Multi-variate Polynomials and Newton-Puiseux Expansions

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Abstract. The classical Newton polygon method for the local resolution of algebraic equations in one variable can be extended to the case of multi-variate equations of the type $f(y) = 0$, $f \in \mathbb{C}[x_1, \ldots, x_N][y]$. For this purpose we will use a generalization of the Newton polygon - the Newton polyhedron - and a generalization of Puiseux series for several variables.

1 Introduction

Starting with the local analysis of non linear differential equations[2], we were induced to study equations $f(y) = 0$, where f is a polynomial in y, whose coefficients involve logarithms and exponentials[4].To solve such equations, we proposed to consider each *independent* transcendental function as a new variable, and so we were brought to solve equations where the coefficients of f are polynomials (or formal Laurent series) in several variables.
For this, we have used the well-known Newton method in one variable in an iterative way. For example, let $f \in \mathbb{C}[[u, v]][y]$, we can apply the process considering $f \in \mathbb{C}((u))((v))[y]$ and obtain solutions in $\mathbb{C}((u^{\frac{1}{m}}))((v^{\frac{1}{q}}))$, $m, q \in \mathbb{N}^*$, or considering $f \in \mathbb{C}((v))((u))[y]$ and obtain solutions in $\mathbb{C}((v^{\frac{1}{m}}))((u^{\frac{1}{q}}))$ (for some other m and q). In both cases, the ramifications are bounded, but in the first case, the series solutions can involve arbitrary large negative powers of $u^{\frac{1}{m}}$, and of $v^{\frac{1}{q}}$ in the second. This leads to some difficulties to define a notion of convergence for such objects.
In this paper, we propose an other approach to solve polynomial equations in N variables ($f(y) = 0$, with $f \in \mathbb{C}[x_1, \ldots, x_N][y]$), which gives a construction of series expansions with exponents in a cone. This construction is based on the Newton polyhedron of f and the algorithm exposed in [5] and has been implemented in *Maple*. The major improvment lies in the fact that the algorithm we will present provides a full set of solutions without the choose of an irrational vector. Furthermore, the cones in which lie the series solutions are known from the beginning on. Finally, we will explain how this construction can be used to reduce the non linearity of differential equations $f(t, y, y') = 0$ with $f \in \mathbb{C}[[t]][y, y']$.

2 Regular case for multi-variate equations

In this section, we will extend the notion of *Regular case* for equations in one variable [4] to multi-variate equations:

$$f(y) = 0 \text{ with } f \in \mathbb{C}[[x_1, \ldots, x_N]][y]. \tag{1}$$

We note $\mathbf{x} = (x_1, \ldots, x_N)$, $\mathbb{C}[[\mathbf{x}]] = \mathbb{C}[[x_1, \ldots, x_N]]$ and we define the expression $f \mod \mathbf{x}$ by $f \mod \mathcal{I}$ where $\mathcal{I}$ is the ideal generated in the ring $\mathbb{C}[[\mathbf{x}]][y]$ by $x_1, \ldots, x_N$. In the same way, for all $n \geq 1$, we define $f \mod \mathbf{x}^n$ by $f \mod \mathcal{I}^n$. With these definitions we will say that the equation (1) satisfies the regular case if there exists $y_0 \in \mathbb{C}$ such that : $\begin{cases} f(y_0) = 0 \mod \mathbf{x} \\ f'(y_0) \neq 0 \mod \mathbf{x}. \end{cases}$

Then we can define: $\forall n \in \mathbb{N}^*, \quad y_n = y_{n-1} - \frac{f(y_{n-1})}{f'(y_{n-1})} \mod \mathbf{x}^{2^n}$. By induction on n, it can be proved that: $\forall n \in \mathbb{N}, \quad f(y_n) = 0 \mod \mathbf{x}^{2^n}$.

Conclusion: if $\hat{y}$ is the element of $\mathbb{C}[[\mathbf{x}]]$ such that $\hat{y} = y_n \mod \mathbf{x}^{2^n}, \forall n \in \mathbb{N}$, then $f(\hat{y}) = 0$.

This result, also known as lifting theorem [14] is interesting for two reasons: first, it proves the existence of a solution of the equation $f(y) = 0$ as a formal power series in the variables $x_1, \ldots, x_N$ and second, it produces a practical algorithm for computing the coefficients of the series in an efficient way. Of course, there is no reason for having exactly d solutions of this form, if d is the degree of the polynomial equation. So, in order to obtain a complete set of solutions, we have to look for more general solutions and to introduce a generalization of both the Newton polygon and the Puiseux series.

3 A first resolution algorithm

The first resolution algorithm appearing in the literature using a polytope in $\mathbb{R}^{N+1}$ is due to John McDonald in [5]. In order to understand our algorithm's improvment, we first summarize this work. In this section, f is supposed to be polynomial in all the variables, $f \in \mathbb{C}[\mathbf{x}][y]$ and we will use the following notations:

$$f(x_1, \ldots, x_N, y) = \sum_{k=0}^{d} \overline{a}_k y^k, \text{ where } \overline{a}_d \neq 0,\ \overline{a}_k = \sum_{\mathbf{K} \in \mathbb{N}^N} a_{\mathbf{K}}^k \mathbf{x}^{\mathbf{K}},\ a_{\mathbf{K}}^k \in \mathbb{C}$$

$$\text{and} \qquad \mathbf{x}^{\mathbf{K}} = x_1^{k_1} \ldots x_N^{k_N}, \text{ if } \mathbf{K} = (k_1, \ldots, k_N).$$

The ring of fractional power series in the variables $x_1, \ldots, x_N$ with exponents in the cone σ will be denoted by $\mathbb{C}[[\mathcal{S}_\sigma]]$, and more generally, the ring of fractional power series with exponents in some translate of σ by $\mathbb{C}((\mathcal{S}_\sigma))$.

