Algebraic Representation, Expansion and Simplification in Automated Geometric Theorem Proving

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In this talk we consider the problem of multiple algebraic representations and multiple elimination results, and the solution by combining expansion and simplification techniques, in automated geometric theorem proving with Grassmann-Cayley algebra, Clifford algebra and their related bracket algebras.

To better understand the situation of the problem, we divide the content into three parts.

1 Projective geometry with Grassmann-Cayley algebra and bracket algebra

Multiple algebraic representations usually occur in problems related to conics. The basic constraint that six points in the projective plane are on the same conic has 15 different representations by bracket equalities. In proving a theorem involving such a constraint in either its hypothesis or conclusion, the proving procedure can be made drastically different (either much simplified or much more difficult) by choosing different representations. In general all possible representations of a geometric constraint must be considered, and techniques on optimal selection should be developed.

Multiple elimination results come from different expansions of the same Cayley expression into bracket polynomials. For example, let $\mathbf{a}, \mathbf{b}, \mathbf{c}$ be three points in the projective plane, let $\mathbf{a}$ be the intersection of lines $\mathbf{12}$ and $\mathbf{34}$, let $\mathbf{b}$ be the intersection of lines $\mathbf{1'2'}$ and $\mathbf{3'4'}$, and let $\mathbf{c}$ be the intersection of lines $\mathbf{1''2''}$ and $\mathbf{3''4''}$. In Grassmann-Cayley algebra the constructions of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are represented by the following Cayley equalities:

$$\mathbf{a} = \mathbf{12} \wedge \mathbf{34}, \quad \mathbf{b} = \mathbf{1'2'} \wedge \mathbf{3'4'}, \quad \mathbf{c} = \mathbf{1''2''} \wedge \mathbf{3''4''}.$$

Here the juxtaposition $\mathbf{12}$ denotes the join of vectors $\mathbf{1}, \mathbf{2}$, and the wedge symbol denotes the meet of two bivectors. If we want to eliminate points $\mathbf{a}, \mathbf{b}, \mathbf{c}$ from the bracket $[\mathbf{abc}]$, we only need to substitute the corresponding Cayley expressions of the three points into the bracket, and we get

$$[\mathbf{abc}] = [(\mathbf{12} \wedge \mathbf{34})(\mathbf{1'2'} \wedge \mathbf{3'4'})(\mathbf{1''2''} \wedge \mathbf{3''4''})].$$

The right side of the equality has 16847 different expansions into bracket polynomials. Choosing different expansions has considerable influence upon the computational complexity of later algebraic manipulations. Thus, techniques on optimal selection of expansions should be developed.

In the polynomial ring of coordinates, simplification techniques of a polynomial expression contain expansions, collections, factorizations, etc. In the bracket ring, similar simplification techniques must be developed in order to carry out efficient computation. The corresponding techniques are called Cayley expansions, contractions and Cayley factorizations respectively.

2 Affine geometry with affine Grassmann-Cayley algebra, affine bracket algebra and heterogeneous Grassmann-Cayley algebra

The most common model of an affine space is a hyperplane away from the origin in a vector space with one more dimension. The hyperspace composed of all displacement vectors fully characterizes the affine space in that a vector in the surrounding vector space represents an affine point if and only if it does not belong to the hyperspace. Vectors in the hyperspace represents points at infinity of the affine space. In this model we can simply adopt the Grassmann-Cayley algebra and bracket algebra for the projective geometry of all the affine points and the points at infinity, under the modification that a new operator should be developed to distinguish affine points from points at infinity. This operator, known as the boundary operator in elementary topology, is simply a bracket operator in bracket algebra. The modified version of Grassmann-Cayley algebra and bracket algebra are called affine Grassmann-Cayley algebra and affine bracket algebra respectively, and a parallel development of the expansion and simplification techniques is necessary.

By discarding the homogeneous feature of the affine Grassmann-Cayley algebra, we get another version of the algebra, which generalizes the classical Grassmann algebraic representation of affine geometry.

3 Metric geometry with different Clifford algebras and Clifford bracket algebras

For the same metric geometry, there are usually several models to realize them in inner product spaces. For Euclidean geometry, for example, one of the most useful models is to realize the Euclidean space in the null cone of a Minkowski space with two more dimensions. For each model there is a Clifford algebra generated by the corresponding inner product space. Each algebra presents a different algebraization of the same geometric problem.

The techniques of expansion and simplification can be further developed in Clifford algebra. In fact, they constitute the main contents of the so-called expansion theory of general Clifford algebras and Clifford bracket algebra. Powerful new computational tools for metric geometries can be developed in this way.

In this workshop, we intend to talk about our recent research work on the above three topics: algebraic representation, expansion and simplification in applying Grassmann-Cayley algebra and Clifford algebra in automated geometric theorem proving.