ANALYSIS OF GEOMETRICAL THEOREMS IN COORDINATE-FREE FORM BY USING ANTICOMMUTATIVE GRÖBNER BASES METHOD

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Abstract. In this paper we consider the Gröbner bases of Grassman algebra and its application to the algebraic geometry. Geometrical statements of constructive type should be given in the coordinate-free form.

1. Introduction

We will consider theorems of elementary geometry. Let $A_1, A_2, A_3, \dots, A_n$ be points in a finite-dimensional space. We treat these points as vectors drawn from the origin 0.

It is known that it is possible to prove algebraic geometry task using computer algebra methods. The method of anticommutative Gröbner bases will be used.

Primarily, let us describe objects and tasks which will be regarded. We consider geometrical theorems in m-dimensional space $\mathbb{R}^m$, where $m \in \mathbb{N}_0$. All our theorems deal with statements of constructive type in the coordinate-free form. Data of a theorem contain a finite number of points $A_1, \dots, A_n$ and finite number of k - dimensional subspaces of $\mathbb{R}^m$ and their properties. Let be $m \geq (n-1)$, because in general case n points define a $(n-1)$ - space, and if we consider a space with $m < (n-1)$, we have some limitations for initial independent points.

Then, geometrically, the outer product of two vectors A and B is a bivector corresponding to the parallelogram obtained by sweeping the vector A along the vector B. The parallelogram obtained by sweeping B along A differs from the parallelogram obtained by sweeping A along B only in the orientation. Consider the algebra generated by points $A_1, A_2, A_3, \dots, A_n$ with an external product $A \wedge B$, which is associative and anticommutative: $A \wedge B = -B \wedge A$. This algebra is called *Grassman algebra*. It can be easily proven, that the monomial would be equal to zero, if it involves a variable in the power of two or more. Dimension of this algebra is equal to 2^n (the number of all not null monomials).

In Grassman algebra some geometrical statement may be formulated in these terms as polynomials.

Let the theorem consists of a number of hypotheses of the constructive type $\mathbb{H}_1, \dots, \mathbb{H}_s$ and a conclusion of the constructive type $\mathbb{C}onc$. Then, these geometrical statements correspond to polynomials $h_1, \dots, h_s \in Gr$ and $conc \in Gr$.

In coordinate Gröbner bases method in general case we have to consider $(s+1)\cdot m$ polynomials, because all the statements are formulated for each coordinate. And let

$$F_{Ant} =< X_1, \dots, X_n| \quad X_1X_2 + X_2X_1, \dots, X_{n-1}X_n + X_nX_{n-1} >$$

be a free algebra with generators $X_1, \dots, X_n$ and relations of anticommutativity on the generators. This algebra is isomorphic to the factor-algebra $F_{Ant} \simeq F/I_{Ant}$, where

$$F =< X_1, \dots, X_n >$$

is a free algebra, and

$$< X_1X_2 + X_2X_1, \dots, X_{n-1}X_n + X_nX_{n-1} >= I_{Ant} \subset F$$

is an two-side ideal in F. Also, we can formulate our statement in the terms of this algebra. So, we get $H_1, \dots, H_s \in F_{Ant}$ and $Conc \in F_{Ant}$.

In this work we are going to show, that Gröbner bases in F_{Ant} are not equivalent to Gröbner bases in Gr. And we have to use an anticommutative Gröbner bases in Gr, but not noncommutative Gröbner bases in F_{Ant} to prove our theorems.

2. Statements of the constructive type in coordinate-free form

In Chou's collection of examples [2] of the two-dimensional geometrical tasks there are some geometrical statements of constructive type, which can be written as polynomials of their coordinates. And the first question is: which of these statements can be rewritten in coordinate free form?

It is easy to check that we get the following:

1. the three points A_1, A_2, A_3 are collinear iff $(A_1 - A_2) \wedge (A_1 - A_3) = 0$;
2. the lines A_1A_2 and A_3A_4 are parallel iff $(A_1 - A_2) \wedge (A_3 - A_4) = 0$;
3. the point A_3 divides the interval $[A_1; A_2]$ in the ratio $n : m$ iff $m(A_3 - A_1) = n(A_2 - A_3) = 0$.

