SUBRESULTANTS AND DISCRIMINANT SEQUENCES

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Abstract: In this paper we rebuild the discriminant sequence of a polynomial with indeterminates in the framework of the subresultant and polynomial remainder sequences. In section 3 we prove an invariance property of discriminants. In section 4 we show an example to construct new inequalities using discriminants.

Key Words: subresultant, polynomial remainder sequence, Sturm Theorem, Tarski sequence, discriminant sequence, Newton's inequality.

§ 1. Introduction

Many problems about automated proving for inequalities concerning certain real geometric configurations lead to finding the real zeros of a system of polynomials. Sometimes we need only to examine the existence of real zeros. As there are various elimination methods to transform a system of polynomial in general form to a triangular form:

$$
\begin{aligned}
&f_1(x_1) = 0,\\
&f_2(x_1, x_2) = 0,\\
&\cdots\cdots\\
&f_r(x_1, x_2, \cdots, x_r) = 0,
\end{aligned}
$$

it is of the first importance to clarify the structure of a univariate polynomial. For a polynomial with constant coefficients, using the well-known Sturm's Theorem and Euclidean successive polynomial division can easily do this. Contrarily if the coefficients are indeterminates or polynomials, say,

$$f_1(x_1) = a_n x_1^n + a_{n-1} x_1^{n-1} + \cdots + a_1 x_1 + a_0$$

and $a_n, a_{n-1}, \cdots, a_1, a_0 \in R[u_1, u_2, \cdots, u_s]$, then the Euclidean successive polynomial division cannot be performed in the coefficient domain. In this case, there are classical works on computing polynomial remainder sequences of two polynomials through subresultant chain (see [1-4], [6] and [7]). This makes it possible to construct a Sturm sequence via the subresultants of a polynomial and its derivative in a recursive way, and then compute the number of real roots for the polynomial by counting the variation in signs of the leading coefficients of the obtained Sturm sequence(see [5] for an example). This process becomes much complicated if the subresultant chain is defective. It would be more convenient if one could simply take the principal subresultants coefficients for counting the variation in signs. This idea is actually feasible. Yang Lu and his cooperators have proved (see [11-12]) that Sturm's theorem can be translated to the principal minor determinants (called "discriminant sequence") of a slightly modified Sylvester resultant matrix by way of the variation in sign of a so-called "revised sign list". This quite surprising result has been found useful in many applications. But the original proof seems a bit isolated. The goal of this paper is to provide a constructive description to this work in the accepted context of subresultants and polynomial remainder sequences. In Section 3 we will prove an invariance property of discriminant sequences. In section we will show an example about the application of discriminant sequences.

§ 2. Link between Subresultant Chain and Discriminant Sequence

In this section, we rebuild the discriminant sequence for a polynomial start from the Fundamental Theorem of p.r.s.'s(polynomial remainder sequences). In the beginning we shall make a brief recall to subresultants with the notation used in [8].

Definition. Let $A_i = \sum_{j=0}^{n_i} a_{ij} x^j, 1 \le j \le k,$ be a sequence of polynomials over an integral domain I. Then the $k \times l$ *matrix associated with* $A_1, A_2, \cdots, A_k$ is

$$mat(A_1, A_2, \cdots, A_k) = (a_{i,l-j}),$$

where $l = 1 + \max_{1 \le i \le k}(n_i)$.

Definition. Let M be a $k \times l$ matrix over I, $k \le l$. By $M^{(j)}$ denotes the submatrix of M consisting of the first $k-1$ columns and the jth column of

M , $k \le j \le l$. The *determinant polynomial* is:

$$\det pol(M) = |M^{(k)}| x^{l-k} + \cdots + |M^{(l)}|.$$

If $M = mat(A_1, \cdots, A_k)$, we write $\det pol(A_1, \cdots, A_k)$ instead of $\det pol(M)$.

Definition. Let $f, g \in I[x]$ with $\deg(f) = m > 0$ and $\deg(g) = n > 0$. For k , $0 \le k < \min(m, n)$, set

$$M_k = mat(x^{n-k-1} f(x), \cdots, f(x), x^{m-k-1} g(x), \cdots, g(x)).$$

Then $S_k = sres_k(f, g) = \det pol(M_k)$ is the k th subresultant of f and g . Note that S_0 is a polynomial of degree 0 which is the Sylvester resultant of f and g .