3.1 A total order on $\mathbb{Q}^N$

The first important idea in [5] is to put a total order on $\mathbb{Q}^N$, by fixing an irrational vector $w \in \mathbb{R}^N$. If $\alpha \in \mathbb{Q}^N$, we define the order of α

$$\bigcirc_w(\alpha) = \langle w, \alpha \rangle = \sum_{i=1}^{N} w_i \alpha_i.$$

This order induces a total order on the monomials $\mathbf{x}^\alpha$, $\alpha \in \mathbb{Q}^N$, and induces the definition of the order of a non null series $\xi = \sum_{\alpha \in \mathbb{Q}^N} c_\alpha \mathbf{x}^\alpha$:

$$\bigcirc_w(\xi) = \inf_{c_\alpha \neq 0} \bigcirc_w(\alpha) \text{ and } \bigcirc_w(0) = -\infty.$$

3.2 The Newton polytope

The algorithm that we will describe in the next paragraph is based on $\mathcal{P}(f)$, the Newton polytope of f, which is the convex hull of its support (in $\mathbb{R}^{N+1}$).
We will use the vocabulary defined in [5] and add the definition of some terms which will be usefull in the description of the resolution algorithm.

- The characteristic polynomial, Φ_e, of an edge, e, will be defined as in [2].
- The barrier cone associated to a monotone coherent edge path is by definition the cone: $\mathcal{C}_E = \sum_{i=1}^{r} \mathcal{C}_{e_i}$, where $\mathcal{C}_{e_i}$ denotes the barrier cone of the edge e_i.
- An admissible edge of the Newton polytope is said simple if its characteristic polynomial has only simple roots. A monotone edge path of the Newton polytope is said simple if the edges which compose it are all simple.

3.3 The Newton algorithm

In fact the principle of this algorithm is very close to the one exposed in [3] for the case of the algebraic equations in two variables. The common idea of these algorithms is to look for solutions under the form $y = c\mathbf{x}^\alpha + \tilde{y}$ where α is the opposite of the slope, $\mathbf{s}_e = (\frac{p_1}{q_1}, \ldots \frac{p_N}{q_N})$, of an admissible edge e of the Newton polytope of f and c a root of the associated characteristic polynomial $\Phi_e(X^q)$ where $q = lcm(q_i)$. We consider then the following change of unknown $\tilde{f}(\mathbf{x}, \tilde{y}) = f(\mathbf{x}, c\mathbf{x}^\alpha + \tilde{y})$, and search a series $\tilde{y}$ solution of $\tilde{f}(\mathbf{x}, \tilde{y}) = 0$ such that $\bigcirc_w(\alpha) < \bigcirc_w(\tilde{y})$.
The first result of [5] is to say that this process can be iterated infinitly if the given irrational vector $w \in \mathcal{C}_e^*$.
By this way it is possible to associate to each admissible edge e of $\mathcal{P}(f)$ and to each irrational vector $w \in \mathcal{C}_e^*$ a strongly convex polyhedral rational cone $\mathcal{C}_{w,e}$ such that $w \in \mathcal{C}_{w,e}^*$ and l_e solutions (counting multiplicity) $\phi \in \mathbb{C}((\mathcal{S}_{\mathcal{C}_{w,e}}))$ such that $\bigcirc_w(\phi) = -\mathbf{s}_e$.
Then it is shown in [5] that for each monotone coherent edge path $E = \{e_1, \ldots, e_r\}$ and for each irrational vector $w \in \cap_{i=1}^{r} \mathcal{C}_{e_i}^*$, we can find a strongly convex polyhedral rational cone $\mathcal{C}_w$ such that $w \in \mathcal{C}_w^*$ and d solutions (counting multiplicity) in $\mathbb{C}((\mathcal{S}_{\mathcal{C}_w}))$.
Furthermore, if $\mathcal{P}(f)$ possesses a simple coherent edge path, $\mathcal{C}_w = \sum_{i=1}^{r} \mathcal{C}_{e_i}$ is suitable in the result above.

Example Let us consider the following equation[3]:

$$f(\mathbf{x}, y) = y^2 + 2x_1y^2 + 2x_1y + x_1^2y^2 + 2x_1^2y + x_1^2 + x_2^2. \tag{2}$$

The associated Newton polytope is represented on the figure 1 below.
It has four coherent edge paths $E_1 = \{e_1\}$, $E_2 = \{e_2\}$, $E_3 = \{e_3\}$, $E_4 = \{e_4\}$.

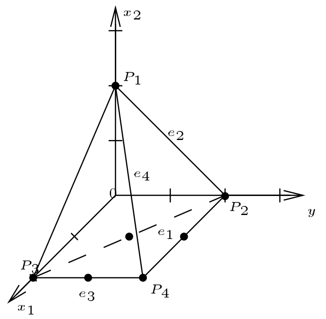

Fig. 1. Newton polytope of f.

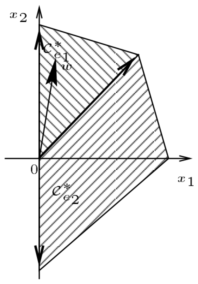

Fig. 2. Choice of w.

Suppose that we are interested in the computation of the series solutions associated to the path E_2. We have then to choose a vector w in the dual of the barrier cone of the admissible edge e_2, $w \in \mathcal{C}^*_{e_2}$. The associated characteristic polynomial is $\Phi_{e_2}(X) = X^2 + 1$. This polynomial has two simple roots $\alpha_1 = i$ and $\alpha_2 = -i$. So we can define two begining of solutions $\psi_1 = ix_2$ and $\psi_2 = -ix_2$ ($\mathbf{s}_{e_2} = (0, -1)$, $q = 1$) and compute the solutions with the algorithm presented in the previous section

$$\begin{cases} \phi_1 = ix_2 - x_1 - ix_1x_2 + x_1^2 + ix_1^2x_2 + \ldots \\ \phi_2 = -ix_2 - x_1 + ix_1x_2 + x_1^2 - ix_1^2x_2 + \ldots \end{cases}$$

Furthermore, since the edge e_2 is simple, we can make precise that $\phi_i \in x_2\mathbb{C}[[\mathcal{S}_{\mathcal{C}_{e_2}}]]$ with $\mathcal{C}_{e_2} = Pos\big((1,0),(1,-1)\big)$.
Let us now have a look at the edge e_1. Let us choose an irrational vector w in the dual of the barrier cone of e_1.We can choose it as on the figure 2.

