Generation of the Quasi-Solitons in the Lasers: Computer Algebra Approach to an Analysis

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1 Introduction

The recent progress in the generation of the extremely short laser pulses has allowed to reach the pulse durations of few femtoseconds and the peak intensities higher than 10^{15} W/cm^2 (for review see [1]). Their applications cover the range from the technology and the medicine to the sophisticated spectroscopical researches and the nuclear fusion. The femtosecond lasers possess the rich nonlinear properties, which dramatically complicate the field dynamics. There are two main approaches to the theoretical analysis of the femtosecond pulse dynamics. In fact, these approaches reproduce the mainstreams of the nonlinear physics in general.

The former is based on the numerical simulations of the field dynamics and the field-matter interaction. Even a comparatively simple model of this type needs the computations taking about of one week (1.2 GHz Athlon) for the simulation of the real lasing taking less than 10 μs. The main obstacle in the direct numerical simulations is not only the limits of the computer performance, but also the large set of the parameters governing the laser dynamics. This excludes the direct enumeration for the analysis of the system's global properties. Moreover, the obtained results, as a rule, are very complicated for the clear physical comprehension.

On the other hand, the use of the analytical and semi-analytical methods can give us the Ariadna's thread in the labyrinth of the nonlinear phenomena. The basis of the analytical approach to the research of the femtosecond laser dynamics is the so-called soliton model [2]. In the framework of this approach, the nonlinear evolution is approximated by the distributed model, which, as a rule, leads to the analog of the Landau-Ginzburg's type equation [3]. The stable soliton-like solutions of this equation are considered as the ultrashort pulses (see, for example, [2, 4]). This approach reduces the problem to the algebraic one and allows the computer algebra realization enhancing the physical comprehension of the underlying phenomena and the result's reproducibility.

However, the main obstacle here is the lack of the exact solutions describing a non-steady-state evolution. Moreover, a lot of the important problems lies at the border of the basic approximations validity that demands the additional analysis involving the numerical simulations.

Here we present the general approach to the analysis of the ultrashort pulse dynamics in the solid-state lasers, which is based on the analytical and semi-analytical consideration and allows the computer algebra realization. The comparisons with the experiment and the numerical results demonstrate the validity of the models under consideration and give the clear and comparatively simple understanding of the underlaying phenomena.

2 Dimension reduction

As it was mentioned, the analytical approach, as a rule, is based on the soliton model of the ultrashort pulse generation. Here we shall demonstrate that the dimension reduction allows to obtain the physically important results on the basis of the relatively simple maps.

Let us consider the four-dimensional model taking into account the evolution of the pulse peak intensity, the wave front radius of curvature, the laser beam diameter and the intracavity net-gain [5].

During the past years a great deal of attention has been paid to the method of the ultrashort pulse generation called as the Kerr-lens mode locking. The basic idea here is the saturation of the diffraction loss by the high-intensive field due to its self-focusing in the laser active medium [2]. The propagation of the field $a(z, r, t)$ (z is the longitudinal coordinate, r is the radial coordinate for the cylindrically symmetric laser field, t is the local time) through nonlinear medium can be described on the basis of the nonlinear parabolic equation [6]:

$$\left(\frac{\partial}{\partial z} + i\frac{\Delta_\perp}{2k} + i\beta|a|^2\right) a = 0, \tag{1}$$

where the laser beam has the Gaussian radial profile:

$$a(r) = E_0\rho^{-1/2} exp\left[\frac{-k_0 r^2}{2\rho} - \frac{ikr^2}{2R}\right]. \tag{2}$$

R is the radius of the wave front curvature, $\rho = \pi w^2/\lambda$ is related to the laser beam diameter w, k=$2\pi/\lambda$, λ is the wavelength, $\Delta_\perp = (1/r^2)(\partial/\partial r)[r^2(\partial/\partial r)]$, $\beta = k\, n_2/n$ is the coefficient of the Kerr-type nonlinearity, n_2 and n are the nonlinear and linear refraction coefficients, respectively. We shall suppose that $|a|^2$ has the dimension of the field intensity I.

