A Special Central Configuration

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Abstract

In this paper, we consider the flat central configurations of bodies using the characteristic set method. We solve a symmetry four bodies problem.

1 Introduction

The problem of determining the central configuration of n-body motions is classical. In fact, as early as in the 18th century, Lagrange and Euler already solved the problem of central configurations of three bodies, namely, three collinear solutions and one equilateral triangle solution[1]. For the case where the particles are collinear, Moulton[2] solved this problem for arbitrary n particles with distinct masses. This result was reverified by Smale and his followers by using topological method(see [3] and [4] for more details). Wintner[1] gave a conjecture which said the configurations with any number of particles with any masses(see also [5]). Waldvogel[7] showed that for any n ($n > 4$) there exists a group of n particles possessing a non-flat central configuration,which showed that the number of central configuration were larger than that of flat central configurations. Wu[5] reduced this problem to a problem of polynomial equation-solving,and reverified the case of three particles using the characteristic set method.In this paper, we find relationship between two unknown masses of a special four planets by characteristic set method.

2 Notations

Let $m_1, m_2, \cdots, m_n$ be the masses of n particles moving under mutual Newtonian gravitational attractions, and $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ be the positions of these masses at a certain moment, with $\mathbf{r}_i \neq \mathbf{r}_j$ for $i \neq j$. Let

$$\begin{bmatrix} m_1, & m_2, & \cdots, & m_n \\ \mathbf{r}_1, & \mathbf{r}_2, & \cdots, & \mathbf{r}_n \end{bmatrix}$$

denote a configuration formed by masses m_i at position $\mathbf{r}_i$ $(1 \leq i \leq n)$

Definition 2.1 The center of n particles $m_1, m_2, \cdots, m_n$ at the positions $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ is defined to be the position $\sum_{i=1}^n m_i \mathbf{r}_i / \sum_{i=1}^n m_i$.

Definition 2.2 A configuration

$$\begin{bmatrix} m_1, & m_2, & \cdots, & m_n \\ \mathbf{r}_1, & \mathbf{r}_2, & \cdots, & \mathbf{r}_n \end{bmatrix}$$

is called a central configuration w.r.t. the masses $m_1, m_2, \cdots, m_n$, if there are initial velocities of the masses m_i such that under the Newtonian gravitational attractions the configurations formed by the masses during the motion remain the same as the initial ones.

From this definition, we know that a central configuration defines a class of configurations. As a direct consequence of Newtonian mechanics, for a central configuration

$$\begin{bmatrix} m_1, & m_2, & \cdots, & m_n \\ \mathbf{r}_1, & \mathbf{r}_2, & \cdots, & \mathbf{r}_n \end{bmatrix}$$

the center of mass of the masses $m_1, m_2, \cdots, m_n$ is fixed during the motion.

Definition 2.3 The inertial coordinate system associated with a central configuration

$$\begin{bmatrix} m_1, & m_2, & \cdots, & m_n \\ \mathbf{r}_1, & \mathbf{r}_2, & \cdots, & \mathbf{r}_n \end{bmatrix}$$

is defined to be a cartesian coordinate system whose origin is at the fixed center of the masses $m_1, m_2, \cdots, m_n$.

In the following, we assume that $\mathbf{r}_1, \mathbf{r}_2, \cdots, \mathbf{r}_n$ are the initial coordinates of the particles $m_1, m_2, \cdots, m_n$, respectively.

By the definitions of the center and the inertial coordinate system, we have

$$\sum_{i=1}^{n} m_i \mathbf{r}_\mathrm{i} = 0$$

.

Finding all central configurations for an arbitrary number n of points is a difficult problem, which is still open. Leaving the trivial case $n = 2$, we list results known presently in this field.

For $n = 3$ the only noncollinear central configuration is formed by the vertices of an equilateral triangle (Lagrange). For $n = 4$ the unique nonflat configuration is given by the vertices of a regular tetrahedron. The collinear central configurations are described by Moulton: there are at least $n!/2$ distinct collinear central configurations. For $n = 3$ there are exactly three such configurations, discovered by Euler. In [8], the author discussed two special four planets : rhombus and square cases, and present all the solutions. For more general cases we have the following conjecture.

