none of the following polynomials vanish $dis_R(d_4, d_3, r_2)$, $lc_R(d_4, d_3, r_2)$, $Hyp_{Z=0}(d_4, d_3, r_2)$, $Hyp_{R-Z=0}(d_4, d_3, r_2)$, $lc_Z(d_4, d_3, r_2)$, $lc_t(d_4, d_3, r_2)$, the system 2 has a constant number of real solutions.

The best way for representing such cells is now to compute a partial CAD (Cylindrical Algebraic Decomposition - see [3]) of $\mathbb{R}^3$ adapted to this set of polynomials. For practical reasons, we are only interested in finding one point or a bowl in the cells of higher dimension, embedding the other cells inside algebraic subsets of the parameters' space. This make much more easier the projection (much less resultant computations) and lifting phases (no computations with real algebraic numbers) of the CAD.

We mix several technics for computing this partial CAD ([5] for reducing the number of polynomials to be computed, [2] for testing is some algebraic sets have real roots or not), and we finally obtain at least one point with positive coordinates in the interior of each cell of higher dimension and a set of algebraic sets that contains the other cells of the full CAD.

It is then sufficient, for each sample point, to solve the zero-dimensional system 3 (counting the number of real roots) after specialization, adding the equations $T_1 Z^2 - 1 = 0$ and $T_2(R - Z)^2 - 1 = 0$ to discriminate the admissible solutions, to get the number of cusps corresponding to the selected set of parameters. This can easily be done using algorithmic solutions proposed in [7], [8] and [4].

5 Results

The final result of the full computation is a partial cellular decomposition of the parameter's space so that for each point taken in the interior of any cell, the number of solutions to the system 2.

Precisely, we have computed :

- at least on point in each cell, as far as possible from the boundaries of the cell;

- the equations of the algebraic sets that bounds these cells;

In practice, we provided 6 polynomials and 105 sample points which represents a reasonable output since it allows roboticians to analyse the results.

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