From Eqs. (1, 2) the maps for R and ρ can be obtained:

$$\rho_{z+1} = \frac{(C_1 x + C_2)^2 + C_3}{C_1}, \tag{3}$$

$$R_{z+1} = \frac{\rho_{z+1}}{C_1 x + C_2} \tag{4}$$

where z is the field's round-trip number (i.e. we use the normalization of the propagation distance to the laser cavity length), x is the active crystal length, $C_1 = \rho_z/R_z^2 + C_3/\rho_z$, $C_2 = \rho_z/R_z$ and $C_3 = 1 - 2\beta\rho_z I_z$.

Then the maps for the field intensity and the dimensionless gain in the active medium are:

$$I_{z+1} = I_z \exp\left(g_z - l - kr^2/D'\right), \tag{5}$$

$$g_{z+1} = g_z + (g_m - g_z)\,P - \frac{2g_z I_z t_p}{E_g} - \frac{g_z T_{cav}}{T_r}, \tag{6}$$

where g and l are the gain and loss coefficients, respectively, g_m is the maximal gain corresponding to the full population inversion of the quasi-two level active medium, T_{cav} is the cavity period, T_r is the gain relaxation time, the E_g parameter has a sense of the gain saturation energy. The saturable diffraction loss is described by the last term in Eq. (5) (D' is the diameter of the intracavity aperture). The key control parameter is $P{=}\sigma_a I_p T_{cav}/(h\nu_a)$ (here σ_a is the pump absorption cross-section, I_p is the pump intensity, ν_a is the frequency of the pumping field), which is the dimensionless pump energy stored during T_{cav}. It is very conventional to normalize the field intensity to the nonlinear parameter β. Then the third term in Eq. (6), which contains the definition of the soliton-like ultrashort pulse energy $2I_z t_p$ (t_p is the pulse width), has to be rewritten as $2\tau I_z$ ($\tau{=}t_p/(\beta E_g)$). Let us confine ourself to the picosecond pulse generation. In this case the pulse duration is defined by the net-group-delay dispersion and the geometrical configuration of the laser cavity, so that the pulse duration is approximately constant and is not mapped.

The result of the simulations on the basis of this simplest model is shown in Fig. 1 for the parameters corresponding to the picosecond Ti:sapphire laser. One can see, that the variation of the P parameter causes the dramatic changes in the field intensity behavior (left axes) and in the corresponding Lyapunov's parameter Λ (right axes). There exist the regions of the stable operation (A), which change into the regions with the oscillatory and chaotical behavior (B, C and D). So, the region of the pump providing the ultrashort pulse generation has the pronounced heterogenic character with the steady-state, chaotic and period multiplication zones. It should be noted, that the period-multiplication behavior is different for B and C regions, which correspond to Ruelle-Takens and Fiegenbaum scenarios, respectively.

It is surprisingly, that such simple scheme allowing the computer algebra realization is in the good agreement with the exact numerical simulation based on the detailed model described in Ref. [7] and agrees with the experiment reported in Ref. [8].

However, it is clear that the transition to the femtosecond generation demands to map not only the pulse intensity, but also another ultrashort pulse parameters. This leads to the increase of the mapping dimension. The most appropriate way is to be based on the distributed laser model, which takes into consideration the saturable gain and loss action, the linear and spectral dissipation, the phase nonlinearity and the group-delay dispersion. As it was mentioned above, this leads to the generalized Landau-Ginzburg's equation:

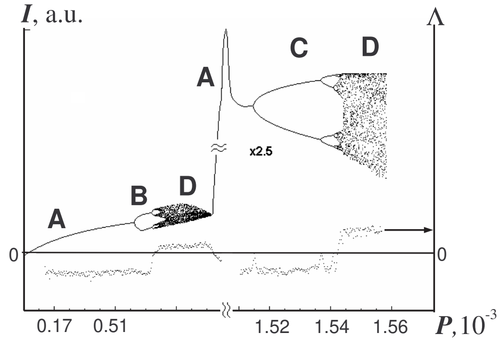

Fig. 1. The ultrashort pulse intensity I and the Lyapunov's parameter Λ versus the dimensionless pump P. T_{cav}=10 ns, t_p=1 ps.

$$\frac{\partial a(z,t)}{\partial z} = \left[g - l + \left(t_f^2 + id\right)\frac{\partial^2}{\partial t^2} - i\beta \left|a(z,t)\right|^2 + \hat{L}(|a|^2)\right] a(z,t), \qquad (7)$$

where we excluded the explicit dependence on r by means of the assumption that the spatial evolution contributes only to the nonlinear operator $\hat{L}(|a|^2)$ causing the change of the field loss on the aperture due to self-focusing. The last process is governed by the field intensity that validates our approximation. Moreover, we introduced the action of the spectral filter with the inverse bandwidth t_f and the group-delay dispersion with the corresponding coefficient d.