Wintner Conjecture The number of central configurations for n points is finite.

3 Reduction to Polynomial-Solving

We take a planar inertial coordinate system (x, y) with origin O at the center of masses. Let $\mathbf{r}_i = (x_i, y_i)$, $x_{ij} = x_i - x_j$, $y_{ij} = y_i - y_j$, $r_{ij} = \sqrt{x_{ij}^2 + y_{ij}^2}$, $(1 \le i, j \le n, i \ne j)$. By Newtonian mechanics, we have a general system of equation[5]

$$\omega^2 x_i = \sum_{j=1, j\ne i}^{n} m_j \frac{x_{ij}}{r_{ij}^3}, \quad \omega^2 y_i = \sum_{j=1, j\ne i}^{n} m_j \frac{y_{ij}}{r_{ij}^3} \qquad (1 \le i \le n),$$

where ω is an angular velocity, which is a constant.

Let $u_i = x_j + \mathbf{i} y_j$, $v_j = x_j - \mathbf{i} y_j \quad (1 \le j \le n)$, where $\mathbf{i} = \sqrt{-1}$, and $m_0 = -\omega^2$. Then from the above equations, we obtain the following system of equations

$$\begin{array}{lll}
m_0 u_i + \displaystyle\sum_{j=1, j\ne i}^{n} \frac{m_j}{(v_i - v_j) r_{ij}} = 0 & (1 \le i \le n), & \text{(I)} \\
m_0 v_i + \displaystyle\sum_{j=1, j\ne i}^{n} \frac{m_j}{(u_i - u_j) r_{ij}} = 0 & (1 \le i \le n), & \text{(II)} \\
r_{ij}^2 - (u_i - u_j)(v_i - v_j) = 0 & (i \ne j;\, 1 \le i \le n), & \\
r_{ij} = r_{ji} & (i \ne j;\, 1 \le i \le n). &
\end{array}$$

Let $p_1, p_2 \cdots, p_n; p_1', p_2', \cdots, p_n'$ be the polynomials obtained from the left-hand side of (I),(II), resp., of the equation system above by clearing the fractions; let $u_{ij} = u_i - u_j$, $v_{ij} = v_i - v_j \, (i \ne j)$ and $p_{ij} = r_{ij}^2 - u_{ij} v_{ij} \, (i < j)$.

Suppose PS is a set of polynomials in

$$m_k, r_{ij}, m_0, u_k, v_k, u_{ij}, v_{ij} \ (i < j, \ 1 \le i, j, k \le n).$$

Let $\mathrm{Zero}_{rc}(PS)$ be the zero set of PS, of which the variables are r_{ij}, u_{ij}, v_{ij}, u_i, v_i, such that

$$r_{ij} > 0, m_k > 0, \ (i < j, \ 1 \le i, j, k \le n), \ m_0 < 0,$$

which are called the restricted zero sets of PS.

Hence, the problem of determining the flat central configurations is reduced to that of determining the restricted zero set of the polynomial system

$$\{p_k, p_k', p_{ij} \,|\, i < j, 1 \le i, j, k \le n\} \qquad (**)$$

with variables u_{ij}, v_{ij}, r_{ij} $(i < j, 1 \le i, j, k \le n)$, for which $r_{ij} > 0$.

Since $m_0 \ne 0$ and

$$\sum_{i=1}^{n} m_i (m_0 u_i + \sum_{j=1, j\ne i}^{n} \frac{m_j}{(v_i - v_j) r_{ij}}) = m_0 (\sum_{i=1}^{n} m_i u_i),$$

$$\sum_{i=1}^{n} m_i (m_0 v_i + \sum_{j=1, j\ne i}^{n} \frac{m_j}{(u_i - u_j) r_{ij}}) = m_0 (\sum_{i=1}^{n} m_i v_i),$$