Then, the second question is: which statements of an m -dimensional space can be written as polynomials in Grassman algebra? And the third is: what kind of polynomials of Grassman algebra may be treated as statements of algebraic geometry?

The only homogeneous polynomials may have geometrical sense in Grassman algebra.

So, we can formulate the following statements:

1. the $(k+1)$ points $A_1, \dots, A_{k+1}$, $k+1 \leq m$ belong to the same k -dimensional subspace $\mathbb{R}^k$ of $\mathbb{R}^m$: $(A_2 - A_1) \wedge \dots \wedge (A_{k+1} - A_1) = 0$ or in other words point A_i belongs to k -subspace defined by the points $A_1, \dots, A_{i-1}, A_{i+1}, \dots A_{k+1}$ for each $1 \leq i \leq k+1$
2. the $(k+1)$ points $A_1, \dots, A_{k+1}$, $k+1 \leq m$ belong to the same k -dimensional subspace $\mathbb{R}^k$ of $\mathbb{R}^m$ and we know information about proportion: $\alpha_1 A_1 + \dots + \alpha_{k+1} A_{k+1} = 0$, where $\alpha_1, \dots, \alpha_{k+1} \in \mathbb{R}$

(3) as the generalisation of the previous expressions, that there is a linear dependency among a finite number ok k -vectors
finite number of k -vectors is dependent linearly:

$$\sum_{(i_1,\dots,i_k)\in\mathbb{R}^k} \alpha_{(i_1,\dots,i_k)} A_{i_1}\wedge\cdots\wedge A_{i_k}=0$$

The statement "*two points are equal*$(A-B=0)$" , can be means that two points belong to the one 0-dimensional subspace is a particular case of this kind of statements. We are not able to formulate conditions concerning angles and circles in the terms of Grassman algebra.

3. Gröbner bases in Grassman algebra

Let Gr be Grassman algebra of the variables $x_1,\dots,x_n$ over the field k. This is the assosiative anticommutative algebra, which consists of Grassman polynomials $\sum_{(i_1,\dots,i_k)}\alpha_{i_1,\dots,i_k}x_{i_1}\wedge\cdots\wedge x_{i_k}$. For all homogeneous polynomials $f,g\in Gr$ of degree 1 we have $f\wedge g=-g\wedge f$ and $f\wedge f=0$. The maximum degree of monomials in Gr is equal to n and dimencion of this algebra equals 2^n.

So, we have

$$Gr=<x_1,x_2,\dots,x_n,\dots,x_1\wedge\cdots\wedge x_n> .$$

As in ring of commutative polynomials $k[x_1,\dots,x_n]$, we can consider ideals in Gr and Gröbner bases of the ideals.

Definition. Set $I\subset Gr$ is to be an *ideal* in Gr iff:

(1) $0\in I$
(2) $\forall f\in Gr\ \forall g\in I$, then $f\wedge g\in Gr$ and $g\wedge f\in Gr$
(3) $\forall f,g\in I$, then $(f+g)\in I$

Definition. *Unsigned monomial* u in Gr is the combination $u=x_{i_1}\wedge\cdots\wedge x_{i_k}=x_1^{u_1}\wedge\cdots\wedge x_n^{u_n}$ with $i_1<\cdots<i_k$.

By analogy with the commutative case, where the monomials can be treated as points in $\mathbb{Z}^n$, unsigned monomials of Gr can be treated as points $(u_1,\dots,u_n)\in\{0,1\}^n$ for $u=x_{i_1}\wedge\cdots\wedge x_{i_k}=x_1^{u_1}\wedge\cdots\wedge x_n^{u_n}$ with non-zero members on the places.

Definition. *Term* is a product of a coefficient and an unsigned monomial.

