C^1 Spline Implicitization of Planar Curves

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Abstract

In CAGD, there are two possibilities to define planar curves: the implicit form $f(x, y) = 0$, or by parametric equations $x = x(t)/w(t)$ and $y = y(t)/w(t)$ where $x(t), y(t)$, and $w(t)$ are polynomials. Both the parametric and implicit representation have its own advantages. The availability of both often results in simpler computation. For example, if both implicit and parametric representation are available, the intersection of two curves is obtained easier than if only implicit or parametric representation is available.

¿From classical algebraic geometry, it is known that each rational parametric curve has an implicit representation, while the converse is not true. The process of converting the parametric equation into implicit form is called *implicitization*. A number of established methods for *exact* implicitization exists: Resultants [4], Gröbner bases [1], and Moving curve and surface [9].

However, *exact* implicitization has not found widespread use in CAGD. This is in part due to the following facts:

- Exact implicitization often produces large data volumes.
- Exact implicitization process is relatively complicated, especially, in the case of high polynomial degree.

- A single exact implicitized parametric curve may have unwanted components or self intersections.

For these reasons, *approximate implicitization* has been proposed. A number of methods are available for approximate implicitization: Montaudouin and Tiller [8] employed a power series method to obtain local explicit approximation (about a regular point) to polynomial parametric curves and surfaces. Chuang and Hoffmann [3] extend this method using what they called "implicit approximation". Dokken [5] proposed a new way to approximate the parametric curve and surface, globally, in the sense that the approximation is valid within the whole domain of the curve segment or surface patch. Sederberg et al. [10] employed monoid curves and surfaces to find an approximate implicit equation and approximate inversion map of a planar rational parametric curve or a rational parametric surface .

This paper discusses the problem of constructing what we call a spline implicitization for planar curves: a partition of the plane into polygonal pieces, and an implicit polynomial for each piece. On the boundaries, these polynomial pieces are joined together with C^m continuity for suitable choice of m.

Recall that the parametric and implicit representations of a planar curve have the same polynomial degree n. However, the number of the coefficients in the parametric case is $2(n+1)$ while it is $(n+1)(n+2)/2$ in the implicit case, i.e., in the implicit case, high polynomial degree will lead to expensive computation.

Therefore, the main goal of this paper is to find a C^1 *low* degree spline implicit representation of a given parametric planar curve. We extend our technique proposed in [6]. First, to ensure the low degree condition, quadratic B-splines are used to approximate the given curve via orthogonal projection in Sobolev spaces. Adaptive knot removal, which is based on spline wavelets [2], is used to reduce the number of segments. The resulting quadratic B-spline segments are implicitized. These implicitized quadratic segments are joined together with C^0 continuity along suitable transversal lines. Using results from classical differential geometry [7], the asymptotic behavior of these transversal lines for small step size is analyzed. We showed that these lines are always well behaved, except at inflections of the original curve.

Finally, by multiplying with suitable polynomial factors, the segments are joined with C^1. The main problem was to "localize" the construction, as otherwise the degree of the implicit representation would depend on the number of segments. In order to localize the construction, we introduce some additional partition lines and multiply each segment by a piecewise quadratic multiplier, defined as a quadratic polynomial in each sub-patch. In this case, we get a global C^1 spline curve of degree 4. As an advantage, we have a smaller data volume compared with the case of *exact* implicitization.

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