Since M_k has $m + n - 2k$ rows and $m + n - k$ columns,

$$\deg S_k \le (m + n - k) - (m + n - 2k) = k,$$

and S_k can be regarded as a polynomial of formal degree k with the principal minor determinant

$$R_k = psc_k(f, g) = \det(mat(x^{n-k-1} f(x), \cdots, f(x), x^{m-k-1} g(x), \cdots, g(x))$$

as formal leading coefficient (called "*the principal subresultant coefficient*" in [8]). By $lc(S_k)$ denote the leading coefficient of S_k, $0 \le k < \min(m, n)$. If $m = n + 1$, we may extend the definition of subresultant to $S_{n+1} = f$, $S_n = g$, and set

$$R_n = psc_n(f, g) = lc(g), R_{n+1} = psc_{n+1}(f, g) = 1.$$

In case $\deg S_k = r < j$, that is, $R_k = 0$ and $R_k \ne lc(S_k)$ or $S_k = 0$, we call S_k *defective* of degree r and otherwise *regular*. The subresultant chain is regular if all its elements are regular and otherwise defective.

The next theorem describes the relation of any polynomial remainder sequence to the subresultant chain starting with the same polynomials.

Fundamental Theorem of p.r.s. Let $f_1, f_2, \cdots, f_r$ be a polynomial remainder sequence over I satisfying for $\alpha_i, \beta_i \in I$, both non-zero

$$\alpha_i f_i = q_i f_{i+1} + \beta_i f_{i+2} \quad (1 \le i \le r-2),$$

Let $n_1, n_2, \cdots, n_r$ be the degree sequence and let $c_1, c_2, \cdots, c_r$ be the leading coefficient $lc(f_i)$ sequence of (f_1, f_2). For any $j, 2 \le j \le r-1$,

$$\begin{aligned}
&S_k = 0, \quad 0 \le k < n_r, \quad n_{j+1} < k < n_j - 1, \\
&\left\{\prod_{i=1}^{j-1} \alpha_i^{n_{i+1}-n_j+1}\right\} S_{n_j-1} = \left\{\prod_{i=1}^{j-1} (-1)^{(n_i-n_j+1)(n_{i+1}-n_j+1)} \beta_i^{n_{i+1}-n_j+1} c_{i+1}^{n_i-n_{i+2}}\right\} \times c_j^{-n_j+n_{j+1}+1} f_{j+1}, \\
&\left\{\prod_{i=1}^{j-1} \alpha_i^{n_{i+1}-n_{j+1}}\right\} S_{n_{j+1}} = \left\{\prod_{i=1}^{j-1} (-1)^{(n_i-n_{j+1})(n_{i+1}-n_{j+1})} \beta_i^{n_{i+1}-n_{j+1}} c_{i+1}^{n_i-n_{i+2}}\right\} \times c_{j+1}^{n_j-n_{j+1}-1} f_{j+1}.
\end{aligned}$$

Now we focus on a specific class of polynomial remainder sequences. Assume I is ordered. A p.r.s $f_1, f_2, \cdots, f_{r-1}, f_r \in I[x]$ is called a negative or generalized Sturm sequence if there exists $\alpha_i, \beta_i \in I$ such that

$$\alpha_i f_i = q_i f_{i+1} + \beta_i f_{i+2}, \quad \alpha_i \beta_i < 0, \quad 1 \le i \le r-1.$$

An instance is *Tarski's remainder sequence* (see [10]) with

$$\beta_i = -1, \quad \alpha_i = c_{i+1}^{2(n_i - n_{i+1}+1)}.$$

We search further simplification on the relation of Tarski's remainder sequence to the subresultants. Substituting these specific coefficients into the third formula of Foundamental Theorem of p.r.s., we have

$$\left\{\prod_{i=1}^{j-1} c_{i+1}^{2(n_i-n_{i+1}+1)(n_{i+1}-n_{j+1})}\right\} S_{n_{j+1}} = (-1)^{\Sigma_{j+1}} \prod_{i=1}^{j-1} c_{i+1}^{n_i-n_{i+1}} \times c_{j+1}^{n_j-n_{j+1}-1} f_{j+1}, \quad 1 < j < r,$$

where

$$\Sigma_{j+1} = \sum_{i=1}^{j-1} (n_i - n_{j+1} + 1)(n_{i+1} - n_{j+1}).$$

Change the subscript $j+1$ in the above formula to j, then we obtain

$$\left\{\prod_{i=1}^{j-2} c_{i+1}^{2(n_i-n_{i+1}+1)(n_{i+1}-n_j)}\right\} S_{n_j} = (-1)^{\Sigma_j} \prod_{i=1}^{j-2} c_{i+1}^{n_i-n_{i+1}} \times c_j^{n_{j-1}-n_j-1} f_j, \quad 2 < j < r,$$