We can also regard *signed monomials* m as a product of unsigned monomial um and $(-1)^{\sigma_m}$. Then $m=(-1)^{\sigma_m}\cdot um$, as a partial case of terms. And any term can be regarded as product of signed monomial and a coefficient $t=\alpha\cdot m$, where $m=(-1)^{\sigma}um$, and um is unsigned monomial.

Definition. Signed monomial $a=x_1^{a_1}\wedge\cdots\wedge x_n^{a_n}$ with non-zero members on the places $i_1<\cdots<i_k$ *is divided by* signed monomial $b=x_1^{b_1}\wedge\cdots\wedge x_n^{b_n}$ with non-zero members on the places $j_1<\cdots<j_l$, iff for two vectors $(a_1,\dots,a_n)$, $(b_1,\dots,b_n)\in\{0,1\}^n$ there are inequality $(a_1,\dots,a_n)\geq(b_1,\dots,b_n)$. This means that $\forall i\in\{1,\dots,n\}$ $a_i\geq b_i$. And so $a=(-1)^{\sigma}m\wedge b$, where m is monomial with multidegree $(m_1,\dots,m_n)=(a_1-b_1,\dots,a_n-b_n)$ with non-zero members on the places $h_1<\cdots<h_s$. and σ is the sign of alternating $(h_1,\dots,h_s,j_1,\dots,j_l)$.

Definition. *Term* $t_a = \alpha_a \cdot a$ *is divided by term* $t_b = \alpha_b \cdot b$, where a, b are signed monomials, iff $\alpha_b \neq 0$ and signed monomial a is divided by signed monomial b. This means, that there exists an unsigned monomial m, so that $a = (-1)^\sigma m \wedge a$. Then exists term $t_l = \alpha \cdot m$, $\alpha = \alpha_a/\alpha_b$ so that $t_a = t_l \wedge t_b$. We say that t_l is left divider of t_a. It can be seen, the right divider of t_a shall be $t_r = (-1)^{\sigma_r}\alpha \cdot m$, where σ_r is the sign of alternating $(j_1, \ldots, j_l, h_1, \ldots, h_s)$ and $t_a = t_b \wedge t_r$.

From this definition, we know how to find **LeftDivider**(*term*) and **RightDivider**(*term*).

Definition. *Admissible monomial order* $\prec$ on Gr is the order on unsigned monomials in Gr with the properties:

(1) $\prec$ is linear
(2) if $m_1 \prec m_2$ then $\forall\ m_3, m_4$ so that $m_3 \wedge m_1 \wedge m_4 \neq 0$ and $m_3 \wedge m_2 \wedge m_4 \neq 0$ then $m_3 \wedge m_1 \wedge m_4 \prec m_3 \wedge m_2 \wedge m_4$
(3) each not-empty subset of monomials has its minimal element

When we have to compare two signed monomials, we shall compare them as the terms.

By analogy with the commutative case, after setting order on all monomials in Gr, for each polynomial $Gr \ni p = \sum_{(i_1,\ldots,i_n)} \alpha_{(i_1,\ldots,i_n)} x_1^{i_1} \wedge \cdots \wedge x_n^{i_n}$ we can define the *leader monomial* $\mathrm{lm}(p)$ as a greater monomial of the finite number of monomials from p. So, we can define *leader coefficient* $\mathrm{lcoeff}(p)$, and *leader term* $\mathrm{lterm}(p) = \mathrm{lcoeff}(p) \cdot \mathrm{lm}(p)$. By knowing the leader monomial in polynomial, we can introduce the same division algorithm of one polynomial by the second.

Theorem. For polynomials $p, p_1, \ldots p_k \in Gr$ an algorithm of obtaining representation $p = a_1 \wedge p_1 + \cdots + a_k \wedge p_k + r$ can be introduced, where $a_i, r \in Gr$ and $r = 0$ or r is a linear combination of monomials, which are not divisible by any monomials $\mathrm{lm}(p_1), \ldots, \mathrm{lm}(p_k)$. Moreover, if $a_i \wedge p_i \neq 0$, then $deg(p) \geq deg(a_i \wedge p_i)$.

