Rational Parametrizations of the Minkowski Sum of two Quadrics in 3-space

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We study the Minkowski sum S of two regular quadrics Q_1, Q_2 in real affine 3-space A^3 and will answer the question, under which conditions the Minkowski sum S admits real rational parametrizations and how they can be computed. A regular quadric Q is given by a quadratic equation

$$X^t M X = 0,$$

where M is a real symmetric 4x4 matrix and $X = (1, x, y, z)^t$ is a vector containing the unknowns x, y, z. It is assumed that M has full rank and that Q possesses real points which implies that Q is either an ellipsoid, a paraboloid or a one or two sheet hyperboloid. It is known that any quadric Q is birationally equivalent to the projective plane P^2. By stereographic projection $\sigma : P^2 \to Q \subset P^3$ it is possible to construct rational parametrizations for Q. Quadrics in affine space A^3 possess same properties and $\sigma : A^2 \to Q \subset A^3$ parametrizes $Q - \{e, f\}$, where e, f are a pair of real or conjugate complex intersecting lines.

The Minkowski sum S of two sets A, B is defined by

$$S = a + b, a \in A \text{ and } b \in B, a, b \in A^3.$$

It is known that if A, B are convex then S is convex too and the boundary of S is given by boundary points $a \in \partial A$ and $b \in \partial B$ whose oriented tangent planes T_a and T_b at a and b respectively, are parallel. If A and B are not convex, the boundary ∂S of S is at least a subset of $\partial A + \partial B$. In the following we assume A and B to be smooth surfaces, in particular quadrics and concentrate on the computation of the Minkowski sum S of these surfaces.

A pointwise construction of S can be obtained as follows: We look for points $a \in A$ and $b \in B$ whose oriented tangent planes T_a, T_b are parallel. Then, at least in the convex case, $s = a + b$ is a point of the boundary surface of the Minkowski sum S and the tangent plane T_s at s is parallel to T_a, T_b.

Our problem can be reformulated as follows: Given two quadrics Q_1, Q_2, is it possible to rationally parametrize both of them in a way that the oriented tangent planes T_{q_1} and T_{q_2} are parallel. More precisely we ask for rational parametrizations $q_1(u, v)$ and $q_2(u, v)$, where q_i parametrizes Q_i, such that the oriented tangent planes to corresponding parameter values (u, v) are parallel. The parameter domain is assumed to be a subset of the affine plane A^2.

At first we notice that if there exists a similarity mapping α (the corresponding linear mapping is a scalar multiple of the identity I) which maps Q_1 onto Q_2, the problem is rather simple since corresponding points $q_1, q_2 = \alpha(q_1)$ with respect to α possess parallel tangent planes T_{q_1}, T_{q_2}. In particular this holds for spheres Q_1, Q_2.

In some kind similar is the case where Q_1, Q_2 are both paraboloids. These are quadrics which are tangent to the plane at infinity in the projective extension P^3 of A^3. It can be shown that the computation of rational parametrizations q_1, q_2 for which the parallelity of T_{q_1} and T_{q_2} holds, is a linear problem.

At second we observe that the problem is invariant with respect to affine transformations. Suppose that Q_2 is an ellipsoid, we assume it to be given by the equation

$$Q_2 : x^2 + y^2 + z^2 - 1 = 0.$$

From a Euclidean point of view Q^2 equals the unit sphere S^2 and the Minkowski sum S is nothing else but the outer offset surface (outer parallel surface) of Q_1. It has been proved earlier that the offset surfaces of quadrics admit rational parametrizations, see [4]. Further, this indicates a kinematic generation of the Minkowski sum S. It has to be noticed that the outer offset S is not a rational surface in the common sense (birationally equivalent to P^2), since its algebraic closure contains in general also the inner offset as a second component. Nevertheless, there exist real rational parametrizations of the outer offset S which shall be called improper, since the inverse mapping is in general no longer rational.

Theorem 1 *The Minkowski sum S of a regular quadric Q_1 and an ellip-*

soid Q_2 is affinely equivalent to an outer offset of Q_1 and it admits rational parametrizations.

If Q_1 and Q_2 are both regular ruled quadrics (which contain one dimensional subspaces) it is already proved in [3] that their Minkowski sum $S = Q_1 + Q_2$ admits rational parametrizations. Moreover, this property holds for general rational non developable ruled surfaces Q_1, Q_2.

Finally it shall be proved that the more general statement holds:

Theorem 2 *The Minkowski sum S of two regular quadrics Q_1, Q_2 in affine space A^3 admits rational parametrizations.*

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