where

$$\Sigma_j = \sum_{i=1}^{j-2}(n_i - n_j + 1)(n_{i+1} - n_j).$$

We observe that

$$\begin{aligned}
\Sigma_{j+1} - \Sigma_j &= \sum_{i=1}^{j-1}\left[(n_i - n_{j+1} + 1)(n_{i+1} - n_{j+1}) - (n_i - n_{j+1} + 1)(n_{i+1} - n_{j+1})\right] \\
&= \sum_{i=1}^{j-1}\left[n_i(n_j - n_{j+1}) + n_{i+1}(n_j - n_{j+1})\right] (\operatorname{mod} 2) \\
&= \left\{\sum_{i=1}^{j-1}(n_i - n_{i+1})\right\}(n_j - n_{j+1}) \ (\operatorname{mod} 2) \\
&= (n_1 - n_j)(n_j - n_{j+1});
\end{aligned}$$

$$\begin{aligned}
\prod_{i=1}^{j-1} c_{i+1}^{n_i - n_{i+2}} &= \prod_{i=1}^{j-2} c_{i+1}^{n_i - n_{i+2}} \times c_j^{n_{j-1} - n_{j+1}} \\
&= \prod_{i=1}^{j-2} c_{i+1}^{n_i - n_{i+2}} \times c_j^{n_{j-1} - n_j - 1} \times c_j^{n_j - n_{j+1} - 1} \times c_j^2.
\end{aligned}$$

For the purpose to calculate the number of real roots using Sturm's theorem, we may multiply any polynomial f_j in the generalized Sturm sequence by a positive factor a^2 $(a \in I, a \neq 0)$ without effect on result. Parallel to the concept of polynomial similarity in the definition of p.r.s., we call $A, B \in I[x]$ "positive similar" if there exists $a, b \in I,\ ab \neq 0$ such that $a^2 A = b^2 B$, and write $A \approx B$. Thus

$$\begin{aligned}
S_{n_{j+1}} &\approx (-1)^{\Sigma_j + (n_1 - n_j)(n_j - n_{j+1})} \prod_{i=1}^{j-2} c_{i+1}^{n_i - n_{i+1}} \times c_j^{n_{j-1} - n_j - 1} \times (c_j c_{j+1})^{n_j - n_{j+1} - 1} f_{j+1}, \\
S_{n_j} &\approx (-1)^{\Sigma_j} \prod_{i=1}^{j-2} c_{i+1}^{n_i - n_{i+1}} \times c_j^{n_{j-1} - n_j - 1} \times f_j,
\end{aligned}$$

and multiplying them we obtain the following recursive relation:

$$S_{n_j} S_{n_{j+1}} \approx (-1)^{(n_1 - n_j)(n_j - n_{j+1})} (c_j c_{j+1})^{n_j - n_{j+1} - 1} f_j f_{j+1}, \quad 2 < j < r,$$

This relation holds also for $j = 1,2$ if $f_2 = f_1'$. It is obvious when $j = 1$. For $j = 2$, substitute $\alpha_1 = c_2^{2(n_1 - n_2 + 1)}$, $\beta_1 = -1$ and $n_2 = n_1 - 1$ into the third formula of the Fundamental Theorem of p.r.d., then we get

$$c_2^{2(n_1-n_2+1)(n_2-n_3)} S_{n_3} = (-1)^{(n_1-n_3)(n_2-n_3)} (-1)^{(n_2-n_3)} c_2^{n1-n3} c_3^{n2-n3-1} f_3,$$
$$S_{n_3} \approx (-1)^{n_2-n_3} (c_2 c_3)^{n_2-n_3-1} f_3,$$
$$S_{n_2} S_{n_3} \approx (-1)^{(n_1-n_2)(n_2-n_3)} (c_2 c_3)^{n_2-n_3-1} f_2 f_3.$$

