# Symmetries of Second- and Third-Order Ordinary Differential Equations

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**Abstract.** In order to apply Lie's symmetry theory for solving a differential equation it must be possible to identify the group of symmetries leaving the equation invariant. The answer is obtained in two steps. At first a classification of the possible symmetries of equations of the respective order is determined. Secondly a decision procedure is provided which allows to identify the symmetry type within this classification. For second-order equations the answer has been obtained by Lie himself. In this article the complete answer for quasilinear equations of order three is given. An important tool is the Janet base representation for the determining system of the symmetries.

## 1 Introduction

In general there are no solution algorithms available for solving nonlinear ordinary differential equations (ode's) in closed form. Usually several heuristics are applied in order to find a solution without any guarantee however to find one, or even to assure its existence. Stimulated by Galois' ideas for solving algebraic equations, Sophus Lie got interested in these problems around the middle of the $19^{th}$ century. His most significant recognition was that the transformation properties of an ode under certain groups of continuous transformations play a fundamental role for answering this question, very much like the permutations of the solutions of an algebraic equation furnish the key to understanding its solution behavior.

With the insight gained by analysing simple examples of solvable equations, Lie developed a solution scheme for ode's that may be traced back to the following question: Are there any transformations leaving the form of the given equation invariant? If the answer is affirmative these transformations are called its *symmetries*. In general, equations that may be transformed into each other are called *equivalent*, they form an *equivalence class* and share the same *symmetry type*. The union of all equivalence classes sharing the same symmetry type is called a *symmetry class*. If the solution of a canonical representative within an equivalence class may be obtained, and moreover it is possible to determine the transformation of the given equation to this canonical form, the original problem is solved.

* Supported in part by INTAS Grant 99-0167

Lie himself has worked out his theory in full detail for ode's of order two [Lie 1891]. However in many fields of application there occur ode's of order three or higher. Therefore it appears to be highly desirable to provide the full answer obtained by his theory at least for equations of order three. To this end a complete classification of the possible symmetries for these equations required. It is provided in this article. Furthermore it is shown how the various symmetry types may be obtained from the Janet base for its determining system.

Two fundamental discoveries that were made within about twenty years after Lie's death turned out to be of fundamental importance for applying his theories. The first is Loewy's theory of linear ode's [Loewy 1906], and its generalization to certain systems of linear pde's. The second is Janet's theory of linear pde's [Janet 1920] and in particular the canonical form he introduced for any such system which is known as a *Janet base* today. This concept is closely related to Buchbergers *Gröbner base* in polynomial ideal theory, and turned out to be fundamental in the realm of differential equations. The importance of these results for Lie's theory originates from the fact that the symmetries of any differential equation are obtained from its so-called *determining system*, a system of linear homogeneous pde's. Representing it as a Janet base allows to identify the type of its symmetry from its coefficients.

In this article ode's are polynomial in the derivatives, linear in the highest derivative, with coefficients that are rational functions of the dependent and the independent variables, it is called the *base field*. The overall scheme for solving ode's based on its possible symmetries decomposes in three major steps.

→ Determine the symmetry class of the given equation.
→ Transform the equation to a canonical form corresponding to its symmetry type.
→ Solve the canonical form and obtain the solution of the given equation from it.

It is the purpose of this article to provide a complete answer for the first step for equations of order two or three. The remaining two steps and many more details may be found in a forthcoming publication [Schwarz 2002].

The subsequent Section 2 deals with linear partial differential equations (pde's), and the concept of a Janet base is introduced for them. In Section 3, those results from the theory of continuous groups of the plane are presented that are relevant for the rest of this article. In Section 4 various geometric and algebraic properties of groups are expressed in terms of a Janet base for its defining equations. Section 5 contains the main result. Several theorems are proved allowing to identify the symmetry type to which a given quasilinear equation of order two or three belongs.

## 2 Linear Partial Differential Equations

The theory of systems of linear homogeneous pde's is of interest for its own right, independent of its applications for finding symmetries and invariants of differential equations. Any such system may be written in infinitely many ways by adding linear combinations of its members or derivatives thereof without changing its solution set. In general it is a difficult question whether there exist nontrivial solutions at all, or what the degree of arbitrariness of the general solution is. It may be a finite set of constants, in this case there is a finite dimensional solution space as it is true for the systems considered in this article. More general, one or more functions depending on a differing number of arguments may be involved.

These questions were the starting point for Maurice Janet. He introduced a unique representation for such systems of pde's similar to a Gröbner base representation of a system of algebraic equations, it is called a *Janet base.* There are two fundamental achievements of a Janet base representation for a system of linear homogeneous pde's. The most important quantity characterizing such systems, the degree of arbitrariness of its general solution, is obvious from its leading terms. Therefore the pattern of leading terms is called the *type of a Janet base.* Secondly due to the uniqueness of the coefficients of a Janet base, additional invariants may be obtained from them providing further information on the solutions of the corresponding system of pde's.

The systems of pde's considered in this article are linear and homogeneous in a finite set of differential indeterminates $u^1, \ldots, u^m$ over the ring of differential operators $\mathbf{Q}(x_1, \ldots, x_n)[\partial_{x_1}, \ldots, \partial_{x_n}]$. The $u^\alpha$ are also called *dependent variables* or *functions*, the $x_i$ are called *independent variables.* A *ranking* of derivatives is a total ordering such that for any two derivatives $u$ and $v$ and any derivation $\delta$ there holds $u \le v \Rightarrow \delta u \le \delta v$. A term $t_{i,j}$ in a system of linear homogeneous pde's is a function $u^k$ or a derivative of it, multiplied by an element of the base field $\mathbf{Q}(x_1, \ldots, x_n)$. The ranking of derivatives defined above generates a linear order for the terms. A *lex* order arranges derivatives according to some lexicographical order of the variables involved. In a *grlex* order, at first the order of the derivatives are compared. A linear homogeneous system of pde's with terms $t_{i,j}$ is always arranged in the form

$$\begin{array}{l}
\boxed{t_{1,1}} > t_{1,2} > \ldots > t_{1,k_1} \\
\;\;\wedge \\
\boxed{t_{2,1}} > t_{2,2} > \ldots > t_{2,k_2} \\
\;\;\wedge \\
\;\;\vdots \\
\;\;\wedge \\
\boxed{t_{N,1}} > t_{N,2} > \ldots > t_{N,k_N}
\end{array}$$

such that the ordering relations are valid as indicated. Each line of this scheme corresponds to an equation of the system of pde's, its terms are arranged in decreasing order from left to right. In order to save space, sometimes several equations are arranged into a single line. In these cases, in any line the leading terms *increase* from left to right. The terms in the rectangular boxes are the *leading terms* containing the *leading derivative*, i. e. the leading term is that term preceding any other term in the respective equation, its coefficient is usually assumed to be one. For any given system of pde's there is a finite number of term orderings. In any term ordering there is a *lowest term*, it is always one of the indeterminates itself.

The main steps for generating a Janet base for any system of pde's will be outlined now. The first fundamental operation is the *reduction*. Given any pair of equations $e_1$ and $e_2$ of the system, it may occur that the leading derivative of the latter equation $e_2$, or of a suitable derivative of it, equals some derivative in the former equation $e_1$. This coincidence may be applied to remove the respective term from $e_1$ by an operation which is called a *reduction step*. To this end, multiply the proper derivative of $e_2$ by the coefficient of the term in $e_1$ that is to be removed, and subtract it from $e_1$ multiplied by the leading coefficient of $e_2$. In general reductions may occur several times. However, due to the properties of the term orderings described above and the genuine lowering of terms in any reduction step it is assured that reductions always terminate after a finite number of iterations. After its completion there are no further reductions possible. In this case $e_1$ is called *completely reduced* with respect to $e_2$.

For any given system it will in general be possible to perform various reductions w.r.t. one or more equations of the system. It is required that all possible reductions be performed. A system with the property that there is no reduction possible of any equation of the system w.r.t. to any other equation is called an *autoreduced system*.

Autoreduced systems bear two major shortcomings which disqualify them from being considered as canonical form of a given system of pde's. In the first place, the result of autoreduction is non-unique in general. Furthermore, by proper differentiation of two equations of a system and subsequent reduction it may be possible to obtain two different equations with the same leading derivatives. Either of them may be considered as a representation of this leading derivative with a different answer. This apparent inconsistency holds already the clue how to avoid them. To this end, choose two equations with like leading function and solve both equations with respect to its leading derivative. By suitable cross differentiation it is always possible to obtain two new equations such that the derivatives at the left hand sides are identical to each other. Intuitively it is expected that the same should hold true for the right hand sides if all reductions w.r.t. to the remaining equations of the system are performed. If this is not true their difference is called an *integrability condition*. It has to be added to the system as a new equation. It can be shown that this proceeding terminates after a finite number of steps. An autoreduced system satisfying all integrability conditions is called a *Janet base* or a *coherent system*.

The following algorithm due to Janet [Janet 1920] accepts any linear homogeneous system of pde's and transforms it into a new system such that all integrability conditions are satisfied.

**Algorithm** *JanetBase*$(S)$. Given a linear homogeneous system of pde's $S = \{e_1, e_2, \ldots\}$, the Janet base corresponding to $S$ is returned. As usual let $v_i$ be the leading term of equation $e_i$.

$S1$ : *Autoreduction.* Set $S := Autoreduce(S)$.

$S2$ : *Completion.* Set $S := CompleteSystem(S)$.

$S3$ : *Find Integrability Conditions.* Find all pairs of leading terms $v_i$ and $v_j$ such that differentiation w.r.t. a non-multiplier $x_{i_k}$ and multipliers $x_{j_1}, \ldots x_{j_l}$ respectively leads to

$$\frac{\partial v_i}{\partial x_{i_k}} = \frac{\partial^{p_1+\ldots+p_l} v_j}{\partial x_{j_1}^{p_1} \ldots \partial x_{j_l}^{p_l}}$$

and determine the integrability conditions

$$c_{i,j} = LeadingCoefficient(e_j) \cdot \frac{\partial e_i}{\partial x_{i_k}} - LeadingCoefficient(e_i) \cdot \frac{\partial^{p_1+\ldots+p_l} e_j}{\partial x_{j_1}^{p_1} \ldots \partial x_{j_l}^{p_l}}$$

$S4$ : *Reduce Integrability Conditions.* For all $c_{i,j}$ set $c_{i,j} := Reduce(c_{i,j}, S)$

$S5$ : *Termination?* If all $c_{i,j}$ are zero return $S$ otherwise set $S := S \cup \{c_{i,j} | c_{i,j} \neq 0\}$, reorder $S$ properly and go to $S1$.

¿From now on any system of linear homogeneous pde's will be assumed to be a Janet base that has been generated by the algorithm described above. Furthermore it is assumed that the dimension of its solution space is finite, it is called the *order* of the system. Both the number dependent and independent variables are assumed to be not greater than two. The reason is that these constraints are *a priori* known to be true for the applications later in this article.

It turns out that systems with these properties allow a complete classification. For the corresponding Janet base types a unique notation $\mathcal{J}_{r,k}^{(m,n)}$ is introduced. Here $m$ and $n$ denote the number of dependent and independent variables respectively, $r$ denotes the order, and $k$ is a consecutive enumeration within this set. The symbol $\mathcal{J}^{(m,n)}$ without the lower indices denotes collectively all Janet base types corresponding to the given values of $m$ and $n$.

In Section 5 Janet bases of type $\mathcal{J}^{(2,2)}$ represent the determining systems of the symmetries. Those types occuring in this analysis are introduced in the remaining part of this section. The two dependent variables are denoted by $\xi$ and $\eta$, the independent ones by $x$ and $y$. In *grlex* term ordering with $\eta > \xi$ and $y > x$, the derivatives of order not higher than two are arranged as follows.

$$\ldots \eta_{yy} > \eta_{yx} > \eta_{xx} > \xi_{yy} > \xi_{yx} > \xi_{xx} > \eta_y > \eta_x > \xi_y > \xi_x > \eta > \xi.$$

For a $r$-dimensional solution space, $r$ derivatives must be selected from this arrangement such that none of it may be obtained by differentiation from any of the remaining ones. For $r = 1$, 2 or 3 the possible selections are as follows.

$$
\begin{aligned}
r=1: &\quad \{\eta\}, \{\xi\},\\
r=2: &\quad \{\eta_x,\eta\}, \{\eta_y,\eta\}, \{\eta,\xi\}, \{\xi_y,\xi\}, \{\xi_x,\xi\},\\
r=3: &\quad \{\eta_{xx},\eta_x,\eta\}, \{\eta_y,\eta_x,\eta\}, \{\eta_{yy},\eta_y,\eta\}, \{\eta_x,\eta,\xi\},\\
&\quad \{\eta_y,\eta,\xi\}, \{\xi_x,\eta,\xi\}, \{\xi_y,\eta,\xi\}, \{\xi_{yy},\xi_y,\xi\}, \{\xi_y,\xi_x,\xi\}, \{\xi_{xx},\xi_x,\xi\}.
\end{aligned}
$$

The corresponding classification of Janet bases is given next. There are two Janet base types of order 1.

$$\mathcal{J}^{(2,2)}_{1,1}: \quad \xi=0,\ \eta_x+a\eta=0,\ \eta_y+b\eta=0. \tag{1}$$

$$\mathcal{J}^{(2,2)}_{1,2}: \quad \eta+a\xi=0,\ \xi_x+b\xi=0,\ \xi_y+c\xi=0. \tag{2}$$

There are five Janet base types of order 2.

$$\mathcal{J}^{(2,2)}_{2,1}: \quad \xi=0,\ \ \eta_y+a_1\eta_x+a_2\eta=0,\ \ \eta_{xx}+b_1\eta_x+b_2\eta=0. \tag{3}$$

$$\mathcal{J}^{(2,2)}_{2,2}: \quad \xi=0,\ \ \eta_x+a_1\eta=0,\ \ \eta_{yy}+b_1\eta_y+b_2\eta=0. \tag{4}$$

$$\mathcal{J}^{(2,2)}_{2,3}: \begin{array}{l} \xi_x+a_1\eta+a_2\xi=0, \quad \xi_y+b_1\eta+b_2\xi=0,\\ \eta_x+c_1\eta+c_2\xi=0, \quad \eta_y+d_1\eta+d_2\xi=0. \end{array} \tag{5}$$

$$\mathcal{J}^{(2,2)}_{2,4}: \quad \eta+a_2\xi=0,\ \ \xi_y+b_1\xi_x+b_2\xi=0,\ \ \xi_{xx}+c_1\xi_x+c_2\xi=0. \tag{6}$$

$$\mathcal{J}^{(2,2)}_{2,5}: \quad \eta+a_1\xi=0,\ \ \xi_x+b_1\xi=0,\ \ \xi_{yy}+c_1\xi_y+c_2\xi=0. \tag{7}$$

There are ten Janet base types of order 3.