For the pulse-like solutions the boundary conditions correspond to the vanishing at $t \to \pm\infty$ field. However, in the case of the numerical simulations we have to put obviously the periodic boundary condition through the identification of $t = -T_{cav}/2$ with $t = T_{cav}/2$.

The form of $\hat{L}$ can be different. In the weak-nonlinear limit we have $\hat{L}$=$\sigma \left|a\right|^2$ (cubic nonlinear Landau-Ginzburg's equation). The more realistic approximation is $\hat{L}$= $\sigma \left|a\right|^2$$-\mu \left|a\right|^4$ (quintic nonlinear Landau-Ginzburg's equation). Here the key parameter is σ=$10^{-10} \div 10^{-12}$ cm^2/W, which has a sense of the inverse intensity of the diffraction loss saturation and can be controlled by the cavity alignment [9]. The saturation of the nonlinearity due to the higher-order nonlinear term with the μ parameter prevents from the collapse of the soliton-like solution of Eq. (7) [4].

The cubic Landau-Ginzburg's equation has a soliton-like solution in the form $a(t)$=$a_0 \cosh\left((t-\delta)/t_p\right)^{-1-i\psi} \exp(i\phi)$, where a_0, t_p, δ, ψ and ϕ are the ultrashort pulse amplitude, its width, the time delay, the chirp and the phase delay, respectively. However, the general non-steady-state pulse-like solution is not known even for this simplest form of $\hat{L}$. In order to analyze the field evolution in the general case and for the more complicated nonlinearities we have to perform the numerical simulations.

Nevertheless there is the possibility for the dimension reduction, which is based on the so-called aberrationless approximation [10, 11]: let us suppose that the pulse shape conserves approximately, but the pulse parameters depend on z. Then the backward substitution into Eq. (7) results in the set of the first-order ordinary differential equations for the pulse parameters evolution. Such system allows the comparatively simple consideration in the framework of the computer algebra system.

Let us consider the cubic Landau-Ginzburg's equation, but with the saturation of the nonlinearity to avoid the collapse. The simplest choice is $\hat{L}=\sigma_0(1-\sigma_0 a_0(z)^2/2)\,|a(z,t)|^2$, where the weak-nonlinear coefficient σ_0 is saturated by the evolutive pulse peak intensity $a_0(z)^2$. Then from Eq. (7) we have the four-dimensional mapping for the ultrashort pulse parameters:

$$
\begin{gathered}
t_p(z+1)=t_p(z)+\\
\frac{4-7d\psi(z)-3\psi(z)^2\left(\phi\psi(z)-2\sigma a_0(z)^2-\psi(z)\beta a_0(z)^2\right)t_p(z)^2}{2t_p(z)^2},
\end{gathered}
$$

$$
\begin{gathered}
\psi(z+1)=\left(1-2\sigma a_0(z)^2\right)\psi(z)+\left(\phi-\beta a_0(z)^2\right)\psi(z)^2+\\
\phi-3\beta a_0(z)^2-\frac{5d+3\psi(z)+5d\psi(z)^2+3\psi(z)^2}{t_p(z)^2},
\end{gathered} \tag{8}
$$

$$
a_0(z+1)=a_0(z)\left[1+\frac{d\psi(z)-1+\left(g(z)-l-\sigma a_0(z)^2\right)t_p(z)^2}{t_p(z)^2}\right],
$$

$$
\begin{gathered}
g(z+1)=g(z)\,\exp\left(-2\,\tau\,a_0(z)^2 t_p(z)^2-\frac{T_{cav}}{T_r}-P\right)+\\
\frac{g_m P}{P+T_{cav}/T_r}\left[1-\exp\left(-\frac{T_{cav}}{T_r}-P\right)\right],
\end{gathered}
$$

where $\sigma=\sigma_0(1-\sigma_0 a_0(z)^2/2)$, $\delta=0$, $\phi=\beta a_0(z)^2+(d+\psi(z))/t_p(z)^2$ and $\tau=t_f/(\beta E_g)$ (note the replacement of t_p by t_f). As the initial conditions we can take the pulse parameters corresponding to the quasi-soliton solution of the cubic Landau-Ginzburg's equation.