The characteristic polynomial associated to e_1 is $\Phi_{e_1}(X) = 1 + 2X + X^2 = (1 + X)^2$. This polynomial has -1 as root of multiplicity 2. This path is not simple. So we define $\psi_1 = -x_1$ ($\mathbf{s}_{e_1} = (-1, 0)$, $q = 1$) and compute

$$f_1(x_1, x_2, y) = f(x_1, x_2, \psi_1 + y) = y^2 - 2x_1^2y + 2x_1y^2 + x_1^4 - 2x_1^3y + x_1^2y^2 + x_2^2$$

The Newton polytope associated to f_1 (fig. 3) has again four coherent edge paths: two composed of an unique simple edge $e_{1,2}$ and $e_{1,4}$, and two others composed of an edge, $e_{1,1}$ and $e_{1,3}$, whose characteristic polynomial has a root of multiplicity two. The barrier cones are represented on figure 4.

[3] Of course, we know a "closed-form" of the solutions: $y = \frac{\pm ix_2 - x_1}{1+x_1}$

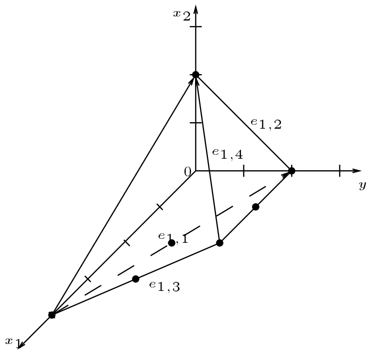

Fig. 3. Newton polytope of f_1.

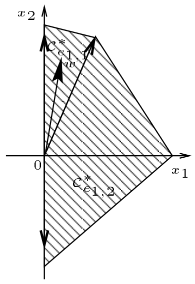

Fig. 4. Barrier cones of the edges $e_{1,1}$, and $e_{1,2}$.

Since $w \in \mathcal{C}^*_{e_{1,1}}$, we continue the process with the edge $e_{1,1}$. This edge is not a simple edge, so we will again have a multiple root for the associated characteristic polynomial and we are not yet able to define the cone of the series solution at this step. We can remark that the normal cone of the edge $e_{1,1}$ has shrunk and it will continue if we carry on the algorithm. At the opposite, the normal cone of the edge $e_{1,2}$ is increasing. So there will be a step n_0 for which the chosen vector w will belong to $\mathcal{C}^*_{e_{n_0,2}}$. This time, we will continue the process with the edge $e_{n_0,2}$ which is simple. The (unique at this step) beginning of solution will split into two simple solutions whose supports are included in a translate of the cone $\mathcal{C}_{e_{n_0,2}}$.
We see through this example that the choice of w can strongly influence the number of steps we will have to compute until we are able to separate the two solutions. In this example, the more the vector w is close to the x_2-axes, the more steps we have to compute in the algorithm.
Suppose for example that $n_0 = 3$: we obtain the set of solutions

$$\begin{cases} \varphi_1 = -x_1 + x_1^2 - x_1^3 + ix_2 + x_1^4 - ix_1x_2 + \dots \\ \varphi_2 = -x_1 + x_1^2 - x_1^3 - ix_2 + x_1^4 + ix_1x_2 + \dots \end{cases}$$

In fact, since $\mathcal{C}_{e_{n_0,2}} \subset \mathcal{C}_{e_2}$ and f has at most two solutions in $\mathbb{C}((\mathcal{S}_{\mathcal{C}_{e_2}}))$, $\{\varphi_1, \varphi_2\} = \{\phi_1, \phi_2\}$.
For example, $\varphi_1 = \phi_1$, only the order of the terms depends on the order on $\mathbb{Q}^N$, that is to say on the choice of the vector w.

The main drawback of this version of the Newton algorithm lies in the fact that we have an infinite possibility to choose the vector w, and in the impossibility of determining a priori if the solutions obtained with two vectors w_1 and w_2 are equal or not.
We will now see that it is possible to improve all points and particularly to determine the cones of the series solution from the begining, even in the case of non simple edges. For that we will use some theoretical results due to Perez ([7]) that we will expose in the next section.

4 Theoretical results on the cones of the solutions

Up to now we consider a strongly convex rational polyhedral cone ρ of dimension N and a non constant polynomial f (of degree d) with coefficients in $\mathbb{C}((\mathcal{S}_\rho))$,