Letting

$$p_0 = \sum_{i=1}^{n} m_i u_i, \quad p_0' = \sum_{i=1}^{n} m_i v_i,$$

then adding these two polynomials p_0, p_0' to polynomial system does not change its restricted zero set, namely,

$$\mathrm{Zero}_{rc}(p_k,\, p_k',\, p_{ij}) = \mathrm{Zero}_{rc}(p_0,\, p_0',\, p_k,\, p_k', p_{ij}), \quad i < j,\, 1 \le i,\, j,\, k \le n$$

Note that p_0 is equivalent to any one of the following

$$t_k = \sum_{i=1}^{n} m_i u_k + \sum_{i=1,\, i \ne k}^{n} m_i (u_i - u_k), \quad 1 \le k \le n$$

and p_0' is equivalent to any one of the following

$$t_k' = \sum_{i=1}^{n} m_i v_k + \sum_{i=1,\, i \ne k}^{n} m_i (v_i - v_k), \quad 1 \le k \le n$$

So that the restricted zero set $\mathrm{Zero}_{rc}(p_0,\, p_0', p_k,\, p_k', p_{ij} \,|\, i < j,\, 1 \le i,\, j,\, k \le n)$ does not change when we delete p_k, p_k' for some k in actual computation.

For the restricted zero set of the system of polynomials($**$),we have

Proposition 3.1[8] There is a one-to-one correspondence between the restricted zero set of polynomial system($**$) with the variables u_i, v_i, r_{ij} $(i < j,\, 1 \le i,\, j,\, k \le n)$ and that of

$$\{p_0,\, p_0',\, q_k,\, q_k',\, p_{ij} \,|\, i < j,\, 1 \le i,\, j,\, k \le n\} \qquad (***)$$

with the variables u_1, v_1, u_{1j}, v_{1j}, r_{ij} $(i < j,\, 1 \le i,\, j \le n)$, where

$$q_k = \sum_{i=1, i \ne k}^{n} f_{ik}\, m_i \Big(\prod_{l=1, l \ne k, l \ne i}^{n} r_{il}\, (v_{1l} - v_{1i}) \Big)$$
$$q_k' = \sum_{i=1, i \ne k}^{n} f_{ik}\, m_i \Big(\prod_{l=1, l \ne k, l \ne i}^{n} r_{il}\, (u_{1l} - u_{1i}) \Big)$$

f_{ik} appearing in q_k and q_k' is

$$f_{ik} = \sum_{j=1}^{n} m_j + m_0 r_{ik}^3.$$

Note that $r_{ik} = r_{ki}$, $f_{ik} = f_{ki}$. In fact, we reduce p_k by t_k, $p_{ij}(1 \le i < j \le n)$ by eliminating variables u_k, $u_{ij}(1 \le i < j \le n)$, the remainder is q_k. Similarly, we reduce p_k' by t_k', $p_{ij}'(1 \le i < j \le n)$ by eliminating variables v_k, $v_{ij}(1 \le i < j \le n)$ in turn, the remainder is q'_k. And $u_{il} = u_{1l} - u_{1i}$, $v_{il} = v_{1l} - v_{1i}$, hence the proposition 3.1 has been proved(see [8] for more details).

By this proposition, to solve the restricted zero set of polynomial system(**), we only need to solve that of polynomial system(* * *). There are two ways to solve the restricted zero set of (* * *):

(**w1**) Solve its characteristic set directly

(**w2**) Divide it into two subsets, and then solve the characteristic sets of each subset independently; after that, combine the characteristics set of these two subsets to form some new systems of polynomials. Solving these system further by **w1**.

In fact, we solve the central configurations of three masses by **w1**, while for $n = 4$, we follow the **w2**[8].

4 A Special Center Configuration of Four Planets

In this case $m_2 = 1$, $m_4 = 1$, $r_{12} = b$, $r_{23} = d$, $r_{34} = d$, $r_{14} = b$, $r_{13} = 1$, $r_{24} = f$, where b, d, f are parameters, we want to know the relationship between two unknown masses m_1 and m_3.