Similarly, we have the following recursive relation for $S_{n_{j-1}}, S_{n_j}$ with Tarski's remainder sequence,

$$S_{n_j} S_{n_j-1} \approx (-1)^{n_1-n_j} f_j f_{j+1}, \quad 1 < j < r,$$

which holds also for $j = 1$ in case $f_2 = f_1'$. From this relation, we can construct a Sturm sequence

$$f_1, f_2 = f', \cdots, f_r$$

for a given polynomial $f_1(x) \in I[x]$ of degree $n+1$ using the subresultant chain

$$S_{n+1} = f_1, S_n = f_1', S_{n-1}, \cdots, S_0 .$$

If the subresultant chain is regular, that is, $n_j = n + 2 - j$, then the induced Sturm sequence is

$$f_j = (-1)^{\sum_{i=1}^{j-2} i} S_{n_j} = (-1)^{\frac{1}{2}(j-2)(j-1)} S_{n+2-j}, \quad 2 < j \le n+2.$$

This motivates the following modified definition to subresultants. Let

$$f_1 = a_{n+1} x^{n+1} + a_n x^n + \cdots + a_0,$$
$$f_2 = f_1' = (n+1) a_{n+1} x^n + n a_n x^{n-1} + \cdots + a_1.$$

For any $k, 0 \le k < n,$ let

$$M_k^* = mat(x^{n-k} f_1, x^{n-k} f_2, x^{n-k-1} f_1, x^{n-k-1} f_2, \cdots, f_1, f_2),$$
$$S_k^* = a_{n+1} \det pol(M_k^*),$$

and let $S_{n+1}^* = f_1, S_n^* = f_2.$ Then matrix M_k^* can be transformed to

$$\begin{pmatrix} a_{n+1} & \cdots \\ 0 & M_k \end{pmatrix}$$

by $1 + 2 + \cdots + (n-k) = \dfrac{1}{2}(n-k)(n-k+1)$ times of interchanging of rows, which implies that

$$S_k^* = (-1)^{\frac{1}{2}(n-k)(n-k+1)} a_{n+1}^2 S_k, \quad 0 \le k < n,$$

and that if $S_{n+1}, S_n, \cdots, S_0$ is regular, then

$$f_1 = S_{n+1}^*, \; f_2 = f_1' = S_n^*, \; f_j = a_{n+1} S_{n+2-j}^* \; (j = 3, \cdots, n+2)$$

is a generalized Sturm sequence. As generalized Sturm sequence is also called negative p.r.s, we may call S_k^* the k th *negative subresultant* of f_1, f_2. S_k^* can be regarded as a polynomial of formal degree k. Let

$$R_k^* = psc_k(S_k^*), 0 \le k \le n,$$

denote the formal leading coefficients of S_k^* and let $R_{n+1}^* = 1$ by convention. Call S_k^* defective if the $R_k^* = 0$ and regular otherwise.

If $f_1, f_2, \cdots, f_{r-1}, f_r \in I[x]$ is a Tarski's remainder sequence of f_1, f_2, then

$$\begin{aligned} S_{n_j}^* S_{n_j-1}^* &\approx (-1)^{(n_1-n_j)} (-1)^{\frac{1}{2}(n-n_j)(n-n_j+1)} (-1)^{\frac{1}{2}(n-n_j+1)(n-n_j+2)} f_j f_{j+1} \\ &= f_j f_{j+1}, \quad 1 < j < r, \\ S_{n_j}^* S_{n_{j+1}}^* &\approx (-1)^{(n_1-n_j)(n_j-n_{j+1})+\frac{1}{2}(n-n_j)(n-n_j+1)+\frac{1}{2}(n-n_{j+1})(n-n_{j+1}+1)} (c_j c_{j+1})^{(n_j-n_{j+1}-1)} f_j f_{j+1} \\ &= (-1)^{\frac{1}{2}(n_j-n_{j+1})(n_j-n_{j+1}-1)} (c_j c_{j+1})^{(n_j-n_{j+1}-1)} f_j f_{j+1}, \quad 2 < j < r. \end{aligned}$$

The two relations hold for $1 \le j < r$ if $f_2 = f_1'$. According to the first relation we can construct a generalized Sturm sequence via negative subresultant chain. (See Appendix for a Maple program).

In what follows we work for extending the Sturm Theorem to the negative subresultant chain.