$$f(y) = 0 \text{ with } f \in \mathbb{C}((\mathcal{S}_\rho))[y]. \tag{3}$$

4.1 Newton ρ-polyhedron

The Newton ρ-polyhedron of a non null series $\phi \in \mathbb{C}((\mathcal{S}_\rho))$ is the Minkowski sum of the convex hull of its exponents and the cone ρ.
The Newton ρ-polyhedron of a polynom $f \in \mathbb{C}((\mathcal{S}_\rho))[y]$, is the Minkowski sum of the convex hull of its exponents and the cone $\rho \times \{0\}$.
The Newton ρ-polyhedron of ϕ (resp. f) is noted $\mathcal{N}_\rho(\phi)$ (resp. $\mathcal{N}_\rho(f)$).
When $\rho = (\mathbb{R}^+)^N$, we will just call it Newton polyhedron of ϕ (resp. f) and simply denote it by $\mathcal{N}(\phi)$ (resp. $\mathcal{N}(f)$).
From an algorithmic point of view, the **construction** of the Newton ρ-polyhedron of a polynom f is based on the fact that it is a polyhedral set in $\mathbb{R}^{N+1}$. This result can be found in [8] in the case $\rho = (\mathbb{R}^+)^N$ and extended in the general case. So it can be represented by a matrix $\mathbf{A}$ with $N+1$ columns and a vector $\mathbf{d}$ such that $z \in \mathcal{N}_\rho(f) \iff \mathbf{A}z \geq \mathbf{d}$. In the particular case of the Newton ρ-polyhedron, we will isolate the last coordinate of z, and separate $\mathbf{A}$ in a matrix $\mathbf{M}$ (the N first columns) and a vector $\mathbf{b}$ (the last column) so that:

$$\mathcal{N}_\rho(f) = \left\{(\mathbf{x}, y) \in \mathbb{R}^N \times \mathbb{R} \mid \mathbf{Mx} + y\mathbf{b} \geq \mathbf{d} \quad (3)\right\}.$$

Each line of the inequation (3) $M_{i1}x_1 + \ldots + M_{iN}x_N + b_i y \geq d_i$ defines a supporting hyperplane of a facet from the Newton ρ-polyhedron, with integer coefficients M_{ij}, b_i, d_i.

4.2 Newton algorithm

The algorithm described briefly in the above section can be transposed in this context and gives the following result:

Theorem 1. *[7] For all irrational vector $w \in \rho^*$, there exists a strongly convex rational cone $\mathcal{C}_w$ of dimension N such that $w \in \mathcal{C}_w^* \subset \rho^*$ and f admits d solutions in $\mathbb{C}((\mathcal{S}_{\mathcal{C}_w}))$.*

More interesting for our purpose is the relation between the cones of the series solutions and the fan of the discriminant of the polynomial f.

4.3 Results on the cones of series solutions

We consider a primitive polynomial of degree $d \geq 1$ and the equation

$$f(\mathbf{x}, y) = \sum_{k=0}^{d} \overline{a}_k y^k = 0 \text{ with } \overline{a}_k \in \mathbb{C}\{\{\mathcal{S}_\rho\}\}. \tag{4}$$

For the discriminant, we use the definition of [10]. If $\mathcal{A}$ is an integral domain, and P in $\mathcal{A}[y]$, the discriminant of P is: $\Delta_y(P) = Resultant_y\,(P, P')$.
A great deal of information concerning the Newton polytope of the discriminant is given in [11]. Here we will use the following precise result:

Theorem 2. *[7] For each cone σ of dimension N of the fan associated to the Newton ρ-polyhedron $\mathcal{N}_\rho(\Delta_y(f))$, f can be decomposed in the ring $\mathbb{C}\{\{\mathcal{S}_{-\sigma^*}\}\}$.*

4.4 Example:

Consider the equation: $f(\mathbf{x}, y) = x_2y^2 + y^2 + x_2^2y + x_1^2y + x_1^2x_2y - x_1^2x_2$.
Its Newton polyhedron is graphically represented on figure 5

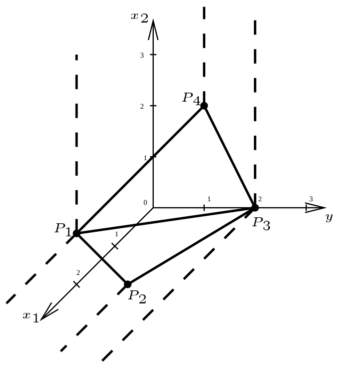

Fig. 5.

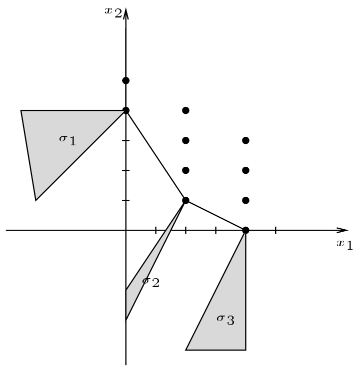

Fig. 6.

and in terms of linear inequalities:
$$\underbrace{\begin{pmatrix} 3 & 2 \\ 1 & 2 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}}_{\mathbf{M}} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + y \underbrace{\begin{pmatrix} 4 \\ 2 \\ 0 \\ 2 \\ 0 \\ 1 \\ 1 \\ -1 \end{pmatrix}}_{\mathbf{b}} \geq \underbrace{\begin{pmatrix} 8 \\ 4 \\ 0 \\ 2 \\ 0 \\ 1 \\ 0 \\ -2 \end{pmatrix}}_{\mathbf{d}} \tag{5}$$

Moreover

$$\Delta_y(f) = -(x_2 + 1)(x_2^4 + 2x_1^2x_2^3 + 6x_1^2x_2^2 + x_1^4x_2^2 + 2x_1^4x_2 + 4x_1^2x_2 + x_1^4).$$

The Newton polyhedron of this *series* and the associated fan is represented on the figure 6. In this case, there are three cones of dimension 2 in the fan, the cones σ_1, σ_2 and σ_3 represented on figure 6. The theorem 2 says us that we can find $d = 2$ series solution with exponents in a translate of each cone $-\sigma_1^*$, $-\sigma_2^*$ and $-\sigma_3^*$. The full set of solutions we are looking for will be composed of three cones and two series solutions with support in a translate of each cone.