Based on the discussion in last sections, we now delete p_4, p'_4 and then compute the characteristic sets of

$$\{p_0, p'_0, p_k, p'_k, p_{ij} \mid 1 \le i < j \le 4, 1 \le k \le 3\}$$

by (**w2**). The initial polynomials are:

$$\begin{aligned}
p_0 &= m_1u_1 + m_2u_2 + m_3u_3 + m_4u_4 \\
p'_0 &= m_1v_1 + m_2v_2 + m_3v_3 + m_4v_4 \\
p_1 &= v_{12}v_{13}v_{14}b\,m_0u_1 + v_{13}v_{14} + v_{12}v_{14}b\,m_3 + v_{12}v_{13} \\
p_2 &= v_{12}v_{23}v_{24}b\,d\,f\,m_0u_2 - v_{23}v_{24}d\,f\,m_1 + v_{12}v_{24}b\,f\,m_3 + v_{12}v_{23}b\,d \\
p_3 &= v_{13}v_{23}v_{34}d\,m_0u_3 - v_{23}v_{34}d\,m_1 - v_{13}v_{34} + v_{13}v_{23} \\
p'_1 &= u_{12}u_{13}u_{14}b\,m_0v_1 + u_{13}u_{14} + u_{12}u_{14}b\,m_3 + u_{12}u_{13} \\
p'_2 &= u_{12}u_{23}u_{24}b\,d\,f\,m_0v_2 - u_{23}u_{24}d\,f\,m_1 + u_{12}v_{24}b\,f\,m_3 + u_{12}u_{23}b\,d \\
p'_3 &= u_{13}u_{23}u_{34}d\,m_0v_3 - u_{23}u_{34}d\,m_1 - u_{13}u_{34} + u_{13}u_{23} \\
p_{12} &= b^2 - u_{12}v_{12} \\
p_{13} &= 1 - u_{13}v_{13} \\
p_{14} &= b^2 - u_{14}v_{14} \\
p_{23} &= d^2 - u_{23}v_{24} \\
p_{24} &= f^2 - u_{24}v_{24} \\
p_{34} &= d^2 - u_{34}v_{34}
\end{aligned}$$

Note that $r_{12} = r_{14}$, $r_{23} = r_{34}$ hence $f_{12} = f_{14}$, $f_{23} = f_{34}$. We denote f_{12} by f_b, f_{13} by f_a, f_{23} by f_d, f_{24} by f_f, then

$$
\begin{aligned}
f_a &= (2 + m_1 + m_3) + m_0 \\
f_b &= (2 + m_1 + m_3) + m_0 b^3 \\
f_d &= (2 + m_1 + m_3) + m_0 d^3 \\
f_f &= (2 + m_1 + m_3) + m_0 f^3
\end{aligned}
$$

By proposition 3.1, we have

$$
\begin{aligned}
q_1 &= -f_b v_{13} v_{14} - f_b v_{12} v_{13} - v_{12} v_{14} b\, m_3 f_a \\
q_2 &= -(v_{13} - v_{12})(v_{14} - v_{12}) d\, f\, m_1 f_b + v_{12}(v_{13} - v_{12}) b\, d\, f_f + \\
&\quad v_{12}(v_{14} - v_{12}) b\, f\, m_3 f_d \\
q_3 &= -f_d v_{13}(v_{14} - v_{13}) + f_d v_{13}(v_{13} - v_{12}) - (v_{13} - v_{12})(v_{14} - v_{13}) d\, m_1 f_a \\
q_1' &= -f_b u_{13} u_{14} - f_b u_{12} u_{13} - u_{12} u_{14} b\, m_3 f_a \\
q_2' &= -(u_{13} - u_{12})(u_{14} - u_{12}) d\, f\, m_1 f_b + u_{12}(u_{13} - u_{12}) b\, d\, f_f + \\
&\quad u_{12}(u_{14} - u_{12}) b\, f\, m_3 f_d \\
q_3' &= -f_d u_{13}(u_{14} - u_{13}) + f_d u_{13}(u_{13} - u_{12}) - (u_{13} - u_{12})(u_{14} - u_{13}) d\, m_1 f_a
\end{aligned}
$$

In general, the restricted zero set of above is divided into three cases according to how many f_{ij} it has (see [8]).

Case 1 If there is exactly one f_{ij} being zero, then polynomial system has exactly collinear solutions;
Case 2 If there are two f_{ij} being zero, then in reality polynomial system has no solutions in reality;
Case 3 If there is no f_{ij} being zero, then polynomial system has exactly non-collinear solutions.