$$\mathcal{J}^{(2,2)}_{3,1}: \xi=0,\ \ \eta_y+a_1\eta_x+a_2\eta=0,\ \ \eta_{xxx}+b_1\eta_{xx}+b_2\eta_x+b_3\eta=0. \tag{8}$$

$$\mathcal{J}^{(2,2)}_{3,2}: \begin{array}{c} \xi=0,\ \ \eta_{xx}+a_1\eta_y+a_2\eta_x+a_3\eta=0,\ \ \eta_{xy}+b_1\eta_y+b_2\eta_x+b_3\eta=0,\\ \eta_{yy}+c_1\eta_y+c_2\eta_x+c_3\eta=0. \end{array} \tag{9}$$

$$\mathcal{J}^{(2,2)}_{3,3}: \xi=0,\ \ \eta_x+a_1\eta=0,\ \ \eta_{yyy}+b_1\eta_{yy}+b_2\eta_y+b_3\eta=0. \tag{10}$$

$$\mathcal{J}^{(2,2)}_{3,4}: \begin{array}{c} \xi_x+a_1\eta+a_2\xi=0,\ \ \xi_y+b_1\eta+b_2\xi=0,\\ \eta_y+c_1\eta_x+c_2\eta+c_3\xi=0,\ \ \eta_{xx}+d_1\xi_x+d_2\eta+d_3\xi=0. \end{array} \tag{11}$$

$$\mathcal{J}^{(2,2)}_{3,5}: \begin{array}{c} \xi_x+a_1\eta+a_2\xi=0,\ \ \xi_y+b_1\eta+b_2\xi=0,\\ \eta_x+c_1\eta+c_2\xi=0,\ \ \eta_{yy}+d_1\eta_y+d_2\eta+d_3\xi=0. \end{array} \tag{12}$$

$$\mathcal{J}^{(2,2)}_{3,6}: \begin{array}{c} \xi_y+a_1\xi_x+a_2\eta+a_3\xi=0,\ \ \eta_x+b_1\xi_x+b_2\eta+b_3\xi=0,\\ \eta_y+c_1\xi_x+c_2\eta+c_3\xi,\ \ \xi_{xx}+d_1\xi_x+d_2\eta+d_3\xi=0. \end{array} \tag{13}$$

$$\mathcal{J}^{(2,2)}_{3,7}: \begin{array}{c} \xi_x + a_1\eta + a_2\xi = 0, \ \ \eta_x + b_1\xi_y + b_2\eta + b_3\xi = 0, \\ \eta_y + c_1\xi_y + c_2\eta + c_3\xi = 0, \ \ \xi_{yy} + d_1\xi_y + d_2\eta + d_3\xi = 0. \end{array} \tag{14}$$

$$\mathcal{J}^{(2,2)}_{3,8}: \eta + a_1\xi = 0, \ \ \xi_x + b_1\xi = 0, \ \ \xi_{yyy} + c_1\xi_{yy} + c_2\xi_y + c_3\xi = 0. \tag{15}$$

$$\mathcal{J}^{(2,2)}_{3,9}: \begin{array}{c} \eta + a_1\xi = 0, \ \ \xi_{xx} + b_1\xi_y + b_2\xi_x + b_3\xi = 0, \\ \xi_{xy} + c_1\xi_y + c_2\xi_x + c_3\xi = 0, \ \ \xi_{yy} + d_1\xi_y + d_2\xi_x + d_3\xi = 0. \end{array} \tag{16}$$

$$\mathcal{J}^{(2,2)}_{3,10}: \eta + a_1\xi = 0, \ \ \xi_y + b_1\xi_x + b_2\xi = 0, \xi_{xxx} + c_1\xi_{xx} + c_2\xi_x + c_3\xi = 0. \tag{17}$$

There are twenty Janet base types of order 4, only the following five types are required for the symmetry analysis in Section 5.

$$\mathcal{J}^{(2,2)}_{4,2}: \begin{array}{c} \xi = 0, \ \ \eta_{xy} + a_1\eta_{xx} + a_2\eta_y + a_3\eta_x + a_4\eta = 0, \\ \eta_{yy} + b_1\eta_{xx} + b_2\eta_y + b_3\eta_x + b_4\eta = 0, \\ \eta_{xxx} + c_1\eta_{xx} + c_2\eta_y + c_3\eta_x + c_4\eta = 0. \end{array} \tag{18}$$

$$\mathcal{J}^{(2,2)}_{4,9}: \begin{array}{c} \xi_y + a_2\xi_x + a_2\xi_x + a_3\eta + a_4\xi = 0, \ \ \eta_y + b_1\eta_x + b_2\xi_x + b_3\eta + b_4\xi = 0, \\ \xi_{xx} + c_1\eta_x + c_2\xi_x + c_3\eta + c_4\xi = 0, \ \ \eta_{xx} + d_1\eta_x + d_2\xi_x + d_3\eta + d_4\xi = 0. \end{array} \tag{19}$$

$$\mathcal{J}^{(2,2)}_{4,10}: \begin{array}{c} \xi_y + a_1\eta_y + a_2\xi_x + a_3\eta + a_4\xi = 0, \ \ \eta_x + b_1\eta_y + b_2\xi_x + b_3\eta + b_4\xi = 0, \\ \xi_{xx} + c_1\eta_y + c_2\xi_x + c_3\eta + c_4\xi = 0, \ \ \eta_{yy} + d_1\eta_y + d_2\xi_x + d_3\eta + d_4\xi = 0. \end{array} \tag{20}$$

$$\mathcal{J}^{(2,2)}_{4,12}: \begin{array}{c} \xi_x + a_1\eta_y + a_2\xi_y + a_3\eta + a_4\xi = 0, \ \ \eta_x + b_1\eta_y + b_2\xi_y + b_3\eta + b_4\xi = 0, \\ \xi_{yy} + c_1\eta_y + c_2\xi_y + c_3\eta + c_4\xi = 0, \ \ \eta_{yy} + d_1\eta_y + d_2\xi_y + d_3\eta + d_4\xi = 0. \end{array} \tag{21}$$

$$\mathcal{J}^{(2,2)}_{4,14}: \begin{array}{c} \eta_x + a_1\xi_y + a_2\xi_x + a_3\eta + a_4\xi = 0, \ \ \eta_y + b_1\xi_y + b_2\xi_x + b_3\eta + b_4\xi = 0, \\ \xi_{xx} + c_1\xi_y + c_2\xi_x + c_3\eta + c_4\xi = 0, \ \ \xi_{xy} + d_1\xi_y + d_2\xi_x + d_3\eta + d_4\xi = 0, \\ \xi_{yy} + e_1\xi_y + e_2\xi_x + e_3\eta + e_4\xi = 0. \end{array} \tag{22}$$

$$\mathcal{J}^{(2,2)}_{4,17}: \begin{array}{c} \eta + a_4\xi = 0, \ \ \xi_{xy} + b_1\xi_{xx} + b_2\xi_y + b_3\xi_x + b_4\xi = 0, \\ \xi_{yy} + c_1\xi_{xx} + c_2\xi_y + c_3\xi_x + c_4\xi = 0, \\ \xi_{xxx} + d_1\xi_{xx} + d_2\xi_y + d_3\xi_x + d_4\xi = 0. \end{array} \tag{23}$$

$$\mathcal{J}^{(2,2)}_{4,19}: \begin{array}{c} \eta = 0, \ \ \xi_{xx} + a_2\xi_y + a_3\xi_x + a_4\xi = 0, \\ \xi_{xy} + b_2\xi_y + b_3\xi_x + b_4\xi = 0, \\ \xi_{yyy} + c_1\xi_{yy} + c_2\xi_y + c_3\xi_x + c_4\xi = 0. \end{array} \tag{24}$$

Finally there are three Janet base types of order five required.

$$\mathcal{J}^{(2,2)}_{5,1}: \begin{array}{c} \xi_x + a_1\xi = 0, \ \ \xi_y = 0, \\ \eta_{xy} + c_1\eta_x + c_2\eta + c_3\xi = 0, \ \ \eta_{yy} + d_1\eta_y + d_2\eta + d_3\xi = 0, \\ \eta_{xxx} + e_1\eta_{xx} + e_2\eta_y + e_3\eta_x + e_4\eta + e_5\xi = 0. \end{array} \tag{25}$$

$$\mathcal{J}^{(2,2)}_{5,2}: \begin{array}{c} \eta_x = 0, \ \ \eta_y + b_1\eta = 0, \\ \xi_{xx} + c_1\xi_x + c_2\eta + c_3\xi = 0, \ \ \xi_{xy} + d_1\xi_y + d_2\eta + d_3\xi = 0, \\ \xi_{yyy} + e_1\xi_{yy} + e_2\xi_y + e_3\xi_x + e_4\eta + e_5\xi = 0. \end{array} \tag{26}$$

$$\mathcal{J}^{(2,2)}_{5,3}: \begin{array}{c} \eta_x + a_1\xi_x + a_2\eta + a_3\xi = 0, \ \ \eta_y + b_1\xi_y + b_2\eta + b_3\xi = 0, \\ \xi_{xy} + c_1\xi_{xx} + c_2\xi_y + c_3\xi_x + c_4\eta + c_5\xi = 0, \\ \xi_{yy} + d_1\xi_{xx} + d_2\xi_y + d_3\xi_x + d_4\eta + d_5\xi = 0, \\ \xi_{xxx} + e_1\xi_{xx} + e_2\xi_y + e_3\xi_x + e_4\eta + e_5\xi = 0. \end{array} \tag{27}$$

The coefficients $a_1, a_2, \ldots$ occuring in these Janet bases have to obey certain differential constraints in order to assure coherence. They are rather complicated and are given explicitly in [Schwarz 2002].

## 3 Lie Groups and Lie Algebras

The discussion of groups in this article is limited to the finite groups of the plane. By definition their elements may be specified by a finite number of parameters. Let a set of transformations in the $x-y-$plane depending on a continuous parameter $a$ be given by

$$\bar{x} = f(x, y, a), \qquad \bar{y} = g(x, y, a). \tag{28}$$

Assume that the dependence on this parameter is such that a subsequent transformation

$$\bar{\bar{x}} = f(\bar{x}, \bar{y}, b), \qquad \bar{\bar{y}} = g(\bar{x}, \bar{y}, b) \tag{29}$$

may be replaced by a single one of the same set (28)

$$\bar{\bar{x}} = f(x, y, c), \qquad \bar{\bar{y}} = g(x, y, c) \tag{30}$$

such that the parameter $c$ may be expressed through $a$ and $b$ *alone*, i.e. there is a relation $c = \phi(a, b)$ with $\phi$ *independent* of $x$ and $y$. The functions $f$ and $g$ are supposed to be invertible such that the action (28) may be reversed, i. e. it is required that the functional determinant $\dfrac{D(f,g)}{D(x,y)} \neq 0$. Furthermore a parameter value corresponding to no action at all is supposed to exist. If not stated otherwise, it is always assumed that this latter value is $a = 0$, i. e. $f(x, y, 0) = x$ and $g(x, y, 0) = y$.

The definition of a one-parameter group may be generalized in a fairly straightforward manner to groups depending on any number $r > 1$ of parameters. The transformation (28) is replaced by the more general one

$$\bar{x} = f(x, y, a_1, a_2, \ldots, a_r), \qquad \bar{y} = g(x, y, a_1, a_2, \ldots, a_r). \tag{31}$$

If

$$\bar{\bar{x}} = f(\bar{x}, \bar{y}, b_1, b_2, \ldots, b_r), \qquad \bar{\bar{y}} = g(\bar{x}, \bar{y}, b_1, b_2, \ldots, b_r)$$

is a second transformation it is required that it may be represented in the form

$$\bar{\bar{x}} = f(x, y, c_1, c_2, \dots, c_r), \qquad \bar{\bar{y}} = g(x, y, c_1, c_2, \dots, c_r)$$

such that the parameters $c_k$ may be expressed as $c_k = \phi_k(a_1, \dots, a_r, b_1, \dots, b_r)$. The identity is supposed to correspond to $a_1 = a_2 = \dots = a_r = 0$, and the existence of the inverse requires the functional determinant of $f$ and $g$ with respect to $x$ and $y$ to be non-vanishing.

A first major result of Lie was to characterize those functions in (31) that define a group of transformations. The answer is essentially obtained in terms of a system of differential equations of a special form that must be satisfied by $f$ and $g$, it is described next.

**Theorem 1.** *(Lie's First Fundamental Theorem). If the equations*

$$\bar{x} = f(x, y, a_1, a_2, \dots, a_r), \qquad \bar{y} = g(x, y, a_1, a_2, \dots, a_r)$$

*define a r-parameter group, $\bar{x}$ and $\bar{y}$, considered as functions of $x$, $y$ and the parameters, satisfy a system of pde's of the form*

$$\frac{\partial \bar{x}}{\partial a_k} = \sum_{j=1}^{r} \psi_{jk}(a_1, \dots, a_r)\xi_j(\bar{x}, \bar{y}), \quad \frac{\partial \bar{y}}{\partial a_k} = \sum_{j=1}^{r} \psi_{jk}(a_1, \dots, a_r)\eta_j(\bar{x}, \bar{y}) \tag{32}$$

*for $k = 1, \dots, r$. The determinant of the $\psi_{jk}$ does not vanish identically. On the other hand, if two function $f$ and $g$ satisfy a system of differential equations of this form and there are parameter values $a_1^0, \dots, a_r^0$ such that*

$$f(x, y, a_1^0, \dots, a_r^0) = x, \quad g(x, y, a_1^0, \dots, a_r^0) = y$$

*and the determinant of the $\psi_{jk}$ is finite and different from zero for these parameter values, then they define a transformation group. The non-vanishing of the determinant of the $\psi_{jk}$ allows solving the system (32) for the $\xi's$ and the $\eta's$, i. e. there holds also*

$$\xi_j = \sum_{k=1}^{r} \alpha_{jk} \frac{\partial \bar{x}}{\partial a_k}, \quad \eta_j = \sum_{k=1}^{r} \alpha_{jk} \frac{\partial \bar{y}}{\partial a_k}. \tag{33}$$

The proof of this theorem may be found in [Lie 1888], vol. I, Kapitel 2, page 27-34 and vol. III, Kapitel 25, page 545-564.