4.5 Full set of solutions

By definition, the full set of solutions of the equation (4) is the union, when σ is any cone of dimension N of the fan of the ρ-polyhedron $\mathcal{N}_\rho(\Delta_y(f))$, of the sets of the d solutions computed in $\mathbb{C}((\mathcal{S}_{-\sigma^*}))$. We will now justify the term of "full set" of solutions.[4] For this purpose, we state the following lemma:

Lemma 1 *Let $\sigma \subset \mathbb{R}^N$ be a rational cone. The following propositions are equivalent:*

- *(a) There exists an irrational vector, w, in σ.*
- *(b) $dim(\sigma) = N$.*

This lemma permits to demonstrate the theorem:

Theorem 3. *Let $w \in \rho^*$ be an irrational vector. Denote by $\{\phi_i\}$ the set of d solutions existing in $\mathbb{C}((\mathcal{S}_{\mathcal{C}_w}))$ (Th 1).*
Then, for $i = 1 \ldots d$, ϕ_i appears in the full set of solutions.

Proof: Since the fan $\Sigma(\mathcal{N}_\rho(\Delta_y(f))) = -\rho^*$, there exists a rational cone $\sigma \in \Sigma(\mathcal{N}_\rho(\Delta_y(f)))$ such that $w \in -\sigma$. By the previous lemma, this cone is of dimension N, and we denote by $\{\varphi_i\}$ the set of d solutions in $\mathbb{C}((\mathcal{S}_{-\sigma^*}))$ (Th. 2). By definition, the solutions φ_i belong to the full set of solutions.
The fact that $w \in (-\sigma) \cap \mathcal{C}_w^*$ implies with the previous lemma that $dim((-\sigma) \cap \mathcal{C}_w^*) = N$. That means that the cone $\tau = ((-\sigma) \cap \mathcal{C}_w^*)^* = (-\sigma)^* + \mathcal{C}_w$ is strongly convex ([9]). But $\{\phi_i\} \subset \mathbb{C}((\mathcal{S}_\tau))$ and $\{\varphi_i\} \subset \mathbb{C}((\mathcal{S}_\tau))$ and then we can conclude that $\{\phi_i\} = \{\varphi_i\}$. $\square$

5 Computation of the full set of solutions

In this section, we consider an equation $f(\mathbf{x}, y) = 0$ where $f \in \mathcal{C}[[\mathcal{S}_\rho^\nu]][y]$, where $\nu \in \mathbb{N}^*$ and $\mathcal{S}_\rho^\nu = \rho \cap \frac{1}{\nu}\mathbb{Z}^N$.

5.1 Paths associated to a cone of the fan of $\mathcal{N}_\rho(\Delta_y(f))$.

Let us consider σ an N dimensional cone of the fan of $\mathcal{N}_\rho(\Delta_y(f))$. Let us take w an irrational vector in $-\sigma$. We know ([7]) that this vector defines a monotone coherent edge path E in $\mathcal{N}_\rho(f)$, which permits to compute d solutions of the equation with exponents in a translate of a cone $\mathcal{C}_w$.
We give now a criterion which associates to each cone σ a set of monotone coherent edge paths capable of furnishing solutions in $\mathbb{C}((\mathcal{S}_{-\sigma^*}))$.
The monotone coherent edge path E with barrier cone $\mathcal{C}_E$ is associated to the cone $\sigma \in \Sigma(\mathcal{N}_\rho(\Delta_y(f)))$ if:

$$dim((\mathcal{C}_E + (-\sigma)^*)^*) = dim(\mathcal{C}_E^* \cap (-\sigma)) = N$$

Theorem 4. *A path E with barrier cone $\mathcal{C}_E$, associated to the cone σ of the fan $\Sigma(\mathcal{N}_\rho(\Delta_y(f)))$, will permit to compute d solutions of the considered equation in $\mathbb{C}((\mathcal{S}_{-\sigma^*}))$.*

[4] Notice that this term does not have the same meaning as in [5]

Proof: Let w be an irrational vector in $(\mathcal{C}_E + (-\sigma)^*)^*$. Since $\mathcal{C}_E + (-\sigma)^*$ is a strongly convex rational polyhedral cone, containing ρ, we have that $f \in \mathbb{C}((\mathcal{S}_{\mathcal{C}_E+(-\sigma)^*}))$. Then we apply theorem 1, $\exists \mathcal{C}_w$ such that $(-\sigma)^* + \mathcal{C}_E \subset \mathcal{C}_w$ with d solutions of the equation in $\mathbb{C}((\mathcal{S}_{\mathcal{C}_w}))$. Since $-\sigma^* \subset \mathcal{C}_w$ we conclude with theorem 1 that these d solutions computed in $\mathbb{C}((\mathcal{S}_{\mathcal{C}_w}))$ are the d solutions existing in $\mathbb{C}((\mathcal{S}_{-\sigma^*}))$. □

In the next section we will detail the resolution process for an edge. We will use the same notations as in section 5.

5.2 Solutions associated to an edge

The main idea is to bring the equation to the regular case through iterative changes of unknown and changes of variables. After what we can easily find, as we have seen it in section 2, the associated series solution.

1. Consider an **admissible edge** e of the Newton ρ-polyhedron, $\mathcal{N}_\rho(f)$. By definition of an admissible edge, the line supporting the edge intersects the null hyperplane: let B be this intersection point, $B = (\beta, 0)$ with $\beta = (\beta_1, \ldots, \beta_N)$. So the edge is supported by the line of equation: $\mathbf{x} = \beta + y\mathbf{s}_e$. Besides this line is also the intersection of supporting hyperplanes of the Newton ρ-polyhedron. So we can extract from the matrix and vectors defining $\mathcal{N}_\rho(f)$ a full rank matrix $\mathbf{M}_e$, and two vectors $\mathbf{b}_e$ and $\mathbf{d}_e \in \mathbb{R}^N$ such that a point $P = (\mathbf{K}, h)$ belongs to the edge if and only if $\mathbf{M}_e\mathbf{K} + h\mathbf{b}_e = \mathbf{d}_e$.
   The fact that B is the intersection with the null hyperplane can be expressed by $\mathbf{M}_e\beta = \mathbf{d}_e$ and the barrier cone $\mathcal{C}_e$ of the edge is the set of points $\mathbf{K} \in \mathbb{R}^N$ such that $\mathbf{M}_e\mathbf{K} \geq 0$.