The stable stationary points of this system correspond to the steady-state generation of the ultrashort pulse. But there exists also the unsteady-state operation, which has regular as well as irregular character. The corresponding regions of the group-delay dispersion are shown in Fig. 2, which depicts the dependance of the pulse duration t_p and its chirp ψ on the d coefficient for the steady-state regime. The A region corresponds to the chaotic behavior of the pulse parameters; B shows the region, where there is no the quasi-soliton solution (divergent mapping); in the C region there are the periodical oscillations of the pulse parameters (most pronounced for a_0^2).

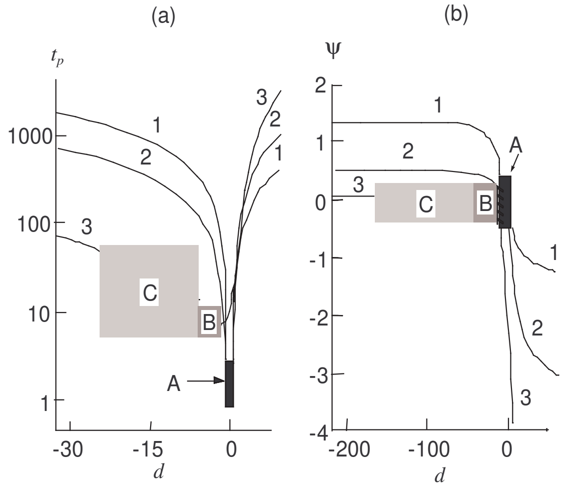

Fig. 2. The steady-state pulse duration (a) and the chirp (b) versus the d coefficient for the time scale and the intensity normalized to t_f and β, respectively. σ_0=10 (curve 1), 1 (curve 2), 0.1 (curve 3). l=0.05, g_m=0.5, P=4×10^{-4}, x=5 mm, λ=800 nm, T_{cav}=10 ns, T_r=3 μs.

The appearance of the regular as well as irregular oscillations, as a result of the $\mid d \mid$ decrease, was observed experimentally [8, 11] and their basic characteristics are in the good agreement with the theoretically predicted on the basis of our simple mapping procedure. Unfortunately, the attempt to describe the B region on the basis of the model under consideration fails due to the strong deviation from the soliton model, i. e. due to the invalidity of the abberationless approximation. The numerical simulations based on Eq. (7) with $\hat{L}$=-$\gamma/(1+\sigma\,|a|^2)$ (the arbitrary strong amplitude nonlinearity) demonstrate the multiple pulse operation in this region (Fig. 3). The main features of the transition to the multiple pulse generation are: 1) there exists the σ parameter producing the minimal extension of the A and B regions; 2) the decrease or the increase of σ relatively to this optimal value expands the region of the multiple pulse operation.

The similar behavior is typical also for the femtosecond lasers with the so-called slow saturable absorber, i. e. the absorber with the energy-dependent nonlinearity: $\hat{L}$=-$\gamma/\left(1+\Sigma\int\limits_{-\infty}^{t}|a(z,t')|^2dt'\right)$, where $\Sigma=1/E_a$, E_a is the loss sat-

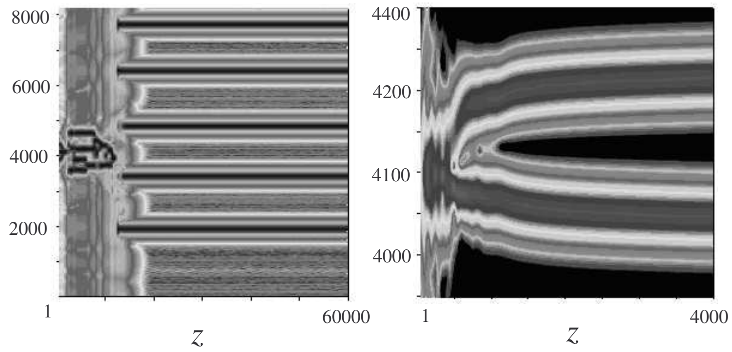

Fig. 3. Contour-plots of the field intensity logarithm below (a) and above (b) optimal σ. Vertical axes is the number of the simulation lattice points (their total number is 2^{13} that corresponds to the 102 ps time window).

uration energy and we do not take into account the longitudinal relaxation in the absorber (the corresponding relaxation time is much larger than t_p) [12].