It is obvious that the number of real roots for $f_1 \in I[x]$ can be calculated by the variation in signs of the (formal) leading coefficients $R_{n+1}^*, R_n^*, \cdots, R_0^*$ if negative subresultants $S_{n+1}^*, S_n^*, \cdots, S_0^*$ are regular. Let $N(f_1)$ be the number of real roots of f_1. Let $V(+\infty)$ be the variation in sign of $S_{n+1}^*(x), S_n^*(x), \cdots, S_0^*(x)$ when $x \to +\infty$, $V(-\infty)$ be the variation in sign of $S_{n+1}^*(x), S_n^*(x), \cdots, S_0^*(x)$ when

$x \to -\infty$. Then $V(+\infty)$ equals the variation in signs of sequence $R_{n+1}^*, R_n^*, \cdots, R_0^*$, and

$$
\begin{aligned}
V(-\infty) - V(+\infty) &= N(f_1), \\
V(-\infty) + V(+\infty) &= n + 1,
\end{aligned}
$$

therefore,

$$n_1 - 2V(+\infty) = N(f_1).$$

If $S_{n+1}^*, S_n^\#, \cdots, S_k^*$ are regular and $S_{k-1}^*, \cdots, S_0^*$ are defective, then polynomials

$$f_1 = S_{n+1}^*,\ f_2 = f_1' = S_n^*,\ f_j = a_{n+1} S_{n+2-j}^*\ (j = 3, \cdots, n+2-k)$$

constitute a generalized Sturm sequence, $n_j = \deg(f_j) = n + 2 - j$. Let $N(f_1)$ be the number of distinct real roots of f_1. Then we have

$$
\begin{aligned}
V(-\infty) - V(+\infty) &= N(f_1), \\
V(-\infty) + V(+\infty) &= n + 1 - k,
\end{aligned}
$$

according to Sturm Theorem. Let $r = n + 2 - k$, then $n_r = k$ and

$$n_1 - n_r - 2V(+\infty) = N(f_1),$$

where $V(+\infty)$ equals to the variation in sign for $R_{n+1}^*, R_n^*, \cdots, R_r^*, R_{r-1}^*, \cdots, R_0^*$, among them $R_{r-1}^* = \cdots = R_0^* = 0$.

What is for the other cases? We are going to prove that there is an appropriate way to revise the variation in signs for a sequence of real number so that the above formula still holds for all defective negative subresultant chain.

Observe that the formal leading coefficients $R_{n+1}^*, R_n^*, R_{n-1}^*, \cdots, R_1^*, R_0^*$ can be written as

$$R_{n_1}^*, R_{n_2}^*, \cdots, R_{n_j}^*, 0, \cdots 0, R_{n_{j+1}}^*, \cdots, R_{n_{r-1}}^*, 0, \cdots 0, R_{n_r}^*, 0, \cdots, 0,$$

where $R_{n_1}^*, R_{n_2}^*, \cdots, R_{n_j}^*, R_{n_{j+1}}^*, R_{n_{r-1}}^*, R_{n_r}^*$ are all not zeros, and the number of zeros between $R_{n_j}^*$ and $R_{n_{j+1}}^*$ equals to $n_j - n_{j+1} - 1$.

Let $v(a, b; l)$ be an enumeration function defined for real numbers a, b and nonnegative integer l, satisfying that if $l=0$, then $\nu(a,b;0)$ is the variation in signs of a,b. Let

$$V_{+\infty}=\sum_{j=1}^{r}\nu(R_{n_j}^*,R_{n_{j+1}}^*;n_j-n_{j+1}-1),$$

and call it the modified number of variations in sign of $R_{n+1}^*,R_n^*,R_{n-1}^*,\cdots,R_1^*,R_0^*$. Let $N(f_1)$ be the number of distinct real roots of f_1 as before. We expect

$$n_1-n_r-2V_{+\infty}=N(f_1)$$

holds under certain appropriate definition of $\nu(a,b;l)$.

Let $f_1(x),f_2(x),\cdots,f_m(x)$ be any generalized Sturm sequence for $f_1,f_2=f_1'$. Then $m=r$. Let $n_1,n_2,\cdots,n_r$ be the sequence of degree, and $c_1,c_2,\cdots,c_r$ be the leading coefficients of $f_1,f_2,\cdots,f_r$. Then, according to Sturm Theorem, the number of distinct real roots of f_1 can be calculate by the variation in sign of real number sequence $c_1,c_2,\cdots,c_r$:

$$\begin{aligned}N(f_1)&=V(-\infty)-V(+\infty)\\&=\sum_{j=1}^{r-1}\frac{1}{2}\left[1-\mathrm{sgn}((-1)^{n_j+n_{j+1}}c_jc_{j+1})\right]-\sum_{j=1}^{r-1}\frac{1}{2}\left[1-\mathrm{sgn}(c_jc_{j+1})\right]\\&=\sum_{j=1}^{r-1}\frac{1}{2}(1+(-1)^{n_j-n_{j+1}-1})\,\mathrm{sgn}(c_jc_{j+1}).\end{aligned}$$