2. Let us calculate the associated **characteristic polynomial** Φ_e.
3. Let c be **a (non null) root** of $\Phi(X^q)$ in $\mathbb{C}$, of multiplicity r. We put: $\Phi(X^q) = (X - c)^r\psi(X)$, $\psi \in \mathbb{C}[X]$ and $\psi(c) \neq 0$.
4. We make a **change of unknown**, putting $y = \mathbf{x}^{-\mathbf{s}_e}(c + \tilde{y})$ and we calculate $\tilde{f}(\tilde{y}) = \mathbf{x}^{-\beta} f(\mathbf{x}^{-\mathbf{s}_e}(c + \tilde{y}))$. In this transformation, a monomial $a^h_{\mathbf{K}}\mathbf{x}^{\mathbf{K}}y^h$ becomes
   $\sum_{m=0}^{h} \binom{h}{m} a^h_{\mathbf{K}} c^m \mathbf{x}^{\mathbf{K}-\beta-h\mathbf{s}_e}\tilde{y}^{h-m}$. Geometrically, this means that the point $P = (\mathbf{K}, h)$ gives rise to $h+1$ points of coordinates $(\mathbf{K} - \beta - h\mathbf{s}_e, h - m)$. In the following, we will denote by $P' = (\mathbf{K}', h')$ an arbitrary point obtained from P by the above computation.
   If P belongs to the edge e, P' lies on the y-axis. Else, $\mathbf{M}_e\mathbf{K} + h\mathbf{b}_e \geq \mathbf{d}_e$ implies that $\mathbf{M}_e\mathbf{K}' \geq 0$.
   This means that $\tilde{f} \in \mathbb{C}[[\mathcal{S}^{\nu q}_{\mathcal{C}_e}]][y]$. Moreover, for all w irrational vector in $\mathcal{C}_e^*$, the terms of null order with respect to w are coming from
   $$\sum_{(\mathbf{K},h)\in e\cap\mathbb{S}(f)} a^h_{\mathbf{K}}(c+\tilde{y})^h = (c+\tilde{y})^j\Phi((c+\tilde{y})^q) = (c+\tilde{y})^j\tilde{y}^r\psi(c+\tilde{y}).$$
   Then we can write $\tilde{f} = \overline{b}_0 + \overline{b}_1\tilde{y} + \ldots + \overline{b}_d\tilde{y}^d$,
   with $\bigcirc_w(\overline{b}_i) > 0$ for $i = 0, \ldots, r-1$ and $\bigcirc_w(\overline{b}_r) = 0$.

So two cases are to be considered:

– c is a simple root: we do a **change of variables**, introducing new variables $\tilde{x}_1, \ldots, \tilde{x}_N$ such that $\tilde{f} \in \mathbb{C}[[\tilde{x}_1, \ldots, \tilde{x}_N]][\tilde{y}]$.
  This transformation can be built in two steps:
  - first we introduce the intermediate variables $t_i = x_i^{\frac{1}{\nu q}}$; a monomial $\mathbf{x}^{\mathbf{K}'}$ becomes $\mathbf{t}^{q\mathbf{K}'}$, so that the exponents are integers.
  - next we calculate the Hermite normal form of $\mathbf{M}_e^t$, that is to say an upper triangular matrix $\mathbf{H}$ and a unimodular matrix $\mathbf{U}$ such that:

$$\begin{cases} \mathbf{H} = \mathbf{U}^t \mathbf{M}_e^t \\ \mathbf{H}_{ii} \geq 0 & \forall\, 1 \leq i \leq N \\ \mathbf{H}_{ij} < 0 & \forall\, i < j. \end{cases}$$

  Now we ask the monomial $\mathbf{t}^{\mathbf{K}}$ to become $\tilde{\mathbf{x}}^{\tilde{\mathbf{K}}}$, where $\mathbf{K} = \mathbf{U}\tilde{\mathbf{K}}$. For that, we have to put $\tilde{x}_i = t_1^{a_{1i}} \ldots t_N^{a_{Ni}}$, where $(a_{1i}, \ldots, a_{Ni})$ is the ith column of $\mathbf{U}$.
  Finally, we can see easily that $\mathbf{M}_e \mathbf{K} \geq 0$ implies $\mathbf{H}^t \tilde{\mathbf{K}} \geq 0$, and then that all coordinates of $\tilde{\mathbf{K}}$ are positive integers.

  Conclusion: the vectors of exponents $\tilde{\mathbf{K}}$ are in $\mathbb{N}^N$.
  Then $\tilde{f} \in \mathbb{C}[[\tilde{\mathbf{x}}]][\tilde{y}]$. Moreover $\tilde{y}_0 = 0$ satisfies the hypothesis of the **regular case** for $\tilde{f}$:
  - $\tilde{f}(\tilde{y}_0) = 0 \mod \tilde{\mathbf{x}}$
  - $\tilde{f}'(\tilde{y}_0) \neq 0 \mod \tilde{\mathbf{x}}$.

  So we can compute a solution $\tilde{y} \in \mathbb{C}[[\tilde{\mathbf{x}}]]$ of the equation $\tilde{f}(\tilde{\mathbf{x}}, \tilde{y}) = 0$, which gives rise to a solution $y = \mathbf{x}^{-\mathbf{s}_e}(c + \tilde{y}) \in \mathbf{x}^{-\mathbf{s}_e}\mathbb{C}[[\mathcal{S}_{\mathcal{C}_e}^{\nu q}]]$ of $f(\mathbf{x}, y) = 0$.
  Remark: the matrix M_e is not unique (up to the order of the lines), so the matrix U is not unique. The different choices lead to different orderings of the terms of the solution $\tilde{y}$, but do not induce the computation of different solutions. This part of the algorithm could have been replaced by a choice of an irrational vector w in $\mathcal{C}_e$ and the developpement of the solution with respect of $\bigcirc_w$. In the same manner, the choice of w is not unique and leads to different orderings of the terms of the solution.