Therefore we only consider case 3 in our problem. Namely, $f_a \neq 0$, $f_b \neq 0$, $f_d \neq 0$, $f_f \neq 0$.

Proposition 4.1

1) Computing $p_0,\, q_k, p_{ij} \,|\, i < j,\, 1 \leq i,\, j,\, k \leq 4$, we obtain four components:

$$PS_1,\, PS_2,\, PS_3,\, PS_4$$

with

$$\mathrm{Zero}_{rc}(p_0,\, q_k,\, p_{ij} \,|\, i < j,\, 1 \leq i,\, j,\, k \leq 4) = \bigcup_{i=1}^{4} \mathrm{Zero}_{rc}(PS_i)$$

2) Computing $p_0',\, q_k', p_{ij} \,|\, i < j,\, 1 \leq i,\, j,\, k \leq 4$, we also obtain four components:

$$PS_1',\, PS_2',\, PS_3',\, PS_4'$$

with

$$\mathrm{Zero}_{rc}(p_0,\, q_k,\, p_{ij}\,|\, i<j,\, 1\le i,\, j,\, k\le 4) = \bigcup_{i=1}^{4} \mathrm{Zero}_{rc}(PS'_i)$$

proof we reduce $q_1,\, q_2,\, q_3$ by eliminating $v_{12},\, v_{13}$ and factoring the remainders we obtain 1). Similarly, we can get 2).

By the proposition we have

$$\begin{aligned}
&\mathrm{Zero}_{rc}(p_0,\, p'_0,\, q_k,\, q'_k,\, p_{ij}) \\
&\quad = \mathrm{Zero}_{rc}(p_0,\, q_k,\, p_{ij}) \bigcap \mathrm{Zero}_{rc}(p'_0,\, q'_k,\, p_{ij}) \\
&\quad = \left(\bigcup_{i=1}^{4} \mathrm{Zero}_{rc}(PS_i)\right) \bigcap \left(\bigcup_{j=1}^{4} \mathrm{Zero}_{rc}(PS'_j)\right) \\
&\quad = \bigcup_{i,\, j=1}^{4} \left(\mathrm{Zero}_{rc}(PS_i) \bigcap \mathrm{Zero}_{rc}(PS'_j)\right) \\
&\quad = \bigcup_{i,\, j=1}^{4} \mathrm{Zero}(PS_i \bigcup PS'_j)
\end{aligned}$$

Where $i<j,\, 1\le i,\, j,\, k\le 4$. We denote $PS_i \cup PS'_j$ by $P_{ij},\, i,\, j=1,2,3,4$, then we have 16 components. After checking each P_{ij}, we find that we only need to compute $PS_i \cup PS'_j,\ i\le j=1,2,3,4$, namely,

$$\bigcup_{i,\, j=1}^{4} \mathrm{Zero}\left(PS_i \bigcup PS'_j\right) = \bigcup_{1\le i\le j\le 4}^{4} \mathrm{Zero}_{rc}\left(PS_i \cup PS'_j\right)$$

We compute every component above, and obtain five polynomials $s_1,\, s_2,\, s_3,\, s_4,\, s_5$. For the sake of saving space, we only give two small size polynomials $s_1,\, s_2$, here.