The functions $\xi$ and $\eta$ occuring at the right hand side of (32) have another important meaning, they occur as coefficients of the infinitesimal generators of the group (28) which are defined next. Let $\bar{x} = f(x, y, a)$, $\bar{y} = g(x, y, a)$ define a one-parameter group in the $x - y -$plane, $f(x, y, 0) = x$, $g(x, y, 0) = y$. Expanding $f$ and $g$ at the value $a = 0$ and retaining only the terms linear in $a$ yields

$$\begin{aligned} \bar{x} &= f(x, y, 0) + a\frac{\partial f(x, y, a)}{\partial a}|_{a=0} + o(a^2) \equiv x + a\xi(x, y) + o(a^2), \\ \bar{y} &= g(x, y, 0) + a\frac{\partial g(x, y, a)}{\partial a}|_{a=0} + o(a^2) \equiv y + a\eta(x, y) + o(a^2). \end{aligned} \tag{34}$$

The infinitesimal generator $U$ is defined by

$$U = \xi(x,y)\partial_x + \eta(x,y)\partial_y. \tag{35}$$

For an $r-$parameter group, to each parameter $a_i$, with $a_j = 0$ for $j \neq i$, there corresponds an infinitesimal generator $U_i$ with coefficients $\xi_i$ and $\eta_i$

$$U_i = \xi_i(x,y)\partial_x + \eta_i(x,y)\partial_y, \qquad i = 1, \ldots, r. \tag{36}$$

An important operation for infinitesimal generators is to form its *commutator.* Let $U_1$ and $U_2$ be given by (36). Then the definition of its commutator

$$[U_1, U_2]f = U_1U_2f - U_2U_1f \equiv U_3f$$

implies for the coefficients of $U_3$

$$\begin{aligned} \xi_3 &= \xi_1\xi_{2,x} - \xi_{1,x}\xi_2 + \eta_1\xi_{2,y} - \xi_{1,y}\eta_2, \\ \eta_3 &= \xi_1\eta_{2,x} - \eta_{1,x}\xi_2 + \eta_1\eta_{2,y} - \eta_{1,y}\eta_2. \end{aligned} \tag{37}$$

The importance of the infinitesimal generators of a group arises from the fact that a group is uniquely determined by them. This relation is established in the following Theorem due to Lie [Lie 1888].

**Theorem 2.** *(Lie's Second Fundamental Theorem). Any r-parameter group*

$$\bar{x} = f(x, y, a_1, a_2, \ldots, a_r), \qquad \bar{y} = g(x, y, a_1, a_2, \ldots, a_r)$$

*of transformations contains $r$ infinitesimal generators*

$$U_i = \xi_i\partial_x + \eta_i\partial_y \qquad for \;\; i = i, \ldots, r$$

*which satisfy $r(r-1)$ relations of the form*

$$[U_i, U_j] = \sum_{k=1}^{r} c_{ij}^k U_k \tag{38}$$

*with constant coefficients $c_{ij}^k$. Conversely, $r$ independent generators $U_1, \ldots, U_r$ satisfying relations (38) generate a r-parameter group of finite transformations.*

Starting with the vector fields forming the infinitesimal generators of a group, a third representation in terms of the *defining equations* as Lie [Lie 1888], vol. I, page 185 called them may be obtained. By definition, these equations are such that its general solution generates the same vector space as the coefficients of the given generators. This representation is particularly important if the groups to be considered occur as symmetry groups of differential equations because they are usually defined in terms of systems of pde's. Let the infinitesimal generators of a group be

$$U_i = \xi_i(x,y)\partial_x + \eta_i(x,y)\partial_y, \quad i = 1, \ldots, r$$

and $\xi = c_1\xi_1 + \dots c_r\xi_r$, $\eta = c_1\eta_1 + \dots c_r\eta_r$ its general element. A system of linear homogeneous pde's which has $\xi_i$ and $\eta_i$ as a base for its solution space is called the system of *defining equtions*.

The existence of the defining equations has been shown by Lie and a constructive method for finding them has been given. A Janet base for these defining equations is called a Janet base for the group. In general it is a much harder problem to obtain the vector field coefficients from a given set of defining equations, because this comes down to determining the general solution of these latter equations. The relation between these representations of a group and how to turn from one to another is illustrated by the following diagram. The operations corresponding to the downward arrows at the left can always be performed algorithmically. Contrary to that, there is no generally valid algorithm for the operations at the right corresponding to the upward arrows.

$$
\begin{tikzcd}
\boxed{\text{Group Equations}} \arrow[d, shift right=2ex, "\text{Differentiation}"] \\
\boxed{\text{Vector Fields}} \arrow[u, shift right=2ex, "\text{Solving } 1^{st} \text{ Order Ode}"] \arrow[d, shift right=2ex, "\text{Solving Linear Algebraic Equations}"] \\
\boxed{\text{Defining Equations}} \arrow[u, shift right=2ex, "\text{Solving Linear Pde's}"]
\end{tikzcd}
$$

By abstraction from its geometric origin, the algebraic objects obeying commutation relations like (38) are studied as a subject of interest of its own right, they have been called *Lie algebras* by Hermann Weyl. First of all they have to be defined precisley.

**Definition 1.** *(Lie algebra). A Lie algebra $\mathcal{L}$ is a vector space over a field $F$ equipped with a binary operation called the commutator satisfying the following constraints for any $U, V, W \in \mathcal{L}$ and $\alpha, \beta \in F$.*

*i)* $[U, V] = -[V, U]$ *(anti-commutativity)*

*ii)* $[\alpha U + \beta V, W] = \alpha[U, W] + \alpha[V, W]$ *(bilinearity)*

*iii)* $[U, [V, W]] + [V, [W, U]] + [W, [U, V]] = 0$ *(Jacobi identity)*

In this article always finite dimensional Lie algebras are considered, and it is always assumed that the field $F$ has characteristic zero, usually it will be the rational number field $\mathbf{Q}$ or some algebraic extension $\bar{\mathbf{Q}}$ of it. The structure constants contain the complete information on the group from which they originate as the following result shows.

**Theorem 3.** *(Lie's Third Fundamental Theorem). The $r^3$ constants $c_{ij}^k$, $i, j, k = 1, \ldots, r$, determine an $r$-parameter continuous group if and only if they obey the relations $c_{ik}^j + c_{ki}^j = 0$ and*

$$\sum_{h=1}^{r} (c_{ik}^h c_{jh}^l + c_{kj}^h c_{ih}^l + c_{ji}^h c_{kh}^l) = 0.$$

The proof of this theorem may be found in [Lie 1888], vol. III, page 597 ff.

Fundamental problems are to classify all types of composition leading to non-isomorphic Lie algebras, and to determine simpler substructures within a given Lie algebra, i. e. to determine its invariant subalgebras. The *characteristic polynomial*

$$\begin{aligned} \Delta(\omega) \equiv |\sum E_i c_{ij}^k - \delta_{jk}\omega| = \\ \omega^r - \psi_1(E)\omega^{r-1} + \psi_2(E)\omega^{r-2} + \ldots + (-1)^{r-1}\psi_{r-1}(E)\omega \end{aligned} \tag{39}$$

or correspondingly the *characteristic equation* $\Delta(\omega) = 0$ where $E_1, \ldots, E_r$ are $r$ indeterminates turns out to be an important tool for dealing with these problems. It has been introduced by Killing [Killing 1887] in order to answer the question in which two-parameter groups a given one-parameter group is contained as an invariant subgroup. The number of independent coefficients of the characteristic polynomial is called the *rank of the Lie algebra.* If new basis elements are introduced in a Lie algebra by $\bar{U}_i = \sum \alpha_{i,j} U_j$, the coefficients $\psi_k$ in (39) are changed according to

$$\bar{\psi}_k(\bar{E}_1, \ldots, \bar{E}_r) = \psi_k(\sum \alpha_{i,1} E_i, \ldots, \sum \alpha_{i,r} E_i),$$

i. e. the structure of the characteristic polynomial remains essentially unchanged. This feature is an important tool for identifying the isormorphy class of a Lie algebra.

Changing coordinates is one of the most important operations for groups and its infinitesimal generators. To this end it must be known in the first place how the new representation is obtained if a coordinate transformation is applied. Closely related to that is the problem how to decide whether two given groups or vector fields are related by a suitable variable transformation. Two groups are called *similar* if there is a point-transformation that transforms them into each other. This definition establishes an equivalence relation between groups. An equivalence class comprises all groups or its vector fields of infinitesimal generators that may be transformed into each other by a suitable variable change, it is called a *group type.* As soon as equivalence between any pair of groups may be decided, the objects of primary interest are the equivalence classes.

Usually groups of transformations are given in terms of their infinitesimal generators. Therefore the behavior of vector fields under a change of variables must be known in the first place. Let $U = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ be a vector field in the $x - y$-plane. Furthermore let new coordinates $u$ and $v$ be defined by $u = \sigma(x,y)$ and $v = \rho(x,y)$ with the inverse $x = \phi(u,v)$ and $y = \psi(u,v)$. Then the coefficients of the transformed vector field $V = \alpha(u,v)\partial_u + \beta(u,v)\partial_v$ are

$$\alpha(u,v) = U\sigma|_{x=\phi, y=\psi}, \qquad \beta(u,v) = U\rho|_{x=\phi, y=\psi}$$

Of particular importance is the question how the commutator of two vectorfields is transformed under a change of variables. The answer has been given by Lie [Lie 1888], part I, page 84. If three vector fields $U_1$, $U_2$ and $U_3$ in the $x-y-$plane are transformed into $V_1$, $V_2$ and $V_3$ in the $u-v-$plane repectively, its commutator $[U_1, U_2] = U_3$ is transformed into $[V_1, V_2] = V_3$.

Let $\bar{x} = f(x,y,a)$ and $\bar{y} = g(x,y,a)$ be the finite equations of a one-parameter group in the plane with infinitesimal generator $U = \xi(x,y)\partial_x + \eta(x,y)\partial_y$. Under the action of this group an arbitrarily chosen point describes a curve the points of which are parametrized by $a$. Any two points on an individual curve may be transformed into each other by a suitable group element, they are *equivalent* w.r.t. this group. If they are located on different curves such a transformation does not exist. Consequently these curves form equivalence classes of points equivalent w.r.t. to transformations of the group. The curve that is generated by the action of a one-parameter group on an arbitrary point of the plane is called its *path curve* or its *orbit*. The orbits of a one-parameter group generate an *invariant decomposition* of the plane.

Given an $r$-parameter group of the plane, the following question may be raised: Is it possible to transform a given point of the $x-y-$plane into any other point that is arbitrarily chosen by a suitable group element? The answer leads to the following definition. An $r$-parameter group is called *simply transitive* or just *transitive* if amongst its operations at least one may be found that transforms a pair of arbitrarily chosen points of the plane into each other. If such a pair of points does not exist it is called intransitive.

Even if a group is transitive there may exist a more general kind of decomposition. This remarkable property is the basis for the new concept defined next. A transitive group of the plane is called *imprimitive* if there is a one-parameter family of curves that are transformed among themselves by the elements of the group. The corresponding curves are called a *system of imprimitivity* or *s.o.i.* for short. If such a system does not exist, the group is called *primitive*. The curves which are interchanged by the operations of an imprimitive group correspond to the blocks of elements which are permuted under the action of finite permutation groups.

The number of systems of imprimitivity is an important distinguishing feature of any transformation group. For a continuous group of the plane the complete answer has been obtained by Lie [Lie 1893], page 342, as follows. Either it is primitive, or it has exactly one or two systems of imprimitivity, or there is a system of imprimitivity depending on a parameter or a function of one variable.

For later applications a particularly well suited criterion in terms of the Janet base of a group may be described as follows [Schwarz 1995]. Let the infinitesimal generators of a symmetry group of the $x-y-$plane be of the form $U = \xi\partial_x + \eta\partial_y$ where the coefficients solve the corresponding determining system. In order for a system of imprimitivity described by $y' = \omega(x,y)$, different from $x = const.$ or

$y = const$ to exist, an equation

$$\eta_y \omega + \eta_x - \eta \omega_y - \xi_y \omega^2 - \xi_x \omega - \xi \omega_x = 0 \tag{40}$$

with a suitable $\omega \equiv \omega(x, y)$ must be valid as a consequence of the determining system of this symmetry group. If $x = const$ or $y = const$ is a system of imprimitivity, (40) reduces to $\xi_y = 0$ or $\eta_x = 0$ respectively.

The subsequent classification of Lie-algebras of dimension not higher than three is complete without any duplication, i. e. any such Lie algebra is isomorphic to one and only one entry. In some cases there occur parameters. They have to be constraint suitably in order to guarantee this feature. In addition the following general rule is applied: Whenever an essential algebraic property like e. g. the dimension of the derived algebra changes for a special value of a parameter, the algebra corresponding to these parameter values is listed as a separate entry. For dimension higher than three only those algebras are included that occur in the symmetry analysis of second- and third-order ode's.

One-dimensional algebras

$\mathbf{l}_1$: *commutative*, Characteristic equation: $\omega = 0$.

Two-dimensional algebras

$\mathbf{l}_{2,1}$: $[U_1, U_2] = U_1$, $dim\ \mathbf{l}'_{2,1} = 1$, $dim\ \mathbf{l}''_{2,1} = 0$.
Characteristic equation: $\omega^2 - E_2\omega = \omega(\omega - E_2) = 0$.

$\mathbf{l}_{2,2}$: *commutative*. Characteristic equation: $\omega^2 = 0$.

Three-dimensional algebras

$\mathbf{l}_{3,1}$: $[U_1, U_2] = U_1, [U_1, U_3] = 2U_2, [U_2, U_3] = U_3$, $dim\, \mathbf{l}'_{3,1} = dim\, \mathbf{l}''_{3,1} = 3$.
Characteristic equation: $\omega^3 + (4E_3E_1 - E_2^2)\omega = \omega(\omega^2 + 4E_3E_1 - E_2^2) = 0$.

$\mathbf{l}_{3,2}(c)$: $[U_1, U_3] = U_1, [U_2, U_3] = cU_2, c \neq 0$, $dim\ \mathbf{l}'_{3,2} = 2$, $dim\ \mathbf{l}''_{3,2} = 0$.
Characteristic equation: $\omega^3 - (c+1)E_3\omega^2 + cE_3^2\omega = \omega(\omega - E_3)(\omega - cE_3) = 0$.

$\mathbf{l}_{3,3}$: $[U_1, U_3] = U_1, [U_2, U_3] = U_1 + U_2$, $dim\ \mathbf{l}'_{3,3} = 2$, $dim\ \mathbf{l}''_{3,3} = 0$.
Characteristic equation: $\omega^3 - 2E_3\omega^2 + E_3^2\omega = \omega(\omega - E_3)^2 = 0$.

$\mathbf{l}_{3,4}$: $[U_1, U_3] = U_1$, $dim\ \mathbf{l}'_{3,4} = 1$, $dim\ \mathbf{l}''_{3,4} = 0$.
Characteristic equation: $\omega^3 - E_3\omega^2 = \omega^2(\omega - E_3) = 0$.

$\mathbf{l}_{3,5}$: $[U_2, U_3] = U_1$, $dim\ \mathbf{l}'_{3,5} = 1$, $dim\ \mathbf{l}''_{3,5} = 0$.
Characteristic equation: $\omega^3 = 0$.

$\mathbf{l}_{3,6}$: *commutative*. Characteristic equation: $\omega^3 = 0$.

