THE NONSOLVABILITY BY RADICALS OF GENERIC 3-CONNECTED PLANAR GRAPHS

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1. Introduction

A fundamental problem in Computer Aided Design (CAD) is the formulation of effective approximation schemes or algebraic algorithms which solve for the location of points on a plane, given a set of relative separations (dimensions) between them. For CAD applications, an important class of configurations are are those for which the dimensions are just sufficient to ensure that the points are located rigidly with respect to one another. These configurations are known to correspond to maximally independent graphs. A number of algebraic and numerical methods have been proposed for solving these configurations (Owen [**8**], Bouma et al [**1**], Light and Gossard [**7**]) and these have been successfully implemented in CAD programs.

These algebraic methods assemble the solution for complete configurations from the solutions of rigid subcomponents. The assembly process involves only rigid body transformations and the solution of quadratic equations. The simplest subcomponent is a triangle of points and this is solvable by quadratic equations. All the other subcomponents in this process are represented by graphs which are 3-connected (in the usual sense of vertex 3-connected [**11**]) and so the problem of solving any general configuration is reduced to the problem of solving just those configurations which are represented by 3-connected graphs.

We have previously suggested that with generic dimension values a subcomponent which is represented by a 3-connected graph cannot be solved by quadratic equations (Owen [**8**]). This would mean that for generic dimensions the existing algebraic methods solve all configurations which can be solved by quadratic equations. Such configurations are also known as "ruler and compass constructible". Gao and Chou [**4**] have given a procedure for determining in principle if any given

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configuration is ruler and compass constructible. However their analysis is based on the detail of derived elimination equations and they do not address the problem of solvability for general classes of graphs.

We propose to strengthen the suggestion above to the following conjecture:

Conjecture. A generic configuration of dimensioned points on the plane whose graph is 3-connected is not solvable by radicals.

If the conjecture is true this would mean that a generic configuration either has all its solutions in quadratic extensions or all of its solutions are non-radical and so cannot be expressed in terms of any radical expression in the dimensions. In particular the current algorithms solve all generic configurations which can be solved by radicals.

We have proved Conjecture 1 for the class of graphs which have a planar embedding. For this class we establish a reduction scheme in which we can prove that the solution to any rigid 3-connected graph with a planar embedding contains as a special case the solution to the single smallest rigid 3-connected planar graph shown in Figure 1, which we call the doublet. We then use the Maple Mathematical software package to show that there are certain dimension values for which the doublet has non-radical roots.

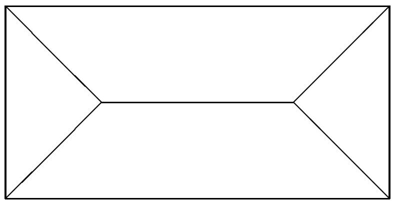


Figure 1.

The proof scheme is both lengthy and eclectic, drawing on new and known results from graph theory, algebraic geometry and Galois theory. In this talk we will give an outline of the main ideas and state the relevant theorems, the proofs of which will appear elsewhere.

2. Generic rigid graphs

In this section we give some results which apply to all generic rigid graphs. In the following section we will focus on planar graphs.

We begin with some definitions which will allow us to refer to a configuration of points and dimensions by their corresponding graph and to be precise about the meaning of generic and rigid. There is a close connection between our formalism and that of the theory of rigid frameworks (Whiteley **[12]**) and in particular the notion of an independent graph is taken from this context.

Definition 2.1. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. Then G is *independent* if $2|V(G)| - |E(G)| \geq 3$ and if every vertex induced subgraph of G (with more than 2 vertices) is also independent. The graph G is *maximally independent* if G is independent and $2|V(G)| - |E(G)| = 3$.

Definition 2.2. Let G be a maximally independent graph with a selected base edge b. A *normalised dimension set* for G is a set of non-negative numbers $\{d_e : e \in E(G)\}$ for which there is a map $\phi : V(G) \to \mathbb{C}^2$ such that $d_b = 1$ and $d_e = ||\phi(v) - \phi(w)||_2^2$ for $e = (v, w) \in E(G)$. A dimension set is *generic* if $\{d_e : e \in E(G), e \neq b\}$ consists of algebraically independent transcendentals.