Algorithm:

Input: points $p_1, \ldots, p_k, p \in Gr$
Output: $a_1, \ldots, a_k, r \in Gr$
begin
  $a_1 := 0; \ldots a_k := 0; \quad r := 0$
  $q := p;;$
  while $q \neq 0$ **do**
    $i := 1;$
    $IsDividing := false;$
    while $i \leq k$ **and** $IsDividing = false$ **do**
      if $\mathrm{lterm}(p_i)$ divides $\mathrm{lterm}(q)$ **then**
        $t :=$ **LiftDivider**$(\mathrm{lterm}(q), \mathrm{lterm}(p_i));$
        $a_i := a_i + t;$
        $q := q - t \wedge p_i;$
      else
        $i := i + 1;$

```
            endif
        endwhile
        if IsDividing = false then
            r := r + lterm(q);
            q := q − lterm(q);
        endif
    endwhile
end
```

Example. For example, using the *lex* order in which
$A_1 < A_2 < A_3 < B_1 < B_2 < B_3 < M_1 < M_2 < M_3$

$$\begin{array}{rcl} p &=& 1 \cdot M_3 \wedge M_2 - 1 \cdot M_3 \wedge M_1 + 1 \cdot M_2 \wedge M_1 \\ p_1 &=& 2 \cdot M_1 - 1 \cdot B_1 - 1 \cdot A_1 \\ p_2 &=& 2 \cdot M_2 - 1 \cdot B_2 - 1 \cdot A_2 \end{array}$$

The outcome of the above algorithm shall be

$$\begin{array}{rcl} a_1 &=& -\frac{1}{2}M_3 + \frac{1}{2}M_2 \\ a_2 &=& \frac{1}{2}M_3 - \frac{1}{4}B_1 - \frac{1}{4}A_1 \\ r &=& \frac{1}{2}M_3 \wedge B_2 - \frac{1}{2}M_3 \wedge B_1 + \frac{1}{2}M_3 \wedge A_2 - \frac{1}{2}M_3 \wedge A_1 + \\ && +\frac{1}{4}B_2 \wedge B_1 - \frac{1}{4}B_1 \wedge A_2 + \frac{1}{4}B_2 \wedge A_1 + \frac{1}{4}A_2 \wedge A_1 \end{array}$$

So, we have a representation $p = a_1 \wedge p_1 + a_2 \wedge p_2 + r$.

Definition. *Gröbner basis* in Gr of the ideal I is a set of polynomials $g_1, \dots, g_s \in I \subset Gr$ that

$$< \mathrm{lterm}(g_1), \dots, \mathrm{lterm}(g_s) >=< \mathrm{lterm}(I) >$$

by analogy with the commutative case. Gr is finite dimensional algebra, therefore any ideal $I \subset Gr$ is the finite dimensional, therefore any ideal has a finite Gröbner basis in Gr.

Theorem. If $G = \{g_1, \dots g_s\}$ is Gröbner basis of $I \subset Gr$ and $f \in Gr$ is any polynomial, then $\exists! r \in Gr$ such that

(1) any monomial of r is not divided by any monomial from $\mathrm{lm}(g_1), \dots \mathrm{lm}(g_s)$
(2) $\exists g \in I$ so that $f = g + r$

Buchberger algorithm for Grassman algebra is the same as in the commutative case.

Example. These polynomials are the Gröbner basis of ideal with respect to *lex* monomial order, generated by themselves:

$$\begin{array}{l} 1 \cdot A_3 \wedge A_2 - 1 \cdot A_3 \wedge A_1 + 1 \cdot A_2 \wedge A_1 \\ 1 \cdot B_2 \wedge B_1 - 1 \cdot B_2 \wedge A_3 + 1 \cdot B_1 \wedge A_3 \\ 1 \cdot B_3 \wedge B_2 - 1 \cdot B_3 \wedge A_1 + 1 \cdot B_2 \wedge A_1 \\ 1 \cdot B_3 \wedge B_1 - 1 \cdot B_3 \wedge A_2 - 1 \cdot B_1 \wedge A_2 \\ 2 \cdot M_1 - 1 \cdot B_1 - 1 \cdot A_1 \\ 2 \cdot M_2 - 1 \cdot B_2 - 1 \cdot A_2 \\ 2 \cdot M_3 - 1 \cdot B_3 - 1 \cdot A_3 \end{array}$$