Some algebras with more than three dimensions

$\mathbf{l}_{4,1}$: $[U_1, U_2] = U_1, [U_1, U_3] = 2U_2, [U_2, U_3] = U_3$, $dim\, \mathbf{l}'_{4,1} = dim\, \mathbf{l}''_{4,1} = 3$.
Characteristic equation: $\omega^4 + (4E_3E_1 - E_2^2)\omega^2 = (\omega^2 + 4E_3E_1 - E_2^2)\omega^2 = 0$.

$\mathbf{l}_{4,4}$: $[U_1, U_4] = 2U_1, [U_2, U_3] = U_1, [U_2, U_4] = U_2, [U_3, U_4] = U_2 + U_3$,
$dim\ \mathbf{l}'_{4,4} = 3$, $dim\ \mathbf{l}''_{4,4} = 1$, $dim\ \mathbf{l}'''_{4,4} = 0$ Characteristic equation:
$\omega^4 - 4E_4\omega^3 + 5E_4^2\omega^2 - 2E_4^3\omega = -\omega(\omega - 2E_4)(\omega - E_4)^2 = 0$.

$\mathbf{l}_{4,5}(c)$: $[U_1, U_4] = cU_1, [U_2, U_3] = U_1, [U_2, U_4] = U_2, [U_3, U_4] = (c-1)U_3$,
$c \neq 1$, $dim\ \mathbf{l}'_{4,5} = 3$, $dim\ \mathbf{l}''_{4,5} = 1$, $dim\ \mathbf{l}''_{4,4} = 0$. Characteristic equation:

$$\omega^4 - 2cE_4\omega^3 + (c^2 + c - 1)E_4^2\omega^2 - c(c-1)E_4^3\omega.$$
$$= \omega(\omega - E_4)(\omega - cE_4)(\omega - (c-1)E_4)) = 0.$$

$\mathbf{l}_{4,10}$: $[U_1, U_2] = U_2, [U_3, U_4] = U_4$, $dim\ \mathbf{l}'_{4,10} = 2$, $dim\ \mathbf{l}''_{4,10} = 0$.
Characteristic equation:

$$\omega^4 + (E_3 + e_1)\omega^3 + E_3 E_1 \omega^2 = \omega^2(\omega + E_1)(\omega + E_3) = 0.$$

$\mathbf{l}_8$: $[U_1, U_3] = U_1$, $[U_1, U_5] = U_2$, $[U_1, U_7] = U_4$, $[U_1, U_8] = U_6 + 2U_3$, $[U_2, U_4] = U_1$, $[U_2, U_6] = U_2$, $[U_2, U_7] = 2U_6 + U_3$, $[U_2, U_8] = U_5$, $[U_3, U_4] = -U_4$, $[U_3, U_5] = U_5$, $[U_3, U_8] = U_8$, $[U_4, U_5] = U_6 - U_3$, $[U_4, U_8] = U_7$, $[U_5, U_6] = U_5$, $[U_6, U_7] = U_7$

Characteristic equation: $\psi_1 = -2e_6$, $\psi_2 = 3e_2e_7 + 6e_1e_8 + e_6^2 + e_6e_3 - 2e_3^2$.

The infinitesimal generators of those groups that occur later on as symmetry groups of differential equations are listed next. In addition the Janet bases for the determining systems are given in *grlex* term ordering with $\eta > \xi$ and $y > x$. The coefficients $\phi_k$ for $k = 1, \ldots, r$ occuring in the group types $\mathbf{g}_{15}$ and $\mathbf{g}_{16}$ are undetermined functions of $x$ from some function field. The coefficients $q_k$ occuring in the Janet base for these groups are defined by

$$q_k = -\frac{W_k^{(2)}(\phi_1, \phi_2)}{W^{(2)}(\phi_1, \phi_2)}$$

for $k = 1, \ldots, r$. The algebraic properties of groups with $r \le 4$ may be obtained from their Lie algebra type which is always given.

One-parameter group

$\mathbf{g}_{27}$: $\{\partial_y\}$. Type: $\mathbf{l}_1$. Janet base: $\{\xi, \eta_x, \eta_y\}$

Two-parameter groups

$\mathbf{g}_4$: $\{\partial_y, y\partial_y\}$. Type: $\mathbf{l}_{2,1}$. Janet base: $\{\xi, \eta_x, \eta_{yy}\}$.

$\mathbf{g}_{15}(r=2)$: $\{\phi_1\partial_y, \phi_2\partial_y\}$. Type: $\mathbf{l}_{2,2}$.
Janet base: $\{\xi, \eta_y, \eta_{xx} + q_1\eta_x + q_2\eta\}$.

$\mathbf{g}_{25}$: $\{\partial_x, x\partial_x + y\partial_y\}$. Type: $\mathbf{l}_{2,1}$. Janet base: $\{\xi_x - \frac{1}{y}\eta, \xi_y, \eta_x, \eta_y - \frac{1}{y}\eta\}$.

$\mathbf{g}_{26}$: $\{\partial_x, \partial_y\}$. Type: $\mathbf{l}_{2,2}$. Janet base: $\{\xi_x, \xi_y, \eta_x, \eta_y\}$.

Three-parameter groups

$\mathbf{g}_5$: $\{\partial_x, \partial_y, y\partial_y\}$. Type: $\mathbf{l}_{3,4}$. Janet base: $\{\xi_x, \xi_y, \eta_x, \eta_{yy}\}$.

$\mathbf{g}_7$: $\{\partial_x, \partial_y, x\partial_x + cy\partial_y\}$ with $c \neq 0, 1$. Type: $\mathbf{l}_{3,2}$.
Janet base: $\{\xi_y, \eta_x, \eta_y - c\xi_x, \xi_{xx}\}$.

$\mathbf{g}_8$: $\{\partial_y, y\partial_y, y^2\partial_y\}$. Type: $\mathbf{l}_{3,1}$. Janet base: $\{\xi, \eta_x, \eta_{yyy}\}$.

$\mathbf{g}_{10}$: $\{\partial_x + \partial_y, x\partial_x + y\partial_y, x^2\partial_x + y^2\partial_y\}$. Type: $\mathbf{l}_{3,1}$. Janet base:
$\{\xi_y, \eta_x, \eta_y + \xi_x + \frac{2}{x-y}(\eta - \xi), \xi_{xx} - \frac{2}{x-y}\xi_x - \frac{2}{(x-y)^2}(\eta - \xi)\}$.

$\mathbf{g}_{13}$: $\{\partial_x, 2x\partial_x + y\partial_y, x^2\partial_x + xy\partial_y\}$. Type: $\mathbf{l}_{3,1}$.
Janet base: $\{\xi_x - \frac{2}{y}\eta, \xi_y, \eta_y - \frac{1}{y}\eta, \eta_{xx}\}$.

$\mathbf{g}_{15}(r=3)$: $\{\phi_1\partial_y, \phi_2\partial_y, \phi_3\partial_y\}$. Type: $\mathbf{l}_{3,6}$.
Janet base: $\{\xi, \eta_y, \eta_{xxx} + q_1\eta_{xx} + q_2\eta_x + q_3\eta\}$.

$\mathbf{g}_{16}(r=2)$: $\{\phi_1\partial_y, \phi_2\partial_y, y\partial_y\}$. Type: $\mathbf{l}_{3,2}$ with $c = 1$.
Janet base: $\{\xi, \eta_{xx} + q_1\eta_x + q_2\eta, \eta_{xy}, \eta_{yy}\}$.

$\mathbf{g}_{17}(l = 1, \rho_1 = 1, \alpha_1 = 0)$: $\{\partial_x, \partial_y, x\partial_y\}$. Type: $\mathbf{l}_{3,5}$.
Janet base: $\{\xi_x, \xi_y, \eta_y, \eta_{xx}\}$.

$\mathbf{g}_{17}(l = 1, \rho_1 = 1, \alpha_1 = 1)$: $\{\partial_x, e^x\partial_y, xe^x\partial_y\}$. Type: $\mathbf{l}_{3,3}$.
Janet base: $\{\xi_x, \xi_y, \eta_y, \eta_{xx} - 2\eta_x + \eta\}$.

$\mathbf{g}_{17}(l = 2, \rho_1 = \rho_2 = 0, \alpha_1 = 0, \alpha_2 = 1)$: $\{\partial_x, \partial_y, e^x\partial_y\}$. Type: $\mathbf{l}_{3,4}$.

Janet base: $\{\xi_x, \xi_y, \eta_y, \eta_{xx} - \eta_x\}$.
$\mathbf{g}_{17}(l=2, \rho_1=\rho_2=0, \alpha_1=1, \alpha_2=c)$: $\{\partial_x, e^x\partial_y, e^{cx}\partial_y\}$, $c \neq 0, 1$.
Type: $\mathbf{l}_{3,2}$. Janet base: $\{\xi_x, \xi_y, \eta_y, \eta_{xx} - (c+1)\eta_x + c\eta\}$.
$\mathbf{g}_{20}(r=1)$: $\{\partial_x, \partial_y, x\partial_x + (x+y)\partial_y\}$. Type: $\mathbf{l}_{3,3}$.
Janet base: $\{\xi_y, \eta_x - \xi_x, \eta_y - \xi_x, \xi_{xx}\}$.
$\mathbf{g}_{24}$: $\{\partial_x, \partial_y, x\partial_x + y\partial_y\}$. Type: $\mathbf{l}_{3,2}$ with $c = 1$.
Janet base: $\{\xi_y, \eta_x, \eta_y - \xi_x, \xi_{xx}\}$.

Selected groups with more than three parameters

$\mathbf{g}_6$: $\{\partial_x, \partial_y, x\partial_x, y\partial_y\}$. Type $\mathbf{l}_{4,10}$. Janet base: $\{\xi_y, \eta_x, \xi_{xx}, \eta_{yy}\}$.
$\mathbf{g}_{14}$: $\{\partial_x, x\partial_x, x^2\partial_x + xy\partial_y, y\partial_y\}$. Type $\mathbf{l}_{4,1}$.
Janet base: $\{\xi_y, \eta_y - \frac{1}{y}\eta, \xi_{xx} - \frac{2}{y}\eta_x, \eta_{xx}\}$.
$\mathbf{g}_{19}$ with $r=2$: $\{\partial_y, x\partial_y, \partial_x, x\partial_x + cy\partial_y\}$. Type: $\mathbf{l}_{4,5}(c)$.
Janet base: $\{\xi_y, \eta_y - c\xi_x, \xi_{xx}, \eta_{xx}\}$.
$\mathbf{g}_{20}$ with $r=2$: $\{\partial_x, \partial_y, x\partial_y, x\partial_x + (x^2+2y)\partial_y\}$. Type: $\mathbf{l}_{4,4}$.
Janet base: $\{\xi_y, \eta_y - 2\xi_x, \xi_{xx}, \eta_{xx} - 2\xi_x\}$.
$\mathbf{g}_{18}(l=2, \rho_1=1, \rho_2=0, \alpha_1=0, \alpha_2=1)$: $\{\partial_y, x\partial_y, e^x\partial_y, y\partial_y, \partial_x\}$.
Janet base: $\{\xi_x, \xi_y, \eta_{xy}, \eta_{yy}, \eta_{xxx} - \eta_{xx}\}$.
$\mathbf{g}_{18}(l=3, \rho_1=\rho_2=\rho_3=0, \alpha_1=0, \alpha_2=1, \alpha_3=c)$, $c \neq 0, 1$:
$\{\partial_y, x\partial_y, e^{cx}\partial_y, y\partial_y, \partial_x\}$.
Janet base: $\{\xi_x, \xi_y, \eta_{xy}, \eta_{yy}, \eta_{xxx} - (c+1)\eta_{xx} + c\eta_x\}$.
$\mathbf{g}_{23}(r=3) = \{\partial_y, x\partial_y, x^2\partial_y, y\partial_y, \partial_x, x\partial_x, x^2\partial_x + 2xy\partial_y\}$.
Janet base: $\{\xi_y, \eta_{xy} - \xi_{xx}, \eta_{yy}, \xi_{xxx}, \eta_{xxx}\}$.
$\mathbf{g}_3$: $\{\partial_x, \partial_y, x\partial_y, y\partial_y, x\partial_x, y\partial_x, x^2\partial_x + xy\partial_y, xy\partial_x + y^2\partial_y\}$.
Janet base: $\{\xi_{yy}, \eta_{xx}, \eta_{xy}, \eta_{yy}, \xi_{xxx}, \eta_{xxy}\}$.

## 4 Lie Systems

In Section 2 general systems of linear homogeneous pde's have been discussed without any assumptions on their origin or special properties of their solutions. The uniqueness of the Janet base representation makes it possible to express additional knowledge in terms of its coefficients. For applications in this article, the determining systems describing the symmetries of ordinary differential equations, this additional knowledge is of threefold origin.

→ *Lie's relations* are always true because they express the general properties of commutators of vector fields as given in (37).
→ *Geometric relations* express geometric properties of particular group types, e. g. the number of systems of imprimitivity.
→ *Algebraic relations* express algebraic properties of particular Lie algebra types, e. g. in terms of the structure of its characteristic polynomial.

The coefficient constraints following from Lie's relations (37) are satisfied for any determining system. Due to its importance a special term is introduced for them. A system of linear homogeneous pde's with the property that $(\xi_3, \eta_3)$ defined by (37) is a solution if this is true for $(\xi_1, \eta_1)$ and $(\xi_2, \eta_2)$ is called a *Lie system*.

In order to distinguish groups by their geometric or algebraic properties, the properties must be expressed in terms of the Janet base coefficients of its defining system. For those Janet bases occuring in the symmetry analysis of second- and third order ode's this is provided by the theorems of this section. At first geometric constraints are considered.

**Lemma 1.** *The Lie systems of types $\mathcal{J}_{1,1}^{(2,2)}$ and $\mathcal{J}_{1,2}^{(2,2)}$ define one-parameter groups allowing a system of imprimitivity depending on an undetermined function.*

*Proof.* The systems of imprimitivity for the two Janet bases (1) and (2) are determined by

$$\omega_y + b\omega + a = 0 \quad \text{or} \quad \omega_x + a\omega_y + c\omega^2 + (a_y - b - c)\omega + a_x - ab = 0.$$

The general solution of either of these pde's for $\omega(x, y)$ contains an undetermined function of a single argument. Consequently they determine systems of imprimitivity depending on an undetermined function. •

**Lemma 2.** *The Lie systems of types $\mathcal{J}_{2,1}^{(2,2)}, \ldots, \mathcal{J}_{2,5}^{(2,2)}$ define vector fields for two-parameter groups with the following systems of imprimitivity.*

*Type* $\mathcal{J}_{2,1}^{(2,2)}$. *Lie system (3) does not allow a one-parameter systems of imprimitivity. It allows two system of imprimitivity iff $a_1 \neq 0$, and a single one iff $a_1 = 0$.*

*Type* $\mathcal{J}_{2,2}^{(2,2)}$. *Lie system (4) allows always two systems of imprimitivity.*

*Type* $\mathcal{J}_{2,3}^{(2,2)}$. *Lie system (5) defines always a one-parameter system of imprimitivity.*

*Type* $\mathcal{J}_{2,4}^{(2,2)}$. *Lie system (6) allows always two systems of imprimitivity.*

*Type* $\mathcal{J}_{2,5}^{(2,2)}$. *Lie System (7) does not allow a one-parameter system of imprimitivity. It allows two system of imprimitivity iff $a_1 \neq 0$ and a single one iff $a_1 = 0$.*

*Proof.* Type $\mathcal{J}_{2,1}^{(2,2)}$. There is always the system of imprimitivity corresponding to $\xi_y = 0$. Any other system of imprimitivity is determined by the equations $a_1\omega - 1 = 0$, $\omega_y + a_2\omega = 0$. If $a_1 \neq 0$ there is one solution, if $a_1 = 0$ the first equation becomes inconsistent, consequently a solution does not exist.