We write f_e for the polynomial $||\phi(v) - \phi(w)||_2^2 - d_e$ and refer to the set of equations $\{f\} = \{f_e = 0 : e \neq b\}$ as the *constraint equations* for the given dimension set. We are ultimately interested in the real realisations $\phi : V(G) \to \mathbb{R}^2$, normalised so that the vertices of b map to $(0, 0)$ and $(1, 0)$, and in the algebraic properties of the real coordinates (x_i, y_i) of the points $\phi(v_i)$ for the remaining (non-base) vertices $v_1, ..., v_n$. However to gain the correct perspective on the proper notion of rigidity it is convenient to embrace algebraically closed fields at the outset, as we do in the next definition. Note in particular the distinction between "real rigid" and "complex rigid" (which we prefer to call zero dimensional) in the example indicated after Theorem 2.9.

With the introduction of complex coordinates (x_i, y_i), $1 \leq i \leq n = |V(G)| - 2$, as above, we can define the complex algebraic variety $W(\{f\})$ for the constraint equations to be the complex variety in $\mathbb{C}^{2n}$ formed by the zeros of these equations. We also refer to points of $W(\{f\})$ as roots, particularly when they are known to be isolated.

Definition 2.3. Let G be a maximally independent graph. A normalised dimension set $\{d_e\}$ for G is said to be *zero dimensional* if the complex algebraic variety $W(\{f\})$ of the corresponding constraint equations is a finite, non-empty set.

The next theorem shows that generic graphs have dimension zero.

Theorem 2.4. *Every generic, maximally independent graph has dimension zero.*

Proof: Laman **[6]** proves that every generic, maximally independent graph has a specialisation which is infinitesimally rigid. It follows immediately that every generic, maximally independent graph is infinitesimally rigid (Whiteley [**12**]) and has dimension zero.

Definition 2.5. A point $\{\phi(v) : v \in V(G)\}$ of a zero dimensional variety $W(\{f\})$ is *non-radical* if at least one of the coordinates of the points $\phi(v)$ does not lie in a radical extension field of the field $Q(\{d_e\})$.

As a useful shorthand, we will describe a dimensioned graph as generic if its dimension set is generic and describe it as zero dimensional if its variety is zero dimensional.

We can immediately state and prove the following two significant theorems which apply to any generic graph. Together they imply that if a generic zero dimensional graph has one non-radical root then all its roots are non-radical.

Theorem 2.6. *Let $W(\{f\}) \subseteq \mathbb{C}^{2n}$ be the complex algebraic variety of the constraint equations for the dimension set $\{d_e\} = \{d_1, ..., d_r\}$ (where $r = |E(G)| - 1$) and let $W_b \subseteq \mathbb{C}^{2n+r}$ be the variety determined by the constraint equations as polynomial functions belonging to*

$$\mathbb{Q}[d_1, ..., d_r, x_1, ..., x_n, y_1, ..., y_n].$$

If $W(\{f\})$ is not empty, then W_b is irreducible.

Proof: The theorem follows since the big variety W_b is defined parametrically by the variables x_i, y_i. (See [**5**].)

Theorem 2.7. *Let $W(\{f\})$ be the algebraic variety of a set of generic constraint equations. If $W(\{f\})$ has dimension zero and a non-radical root then all its roots are non-radical.*

Proof sketch: Since $W(\{f\})$ has dimension zero each of the coordinates of a point in the variety is the root of a single polynomial, say $g = g_i(x_i)$. This polynomial generates the elimination ideal for the variable. Since the big variety W_b is irreducible the polynomial g_i is irreducible over the ring $\mathbb{Q}[\{d_e\}, x_i]$ and therefore over the field $\mathbb{Q}(\{d_e\})[x_i]$. ¿From elementary Galois theory (see Stewart [**10**]) we conclude that if one root of the irreducible polynomial g_i is non-radical over $\mathbb{Q}(d_e)$ then all roots are non-radical. The theorem follows because if $W(\{f\})$ has a non-radical root at least one of the polynomials for the coordinates x_i, y_i has a non-radical root.

The following corollary follows easily from the theorems above and shows that the zeros of the variety $W(\{f\})$ bear an analogy with the roots of an irreducible univariate polynomial over $\mathbb{Q}$.

Corollary 2.8. *The big variety W_b, corresponding to a generic, maximally independent graph, is irreducible. If any root of $W(\{f\})$ is non-radical then all roots are non-radical.*

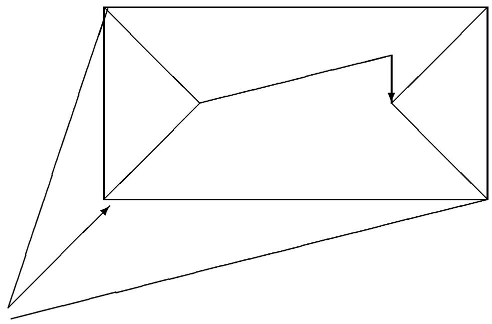
3. Specialisation

In order to develop a reduction scheme for the (infinite) class of all 3-connected graphs we effectively reduce the size of a graph by selecting certain special values for some of the dimensions. This is known as a specialisation [**5**]. The following theorem allows us to make deductions about the generic graph from the properties of a specialised version of the graph.