$$\begin{aligned}
s_1 = \;& 4m_1^5m_3 + 8m_1^4m_3^2 + m_1^3m_3^3 - 5m_1^2m_3^4 - \\
& m_1m_3^5 + m_3^6 + 36m_1^4m_3 + 36m_1^3m_3^2 - 27m_1^2m_3^3 - 18m_1m_3^4 + \\
& 9m_3^5 + 16m_1^4 + 108m_1^3m_3 - 36m_1^2m_3^2 - 89m_1m_3^3 + 39m_3^4 + \\
& 96m_1^3 + 12m_1^2m_3 - 228m_1m_3^2 + 99m_3^3 + 144m_1^2 - 176m_1m_3 + \\
& 220m_3^2 + 64m_1 + 400m_3 + 320 \\
s_2 = \;& 3m_1^6 + m_1^5m_3 - 7m_1^4m_3^2 - m_1^3m_3^3 + 4m_1^2m_3^4 - 4m_1m_3^5 - 4m_3^6 + \\
& 27m_1^5 - 6m_1^4m_3 - 37m_1^3m_3^2 + 16m_1^2m_3^3 - 16m_1m_3^4 - 36m_3^5 + \\
& 105m_1^4 - 31m_1^3m_3 - 32m_1^2m_3^2 - 4m_1m_3^3 - 108m_3^4 + 225m_1^3 - \\
& 12m_1^2m_3 + 24m_1m_3^2 - 108m_3^3 + 216m_1^2 - 40m_1m_3 - 40m_3^2 - \\
& 72m_1 - 192m_3 - 192
\end{aligned}$$

The total degrees of s_3 , s_4 and s_5 are 32, 30,and 94 respectively. The maximal degrees of $m_1,\, m_3$ in s_3 are the same which is 28, and the number of monomials in s_3 is 535.

The maximal degree of m_1 in s_4 is 30, and three more than that of m_3 in s_4. The number of monomials in s_4 is 484. The number of monomials in s_5 is 4318.

Now we will discuss the classification of roots about s_1 and s_2. The same discuss is applicable to s_3, s_4, s_5.

We begin with introducing some definitions and results in [10].

Given a polynomial $g(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_n$, whose derivative is denoted by $g(x) = 0 \cdot x^n + n a_0 x^{n-1} + (n-1) a_1 x^{n-2} + \cdots + a_{n-1}$.

Definition 4.1 The matrix

$$\begin{bmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} & a_n & & \\ 0 & n\,a_0 & (n-1)a_1 & \cdots & a_{n-2} & a_{n-1} & & \\ & a_0 & a_1 & \cdots & 2a_{n-2} & a_{n-1} & a_n & \\ & 0 & n\,a_n & \cdots & 3a_{n-2} & 2a_{n-2} & a_{n-1} & \\ & & & \cdots & \cdots & & & \\ & & & \cdots & \cdots & & & \\ & & & & a_0 & a_1 & \cdots & a_n \\ & & & & 0 & n\,a_n & \cdots & a_{n-1} \end{bmatrix}$$

is called the Sylvester matrix of $g(x)$ with respect to variable x , denoted by Sylvester$(g,\, x)$, whose rank is $2n$. Its determinant is called resultant of g , denoted by $\mathrm{res}(g,\, x)$, namely,

$$\mathrm{res}(g,\, x) = \det(\mathrm{Sylvester}(g,\, x)).$$

And

$$(-1)^{\frac{n(n-1)}{2}} \frac{\mathrm{res}(g,\, x)}{a_0}$$

is said to be the discriminant of $g(x)$, denoted by $\mathrm{discrim}(g, x)$.

Proposition 4.2 There are repeated roots of $g(x)$ if and only if

$$\mathrm{discrim}(g,\, x) = 0$$

.

We give an algorithm to solve the root classification of a polynomial $g(x)$ with a parameter u in coefficients.

step 1 Compute the discriminant of g w.r.t u, discricm$(g9x)$, which is a polynomial with variable u, then compute the solutions of parameter u.

step 2 Putting the constant of $g(x)$ equals zero, and solving this equation, we can obtain the solutions of parameter u.

step 3 All solutions we obtain by step 1 and step 2 divide the real axis into different intervals. we take arbitrary point from those intervals, and compute the solutions of original polynomial when parameter take this value.

By this algorithm we get the following proposition

Proposition 4.3

1. There are no roots of s_1 which satisfy the restricted conditions
2. The roots of s_2, which satisfy the restricted conditions, are simple.
3. There are no roots of s_3 which satisfy the restricted conditions.
4. There is at least simple real root of s_4 that satisfies restricted conditions.
5. There is at least simple real root of s_5 that satisfies restricted conditions.

where the restricted conditions is $m_1 > 0$, $m_3 > 0$.

Then we get the relation between m_1 and m_3 satisfies polynomial s_2, s_4, s_5.

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