Type $\mathcal{J}_{2,2}^{(2,2)}$. There is always the systems of imprimitivity corresponding to $\xi_y = 0$, and a second one that follows from the equations $\omega = 0$, $\omega_y + a_1 = 0$ with the constraint $a_1 = 0$ which is obtained as a consequence of (37).

Type $\mathcal{J}_{2,3}^{(2,2)}$. Any system of imprimitivity is determined by the coherent equations

$$\omega_x - b_2\omega^2 - (a_2 - d_2)\omega + c_2 = 0, \;\; \omega_y - b_1\omega^2 - (a_1 - d_1)\omega + c_1 = 0.$$

They determine always a system of imprimitivity depending on a parameter.

Type $\mathcal{J}^{(2,2)}_{2,4}$. Any system of imprimitivity is determined by the equations

$$b_1\omega^2 + (a_2 b_1 - 1)\omega = 0, \;\; a_2\omega_y - \omega_x + b_2\omega^2 - (a_{2,y} - a_2 b_2)\omega - a_{2,x} = 0.$$

If $b_1 \neq 0$, the first equation determines two systems of imprimitivity because $a_1 b_1 + 1 \neq 0$. If $b_1 = 0$, the condition (37) requires $b_2 = 0$, i. e. there is one system of imprimitivity corresponding to $\xi_y = 0$, and a second one follows from $\omega = 0$.

Type $\mathcal{J}^{(2,2)}_{2,5}$. Any system of imprimitivity is determined by the equations

$$\omega(\omega + a_1) = 0, \;\; a_1\omega_y - \omega_x - (a_{1,y} + b_1)\omega - a_{1,x} + a_1 b_1 = 0.$$

There is always the solution $\omega = 0$. If $a_1 \neq 0$ there is a second system of imprimitivity corresponding to $\omega = -a_1$. •

There are three Janet base types that occur in the symmetry analysis of second- and third order ode's in Section 5, their systems of imprimitivity are considered next.

**Lemma 3.** *The Lie systems of types $\mathcal{J}^{(2,2)}_{3,4}$, $\mathcal{J}^{(2,2)}_{3,6}$ and $\mathcal{J}^{(2,2)}_{3,7}$ define vector fields for three-parameter groups with the following systems of imprimitivity.*

*Type $\mathcal{J}^{(2,2)}_{3,4}$. Lie system (11) defines a three-parameter group allowing two systems of imprimitivity iff $c_1 \neq 0$ and a single one iff $c_1 = 0$. There cannot be a system of imprimitivity depending on a parameter.*

*Type $\mathcal{J}^{(2,2)}_{3,6}$. Lie system (13) defines a three-parameter group allowing a system of imprimitivity depending on a parameter iff $a_1 = b_1 = 0$ and $c_1 = -1$, two systems of imprimitivity iff either $a_i = b_i = 0$ for $i = 1, 2, 3$ or $D \equiv 4a_1 b_1 + (c_1 + 1)^2 \neq 0$, and a single one iff $D = 0$ and $a_1 \neq 0$ or $b_1 \neq 0$ or $c_1 \neq -1$.*

*Type $\mathcal{J}^{(2,2)}_{3,7}$. Lie system (14) defines a two-parameter group which does not allow a system of imprimitivity depending on a parameter. It allows two systems of imprimitivity iff $D \equiv b_1 - \frac{1}{4}c_1^2 \neq 0$ and a single one iff $D = 0$.*

The proof of this lemma and the next one is similar to the proof of Lemma 2, it is therefore omitted. There are two Lie systems of order four for which the systems of imprimitivity have to be determined.

**Lemma 4.** *The Lie systems of types $\mathcal{J}^{(2,2)}_{4,9}$ and $\mathcal{J}^{(2,2)}_{4,14}$ define vector fields for four-parameter groups with the following systems of imprimitivity.*

*Type $\mathcal{J}^{(2,2)}_{4,9}$. Lie system (19) defines a four-parameter group which does not allow a system of imprimitivity depending on a parameter. It allows two systems of imprimitivity iff $a_1 = 0$, $a_2 = a_3 = a_4 = 0$, $b_2 + 1 = 0$ and $b_1 \neq 0$, and a single one iff $a_2 = a_3 = a_4 = b_1 = 0$.*

*Type $\mathcal{J}^{(2,2)}_{4,14}$. Lie system (22) defines a four-parameter group which does not allow a system of imprimitivity depending on a parameter. It allows two systems of imprimitivity iff $a_1 = 0$, $a_2 = a_3 = a_4 = 0$, $b_2 + 1 = 0$ and $b_1 \neq 0$ or $a_2 = 0$, $b_2 + 1 = 0$, $b_1^2 - 4a_1 \neq 0$ and $a_1 \neq 0$ or $a_3 \neq 0$ or $a_4 \neq 0$.*

The third set of constraints for the coefficients of Janet bases occuring in the symmetry analysis of ode's originates from the algebraic properties of their Lie algebras. They are based on the above mentioned property of the characterisitc polynomial w.r.t. to basis transformations of the respective Lie algebra.

**Lemma 5.** *The Lie system (5) of type* $\mathcal{J}^{(2,2)}_{2,3}$ *defines a Lie algebra of vector fields of a certain type as given in the classification on page 122 if its coefficients satisfy the following constraints.*

*Lie algebra* $\mathbf{l}_{2,1}$ *iff the coefficient obey* $a_1 = b_2$ *and* $c_1 = d_2$*.*
*Lie algebra* $\mathbf{l}_{2,2}$ *iff the coefficient obey* $a_1 \neq b_2$ *or* $c_1 \neq d_2$*.*

*Proof.* For the general Lie system of type $\mathcal{J}^{(2,2)}_{2,3}$ the characteristic polynomial has the form $\omega^2 - [(c_1 - d_2)E_2 - (a_1 - b_2)E_1]\omega$. The above coefficient constraints generate the structure of the two two-dimensional Lie algebras $\mathbf{l}_{2,1}$ or $\mathbf{l}_{2,2}$ respectively listed on page 122. •

**Lemma 6.** *The Lie system (11) of type* $\mathcal{J}^{(2,2)}_{3,4}$ *defines a Lie algebra of vector fields of a certain type as given in the classification on page 122 if its coefficients satisfy the following constraints.*

*Lie algebra* $\mathbf{l}_{3,1}$ *iff the coefficient obey*

$$c_1 = 0,\ a_3 - 2c_3 + d_1 = 0,\ a_2 \neq 0 \text{ or } c_{3,x} - c_3^2 + c_3 d_1 - d_2 \neq 0.$$

*Lie algebra* $\mathbf{l}_{3,2}(c)$ *iff*

$$c_{3,x} + \frac{1}{4}a_3(a_3 - 4c_3 + 2d_1) + \frac{1}{4}d_1^2 - d_2 \neq 0$$

*and* $c \neq 1$ *is a solution of*

$$c^2 + \frac{2c_{3,x} + 2c_3^2 - 2c_3 d_1 - 4a_3 c_3 + a_1^2 + d_1^2 - 2d_2}{c_{3,x} - c_3^2 + c_3 d_1 - d_2} c + 1 = 0.$$

*Lie algebra* $\mathbf{l}_{3,3}$ *iff the coefficients obey*

$$a_1 = 0,\ a_2 = 0,\ c_{3,x} + \frac{1}{4}a_3(a_3 - 4c_3 + 2d_1 + \frac{1}{4}d_1^2 - d_2 = 0,\ a_3 - 2c_3 + d_1 \neq 0.$$

*Lie algebra* $\mathbf{l}_{3,4}$ *iff the coefficients obey*

$$a_2 = 0,\ c_{3,x} - c_3^2 + c_3 d_1 - c_1 d_3 - d_2 = 0,\ c_1 \neq 0 \text{ or } a_3 - 2c_3 + d_1 \neq 0.$$

*Lie algebra* $\mathbf{l}_{3,5}$ *iff the coefficients obey*

$$c_1 = 0,\ a_2 = 0,\ a_3 - 2c_3 + d_1 = 0,\ c_{3,x} - c_3^2 + c_3 d_1 - c_1 d_3 - d_2 = 0.$$

*Lie algebra* $\mathbf{l}_{3,6}$ *not possible.*

*Proof.* According to the definition (39) the characteristic polynomial for any three-dimensional Lie algebra has the form $\omega^2 + C_2\omega^2 + C_1\omega$. In terms of a Lie system of type $\mathcal{J}^{(2,2)}_{3,4}$ the coefficients may be expressed as follows.

$$C_2 = c_1E_1 + (a_3 - c_3 + d_1)c_1E_2 - (a_3 - 2c_3 + d_1)E_3,$$

$$\begin{aligned} C_1 = a_2c_1E_1E_2 &- a_2(a_2 - a_3c_1 + c_1c_3 - c_1d_1)E_2^2 - 2a_2E_1E_3 \\ &+[c_{3,x} - c_1d_3 - c_3^2 + c_3d_1 - d_2)c_1 - a_2(a_3 + d_1)]E_2E_3 \\ &-(c_{3,x} - c_1d_3 - c_3^2 + c_3d_1 - d_2)E_3^2. \end{aligned}$$

Imposing the constraints on $C_1$ and $C_2$ that follow from the structure given in the listing on page 122, the above conditions are obtained after some simlifications. The condition for $\mathbf{l}_{3,6}$ follows from the fact that some structure constants are numbers. •

**Lemma 7.** *The Lie system (13) of type $\mathcal{J}^{(2,2)}_{3,6}$ defines a Lie algebra of vector fields of a certain type as given in the classification on page 122 if its coefficients satisfy the following constraints.*

*Lie algebra* $\mathbf{l}_{3,1}$ *iff the coefficients obey*

$$c_1 = 1, \ \ a_3b_1 - a_1b_3 + b_2 - c_3 + d_1 = 0$$

*and*

$$\begin{gathered} a_1b_1 + 1 \neq 0 \ \ \textit{or} \ \ a_1(b_1 - c_3) + a_3 \neq 0 \ \ \textit{or} \ \ a_3b_1 - b_2 + c_3 \neq 0 \\ \textit{or} \ \ a_{3,x} - a_3(a_1b_3 + c_3 - d_1) - a_1d_3 - d_2 \neq 0 \\ \textit{or} \ \ a_{3,x}b_1 + (a_1b_3 + c_3 - d_1)(c_3 - b_2) - b_1d_2 + d_3 \neq 0 \\ \textit{or} \ \ a_{3,x} + \tfrac{1}{2}(a_1b_2 - a_1c_3 - a_3)(a_1b_3 + c_3 - d_1) \neq 0. \end{gathered}$$

*Lie algebra* $\mathbf{l}_{3,2}(c)$ *iff* $c \neq 1$ *is a solution of*

$$c^2 + \frac{2a_1b_1 + c_1^2 + 1}{a_2b_1 + c_1}c + 1 = 0.$$

*Lie algebra* $\mathbf{l}_{3,3}$ *iff the coefficients obey either*

$$\begin{gathered} c_1 + 1 = 0, \ b_1 = 0, \ a_1b_3 + b_2 - c_3 - d_1 = 0, \\ a_{3,x} + a_1(b_2 - c_3)^2 - a_3b_2 - a_1d_3 - d_2 = 0 \end{gathered}$$

*or*

$$\begin{gathered} c_1 + 1 = 0, \ b_1 = 0, \\ a_{3,x}b_1 - \tfrac{1}{4}a_3^2b_1^2 - \tfrac{1}{4}(b_2 - c_1 - d_1)^2 - \tfrac{1}{2}a_3b_1(b_2 + c_3 - d_1) - b_1d_1 = 0 \end{gathered}$$

*or*

$$
\begin{gathered}
4a_1b_1 + (c_1+1)^2 = 0,\\
(c_1+3)(b_2-c_3) + (c_1-1)(d_1-a_1b_3) + (c_1-5)a_1b_1 = 0,\\
a_3(c_1+1)^2(c_1-5) + 4a_1(a_1b_3-d_1)(c_1-1) - 4a_1(c_1+3)(b_2-c_3) = 0,\\
a_{3,x}b_1 - \tfrac{1}{8}(a_3b_3-2d_3)(c_1+1)^2 - \tfrac{1}{4}(b_2-d_1)^2 - \tfrac{1}{4}(a_1^2b_3^2+a_3^2b_1^2) - b_1d_2\\
-\tfrac{1}{2}a_3b_1(b_2+c_3+d_1) - \tfrac{1}{2}a_1b_3(b_2-c_3-d_1) + \tfrac{1}{4}c_3(2b_2-c_3-2d_1) = 0
\end{gathered}
$$

*and* $b_1 \neq 0$.

*Lie algebra* $\mathbf{l}_{3,4}$ *iff the coefficients obey either*

$$a_1=0,\ c_1=0,\ a_3b_1=b_2-c_3,\ a_{3,x}=a_3(c_3-d_1)+d_2,\ c_3-b_2\neq \tfrac{1}{2}d_1$$

*or*

$$
\begin{gathered}
a_1b_1+1=0,\ a_3c_1+a_1(b_2-c_3)=0,\ a_3b_1=b_2-c_3,\\
a_{3,x}-a_3(a_1b_3+c_3-d_1)-a_1d_3-d_2=0,\\
a_1b_3-2(b_2-c_3)\neq d_1,\ a_3+a_1(a_1b_3+c_3-b_2-d_1)\neq 0.
\end{gathered}
$$

*Lie algebra* $\mathbf{l}_{3,5}$ *iff the coefficient obey*

$$
\begin{gathered}
c_1=1,\ a_1b_1+1=0,\ a_1b_3-2(b_2-c_3)-d_1=0,\ a_3+a_1(b_2-c_3)=0,\\
(b_{2,x}-c_{3,x})a_1+(b_2-c_3)(a_1d_1-a_3c_1)+a_1d_3+d_2=0.
\end{gathered}
$$

*Lie algebra* $\mathbf{l}_{3,6}$ *not possible.*

The proof of this lemma and the next one is similar to the proof of Lemma 6 and is therefore skipped.