– if c is a root of multiplicity $r > 1$: we go to step 1 above, with $\tilde{f}$ in place of f, taking into account only the part of the Newton $\mathcal{C}_e$-polyhedron of $\tilde{f}$ located between the hyperplanes $\tilde{y} = 0$ and $\tilde{y} = r$.

5.3 Description of the algorithm

Even if the theoretical algorithm has been presented in the frame of polynomials with power series coefficients, the effective implementation deals withs multi-variate polynomials. So we consider $f(y) = 0$ with $f \in \mathbb{C}[x_1, \ldots, x_N, y]$.

1. First of all, we perform the squarefree factorization of f:

$$f = \prod_{i=1}^{h} f_i^{\alpha_i} \text{ with } f_i \in \mathbb{C}[x_1, \ldots, x_N][y] \text{ and } \alpha_i \in \mathbb{N}^*$$

We will solve separately each simplified equation $f_i = 0$ with f_i squarefree. Each root of the equation $f_i = 0$ will then be a root of multiplicity α_i for the equation $f = 0$. From now on, we suppose that f is **squarefree**.
2. We calculate the discriminant $\Delta_y(f)$ of f with respect to the variable y and construct its Newton polyhedron.
3. We construct the Newton polyhedron of f.
4. To each N-dimensional cone σ of the fan of $\mathcal{N}(\Delta_y(f))$, we associate a (nonempty) set of coherent monotone edge paths. If this set contains a simple path, we will consider this one else we take one path in the set. For the considered path $E = \{e_1, \dots, e_r\}$, we build the set of solutions in the following way: for each edge in the path, we iterate the transformations as described in the above paragraph. Simple roots (if there exists any) give rise to solutions in $\mathbb{C}((\mathcal{S}_{-\sigma^*}))$. We stop the process when the set of solutions associated to the path is complete (that is to say, contains a number of solutions equal to the length of the considered edge). The fact that f is squarefree assures that this situation will be reached after a finite number of steps.

The algorithms presented here can be summarized in the following way:

algorithm *Newton-Puiseux(f, n)*

input :

- $f \in \mathbb{C}[\mathbf{x}][y]$
- n, an integer for the number of terms in the solutions expansions

output :

- A set of cones and associated truncated generalized Puiseux series.

Begin

Squarefree factorization(f)
$d := \mathit{Degree}(f, y)$
$\rho := (\mathbb{R}^+)^N$
$\nu := 1$
$\mathcal{S}ol :=$ **Compute-NP**(f, n, ρ, ν, d)
Return$(\mathcal{S}ol)$

End.

The procedure **Compute-NP** computes the solutions expansions of f by considering only the part of the Newton-ρ-polyhedron located between the two hyperplanes of equation $y = 0$ and $y = r$. This procedure will be used recursively.

procedure *Compute-NP(f, n, ρ, ν, r)*

input :

- $f \in \mathbb{C}[\mathcal{S}_\rho^\nu][y]$
- n, an integer for the number of terms in the solutions expansions
- ρ, the set of vectors $\{u_1, \dots, u_p\}$ defining the cone
- ν, an integer defining the ramification of the expansions solutions
- r, an integer defining the equation of the hyperplane $y = r$

output :

- A set of cones and associated truncated generalized Puiseux series.

Begin
$\mathcal{S}ol := \emptyset$
$\Delta := Discriminant(f, y)$
$\mathcal{N}_\rho(\Delta) :=$ *Newton* ρ*-polyhedron of* Δ
$\mathcal{N}_\rho(f) :=$ *Newton* ρ*-polyhedron of* f *between the hyperplanes* $y = 0$ *and* $y = r$
$E(\mathcal{N}_\rho(f)) :=$ *Set of monotone coherent paths of* $\mathcal{N}_\rho(f)$
$\Sigma(\mathcal{N}_\rho(\Delta)) :=$ *Associated fan of* $\mathcal{N}_\rho(\Delta)$
For each $\sigma \in \Sigma(\mathcal{N}_\rho(\Delta))$ *of dimension* N **do**
  $E(\sigma) :=$ *Paths of* $E(\mathcal{N}_\rho(f))$ *associated to the cone* σ
  Choose a path E *in* $E(\sigma)$ *(simple if possible)*
  For each $e \in E$ **do**
    $\mathcal{S}ol_e :=$ **Compute-Edge-Solution**(f, n, ν, e)
    $\mathcal{S}ol := \mathcal{S}ol$ **union** $\mathcal{S}ol_e$
Return$(\mathcal{S}ol)$
End.

The procedure **Compute-Edge-Solution** computes the solutions associated to an edge by iterating the transformations described in the section 7.2.

procedure *Compute-Edge-Solution*$(f,\ n,\ \nu,\ e)$
input :

- $f \in \mathbb{C}[\mathcal{S}_\rho^\nu][y]$
- n, an integer for the number of terms in the solutions expansions
- ν, an integer defining the ramification of the expansions solutions
- e, the set of the points of the edge

output :

- A set of truncated generalized Puiseux series.