In the next section we shall demonstrate that, in spite of the absence of the analytical solution for the description of the multiple pulses, the computer algebra approach allows the comprehension of their sources.

3 Perturbation analysis

Even beyond the rigorous limits of its validity the linear stability analysis can shed light upon the nature of the nonlinear system instability. We can not exactly describe the solution resulting from the perturbation growth, but can comprehend the character of the processes causing the pulse transformation. Here we try to fill up the gaps in the above presented analysis. We shall consider the destabilizing phenomena in the vicinity of the stability boundaries. As the basic model the ultrashort pulse laser with a slow saturable absorber will be considered.

The simplifying idealization intended for the description of the mode locking region heterogeneity (Fig. 1) and for the understanding of the destabilization scenarios difference (Fig. 3) is to neglect the imaginary terms in Eq. (7). This simplification is valid in the picosecond region of the ultrashort pulse duration and allows to comprehend the basic sources of the phenomena under consideration.

Let the pulse duration be much less than the longitudinal relaxation time in the saturable absorber (for the review of the physical model under consideration see [13]) and much larger than the transversal relaxation time (i. e. we consider the incoherent regime of the light-matter interaction). Then in the weak-nonlinear limit (i. e. when the pulse energy $\int\limits_{-\infty}^{t} |a(t')|^2 \, dt' \ll E_a, E_g$) the master Eq. (7) results in:

$$\frac{\partial a(z,t)}{\partial z} = g_0 \left[1 - \Xi \int\limits_0^t a(z,t')^2 dt' + \left(\Xi \int\limits_0^t a(z,t')^2 dt' \right)^2 \right] a(z,t) - \tag{9}$$

$$\gamma_0 \left[1 - \int\limits_0^t a(z,t')^2 dt' + \left(\int\limits_0^t a(z,t')^2 dt' \right)^2 \right] a(z,t) + t_f^2 \frac{\partial^2 a(z,t)}{\partial t^2} - l a(z,t),$$

where the first and second terms correspond to the gain and loss saturation, respectively; g_0 and γ_0 are the respective saturable gain and loss coefficients at t=0 (t=0 corresponds to the quasi-soliton maximum). The saturation parameter is Ξ=E_a/E_g that results from the normalization of the pulse energy to E_a. The weak-nonlinear limit allows to omit the higher-order terms on the pulse energy in Eq. (9). This equation has the soliton-like solution [14]: $a(t)$=$a_0/\cosh((t-\delta)/t_p)$, where $t_p^2 = 1/(g_0 - l - \gamma_0)$, $a_0^2 = \sqrt{2}\,(g_0 - l - \gamma_0)\,/\sqrt{\gamma_0 - \Xi^2 g_0}$, $\delta = -\sqrt{2}\,(\Xi g_0 - \gamma_0)\,/\sqrt{\gamma_0 - \Xi^2 g_0}$.

Let us consider the stability of this solution in the framework of the linear stability analysis. The seeding perturbation $\xi(t)\exp(\varsigma z)$ obeys after linearization of Eq. (9) the following equation:

$$\frac{\partial^2 \xi(t)}{\partial t^2} + \delta \frac{\partial \xi(t)}{\partial t} +$$

$$\xi(t)\left(\gamma_0 - \Xi g_0\right) \int\limits_0^t a(t')^2 dt' + \xi(t)\left(\Xi^2 g_0 - \gamma_0\right) \left[\int\limits_0^t a(t')^2 dt' \right]^2 + \tag{10}$$

$$a(t)\left(\gamma_0 - \Xi g_0\right) \int\limits_0^t a(t')\xi(t') dt' + \left(g_0 - \gamma_0 - l - \varsigma\right) = 0.$$