**Lemma 8.** *The Lie system (14) of type* $\mathcal{J}_{3,7}^{(2,2)}$ *defines a Lie algebra of vector fields of a certain type as given in the classification on page 122 if its coefficients satisfy the following constraints.*

*Lie algebra* $\mathbf{l}_{3,1}$ *iff the coefficient obey*

$$
\begin{gathered}
c_1=0,\ a_2-\tfrac{1}{2}(c_2+d_1)=0,\\
b_1\neq 0 \text{ or } c_3\neq 0 \text{ or } c_{2,y}+d_{1,y}-\tfrac{1}{2}c_2^2+\tfrac{1}{2}d_1^2-2d_3\neq 0.
\end{gathered}
$$

*Lie algebra* $\mathbf{l}_{3,2}(c)$ *iff* $c\neq 1$ *is a solution of*

$$c^2+\frac{2b_1-c_1^2}{b_1}c+1=0.$$

*Lie algebra* $\mathbf{l}_{3,3}$ *iff the coefficients obey*

$$b_1 - \frac{1}{4}c_1^2 = 0,\ c_3 - \frac{1}{2}c_1(a_2 + 2c_2 + 2d_1) = 0,$$
$$c_{2,y} + d_{1,y} + c_1 d_2 - \frac{1}{2}c_2^2 + \frac{1}{2}d_1^2 - 2d_3 = 0.$$

*Lie algebra* $\mathbf{l}_{3,4}$ *iff the coefficients obey*

$$b_1 = 0,\ c_3 - a_2 c_1 = 0,$$
$$c_{2,y} + d_{1,y} - 2a_2(a_2 + c_2 + d_1) + c_1 d_2 - c_2 d_1 - c_2^2 - 2d_3 = 0,$$
$$c_1 \neq 0 \text{ or } a_2 - \frac{1}{2}(c_2 + d_1) \neq 0.$$

*Lie algebra* $\mathbf{l}_{3,5}$ *iff the coefficients obey*

$$b_1 = 0,\ c_1 = 0,\ c_3 = 0,\ a_2 - \frac{1}{2}(c_2 + d_1) = 0,\ c_{2,y} + d_{1,y} - \frac{1}{2}c_2^2 + \frac{1}{2}d_1^2 - 2d_3 = 0.$$

*Lie algebra* $\mathbf{l}_{3,6}$ *not possible.*

Finally two Lie systems of order four have to be considered.

**Lemma 9.** *The Lie system (19) of type* $\mathcal{J}^{(2,2)}_{4,9}$ *defines a Lie algebra of vector fields of a certain type as given in the classification on page 122 if its coefficients satisfy the following constraints.*

*Lie algebra* $\mathbf{l}_{4,1}$ *iff the coefficient obey* $a_2 = b_1$, $b_2 = 0$ *and* $b_1 \neq 0$.
*Lie algebra* $\mathbf{l}_{4,4}$ *iff the coefficients obey* $b_2 = -2$, $b_1 = a_2 = 0$ *and* $a_4 = \frac{1}{2}c_1$.
*Lie algebra* $\mathbf{l}_{4,5}(c)$ *iff the coefficients obey* $b_2 \neq 0$, $b_2 + 1 \neq 0$. *The parameter is determined by* $c = -b_2$.

**Lemma 10.** *For a Lie system (22) of type* $\mathcal{J}^{(2,2)}_{4,14}$ *to define a Lie algebra of vector fields of type* $\mathbf{l}_{4,4}$ *or* $\mathbf{l}_{4,5}(c)$, *the following necessary constraints have to be satisfied.*

*Lie algebra* $\mathbf{l}_{4,4}$ : $b_2 = -\frac{1}{2}$.
*Lie algebra* $\mathbf{l}_{4,5}(c)$ : $b_2 = -\frac{1}{c}$, $c \neq 0$ *and* $c \neq -2$.

Lemmata 1 to 10 provide the interface between the geometric and algebraic properties of the Lie groups of the plane on the one hand, and the symmetries of ode's expressed in terms of the Janet base of its determining system.

## 5 Symmetries of Differential Equations

The behavior of ode's under various kinds of transformations will be investigated first. Let an ode of order $n$ in the independent variable $x$ and the dependent variable $y \equiv y(x)$ be given as

$$\omega(x, y, y', \ldots, y^{(n)}) = 0. \tag{41}$$

If not stated explicitly otherwise it will be assumed that $\omega$ is a polynomial in the derivatives with coefficients in some *base field* which is usually the field of rational functions in $x$ and $y$, i. e. $\omega \in \mathbf{Q}(x, y)[y', \ldots, y^{(n)}]$. Any other field that occurs later on during the solution procedure is an extension of this base field.

A *point-transformation* between two planes with coordinates $(x, y)$ and $(u, v)$ and dependencies $y \equiv y(x)$ and $v \equiv v(u)$ respectively is considered in the form

$$u = \sigma(x, y), \qquad v = \rho(x, y). \tag{42}$$

A point-transformation is a diffeomorphism in the first place. Depending on the particular situation, the function field in which $\sigma$ and $\rho$ are contained has to be specified.

Let a curve in the $x-y$-plane described by $y = f(x)$ be transformed under (42) into $v = g(u)$. There arises the question how the derivative $y' = \frac{df}{dx}$ corresponds to $v' = \frac{dg}{du}$ under this transformation. A simple calculation leads to the *first prolongation*

$$v' = \frac{dv}{du} = \frac{\rho_x + \rho_y y'}{\sigma_x + \sigma_y y'} \equiv \chi_1(x, y, y'). \tag{43}$$

Similarly the transformation law for derivatives of second order is obtained as

$$v'' = \frac{dv'}{du} = \frac{\chi_{1,x} + \chi_{1,y} y' + \chi_{1,y'} y''}{\sigma_x + \sigma_y y'} \equiv \chi_2(x, y, y', y'').$$

Explicitly in terms of $\sigma$ and $\rho$ it is

$$\begin{aligned} v'' = \frac{1}{(\sigma_x + \sigma_y y')^3} \{ & (\sigma_x \rho_y - \sigma_y \rho_x) y'' + (\sigma_y \rho_{yy} - \sigma_{yy} \rho_y) y'^3 \\ & + [\sigma_x \rho_{yy} - \sigma_{yy} \rho_x + 2(\sigma_y \rho_{xy} - \sigma_{xy} \rho_y)] y'^2 \\ & + [\sigma_y \rho_{xx} - \sigma_{xx} \rho_y + 2(\sigma_x \rho_{xy} - \sigma_{xy} \rho_x)] y' + \sigma_x \rho_{xx} - \sigma_{xx} \rho_x \}. \end{aligned} \tag{44}$$

In general there holds

$$v^{(n)} = \frac{dv^{(n-1)}}{du} \equiv \chi_n(x, y, y', \ldots, y^{(n)})$$

and

$$v^{(n+1)} = \frac{dv^{(n)}}{du} = \frac{\chi_{n,x} + \chi_{n,y} y' + \ldots + \chi_{n,y^{(n-1)}} y^{(n)}}{\sigma_x + \sigma_y y'}.$$

The form of a differential equation is extremely sensitive to a variable change. The following definition introduces a particular term for the exceptional cases where this is not true. Equation (41) is said to be *invariant* under the transformation $x = \phi(u, v)$, $y = \psi(u, v)$ with $v \equiv v(u)$ if it retains its form under this transformation, i. e. if the functional dependence of the transformed equation $\omega(u, v, v', \ldots, v^{(n)}) = 0$ on its arguments is the same as in the original equation (41). Such a transformation is called a *symmetry of the differential equation.* The totality of symmetry transformations of an ode form a continuous group, it is called the *symmetry group* of that equation. The type of this group is called its *symmetry type.* Lie [Lie 1891] has shown that equivalent equations have the same symmetry type. The reverse is not true, in general to any symmetry type there corresponds the union of one or more equivalence classes.

In order to utilize the symmetries of an ode for the solution procedure, the geometrical considerations above must be expressed in analytical terms. Of particular importance is the fact that this may be achieved in terms of the infinitesimal generators of a symmetry, the expressions for its finite transformations are not required. To this end the prolongation of a vector field in $x$ and $y(x)$ is introduced next. Let an infinitesimal generator $U = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ be given and $y \equiv y(x)$ depend on $x$. Its $n-th$ prolongation is

$$U^{(n)} = U + \zeta^{(1)} \frac{\partial}{\partial y'} + \zeta^{(2)} \frac{\partial}{\partial y''} + \ldots + \zeta^{(n)} \frac{\partial}{\partial y^{(n)}}. \tag{45}$$

The functions $\zeta^{(k)}$ are recursively defined by

$$\zeta^{(1)} = D(\eta) - y' D(\xi), \quad \zeta^{(k)} = D(\zeta^{(k-1)}) - y^{(k)} D(\xi) \tag{46}$$

for $k \geq 2$, $D = \partial_x + y'\partial_y + y''\partial_{y'} \ldots$ is the operator of total differentiation with respect to $x$. The three lowest $\zeta's$ are explicitly

$$\begin{aligned}
\zeta^{(1)} &= \eta_x + (\eta_y - \xi_x)y' - \xi_y y'^2, \\
\zeta^{(2)} &= \eta_{xx} + (2\eta_{xy} - \xi_{xx})y' + (\eta_{yy} - 2\xi_{xy})y'^2 - \xi_{yy} y'^3 \\
&\qquad + (\eta_y - 2\xi_x)y'' - 3\xi_y y' y'' \\
\zeta^{(3)} &= \eta_{xxx} + (3\eta_{xxy} - \xi_{xxx})y' + 3(\eta_{xyy} - \xi_{xxy})y'^2 + (\eta_{yyy} - 3\xi_{xyy})y'^3 \\
&\qquad - \xi_{yyy} y'^4 + 3(\eta_{xy} - \xi_{xx})y'' + 3(\eta_{yy} - 3\xi_{xy})y' y'' - 6\xi_{yy} y'^2 y'' \\
&\qquad + (\eta_y - 3\xi_x)y''' - 4\xi_y y' y''' - 3\xi_y y''^2.
\end{aligned} \tag{47}$$

The classification problem of second-order equations has been solved by Lie himself [Lie 1883]. The next theorem and the corresponding results for third order equations are completely contained in part I of [Lie 1883]. They are just a different view of Lie's listing of differential invariants of the groups of the plane.

**Theorem 4.** *(Lie 1883). Any symmetry generator of a second order quasilinear ode is similar to one in canonical variables $u$ and $v$ as given in the following listing. In addition the corresponding Janet base is given where $\alpha(u,v)$ and $\beta(u,v)$ are the coefficients of $\partial_u$ and $\partial_v$ respectively.*

*One-parameter group*

$\mathcal{S}^2_1$ : $\mathbf{g}_{27} = \{\partial_v\}$. *Janet base* $\{\alpha, \beta_u, \beta_v\}$.

*Two-parameter groups*

$\mathcal{S}^2_{2,1}$: $\mathbf{g}_{26} = \{\partial_u, \partial_v\}$. *Janet base* $\{\alpha_u, \alpha_v, \beta_u, \beta_v\}$.

$\mathcal{S}^2_{2,2}$: $\mathbf{g}_{25} = \{\partial_v, u\partial_u + v\partial_v\}$. *Janet base* $\{\alpha_v, \beta_u, \beta_v - \frac{1}{v}\beta, \alpha_{uu}\}$.

*Three-parameter groups*

$\mathcal{S}^2_{3,1}$: $\mathbf{g}_{10} = \{\partial_u + \partial_v, u\partial_u + v\partial_v, u^2\partial_u + v^2\partial_v\}$. *Janet base*
$\{\alpha_v, \beta_u, \beta_v + \alpha_u + \frac{2}{u-v}(\beta - \alpha), \alpha_{uu} - \frac{2}{u-v}\alpha_u - \frac{2}{(u-v)^2}(\beta - \alpha)\}$.

$\mathcal{S}^2_{3,2}$: $\mathbf{g}_{13} = \{\partial_u, 2u\partial_u + v\partial_v, u^2\partial_u + uv\partial_v\}$. *Janet base*
$\{\alpha_u - \frac{2}{v}\beta, \alpha_v, \beta_v - \frac{1}{v}\beta, \beta_{uu}\}$.

$\mathcal{S}^2_{3,3}(c)$: $\mathbf{g}_7(c) = \{\partial_u, \partial_v, u\partial_u + cv\partial_v\}$, $c \neq 1$.
*Janet base* $\alpha_v, \beta_u, \beta_v - c\alpha_u, \alpha_{uu}\}$.

$\mathcal{S}^2_{3,4}$: $\mathbf{g}_{20}(r=1) = \{\partial_u, \partial_v, u\partial_u + (u+v)\partial_v\}$.
*Janet base* $\{\alpha_v, \beta_u - \alpha_u, \beta_v - \alpha_u, \alpha_{uu}\}$.

*Eight-parameter group:*

$\mathcal{S}^2_8$: $\mathbf{g}_3 = \{\partial_u, \partial_v, u\partial_v, v\partial_v, u\partial_u, v\partial_u, u^2\partial_u + uv\partial_v, uv\partial_u + v^2\partial_v\}$.
*Janet base* $\{\alpha_{vv}, \beta_{uu}, \beta_{uv}, \beta_{vv}, \alpha_{uuu}, \beta_{uuv}\}$.

*This listing shows in particular that there does not exist any second order ode allowing a group of point symmetries with 4, 5, 6 or 7 parameters.*

In Section 3 and the above Theorem 4 the groups have been characterized by its Janet bases in *canonical variables*. In applications, the symmetry generators of differential equation and accordingly its Janet base occur in *actual*, i. e. non-canonical variables. Therefore the relation between these two descriptions must be found. This is based on the following observation. Take the Janet base for any group in canonical variables $u$ and $v$, and transform it into actual variables by means of (42). In general this change of variables will destroy the Janet base property. In order to reestablish it, the algorithm *JanetBase* has to be applied. During this process it may occur that a leading coefficient of an equation that is applied for reduction vanishes due to a special choice of the transformation (42). This has the consequence that alternatives occur which may lead to different types of Janet bases. In order to obtain the complete answer, each of these alternatives has to be investigated separately. Finally all Janet bases have to be combined such that a minimal number of *generic cases* is retained, and all those that are discarded may be obtained from them by specialization. It turns out that up to a few exceptional cases all alternatives are due to vanishing first order partial derivatives of the transformation functions $\sigma$ and $\rho$. Based on these observations the following result has been obtained [Schwarz 1996].

**Theorem 5.** *(Schwarz 1996) The following criteria provide a decision procedure for the symmetry type of a second order ode if its Janet base in a grlex term ordering with $\eta > \xi$, $y > x$ is given.*

*One-parameter group*

$\mathcal{S}^2_1$ : *Group* $\mathbf{g}_{27}$, *Janet base type* $\mathcal{J}^{(2,2)}_{1,1}$ *or* $\mathcal{J}^{(2,2)}_{1,2}$.