In view of the recursive relation

$$S_{n_j}^*S_{n_{j+1}}^*\approx(-1)^{\frac{1}{2}(n_j-n_{j+1})(n_j-n_{j+1}-1)}(c_jc_{j+1})^{(n_j-n_{j+1}-1)}f_jf_{j+1},\quad 1\le j<r,$$

if n_j-n_{j+1} is odd, then

$$\begin{aligned}&S_{n_j}^*S_{n_{j+1}}^*\approx(-1)^{\frac{1}{2}(n_j-n_{j+1})(n_j-n_{j+1}-1)}f_jf_{j+1},\\&\mathrm{sgn}(R_{n_j}^*R_{n_{j+1}}^*)=(-1)^{\frac{1}{2}(n_j-n_{j+1})(n_j-n_{j+1}-1)}\mathrm{sgn}(c_jc_{j+1});\end{aligned}$$

and if n_j-n_{j+1} are even,

$$\operatorname{sgn}(R^*_{n_j}R^*_{n_{j+1}}) = (-1)^{\frac{1}{2}(n_j-n_{j+1})(n_j-n_{j+1}-1)}\left[\operatorname{sgn}(c_jc_{j+1})\right]^2$$
$$= (-1)^{\frac{1}{2}(n_j-n_{j+1})(n_j-n_{j+1}-1)}.$$

From this follows immediately that

$$N(f_1) = \sum_{\substack{j=1 \\ n_j-n_{j+1}=1(\mathrm{mod}\,2)}}^{r-1} (-1)^{\frac{1}{2}(n_j-n_{j+1})(n_j-n_{j+1}-1)} \operatorname{sgn}(R^*_{n_j}R^*_{n_{j+1}}).$$

On the other hand, we have

$$n_1 - n_r - 2V_{+\infty} = n_1 - n_r - 2\sum_{j=1}^{r-1} \nu(R^*_{n_j}, R^*_{n_{j+1}}; n_j - n_{j+1} - 1)$$
$$= \sum_{j=1}^{r-1} (n_j - n_{j+1} - 2\nu(R^*_{n_j}, R^*_{n_{j+1}}; n_j - n_{j+1} - 1)).$$

So it is natural we expect that the enumeration function $\nu(a,b;l)$ satisfies

$$n_j - n_{j+1} - 2\nu(R^*_{n_j}, R^*_{n_{j+1}}; n_j - n_{j+1} - 1) = (-1)^{\frac{1}{2}(n_j-n_{j+1})(n_j-n_{j+1}-1)} \operatorname{sgn}(R^*_{n_j}R^*_{n_{j+1}})$$

when $n_j - n_{j+1}$ is odd, and

$$n_j - n_{j+1} - 2\nu(R^*_{n_j}, R^*_{n_{j+1}}; n_j - n_{j+1} - 1) = 0$$
$$= (-1)^{\frac{1}{2}(n_j-n_{j+1})(n_j-n_{j+1}-1)} \operatorname{sgn}(R^*_{n_j}R^*_{n_{j+1}}) - 1$$

when $n_j - n_{j+1}$ is even. This property can be written in a unified form:

$$l + 1 - 2\nu(a,b,l) = (-1)^{\frac{1}{2}l(l+1)} \operatorname{sgn}(ab) + \sum_{k=1}^{l} (-1)^k.$$

This leads to the following definition.

Definition. Let $a_1, a_2, a_3, \cdots, a_m$ be a sequence of real numbers. Then the modified number of variations in sign of this sequence is the final result of the following procedure.

Let $N = 0$.

For i from 1 to $m-1$ do

if $a_i \cdot a_{i+1} < 0$, then $N \leftarrow N + 1$;

if $a_{i+1} = 0$, and a_{i+j} is the first non-zero number in $a_{i+1}, \cdots, a_m$, then

$N \leftarrow N + v(a_i, a_j, j-i-1)$, where

$$v(a,b,l) = \frac{1}{2}(l+1) - \frac{1}{2}\sum_{k=1}^{l}(-1)^k - \frac{1}{2}(-1)^{\frac{1}{2}l(l+1)}\operatorname{sgn}(ab), \quad \text{and}$$

$\operatorname{sgn}(x)$ is the sign function.

return N.

It is clear that if then is no zero in sequence $a_1, a_2, a_3, \cdots, a_m$, or all zero elements in the sequence are adjacent in the end of the sequence, then the modified number of variation in sign equals to that calculated in usual way.