The simplest way is to analyze Eq. (10) in the long- and short-wavelength limits that allows either to neglect the second-order derivative in the comparison with the first-order one or to exclude the integral over the perturbation by virtue of the Riemann-Lebesgue's theorem. In the first case there is no the perturbation damped at $t = \pm\infty$. In the second case we have the solution for the perturbation expressed through the hypergeometric functions, which is too awkward to be written here (see Maple 6 worksheet at Waterloo Maple Application Center [15]). The basic picture is close to that in the quantum mechanics: the loss and gain saturations by the soliton-like pulse form the "potential well", which allows the existence of both the discrete and the continuous spectrum of the perturbations. It should be noted, that our approximation is valid for large "quantum numbers", that is in the vicinity of the continuous spectrum. If the allowed "levels" lie below the critical level with eigenvalue ("energy") $g_0 - \gamma_0$, then $\varsigma > 0$ and there are the asymptotically growing perturbations, i.e. the pulse is unstable in the framework of linear analysis. The undamped perturbation belonging to the discrete spectrum is visible in the numerical simulations as the slowly decaying wings outside the soliton wings (compare solid curve with dashed one in Fig. 4). The amplitude of this perturbation is proportional to the pulse energy.

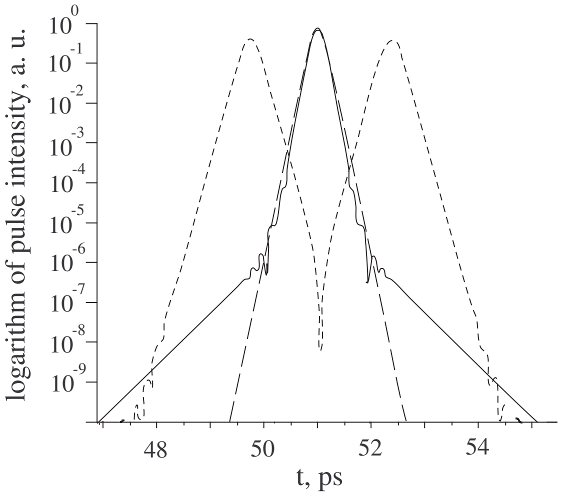

Fig. 4. Logarithm of the numerical profile of the pulse intensity (solid). Dashed - unperturbed soliton profile, Dotted - double pulses generation as a result of the perturbation growth.

The increase of the pulse energy due to the Ξ decrease, i. e. the decrease of E_a, results in the increase of the “bounded” (i. e. belonging to the discrete spectrum) perturbation. This causes the pulse splitting and the multiple pulse generation (dotted curve in Fig. 4 and the right-hand side picture in Fig. 3).

The contribution of the continuous spectrum is quite different. Its appearance results from the increase of the spectral loss for the shortest pulses or from the nonlinear pulse spectrum broadening (the latter takes a place in the presence of the imaginary terms in the master equation). In this case the pulse energy decreases and the amplified background noise appears. The satellite pulses are formed by this noise amplification. This causes the multiple pulse generation (the left-hand side picture in Fig. 3) or the chaotic behavior.

Our relatively simple analytical model explains also the heterogeneous character of the mode locking region (see Fig. 5). The increment of the perturbation growth, which is negative if the soliton is stable, can oscillate around zero. An example of such an oscillation in the form of the “kink” is shown in Fig. 5. As a result, there exist the regions of the ultrashort pulse stability, which alternate with the regions of the pulse instability.

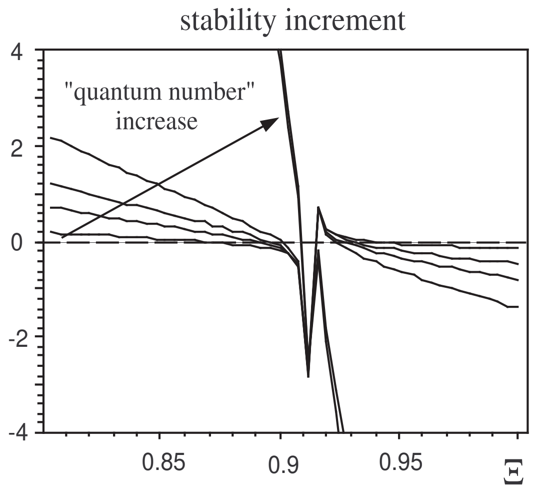

Fig. 5. Stability increment ς vs Ξ for the growing "quantum number" (i. e. the growing order of the perturbation).

Hence, we may completely analyze the main stability properties of the quasi-solitons in the mode-locked lasers, which allows the computer algebra realization.