Theorem 3.1. *Let $W(\{f_e\})$ be the variety for the constraint equations $\{f_e\}$ for a generic maximally independent graph and let $W(\{f_e'\})$ be the variety for the constraint equations arising from a (possibly partial) specialisation $\{d_e\} = \{d_e'\}$. If $W(\{f_e'\})$ has dimension zero and is non-radical over $\mathbb{Q}(\{d'\})$ then $W(\{f_e\})$ has dimension 0 and is non radical over $\mathbb{Q}(\{d\})$.*

Proof: The proof involves algebraic geometry and Galois theory and is to be given in Owen and Power [**9**].

This theorem allows us to conclude that a generic configuration is non-radical if a certain specialised configuration is non-radical. The important point here is that the specialised configuration must have dimension zero over the complex numbers. It is perfectly possible for a configuration which is generically quadratically solvable to have non-radical solutions for certain specialisations if we relax the requirement of dimension zero. This is illustrated in Figure 2 which indicates a specialisation onto the doublet which (as we see below) may give a non-radical root. Although this root is isolated when the coordinates are constrained to be real, the variety $W(\{f_e'\})$ does not have dimension zero. (Figuratively speaking, "real rigidity" is considerably weaker than "complex rigidity".) Proving that the specialised configuration has dimension zero is the essential constraint in the application of Theorem 3.1.



Figures 2. Specialisation of a quadratically solvable graph onto a doublet.

4. On the Proof For Planar Graphs

We now focus on planar graphs by which we mean graphs which have a planar embedding in the usual graph theory sense (Diestel **[2]**). These form an infinite subset of all 3-connected rigid graphs

The next theorem uses the idea of edge contraction in a graph **[2]**, **[11]** and such reductions can be used in tandem with the specialisation theorem above. However in order to preserve zero dimensionality it is esential that the graphs arising in the reduction sequence are once again maximally independent.

If an edge e of a graph G joins vertices x and y then contracting the edge e gives a new graph $G\backslash e$, with $|V(G\backslash e)| = |V(G)| - 1$ by deleting the edge e, combining vertices x and y and deleting any duplicated edges. If e lies in exactly one 3-cycle of G then $|E(G\backslash e)| = |E(G)| - 2$ and in this case $G\backslash e$ is maximally independent whenever G is maximally independent. It turns out that by restricting to planar graphs we can be sure that such a 3-cycle exists. Moreover we are able to obtain the following key theorem.

Theorem 4.1. *Every 3-connected, maximally independent, planar graph G with $|V(G)| > 6$ has either (i) an edge on exactly one 3-cycle which can be contracted to give a 3-connected, maximally independent, planar graph with $|V(G)| - 1$ vertices, or (ii) a vertex induced subgraph which is maximally independent with three vertices of attachment and at least one internal vertex.*

This theorem allows us to generate a sequence of 3-connected maximally independent, planar graphs which terminates in the doublet graph shown in Figure 1. In case (i) the next graph in the sequence is obtained by contracting the edge. In case (ii) of the theorem the vertex induced subgraph is replaced by a 3-cycle to generate a smaller maximally independent graph in the sequence.

In order to connect this sequence with the solvability by radicals of the corresponding algebraic variety we note that contracting an edge in a 3-cycle corresponds to setting the dimension value on the contracted edge to zero and specifying that the dimension values on the other two edges in the 3-cycle are equal. This is a specialisation of the dimension values and we connect it to solvability by using Theorem 3.1.

These theorems together allow us to reduce the question of the radical solvability of a generic, 3-connected planar graph to the existence of a non-radical root for any specialised instance of the doublet graph which has dimension zero. This final step is resolved in the following theorem.

Theorem 4.2. *The doublet graph with (unsquared) dimensions on the triangles of* $10, 13, 13$ *and* $8, 5, 5$ *and connecting edges* $1, 15$ *and* 16 *has dimension zero and a non-radical root.*

Despite the smallness of the graph our proof of the existence of a non-radical doublet requires close examination of its single variable elimination ideals, the generators of which have degree 28. We are not able to determine the Galois group of a polynomial with such a high degree. However, we have found that for the special values given the degree 28 polynomial factors into two degree 8 polynomials and two degree 6 polynomials. Maple is able to determine the Galois group for each of these polynomials and we can verify that none of them is a soluble group. They therefore have all their roots non-radical.

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