We can summarize our discussion above in the following theorem

Theorem (Yang-Hou-Zeng [11]): Let f_1 be a polynomial of degree $n+1$ with determined or indetermined real coefficients. Let $S_{n+1}^*, S_n^*, S_{n-1}^*, \cdots, S_0^*$ be the negative subresultant chain generated by $f_1, f_2 = f_1'$, $R_{n+1}^*, R_n^*, R_{n-1}^*, \cdots, R_0^*$ the formal leading coefficients of negative subresultants, $V_{+\infty}$ the modified number of variation in signs of the sequence. Let $N(f_1)$ be the number of the distinctive real roots of f_1, $n_1 = n+1$, and n_r the degree of the last regular negative subresultant. Then

$$N(f_1) = n_1 - n_r - 2V_{+\infty}.$$

In the remaining part of this section we show the connection between the sequence $R_k^*(k = n+1, n, \cdots, 0)$ and the classical discriminant, and an intuitive algorithm for counting the modified number of variation in sign of real number sequences.

Let

$$D_k = \det(mat(x^{k-1}f_1, x^{k-1}f_1', \cdots_1, f_1, f_1')), \quad 1 \le k \le n+1.$$

Then,

$$\begin{aligned} R_{n+1}^* &= a_{n+1}, \\ R_n^* &= (n+1)a_{n+1} = D_1 / a_{n+1}, \\ R_k^* &= a_{n+1}D_{n+1-k}, \quad k = n-1, \cdots, 0. \end{aligned}$$

Hence the modified number of variation in sign of $R_{n+1}^*, R_n^*, R_{n-1}^*, \cdots, R_0^*$ equal to that of $D_1, D_2, \cdots, D_{n+1}$. Note that

$$D_1 = (n+1)a_{n+1}^2,$$

$$D_2 = R_{n-1}^* / a_{n+1} = \begin{vmatrix} a_{n+1} & a_n & a_{n-1} & a_{n-2} \\ 0 & (n+1)a_{n+1} & na_n & (n-1)a_{n-1} \\ 0 & a_{n+1} & a_n & a_{n-1} \\ 0 & 0 & (n+1)a_{n+1} & na_n \end{vmatrix}$$

$$= a_{n+1}^2(na_n^2 - 2(n+1)a_{n+1}a_{n-1})$$

$$= a_{n+1}^4 \sum_{i<j}(x_i - x_j)^2,$$

$$D_{n+1} = R_0^* / a_{n+1}$$

$$= \det pol(M_0^*) = (-1)^{\frac{1}{2}n(n+1)} a_{n+1} R(f_1, f_1')$$

$$= a_{n+1}^2 D(f_1) = a_{n+1}^{2n+2} \prod_{i<j}(x_i - x_j)^2,$$

where $D(f_1)$ is the discriminant of f_1 and $x_1, \cdots, x_n, x_{n+1}$ are roots of f_1. Call $D_k(k = 1,2,\cdots, n+1)$ the k th discriminant of f_1, and $D_1, D_2, \cdots, D_{n+1}$ the discriminant sequence of f_1.

To conclude this section we give an easy-to-use method to calculate the modified number of variation in sign. Given a sequence $a_1, a_2, a_3, \cdots, a_m$ of real numbers, we can construct $b_1, b_2, b_3, \cdots, b_m$ according to the following process:

(1): If $a_i \neq 0$, then $b_i = a_i$;

(2): If $a_i \neq 0, a_{i+k} = 0, k = 1, \cdots, j-1, a_{i+j} \neq 0$, then

$$b_{i+k} = (-1)^{\frac{1}{2}k(k+1)} a_i, k = 1, \cdots, j-1.$$

(3): If $a_r \neq 0, a_{r+1} = \cdots = a_m = 0$, then $b_{r+1} = \cdots = b_m = 0.$

Then the number of variation in sign for $b_1, b_2, b_3, \cdots, b_m$ equals to the modified number of variation in sign for $a_1, a_2, a_3, \cdots, a_m$.

§ 3. Invariance properties of Discriminant Sequence

In this section we prove the following result.

Theorem. Let $f(x) \in I[x]$ be a polynomial of degree $n+1$ over integral domain I. Then for any $u \in I$, the k th discriminant $D_k(u)$ of polynomial $g(x) = f(x-u)$ equals to the k th discriminant D_k of $f(x)$, $k = 1, \cdots, n, n+1$.