4 Coherent effects

The above presented analytical and semi-analytical models do not take into consideration the coherent nature of the field-matter interaction, which is important for the ultrashort pulse duration shorter than 50 fs. The usual approach here is numerical simulation (see, for example, [16]). Here we evolve the analytical model of the quasi-soliton generation in the lasers with a coherent semiconductor absorber. Our approach allows the direct computer algebra realization [17].

In the presence of the coherent interaction with the absorber the $\hat{L}$ operator in Eq. (7) has to be replaced by the expression

$$\frac{2\pi N z_a \omega \mu}{c} u - \frac{2\pi N z_a \mu}{c} \frac{dv}{dt},$$

where $N = \gamma E_a/(\hbar\omega z_a) \sim 10^{18}$ cm^{-3} is the carrier density in the semiconductor absorber, ω is the field carrier frequency, z_a is the absorber thickness ($\sim$10 nm), μ is the dipole momentum ($\sim 0.28\times e$ coulomb$\times$nanometer for GaAs/AlAs absorber, e is the elementary charge). For the semiconductor absorber $E_a \simeq 10 \div 100$ μJ/cm^2. u and v are the slowly varying envelopes of the polarization quadrature components, which obey the Bloch equations (w is the population difference) [18]:

$$\frac{du}{dt} = (\varDelta - \frac{d\theta}{dt})v + qaw, \qquad \frac{dv}{dt} = -(\varDelta - \frac{d\theta}{dt})u, \qquad \frac{dw}{dt} = -qau, \tag{11}$$

where $\varDelta$ is the mismatch between the optical resonance and the pulse carrier frequency, θ is the instant field phase, $q = \mu/\hbar$. We suppose that the pulse duration is shorter than both longitudinal and transverse relaxation times in the absorber. For the latter we use t_{coh}=50 fs.

Now let us introduce in $\hat{L}$ also the fast absorber term $\sigma|a|^2$ in order to take into account the self-focusing in the active medium and then to make some trick for the sake of the analytical consideration of the soliton-like solution of Eqs. (7, 11). The transition to the field area $\alpha = \int\limits_{-\infty}^{t} a(t')dt'$ yields:

$$\left[(g - l + i\phi)\frac{d}{dt} + \delta\frac{d^2}{dt^2} + (1 + id)\frac{d^3}{dt^3} + \frac{\sigma - i\beta}{\eta^2}\left(\frac{d\psi(t)}{dt}\right)^2 \frac{d}{dt}\right]\alpha(t) - \frac{\gamma}{t_{coh}}\sin(\alpha(t)) = 0. \tag{12}$$

Here we introduced the η parameter, which is governed by the ratio between the size of the generation mode in the active medium and in the semiconductor absorber or by the reflectivity of the upper surface of semiconductor saturable absorber. Also we normalized the time to t_f and the field to qt_f. Then the intensity and time units for the Ti:sapphire laser are 0.2 TW/cm^2 and 2.5 fs, respectively.

The trick $\alpha(t) = x$, $d\alpha(t)/dt = y(x)$ with d=0 and β=0 transforms this third-order equation in the second-order one:

$$\left[\left(\frac{d^2y}{dx^2}\right)y + \left(\frac{dy}{dx}\right)^2 + \delta\frac{dy}{dx} + (g - l)\right]y - \frac{\gamma}{t_{coh}}\sin(x) = 0, \tag{13}$$

which has not the soliton-like solution but allows the approximate solution in the form $y(x) = a_0\sin(x/2) + \sum\limits_{m\geq 1} a_m \sin(\frac{(m+1)\pi}{2})$. In the lowest order this solution corresponds to the generation of the 2π-area pulse (the so-called self-induced transparency pulse) and gives for the pulse amplitude, the time delay and the duration $a_0 = 2\sqrt{2(g-l)}$, $\delta = \gamma/(2(g-l)t_{coh})$ and $t_p = 2/a_0$, respectively. The positivity of $g - l$ imposes the initial absorber loss constraint $\gamma > g - l$. Only in this case there is no background amplification (i. e. there exists no continuous spectrum for the ultrashort pulse perturbations). This condition is obviously disadvantageous because the γ growth increases the lasing threshold. The presence of the phase factors ($d, \beta \neq 0$) does not improve the situation, although in the chirp-free limit this allows to reduce Eq. (12) to the first-order equation:

$$\left[\delta\frac{dy(x)}{dx} + \frac{\beta}{\eta^2 d}y(x)^2 + (\alpha - \gamma - \frac{\phi}{d})\right]y(x) - \frac{\gamma}{t_{coh}}\sin x = 0, \tag{14}$$

which has the same approximated solution, but with the slightly transformed parameters: $\phi = 3\beta a_0^2/(4\eta^2) + d(g-l)$, $\delta = 4\gamma/(t_{coh}a_0^2)$, $a_0 = 2\sqrt{3(g-l)}$, $t_p = 2/a_0$.