and polynomial $f = 1 \cdot M_1 \wedge M_1 - 1 \cdot M_1 \wedge M_3 - 1 \cdot M_2 \wedge M_1 + 1 \cdot M_2 \wedge M_3$ belong to this ideal, and it is easy to verify that $r = 0$ for this polynomial.

4. Gröbner bases method applied to the coordinate-free geometry

To prove those theorems, formulated in the coordinate-free form, the theory of anticommutative Gröbner bases may be applied.

The system of polynomials corresponding to the hypotheses of the theorem and anticommutativity relations for the generators of the Grassman algebra are considered as generators of an ideal in Grassman algebra.

Grassman algebra is generated by points of our theorem $A_1, \ldots, A_n$, $Gr = Gr(A_1, \ldots, A_n)$. Geometrical statements (hypotheses of the theorem) are formulated as polynomials in Gr:

$$h_1(A_1, \ldots, A_n) = 0$$
$$\vdots$$
$$h_s(A_1, \ldots, A_n) = 0$$

Definition. Let $I =< h_1, \ldots, h_s >$, $(A_1^0, \ldots, A_n^0)$ be a partial solution of the system $h_1 = 0, \ldots, h_s = 0$ and $f \in I$, then $f(A_1^0, \ldots, A_n^0) = 0$ and we say that f *generally true* under conditions $h_1, \ldots, h_s$.

This property is equivalent to zero reducibility of this polynomial f.

In the commutative case, we have the same idea, but for the description of hypotheses and conclusions we use the equations for the coordinates of the points.

5. Description of the implemented algorithm

The algorithm for computing the anticommutative Gröbner bases with integer coefficients have been implemented.

The main algorithm in the program is the following:

Input: points $A_1, \ldots, A_n$, order, hypothesis $H_1, \ldots, H_m$, conclusion C

Output: The answer if the theorem is true in general.

```
begin
    data type Gröbner base GB;
    GrBase = new  GB();
    GrBase−> Add(H_1, ..., H_m);
    GrBase−> Calculate(order);
    C−> NForm(GB);
    if (C eq 0)      /* It means that C ∈< H_1, ..., H_m >* /
        return "The theorem is generally true";
    else
        return "The theorem is not generally true";
    endif
end
```

The kernel of the first program processes the general case of noncommutative Gröbner bases. And interface is oriented to proving a specific class of geometrical theorems which can be formulated in terms of statements (1)–(3) in the free algebra with relations of anticommutativity. Not all the theorems, which are generally true could be proven by using this program. It is concerned with the fact, that in any noncommutative free algebra with relations of anticommutativity and in any anticommutative Grassman algebra conseptions of division of monomials are not the same. Therefore, we have implemented a program of calculating anticommutative Gröbner bases for proving geometrical theorems in coordinate-free form.

The new version of the program for noncommutative Gröbner bases with relations of anticommutativity and Gröbner bases in Grassman algebra is written in C++ with the help of K.J.Andreev.

In CAS Maple V, functions which are able to make similar calculations have yet to be revealed.

6. Gröbner bases in free algebra with relations of anticommutativity

Let us consider an associative noncommutative free algebra with unit 1 over the field k with generators $X_1, \ldots, X_k$. Each element of this algebra can be represented in the form

$$\sum_{(j_1,\ldots,j_l)} \alpha^{(j_1,\ldots,j_l)} X_{j_1} \ldots X_{j_l}$$

.