§ 4. Applications: Generalization of Newton's Inequalities

The Newton inequalities, which are indeed due to Newton, say that if $x_1, x_2, \cdots, x_n$ are real numbers, then for $j = 1, 2, \cdots, n-1$

$$E_j(x_1, x_2, \cdots, x_n)^2 \geq E_{j-1}(x_1, x_2, \cdots, x_n) \cdot E_{j+1}(x_1, x_2, \cdots, x_n)$$

and the inequality is strict unless $x_1 = x_2 = \cdots = x_n$, or both side vanish, where

$$E_j(x_1, x_2, \cdots, x_n) = \binom{n}{j}^{-1} s_j(x_1, x_2, \cdots, x_n)$$

and $s_j(x_1, x_2, \cdots, x_n)$ is the elementary symmetric function:

$$s_j(x_1, x_2, \cdots, x_n) = \sum_{1 \leq i_1 < i_2 < \cdots < i_j \leq n} x_{i_1} x_{i_2} \cdots x_{i_j} .$$

(see [9]) In this section we will give a method to construct a set of stronger inequalities using discriminant sequences.

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APPENDIX A Maple Program for Computing Generalized Sturm Sequences via Negative Subresultants

```
with(linalg):

### matrix associated to a sequence of polynomials
mat := proc(plist, x)
local l, i, p, cf, M;
    l := max(op(map(degree, plist, x)));
    M := [];
    for p in plist do
        cf := [subs(x = 0, p)];
        for i to l do cf := [coeff(p, x, i), op(cf)] od;
        M := [op(M), cf]
    od;
    M
end
```

```
### determinant polynomial generated by a matrix
detpol := proc(M, x)
local i, j, k, l, k1, cf, mj;
        k := nops(M);
        l := nops(op(1, M));
        cf := [];
        k1 := seq(i, i = 1 .. k - 1);
        for j from k to l do
                mj := submatrix(M, 1 .. k, [k1, j]);
                cf := [op(cf), det(mj)]
        od;
        mj := 0;
        for i to l - k + 1 do mj := mj + op(i, cf)*x^(l - k + 1 - i)
        od;
        mj
end

### subresultant chain of two polynomials
sres := proc(f, g, x)
local m, n, k, i, p, sk;
        m := degree(f, x);
        n := degree(g, x);
        sk := [];
        for k from 0 to min(m, n) - 1 do
                p := [];
                p := [seq(x^i*g, i = 0 .. m - k - 1),
                        seq(x^i*f, i = 0 .. n - k - 1)];
                p := map(collect, p, x);
                sk := [detpol(mat(p, x), x), op(sk)]
        od;
        sk
end

### negative subresultant chain of polynomials f and g with degree(f,x)=degree(g,x)+1
s_res := proc(f, g, x)
local m, n, k, i, p, sk;
        m := degree(f, x);
        n := degree(g, x);
        sk := [];
        for k from 0 to min(m, n) - 1 do
                p := [];
                for i from 0 to max(m, n) - k - 1 do
                        p := [x^i*f, x^i*g, op(p)]
                od;
```

```
        p := map(collect, p, x);
        sk := [coeff(f, x, m)*detpol(mat(p, x), x), op(sk)]
    od;
    map(primpart, [f, g, op(sk)])
end

### formal leading coefficients of f and diff(f,x)
psc_ := proc(f, x)
local i, n1, sr, r;
    n1 := degree(f, x);
    sr := s_res(f, diff(f, x), x);
    print(sr);
    r := [];
    for i to n1 + 1 do
        r := [op(r), coeff(op(i, sr), x, n1 - i + 1)]
    od;
    r
end

### construct generalized sturm sequence for polynomials with indeterminates coefficients
res_sturm := proc(f, x)
local n1, sn, i, j, j1, r, nj, fj, stm;
    n1 := degree(f, x);
    sn := s_res(f, diff(f, x), x);
    stm := [f, diff(f, x)];
    nj := [];
    for i to n1 + 1 do
        if degree(op(i, sn), x) = n1 - i + 1 then
            nj := [op(nj), n1 - i + 1]
        fi
    od;
    r := nops(nj);
    for j from 3 to r do
        j1 := n1 + 2 - op(j - 1, nj);
        fj := op(j1, sn)*lcoeff(op(j1 - 1, sn), x)*
            lcoeff(op(j - 1, stm), x);
        stm := [op(stm), fj]
    od;
    map(primpart, stm)
end

### an example
> res_sturm(x^5+a*x+b,x);
> [x^5+a*x+b, 5*x^4+a, -4*a*x-5*b, (-256*a^5-3125*b^4)*a^4]
```