*Two-parameter groups*

$\mathcal{S}^2_{2,1}$ : *Group* $\mathbf{g}_{26}$, *Lie algebra* $\mathbf{l}_{2,2}$, *Janet base type* $\mathcal{J}^{(2,2)}_{2,3}$.
$\mathcal{S}^2_{2,2}$ : *Group* $\mathbf{g}_{27}$, *Lie algebra* $\mathbf{l}_{2,1}$, *Janet base type* $\mathcal{J}^{(2,2)}_{2,3}$.

*Three-parameter groups*

$\mathcal{S}^2_{3,1}$ : *Group* $\mathbf{g}_{10}$, *two s.o.i., Lie algebra* $\mathbf{l}_{3,1}$, *Janet base type* $\mathcal{J}^{(2,2)}_{3,6}$ *or* $\mathcal{J}^{(2,2)}_{3,7}$.
$\mathcal{S}^2_{3,2}$ : *Group* $\mathbf{g}_{13}$, *one s.o.i., Lie algebra* $\mathbf{l}_{3,1}$, *Janet base type* $\mathcal{J}^{(2,2)}_{3,4}$, $\mathcal{J}^{(2,2)}_{3,6}$ *or* $\mathcal{J}^{(2,2)}_{3,7}$.
$\mathcal{S}^2_{3,3}(c)$ : *Group* $\mathbf{g}_7(c)$, *two s.o.i., Lie algebra* $\mathbf{l}_{3,2}(c)$, *Janet base type* $\mathcal{J}^{(2,2)}_{3,6}$ *or* $\mathcal{J}^{(2,2)}_{3,7}$.
$\mathcal{S}^2_{3,4}$ : *Group* $\mathbf{g}_{20}(r=1)$, *one s.o.i., Lie algebra* $\mathbf{l}_{3,3}$, *Janet base type* $\mathcal{J}^{(2,2)}_{3,6}$ *or* $\mathcal{J}^{(2,2)}_{3,7}$.

*Eight-parameter group*

$\mathcal{S}^2_8$ : *Group* $\mathbf{g}_3$, *Janet base type* $\{\xi_{yy}, \eta_{xx}, \eta_{xy}, \eta_{yy}, \xi_{xxx}, \xi_{xxy}\}$.

*Proof. One-parameter group.* A one-parameter symmetry is uniquely determined by the Janet base type.

*Two-parameter groups.* The two groups are identified by their Lie algebra via Lemma 5.

*Three-parameter groups.* At first the groups are distinguished by the number of imprimitivity systems they allow by Lemma 3. The two pairs corresponding to one or two systems of imprimitivity are separated w.r.t. to their Lie algebras by Lemma 6, 7 and 8.

*The projective group.* The eight-parameter symmetry is uniquly identified by its Janet base type. •

These results may be extended to higher order equations, although the larger number of symmetry types leads to a much larger number of alternatives that have to be considered. There is another feature that distinguishes equations of order three or higher from second-order equations. Whereas for the latter linear equations form a single equivalence class with canonical form $v'' = 0$, this is not true any more if the order is higher than two. Therefore it appears to be appropriate to distinguish equations that are equivalent to a linear one from those that are not. These latter equations are simply called nonlinear third-order equations, they are the subject of the next two theorems, whereas the linearizable equations are considered thereafter. The equivalent of Theorem 4 for nonlinear third order equations is given first.

**Theorem 6.** *Any symmetry generator of a third order quasilinear ode is similar to one in canonical variables u and v as given in the following listing. In addition the corresponding Janet base is given where $\alpha(u,v)$ and $\beta(u,v)$ are the coefficients of $\partial_u$ and $\partial_v$ respectively.*

*One-parameter group*

$\mathcal{S}^3_1$: $\mathbf{g}_{27} = \{\partial_v\}$. *Janet base* $\{\alpha, \beta_u, \beta_v\}$.

*Two-parameter groups*

$\mathcal{S}^3_{2,1}$: $\mathbf{g}_{26} = \{\partial_u, \partial_v\}$. *Janet base* $\{\alpha_u, \alpha_v, \beta_u, \beta_v\}$.

$\mathcal{S}^3_{2,2}$: $\mathbf{g}_{25} = \{u\partial_u + v\partial_v, \partial_v\}$. *Janet base* $\{\alpha_v, \beta_u, \beta_v - \frac{1}{v}\beta, \alpha_{uu}\}$.

$\mathcal{S}^3_{2,3}$: $\mathbf{g}_4 = \{v\partial_v, \partial_v\}$. *Janet base* $\{\alpha, \beta_u, \beta_{vv}\}$.

$\mathcal{S}^3_{2,4}$: $\mathbf{g}_{15}(r=2) = \{\partial_v, u\partial_v\}$. *Janet base* $\{\alpha, \beta_v, \beta_{uu}\}$.

*Three-parameter groups*

$\mathcal{S}^3_{3,1}$: $\mathbf{g}_{10} = \{\partial_u + \partial_v, u\partial_u + v\partial_v, u^2\partial_u + v^2\partial_v\}$. *Janet base* $\{\alpha_v, \beta_u, \beta_v + \alpha_u + \frac{2}{u-v}(\beta - \alpha), \alpha_{uu} - \frac{2}{u-v}\alpha_u - \frac{2}{(u-v)^2}(\beta - \alpha)\}$.

$\mathcal{S}^3_{3,2}$: $\mathbf{g}_{13} = \{\partial_u, 2u\partial_u + v\partial_v, u^2\partial_u + uv\partial_v\}$. *Janet base* $\{\alpha_u - \frac{2}{v}\beta, \alpha_v, \beta_v - \frac{1}{v}\beta, \beta_{uu}\}$.

$\mathcal{S}^3_{3,3}(c)$: $\mathbf{g}_7 = \{\partial_u, \partial_v, u\partial_u + cv\partial_v\}$, $c \neq 0, 1$. *Janet base* $\alpha_v, \beta_u, \beta_v - c\alpha_u, \alpha_{uu}\}$.

$\mathcal{S}^3_{3,4}$: $\mathbf{g}_{20}(r=1) = \{\partial_u, \partial_v, u\partial_u + (u+v)\partial_v\}$. *Janet base* $\{\alpha_v, \beta_u - \alpha_u, \beta_v - \alpha_u, \alpha_{uu}\}$.

$\mathcal{S}^3_{3,5}$: $\mathbf{g}_5 = \{\partial_u, \partial_v, v\partial_v\}$. *Janet base* $\{\alpha_u, \alpha_v, \beta_u, \beta_{vv}\}$.

$\mathcal{S}^3_{3,6}$: $\mathbf{g}_{17}(l=1, \rho_1 = 1, \alpha_1 = 0) = \{\partial_u, \partial_v, u\partial_v\}$. *Janet base* $\{\alpha_u, \alpha_v, \beta_v, \beta_{uu}\}$.

$\mathcal{S}^3_{3,7}$: $\mathbf{g}_{17}(l=1, \rho_1 = 1, \alpha_1 = 1) = \{e^u\partial_v, ue^u\partial_v, \partial_u\}$. *Janet base* $\{\alpha_u, \alpha_v, \beta_v, \beta_{uu} - 2\beta_u + \beta\}$.

$\mathcal{S}^3_{3,8}$: $\mathbf{g}_{17}(l=2, \rho_1 = \rho_2 = 0, \alpha_1 = 0, \alpha_2 = 1) = \{e^u\partial_v, \partial_v, \partial_u\}$. *Janet base* $\{\alpha_u, \alpha_v, \beta_v, \beta_{uu} - \beta_u\}$.

$\mathcal{S}^3_{3,9}(c)$: $\mathbf{g}_{17}(l=2, \rho_1 = \rho_2 = 0, \alpha_1 = 1, \alpha_2 = c) = \{e^u\partial_v, e^{cu}\partial_v, \partial_u\}$, $c \neq 0, 1$. *Janet base* $\{\alpha_u, \alpha_v, \beta_v, \beta_{uu} - (a+1)\beta_u + a\beta\}$.

$\mathcal{S}^3_{3,10}$: $\mathbf{g}_{24} = \{\partial_u, \partial_v, u\partial_u + v\partial_v\}$. *Janet base* $\{\alpha_v, \beta_u, \beta_v - \alpha_u, \alpha_{uu}\}$.

*Four-parameter groups*

$\mathcal{S}^3_{4,1}$: $\mathbf{g}_6 = \{\partial_u, \partial_v, v\partial_v, u\partial_u\}$. *Janet base* $\{\alpha_v, \beta_u, \alpha_{uu}, \beta_{vv}\}$.

$\mathcal{S}^3_{4,2}$: $\mathbf{g}_{14} = \{\partial_u, u\partial_u, v\partial_v, u^2\partial_u + uv\partial_v\}$. *Janet base* $\{\alpha_v, \beta_v - \dfrac{1}{v}\beta, \alpha_{uu} - \dfrac{2}{v}\beta_u, \beta_{uu}\}$.

$\mathcal{S}^3_{4,3}(c)$: $\mathbf{g}_{19}(r=2) = \{\partial_u, \partial_v, u\partial_v, u\partial_u + cv\partial_v\}$, $c \neq 2$. *Janet base* $\{\alpha_v, \beta_v - c\alpha_u, \alpha_{uu}, \beta_{uu}\}$.

$\mathcal{S}^3_{4,4}$: $\mathbf{g}_{20}(r=2) = \{\partial_u, \partial_v, u\partial_v, u\partial_u + (u^2 + 2v)\partial_v\}$. *Janet base* $\{\alpha_v, \beta_v - 2\alpha_u, \alpha_{uu}, \beta_{uu} - 2\alpha_u\}$.

*Groups with six parameters*

$\mathcal{S}^3_6$: $\mathbf{g}_{12} = \{\partial_u, \partial_v, u\partial_u, v\partial_v, u^2\partial_u, v^2\partial_v\}$. *Janet base* $\{\alpha_v, \beta_u, \alpha_{uuu}, \beta_{vvv}\}$.

As it has been mentioned before, this theorem is an immediate consequence of Lie's classification of differential invariants [Lie 1883]. The next theorem allows to determine the symmetry type of an actually given equation from its Janet base, it is the equivalent of Theorem 5.

**Theorem 7.** *The following criteria provide a decision procedure for the symmetry type of a third order ode if its Janet base in a grlex term ordering with $\eta > \xi$, $y > x$ is given.*

*One-parameter group*

$\mathcal{S}_1^3$ : *Janet base of type* $\mathcal{J}_{1,1}^{(2,2)}$ *or* $\mathcal{J}_{1,2}^{(2,2)}$.

*Two-parameter groups*

$\mathcal{S}_{2,1}^3$ : *Group* $\mathbf{g}_{26}$, *one-parameter s.o.i., Lie algebra* $\mathbf{l}_{2,2}$, *Janet base type* $\mathcal{J}_{2,3}^{(2,2)}$.

$\mathcal{S}_{2,2}^3$ : *Group* $\mathbf{g}_{27}$, *one-parameter s.o.i., Lie algebra* $\mathbf{l}_{2,1}$, *Janet base type* $\mathcal{J}_{2,3}^{(2,2)}$.

$\mathcal{S}_{2,3}^3$ : *Group* $\mathbf{g}_4$, *two s.o.i., Lie algebra* $\mathbf{l}_{2,1}$, *Janet base type* $\mathcal{J}_{2,1}^{(2,2)}$, $\mathcal{J}_{2,2}^{(2,2)}$ *or* $\mathcal{J}_{2,4}^{(2,2)}$.

$\mathcal{S}_{2,4}^3$ : *Group* $\mathbf{g}_{15}(r = 2)$, *one s.o.i., Lie algebra* $\mathbf{l}_{2,1}$, *Janet base type* $\mathcal{J}_{2,1}^{(2,2)}$ *or* $\mathcal{J}_{2,5}^{(2,2)}$.

*Three-parameter groups*

$\mathcal{S}_{3,1}^3$ : *Group* $\mathbf{g}_{10}$, *two s.o.i., Lie algebra* $\mathbf{l}_{3,1}$, *Janet base type* $\mathcal{J}_{3,6}^{(2,2)}$ *or* $\mathcal{J}_{3,7}^{(2,2)}$.

$\mathcal{S}_{3,2}^3$ : *Group* $\mathbf{g}_{13}$, *one s.o.i., Lie algebra* $\mathbf{l}_{3,1}$, *Janet base type* $\mathcal{J}_{3,4}^{(2,2)}$, $\mathcal{J}_{3,6}^{(2,2)}$ *or* $\mathcal{J}_{3,7}^{(2,2)}$.

$\mathcal{S}_{3,3}^3(c)$ : *Group* $\mathbf{g}_7(c)$, *two s.o.i., Lie algebra* $\mathbf{l}_{3,2}(c)$, *Janet base type* $\mathcal{J}_{3,6}^{(2,2)}$ *or* $\mathcal{J}_{3,7}^{(2,2)}$.

$\mathcal{S}_{3,4}^3$ : *Group* $\mathbf{g}_{20}(r = 1)$, *one s.o.i., Lie algebra* $\mathbf{l}_{3,3}$, *Janet base type* $\mathcal{J}_{3,6}^{(2,2)}$ *with* $c_1 \neq$ *or* $\mathcal{J}_{3,7}^{(2,2)}$ *with* $c_1 \neq 0$.

$\mathcal{S}_{3,5}^3$ : *Group* $\mathbf{g}_5$, *two s.o.i., Lie algebra* $\mathbf{l}_{3,4}$, *Janet base type* $\mathcal{J}_{3,4}^{(2,2)}$, $\mathcal{J}_{3,5}^{(2,2)}$, $\mathcal{J}_{3,6}^{(2,2)}$ *or* $\mathcal{J}_{3,7}^{(2,2)}$.

$\mathcal{S}_{3,6}^3$ : *Group* $\mathbf{g}_{17}(l = 1, \rho_1 = 1, \alpha_1 = 0)$, *one s.o.i., Lie algebra* $\mathbf{l}_{3,5}$, *Janet base type* $\mathcal{J}_{3,4}^{(2,2)}$, $\mathcal{J}_{3,6}^{(2,2)}$ *or* $\mathcal{J}_{3,7}^{(2,2)}$.

$\mathcal{S}_{3,7}^3$ : *Group* $\mathbf{g}_{17}(l = 1, \rho_1 = 1, \alpha_1 = 1)$, *one s.o.i., Lie algebra* $\mathbf{l}_{3,3}$, *Janet base type* $\mathcal{J}_{3,4}^{(2,2)}$, $\mathcal{J}_{3,6}^{(2,2)}$ *with* $c_1 = 1$ *or* $\mathcal{J}_{3,7}^{(2,2)}$ *with* $c_1 = 0$.