Begin
$\mathcal{S}ol_e := \emptyset$
$\mathcal{C}_e :=$ *Barrier cone*(e)
$\mathbf{s}_e :=$ *Slope*$(e) := (p_1/q_1, \dots, p_N/q_N)$
$q :=$ *lcm*(q_i)
$\Phi_e :=$ *Characteristic polynomial*(e)
$C :=$ *Set of non null roots of*(Φ_e)
For each $c \in C$ **do**
  Compute $\tilde{f}(\tilde{y})$
  If $(c$ *simple*$)$ **then**
    $\mathcal{S}ol_c :=$ *Regular case*$(\tilde{f},\ n,\ e,\ \nu,\ q)$
  Else
    $r :=$ *Multiplicity*(c)
    $\mathcal{S}ol_c :=$ **Compute-NP**$(\tilde{f},\ n,\ \mathcal{C}_e,\ \nu q,\ r)$
  $\mathcal{S}ol_e := \mathcal{S}ol_e$ *union* $\mathcal{S}ol_c$
Return$(\mathcal{S}ol_e)$
End.

The algorithms of the resolution process are implemented in Maple. For the elementary functions on the cones they use the package Convex developped by Mattias Franz, Universität Konstanz, Germany.

5.4 Example

We now run the algorithm on the equation 2 from the section 3.3.

1. f is squarefree.
2. $\Delta(f) = 4x_2^2(1+x_1)^4$. The Newton polyhedron of this expression and the associated fan are represented on the figure 7 below. The associated fan is in fact composed of one unique cone $\sigma = Pos\big((-1,0),(0,-1)\big)$.
3. The Newton polyhedron of f is given by the figure 8.
4. To the cone σ, there is two monotone coherent edge paths associated: $E_1 = \{[P_1P_2]\}$, $E_2 = \{[P_3P_2]\}$. In the case of this example, the path E_1 is simple and the other is not. To compute the series solutions associated to the cone σ we use the simple path and obtain then the following full set of solutions: the series ϕ_1 and ϕ_2, elements of $\mathbb{C}((\mathcal{S}_{-\sigma^*}))$, computed in section 3.3.

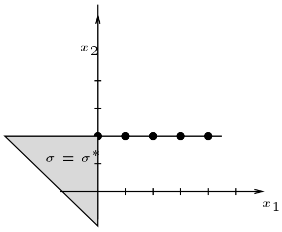

Fig. 7.

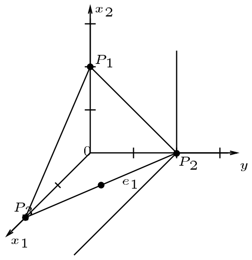

Fig. 8.

6 Link with differential equations

This work is linked with the study of differential equations in the following way: it permits to transform an implicit differential equation of order one into an explicit one. Indeed, consider $f(t,y,y') = 0$, $f \in \mathbb{C}[[t]][y,y']$. It is possible to apply directly a Newton algorithm as explained in [2], but it is also possible to run the above algorithm on $f(t,y,p)$. By this way, we obtain after a finite number of steps, $p = \dfrac{dy}{dt}$ as a series in t and y, with exponents in a cone, or more precisely

$$\begin{array}{c} p = t^{-s_1(1)}y^{-s_1(2)}(c_1 + t^{-s_2(1)}y^{-s_2(2)}(c_2 + \ldots + \tilde{p})) \\ \text{where } \tilde{p} \in \mathbb{C}[[\tilde{t},\tilde{y}]], \text{ (with finitely many } c_i,\ s_i) \ . \end{array} \tag{6}$$

The variables are linked by the following relations :

$$\begin{cases} \tilde{y}^q = t^{\alpha_1}y^{\beta_1} \\ \tilde{t}^q = t^{\alpha_2}y^{\beta_2} \end{cases} \Longleftrightarrow \begin{cases} y = \tilde{t}^{q\alpha'_1}\tilde{y}^{q\beta'_1} \\ t = \tilde{t}^{q\alpha'_2}\tilde{y}^{q\beta'_2} \end{cases},$$

with q, α_i, β_i, α'_i, $\beta'_i \in \mathbb{Z}$. Let us differentiate the previous expressions

$$\begin{cases} q\tilde{y}^{q-1}d\tilde{y} = \alpha_1 t^{\alpha_1-1}y^{\beta_1}dt + \beta_1 t^{\alpha_1}y^{\beta_1-1}dy \\ q\tilde{t}^{q-1}d\tilde{t} = \alpha_2 t^{\alpha_2-1}y^{\beta_2}dt + \beta_2 t^{\alpha_2}y^{\beta_2-1}dy \end{cases}$$

and finally

$$\left(\alpha_2 t^{\alpha_2-1} y^{\beta_2} + \beta_2 t^{\alpha_2} y^{\beta_2-1}\frac{dy}{dt}\right)\tilde{y}^{q-1}\frac{d\tilde{y}}{d\tilde{t}} = \tilde{t}^{q-1}\left(\alpha_1 t^{\alpha_1-1} y^{\beta_1} + \beta_1 t^{\alpha_1} y^{\beta_1-1}\frac{dy}{dt}\right)$$

Since $p = \dfrac{dy}{dt}$ can be expressed as a series in $\tilde{y}$ and $\tilde{t}$ with the relation (6) we can observe that this relation is in fact of the form

$$a(\tilde{t},\tilde{y})\frac{d\tilde{y}}{d\tilde{t}} + b(\tilde{t},\tilde{y}) = 0 \text{ with } a, b \in \mathbb{C}[[\tilde{t},\tilde{y}]]$$

which are the type of differential equations which have been the subject of many investigations. Some results on Newton algorithms and Puiseux series solutions can be found in [13].

7 Conclusion

From the computer algebra point of view, the local resolution of multi-variate equations is achieved. Of course, it will be interesting to complete the algebraic study presented here, by an analytic study of the solutions (definition of the domain of convergence of the series expansions). But we are convinced that the rings $\mathbb{C}((\mathcal{S}_{-\sigma^*}))$ are the good way of generalizing the Puiseux series in several variables, and that this approach for solving multi-variate polynomial equations is very promising and will have many fruitfull developments (local study of differential equations, of systems of multi-variate algebraic equations).

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