The fundamental improvement is caused by $\sigma \neq 0$. In this case there exists the soliton-like solution with *sech*-shaped profile and

$$a_0 = \frac{2}{t_p},\ t_p = \frac{1}{\sqrt{l-g}},\ \delta = \frac{\gamma}{t_{coh}(l-g)},\ \sigma = \eta^2/2. \tag{15}$$

The confinement on the value of σ results from the coherent absorber induced selection of only 2π-pulses, although the pulse amplitude and the duration are defined by the fast saturable absorber, which is produced by the Kerr-lens mode locking. The main advantage is the negative value of $g-l-\gamma$ for the arbitrary γ. This produces the stability against background amplification automatically and allows to reduce the pulse duration down to sub-10 fs for the moderate pumping. Moreover, the pulse parameters mapping obtained on the basis of the abberationless approximation

$$\frac{da_0}{dz} = 2\frac{(g-l)\eta^2 t_p^2 - \eta^2 + 4\sigma}{\eta^2 t_p^3},\ \frac{dt_p}{dz} = 4\frac{\eta^2 - 2\sigma}{a_0\eta^2 t_p^2},$$
$$\theta = \frac{2(d + 4\frac{\beta}{\eta^2})}{a_0 t_p^3}, \delta = 2\frac{\gamma t_p}{t_{coh} a_0}. \tag{16}$$

demonstrates the pulse automodulational stability.

Additionally the presented model describes the *sech*-shaped chirp-free pulses with π-area and the chirped pulses with variable area. The respective substitutions are

$$u(t) = u_0/\cosh(\frac{t}{t_p}),\ v(t) = v_0/\cosh(\frac{t}{t_p}),$$
$$w(t) = \tanh(\frac{t}{t_p}),\ \frac{d\phi(t)}{dt} = \frac{\psi}{t_p}\tanh(\frac{t}{t_p}),$$

where $a_0 = \sqrt{1+\psi^2}/t_p, u_0 = -1/\sqrt{1+\psi^2}, v_0 = \psi/\sqrt{1+\psi^2}$.
As our analysis demonstrates, in the contrast with the propagation in the solely semiconductor layer, the π-pulses can be stable in the laser due to the fast saturable absorber action.

Hence, the pure analytical extension of the soliton model allows to describe the coherent effects. As a result, the computer algebra simulation replaces the cumbersome numerical analysis and allows the very clear physical interpretation.

5 Conclusion

We presented the self-consistent approach to the analysis of the ultrashort pulse generation in the mode-locked lasers, which is based on the dimension reduction and the analysis of the soliton-like states of the dynamical equations describing the action of the main lasing factor in the incoherent as well as the coherent limits. The numerical simulations in the finite dimensional model in the combination with the stability analysis of the soliton-like solutions of the master dynamical equations allow the computer algebra realization including the on-line calculators. This can essentially supplement the full dimensional numerical analysis and partially replaces their, especially, in the problem of the global optimization.

The approach under consideration is realized as the Maple 6 worksheet [15, 19] and takes less than 15 min of calculation a 900 MHz Athlon processor.

For the sake of completeness, let us note some problems, which are still unsolved. The methods under consideration take into account only single pulse regimes or, as a direct but nontrivial generalization, the complexes of the *unbounded* multiple pulses. However, there exist the reach structures of the strongly interacting pulses, which can be considered only numerically. We suppose, that the situation can be improved by the means of the extension of the single soliton-like solution. There are the first attempts in this direction, which are based on the Painlevé analysis [20]. In order to be applicable the cumbersome algebraic calculations taking place in this approach require the computer algebra realization likewise the direct soliton methods (as an example of the computer algebra algorithmization, the Hirota's method is considered in [19]). We plan the further development of the computer algebra approach in this direction.

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