Introducing an order on the generators and the admissible order on the monomials, we can define the leading monomials

$$\mathrm{lm}(u) = \alpha_u^{(j_1,\ldots,j_{l_u})} X_{j_1} \ldots X_{j_{l_u}}$$

and

$$\mathrm{lm}(v) = \alpha_v^{(i_1,\ldots,i_{l_v})} X_{i_1} \ldots X_{i_{l_v}}$$

for any polynomials

$$u = \sum_{(j_1,\ldots,j_{l_u})} \alpha_u^{(j_1,\ldots,j_{l_u})} X_{j_1} \ldots X_{j_{l_u}}$$

and

$$v = \sum_{(i_1,\ldots,i_{l_v})} \alpha_v^{(i_1,\ldots,i_{l_v})} X_{i_1} \ldots X_{i_{l_v}}.$$

Since we have considered a free algebra over a field, we can normalise these polynomials having the leading coefficients become equal to unit. So, we assume that $\mathrm{lm}(u) = X_{j_1} \ldots X_{j_{l_u}}$ and $\mathrm{lm}(v) = X_{i_1} \ldots X_{i_{l_v}}$.

Definition. We can consider all *compositions* of two monomials $\mathrm{lm}(u)$ and $\mathrm{lm}(v)$.Two monomials have a composition $f(u, v)$, iff the end of the first monomial is equal to the beginning of the second one, namely, in our case there exists an integer $m > 0$ and a set of indices $p_1, \ldots, p_m$ such that $\mathrm{lm}(u) = X_{j_1} \ldots X_{j_{l_u-m}} X_{p_1} \ldots X_{p_m}$ and $\mathrm{lm}(v) = X_{p_1} \ldots X_{p_m} X_{i_{m+1}} \ldots X_{i_{l_v}}$. Then, the composition is equal to

$$f(u, v) = X_{j_1} \ldots X_{j_{l_u-m}} X_{p_1} \ldots X_{p_m} X_{i_{m+1}} \ldots X_{i_{l_v}} = \mathrm{lm}(u) v_1 = u_1 \,\mathrm{lm}(v),$$

where $v_1 = X_{i_{m+1}} \ldots X_{i_{l_v}}$ and $u_1 = X_{j_1} \ldots X_{j_{l_u-m}}$.

Since any monomial can be represented as a finite noncommutative product of the generators, there exist at most a finite set of compositions for each pair of monomials.

Definition. Having obtained all compositions of leading monomials of two polynomials, one can write a finite number of *noncommutative S -polynomials*, which can be constructed as

$$S(u, v) = u_1 v - u v_1.$$

Definition. The monomial $x = X_{s_1} \dots X_{s_n}$ *is divisible by* the monomial $y = X_{t_1} \dots X_{t_m}$ iff the monomial y is the substring of the monomial x so that

$$x = y_{left}\ y\ y_{right}.$$

Definition. A polynomial p_1 with the unit leading coefficient *is reducible* with respect to a polynomial p_2 with the unit leading coefficient iff the leading monomial $\operatorname{lm}(p_1)$ is divided by the leading monomial $\operatorname{lm}(p_2)$ so that $\operatorname{lm}(p_1) = \alpha \operatorname{lm}(p_2)\beta$, where α and β are some monomials.

Definition. The *result of reduction* shall be the polynomial

$$p_1' = p_1 - \alpha p_2 \beta.$$

Noncommutative Gröbner bases of an ideal I are determined by analogy with the commutative case, as a complete system of relations, which generate this ideal.

The Buchberger algorithm is the same, however, the definitions of division, reduction and S -polynomial are different.

7. Noncommutative Gröbner bases for the anticommutative algebra

At the beginning, we have tried to apply noncommutative Gröbner bases method to prove such kind of geometrical theorem.

Let $I_{Ant} =< X_i X_j + X_j X_i \quad \forall i < j\ >$ and two-side ideal in F.

Let F/I_{Ant} be the free algebra with relations of anticommutativity

$$F/I_{Ant} =< X_1, \dots, X_n \quad | \quad X_i X_j + X_j X_i \quad \forall i < j\ >$$

and $Gr = Gr(A_1, \dots, A_n)$ be the Grassman algebra with the anticommutative product $\wedge$.