$\mathcal{S}_{3,8}^3$ : *Group* $\mathbf{g}_{17}(l = 2, \rho_1 = \rho_2 = 0, \alpha_1 = 0, \alpha_2 = 1)$, *one s.o.i., Lie algebra* $\mathbf{l}_{3,4}$, *Janet base type* $\mathcal{J}_{3,4}^{(2,2)}$, $\mathcal{J}_{3,6}^{(2,2)}$ *or* $\mathcal{J}_{3,7}^{(2,2)}$.

$\mathcal{S}_{3,9}^3(c)$ : *Group* $\mathbf{g}_{17}(l = 2, \rho_1 = \rho_2 = 0, \alpha_1 = 1, \alpha_2 = c)$, *one s.o.i., Lie algebra* $\mathbf{l}_{3,2}(c)$, *Janet base type* $\mathcal{J}_{3,4}^{(2,2)}$, $\mathcal{J}_{3,6}^{(2,2)}$ *or* $\mathcal{J}_{3,7}^{(2,2)}$.

$\mathcal{S}_{3,10}^3$ : *Group* $\mathbf{g}_{24}$, *one-parameter s.o.i., Lie algebra* $\mathbf{l}_{3,2}(c = 1)$, *Janet base type* $\mathcal{J}_{3,6}^{(2,2)}$.

*Four-parameter groups*

$\mathcal{S}^3_{4,1}$ : *Group* $\mathbf{g}_6$*, two s.o.i., Lie algebra* $\mathbf{l}_{4,10}$*, Janet base type* $\mathcal{J}^{(2,2)}_{4,9}$*,* $\mathcal{J}^{(2,2)}_{4,10}$ *or* $\mathcal{J}^{(2,2)}_{4,14}$*.*

$\mathcal{S}^3_{4,2}$ : *Group* $\mathbf{g}_{14}$*, one s.o.i., Lie algebra* $\mathbf{l}_{4,1}$*, Janet base type* $\mathcal{J}^{(2,2)}_{4,9}$ *or* $\mathcal{J}^{(2,2)}_{4,12}$*.*

$\mathcal{S}^3_{4,3}(c)$ : *Group* $\mathbf{g}_{19}(r=2)$*, one s.o.i., Lie algebra* $\mathbf{l}_{4,5}(c)$*, Janet base type* $\mathcal{J}^{(2,2)}_{4,9}$ *or* $\mathcal{J}^{(2,2)}_{4,14}$*.*

$\mathcal{S}^3_{4,4}$ : *Group* $\mathbf{g}_{20}(r=2)$*, one s.o.i., Lie algebra* $\mathbf{l}_{4,4}$*, Janet base type* $\mathcal{J}^{(2,2)}_{4,9}$ *or* $\mathcal{J}^{(2,2)}_{4,14}$*.*

*Six-parameter group*

$\mathcal{S}^3_6$ : *Janet base of type* $\{\xi_y, \eta_x, \xi_{xxx}, \eta_{yyy}\}$*, type* $\{\xi_y, \eta_y, \xi_{xxx}, \eta_{xxx}\}$*, type* $\{\eta_x, \eta_y, \xi_{xy}, \xi_{xxx}, \xi_{yyy}\}$ *or type* $\{\eta_x, \eta_y, \xi_{yy}, \xi_{xxx}, \xi_{xxy}\}$*.*

*Proof. One-parameter group.* A one-parameter symmetry is uniquely determined by the Janet base type.

*Two-parameter groups.* The first two symmetry classes are singled out by their one-parameter system of imprimitivity by Lemma 2. The further distiction between the two groups is obtained through their Lie algebra by Theorem 5. For Janet base type $\mathcal{J}^{(2,2)}_{2,1}$ the symmetry class is uniquely identified by Lemma 2 through their systems of imprimitivity. If the Janet base type is $\mathcal{J}^{(2,2)}_{2,2}$, $\mathcal{J}^{(2,2)}_{2,4}$ or $\mathcal{J}^{(2,2)}_{2,5}$ the symmetry class is unique.

*Three-parameter groups.* If there is a type $\mathcal{J}^{(2,2)}_{3,5}$ Janet base, the symmetry class $\mathcal{S}^3_{3,5}$ is uniquely identified. For Janetbase type $\mathcal{J}^{(2,2)}_{3,4}$, by Lemma 3 again symmetry class $\mathcal{S}^3_{3,5}$ is uniquely identified by its two systems of imprimitivity. The remaining cases allow a single system of imprimitivity, the symmetry type is identified from its Lie algebra by means of Lemma 6. For Janetbase type $\mathcal{J}^{(2,2)}_{3,6}$, Lemma 3 gives the unique answer $\mathcal{S}^3_{3,10}$ if a one-parameter system of imprimitivity is found, otherwise it combines the symmetry classes into those allowing two systems of imprimitivity, i. e. $\mathcal{S}^3_{3,1}$, $\mathcal{S}^3_{3,3}(c)$ and $\mathcal{S}^3_{3,5}$, and the remaining ones. In the former case Lemma 7 leads to a unique identification by its Lie algebra. In the latter the same is true except for the two symmetry classes $\mathcal{S}^3_{3,4}$ and $\mathcal{S}^3_{3,7}$, both with a $\mathbf{l}_{3,3}$ Lie algebra. The are distinguished by their different connectivity properties which is identified by $c_1 \neq 1$ of $c_1 = 1$ respectively. For Janet base type $\mathcal{J}^{(2,2)}_{3,7}$ the discussion is identical, Lemmata 3 and 8 are applied now. The distinction between symmetry classes $\mathcal{S}^3_{3,4}$ and $\mathcal{S}^3_{3,7}$ is obtained now by the condition $c_1 \neq 0$ or $c_1 = 0$ respectively.

*Four-parameter groups.* The symmetry classes $\mathcal{S}^3_{4,1}$ and $\mathcal{S}^3_{4,2}$ are uniquely identified from the Janet base types $\mathcal{J}^{(2,2)}_{4,10}$ and $\mathcal{J}^{(2,2)}_{4,12}$. For Janet base type $\mathcal{J}^{(2,2)}_{4,9}$, by Lemma 4 symmetry class $\mathcal{S}^3_{4,1}$ is uniquely identified by its two systems of imprimitivity. The remaining three symmetry types are distinguished by their Lie algebra applying Lemma 9. For Janetbase type $\mathcal{J}^{(2,2)}_{4,14}$, by Lemma 4 again symmetry class $\mathcal{S}^3_{4,1}$ is uniquely identified through its two systems of imprimitivity. The distinction between the two remaining cases is obtained from Lemma 10.

*Six-parameter group.* The only symmetry class $\mathcal{S}^3_6$ comprising six parameters is identified from any of the Janet bases of order six without constraints on its coefficients. $\bullet$

As already mentioned before, for order higher than two, linearizable equations comprise more than a single equivalence class. Moreover they combine into several symmetry classes. As usual by now at first a complete survey of all possible symmetry types is provided.

**Theorem 8.** *Any symmetry generator of a third order quasilinear ode is similar to one in canonical variables $u$ and $v$ as given in the following listing. In addition the corresponding Janet base is given where $\alpha(u,v)$ and $\beta(u,v)$ are the coefficients of $\partial_u$ and $\partial_v$ respectively.*

*Four-parameter symmetry*

$\mathcal{S}^3_{4,5}$: $\mathbf{g}_{16}(r=3) = \{\partial_v, u\partial_v, \phi(u)\partial_v, v\partial_v\}$.

*Janet base* $\{\alpha, \beta_{uv}, \beta_{vv}, \beta_{uuu} - \frac{\phi'''}{\phi''}\beta_{uu}\}$.

*Five-parameter symmetry*

$\mathcal{S}^3_{5,1}$: $\mathbf{g}_{18}(l=2, \rho_1=1, \rho_2=0, \alpha_1=0, \alpha_2=1) = \{\partial_v, u\partial_v, e^u\partial_v, v\partial_v, \partial_u\}$.

*Janet base:* $\{\alpha_u, \alpha_v, \beta_{uv}, \beta_{vv}, \beta_{uuu} - \beta_{uu}\}$.

$\mathcal{S}^3_{5,2}(a) \equiv \mathbf{g}_{18}(l=3, \rho_1=\rho_2=\rho_3=0, \alpha_1=0, \alpha_2=1, \alpha_3=a) = \{\partial_v, e^u\partial_v, e^{au}\partial_v, v\partial_v, \partial_u\}$, $a \neq 0, 1$.

*Janet base:* $\{\alpha_u, \alpha_v, \beta_{uv}, \beta_{vv}, \beta_{uuu} - (a+1)\beta_{uu} + a\beta_u\}$.

*Seven-parameter symmetry*

$\mathcal{S}^3_7$: $\mathbf{g}_{23}(r=3) = \{\partial_v, u\partial_v, u^2\partial_v, v\partial_v, \partial_u, u\partial_u, u^2\partial_u + 2uv\partial_v\}$.

*Janet base:* $\{\alpha_v, \beta_{uv} - \alpha_{uu}, \beta_{vv}, \alpha_{uuu}, \beta_{uuu}\}$.

Based on the above classification of symmetries, the subsequent theorem provides algorithmic means for obtaining the symmetry class of any given quasilinear third-order ode.

**Theorem 9.** *The following criteria provide a decision procedure for the symmetry class of a third order ode that is equivalent to a linear one if its Janet base in a grlex term ordering with $\eta > \xi$, $y > x$ is given.*

*Four-parameter symmetry*

$\mathcal{S}^3_{4,5}$ : *Janet base of type $\mathcal{J}^{(2,2)}_{4,2}$, type $\mathcal{J}^{(2,2)}_{4,17}$ or type $\mathcal{J}^{(2,2)}_{4,19}$.*

*Five-parameter symmetries*

$\mathcal{S}^3_{5,1}$ *and* $\mathcal{S}^3_{5,2}(a)$ : *Three Janet base types may occur. In all three cases the parameter $a$ determining the symmetry type is a solution of the equation*

$$P^2(a^2-a+1)^3 + Q^2R(a-\frac{1}{2})^2(a+1)^2(a-2)^2) = 0. \tag{48}$$

*If $a \neq 0$ the symmetry class is $\mathcal{S}^3_{5,2}(a)$ and $\mathcal{S}^3_{5,1}$ otherwise. Its coefficients $P$, $Q$ and $R$ for the various cases are given below.*

*a) Janet base type* $\mathcal{J}^{(2,2)}_{5,1}$.

$$P = -\tfrac{9}{2}(a_{1,x}a_1 + \tfrac{1}{3}a_{1,x}e_1 + c_{3,x} - a_1^3 - \tfrac{2}{3}a_1^2e_1 - 2a_1c_3$$
$$-\tfrac{1}{3}a_1e_1^2 + a_1e_3 + c_1e_2 - \tfrac{2}{27}e_1^3 + \tfrac{1}{3}e_1e_3 - e_4),$$

$$Q = a_{1,x} - 2a_1^2 - a_1e_1 - 3c_3 - \frac{1}{3}e_1^2 + e_3, \quad R = 3Q.$$

*b) Janet base type* $\mathcal{J}^{(2,2)}_{5,2}$.

$$P = -\tfrac{9}{2}(b_{1,y}b_1 + \tfrac{1}{3}b_{1,y}e_1 + d_{2,y} - b_1^3 - \tfrac{2}{3}b_1^2e_1 - 2b_1d_2$$
$$-\tfrac{1}{3}b_1e_1^2 + b_1e_2 + c_1e_3 - \tfrac{2}{27}e_1^3 + \tfrac{1}{3}e_1e_2 - e_5),$$

$$Q = b_{1,y} - 2b_1^2 - b_1e_1 - 3d_2 - \frac{1}{3}e_1^2 + e_2, \quad R = 3Q.$$

*c) Janet base type* $\mathcal{J}^{(2,2)}_{5,3}$.

$$P = \tfrac{9}{2}(a_{3,x}a_3c_1^3 - \tfrac{1}{3}a_{3,x}c_1^2e_1 + a_{3,x}c_1c_3 + c_{5,x} + a_2a_3^2c_1^3 - \tfrac{1}{3}a_2a_3c_1^2e_1$$
$$+a_2a_3c_1c_3 - a_2c_5 + 2a_3^3c_1^4 - a_3^2c_1^3e_1 + 3a_3^2c_1^2c_3 + a_3c_1^3e_2$$
$$+\tfrac{1}{3}a_3c_1^2e_1^2 - a_3c_1^2e_3 + 2a_3c_1c_5 + a_3c_4 - \tfrac{1}{3}c_1^2e_1e_2$$
$$+c_1c_3e_2 - \tfrac{2}{27}c_1e_1^3 + \tfrac{1}{3}c_2e_1e_3 - c_1e_5 - e_4),$$

$$Q = a_{3,x}c_1^2 + a_2a_3c_1^2 + 3a_3^2c_1^3 - a_3c_1^2e_1 + 3a_3c_1c_3 + c_1^2e_2$$
$$+\tfrac{1}{3}c_1e_1^2 - c_1e_3 + 3c_5, \qquad R = \tfrac{3}{c_1}Q.$$

<u>*Seven-parameter symmetry*</u>

$\mathcal{S}^3_7$ : *Janet base of type* $\{\xi_y, \eta_{xy}, \eta_{yy}, \xi_{xxx}, \eta_{xxx}\}$, *type* $\{\eta_x, \xi_{xx}, \eta_{yy}, \xi_{xyy}, \xi_{yyy}\}$ *or type* $\{\eta_y, \xi_{yy}, \eta_{xx}, \xi_{xxx}, \xi_{xxy}\}$.

*Proof.* A Janet base of type $\mathcal{J}^{(2,2)}_{4,2}$, $\mathcal{J}^{(2,2)}_{4,17}$ or $\mathcal{J}^{(2,2)}_{4,19}$ uniquely identifies symmetry class $\mathcal{S}^3_{4,5}$ because these Janet base types do not occur for any other four-parameter symmetry as it may be seen from Theorem 7. If the equations relating the generic coefficients of a Janet base of type $\mathcal{J}^{(2,2)}_{5,1}$, $\mathcal{J}^{(2,2)}_{5,2}$ or $\mathcal{J}^{(2,2)}_{5,3}$ and the transformation functions $\sigma(x,y)$ and $\rho(x,y)$ are transformed into a Janet base, one of the equations is (48) from which the value of $a$ and thereby the symmetry class $\mathcal{S}^3_{5,2}(a)$ may be obtained if $a \neq 0$ or $\mathcal{S}^3_{5,1}$ if $a = 0$. A Janet base allowing a seven-parameter symmetry group identifies uniquely symmetry class $\mathcal{S}^3_7$. •

## 6 Concluding Remarks

The results described in this article are the basis for extending Lie's symmetry analysis of differential equations and the solution procedures based on it to equations of third order. It should be emphasized that the application of the theorems of Section 5 for determining the symmetry type are completely algorithmic. They do not require to extend the base field determined by the given differential equation. Only the next step, when the transformation to canonical form is considered, in general a base field extension will be required. This is discussed in detail in [Schwarz 2002].

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