Let $I \in Gr$ be an ideal in Gr and γ be a natural mapping

$$\gamma : F \mapsto Gr$$

taking X_i as A_i for all $1 \leq i \leq n$. Then define $J \subset F$ as the set $J = \gamma^{-1}(I)$.

We can define a map

$$\delta : Gr \mapsto F, \quad A_{i_1} \wedge \dots \wedge A_{i_k} \mapsto X_{i_1} \dots X_{i_k}, \quad if \quad i_1 \leq \dots \leq i_k$$

Taking as the initial representative of class $\gamma^{-1}(f)$ for $f \in I \subset Gr$ the element $\delta(f)$, it is easy to prove that $J/I_{Ant} \simeq I$, by verifying all operations.

And if $I \subset Gr$ is generated by the polynomials $h_1, \dots, h_s$,and $I_{Ant} \subset F$ is generated by $X_i X_j + X_j X_i \quad \forall i < j$ then

$$F/ < \delta^{-1}(h_1), \dots, \delta^{-1}(h_s),\ X_i X_j + X_j X_i \quad \forall i < j\ > \simeq Gr/I.$$

Let be
$$\bar{J} =< \delta^{-1}(h_1), \dots, \delta^{-1}(h_s),\ X_iX_j + X_jX_i \quad \forall i < j\ > .$$
And we can calculate a noncommutative Gröbner basis for $\bar{J}$. So, if we have hypotheses $H_1, \dots, H_s \in F$ and conclusion $Cons \in F$ and try to check if $Conc$ belongs to the ideal $\bar{J}$, by calculate $NForm(Conc)$ we have
$$\mathbb{C}onc \in< H_1, \dots, H_s,\ X_iX_j{+}X_jX_i \quad \forall i < j\ > \quad \Rightarrow \quad \gamma(\mathbb{C}onc) \in< \gamma(H_1), \dots, \gamma(H_s) >= I$$
but
$$\gamma(Conc) \in< \gamma(H_1), \dots, \gamma(H_s) >= I \quad \not\Rightarrow \quad \mathbb{C}onc \in< H_1, \dots, H_s, Ant_{ij} >$$
Because, conseption of dividing in F and in Gr are not the same and m_1 divide m_2 in Gr do not imply that $M_1 = \delta(m_1)$ divides $M_2 = \delta(m_2)$ in F.

Example.

Let be $m_1 = xz$, and $m_2 = xyz$. In Grassman algebra m_1 divides m_2 and $m_2 = (-1) \cdot y\ m_1$. But in F we obtain $M_1 = XZ$, $M_2 = XYZ$ and M_1 is not substring of M_2, that means that M_1 does not divide M_2 in F. In class of equivalence of M_2 in $F/ < Ant >$ there exists an element YZX which has the substring M_1, however we can not apply techniques of noncommutative Gröbner bases to aur case.

8. The dimension of the task

So, we have a comprehensive space $\mathbb{R}^m$, and the task with its n points and its.

Let Sp_k the set of statements defined that all points of the task belong to the same k -dimensional space (for each set of points $\{A_{i_1}, \dots, A_{i_k}\} \subset \{A_1, \dots, A_n\}$ can be constructed a polynomial by using previous rules). We obtain C_n^k polynomials.

There are two kinds of the theorems, which we consider:

(1) hypotheses of the theorem $H = \{h_1, \dots, h_s\}$ imply the statement, that all points of this theorem lie in the same k -dimensional subspace $\mathbb{R}^k$ of $\mathbb{R}^m$
(2) all other theorem

The first class of theorem, we obtain the following:
$$\{\mathbb{H}_1, \dots, H_s\} \Rightarrow \{Sp_k\}$$
$$\{\mathbb{H}_1, \dots, H_s\} \Rightarrow \mathbb{C}onc$$
It would be noticed, that if we have n point $A_1, \dots, A_n$ in the task, then for any compatible set of hypotheses we have
$$\{\mathbb{H}_1, \dots, \mathbb{H}_s\} \Rightarrow \{Sp_{n-1}\}$$
because any of n points belong to the same $(n-1)$ -dimensional space. And it easy to verify, by using Gröbner bases method.

From the other hand, for any k we have
$$\{Sp_k\} \Rightarrow \{Sp_{k+1}\}$$
and it can be verified by using Gröbner bases method too.

And we have the following:
$$(\{\mathbb{H}_1, \dots, \mathbb{H}_s\} \Rightarrow \mathbb{C}onc) \Rightarrow (\{\mathbb{H}_1, \dots, \mathbb{H}_s, Sp_k\} \Rightarrow \mathbb{C}onc)$$

So, we obtain, if $\{\mathbb{H}_1, \dots, H_s\} \Rightarrow \{Sp_k\}$ and the theorem is generally true, then this theorem is generally true in any subspaces Sp_k for $0 \leq k \leq n-1$.

To verify this property, when the hypotheses of the theorem $\{\mathbb{H}_1, \dots, \mathbb{H}_s\}$ imply the Sp_k, $k < n-1$, we can try to prove the theorems

Algorithm:

For $i = 0$ **to** $n-1$ **do**
 Prove $(\{\mathbb{H}_1, \dots, \mathbb{H}_s\} \Rightarrow \{Sp_i\})$
 if (Theorem is generally true)
  return The Hypotheses imply property Sp_i
 endif
endfor

For the other set of theorem, it is interesting to define dimension of the task as a maximal dimension of the subspace, in which this theorem is generally true.

Definition. *Dimension of the task* is a number k iff

when we add to the set of hypotheses $H = \{h_1, \dots, h_s\}$ properties Sp_k, the new theorem with hypotheses $H \cup Sp_k$ will be generally true, but the another theorem with hypotheses $H \cup Sp_{k+1}$ will not be generally true.

Note, that in 0 -dimentional subspace, all our theorem are true, but not all of them are generally true in Sp_{n-1}. And the "real" dimension of the task $0 \leq k \leq n-1$.

Note, that if

$$\{\mathbb{H}_, \dots, \mathbb{H}_s\, Sp_k\} \Rightarrow \{Conc\},$$

then

$$\{\mathbb{H}_, \dots, \mathbb{H}_s\, Sp_{k-1}\} \Rightarrow \{Conc\},$$

because

$$\{\mathbb{H}_, \dots, \mathbb{H}_s\, Sp_{k-1}\} \Rightarrow \{\mathbb{H}_, \dots, \mathbb{H}_s\, Sp_k\} \Rightarrow \{Conc\}.$$

Example. For example, the Gauss line theorem [2] the hypotheses imply that all points belong to the same plane (it is easy to check). So, in this theorem we have $\{\mathbb{H}_1, \dots, \mathbb{H}_7\}$ hypotheses, and $(\{\mathbb{H}_1, \dots, \mathbb{H}_7\} \Rightarrow \{Sp_2\})$.

But in the Pappus theorem hypotheses do not imply this property.

If we use the coordinate method, we add the properties of the q -dimension space, when we introduce the coordinate system $(e_1, \dots, e_q)$. This is equivalent to adding properties Sp_q to the hypotheses. Therefore, in coordinate-free method we have not to reformulate all hypotheses, when we try to find the dimension of the task.

9. Advantages of the coordinate-free method

a) If the dimension of the task is q, then max degree of all hypotheses and conclusions will be less than or equal to q
b) if the conditions of the task are satisfied, then an equation, which degree is higher than 1, cannot be present in the Gröbner basis (however, this is possible in the coordinate case).
c) it is easy to interpret Grassman polynomials as geometry expressions;

d) for calculate the dimension of the task in coordinate method without reformulation the hypotheses, we have take coordinate system with $q = n - 1$ coordinate orts, and we have much more equations, then in coordinate-free method for the same theorem;
e) all polynomials in the Buchberger algorithm may not have degree much than n (where n is number of points in the task), but it is a problem in commutative case

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