Nonlinear Periodic Waves in Shallow Water

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Abstract. Two classical types of periodic wave motion in polar coordinates has been studied using a computer algebra system. In the case of polar coordinates, the usual perturbation techniques for the nonlinear shallow water equation leads to overdetermined systems of linear algebraic equations for unknown coefficients. The compatibility of the systems is the key point of the investigation. The found coefficients allow to construct solutions to the shallow water equation which are periodic in time. The accuracy of the solutions is the same as of the shallow water equation. Expanding the potential and surface elevation in Fourier series, we express explicitly the coefficients of the first two harmonics as polynomials of Bessel functions. One may speculate that the obtained expressions are the first two terms of an expanded exact three-dimensional solution to the surface wave equations, which describe the axisymmetrical and the simplest unaxisymmetrical surface waves in shallow water.

1 Introduction

Two-dimensional nonlinear shallow water equation

$$f_{xx} + f_{yy} - f_{tt} + \alpha\varepsilon\left(-f_x f_{xt} - f_y f_{yt}\right) + \beta\varepsilon\left(-f_{xx} f_t - f_{yy} f_t\right) = 0 \tag{1}$$

was introduced by Airy to describe the propagation of surface gravity waves. This is the leading approximation for very long waves of finite amplitude. The same equation describes two-dimensional gas dynamics in an isentropic gas flow.

The aim of this paper is to present a series of solutions to this equation in polar coordinates $(\theta,\, r)$. They are periodic in time, and their accuracy is the same as of equation (1).

In his classic book "Hydrodynamics", §§ 191-195, Lamb [4] considers at least two special cases of long **linear** waves in polar coordinates $(\theta,\, r)$ propagating over a horizontal bottom. The first case is axisymmetric waves caused by a periodic source of energy (3). The second case is the simplest unaxisymmetric waves (4) in a circular basin. The solution we have found gives, in particular, nonlinear corrections to these well known linear solutions.

Different versions of the shallow water equations are related mainly to a choice of basic variables (see [5]). We follow the book of Mei [6] and use the potential at bottom and surface elevation as basic variables.

There is a small parameter associated with the shallow water equations: the ratio of amplitude to depth, ε (measure of nonlinearity). As in the classic Airy equations, we retain only the quadratic terms (of orders $O(\varepsilon)$).

The potential at the bottom is expanded in Fourier series in time:

$$\begin{aligned} f(r,\theta,t) = U(r,\theta) &+ S^1(r,\theta)\sin(\omega t) + C^1(r,\theta)\cos(\omega t) \\ &+ S^2(r,\theta)\sin(2\omega t) + C^2(r,\theta)\cos(2\omega t) + \ldots \\ &+ S^m(r,\theta)\sin(m\omega t) + C^m(r,\theta)\cos(m\omega t) + \ldots \end{aligned} \tag{2}$$

The main result of this work consists in the explicit expressions (35) - (42) for functions U, S^1, C^1, S^2 and C^2 up to orders ε, which are homogenous polynomials in Bessel functions $Z_0(\omega r)$ and $Z_1(\omega r)$ and trigonometric functions of angular variable θ. Their coefficients are polynomials in r^{-1} and r. A similar expansion for the surface elevation can be derived. These expressions give a periodic solution to the classic shallow water equation, within the same accuracy as the equations are derived. Therefore, the result can be interpreted as a nonlinear periodic solution of surface wave equations (8) - (11) calculated up to orders ε. The calculations of coefficients and the check of results were performed using the Computer Algebra System "MATHEMATICA".

The two linear solutions used in the book of Lamb [4] are following:

$$f(r,\theta,t) = J_0(\omega r)\sin(\omega t) - Y_0(\omega r)\cos(\omega t) \tag{3}$$

$$f(r,\theta,t) = J_1(\omega r)\cos\theta\sin(\omega t)\,. \tag{4}$$

They describe the axisymmetric wave motion with periodic source located in the center of polar coordinates system and the simplest case of regular unsymmetrical wave motion in a circular basin.

When trying to find the functions $U(r,\theta)$, $S^1(r,\theta)$, $C^1(r,\theta)$, $S^2(r,\theta)$, and $C^2(r,\theta)$, we are forced to solve a Bessel-type non-homogenous differential equation of the second order:

$$\begin{aligned} &Z''(r) + \frac{1}{r}Z'(r) + \left(A + \frac{B}{r^2}\right)Z \\ &= q_{00}J_0(\omega r)Y_0(\omega r) + q_{01}J_0(\omega r)Y_1(\omega r) + q_{10}J_1(\omega r)Y_0(\omega r) + q_{11}J_1(\omega r)Y_1(\omega r), \end{aligned} \tag{5}$$

where q_{ij} are polynomials of r, r^{-1}.

We seek a solution $Z(r)$ in the form

$$Q_{00}J_0(\omega r)Y_0(\omega r) + Q_{01}J_0(\omega r)Y_1(\omega r) + Q_{10}J_1(\omega r)Y_0(\omega r) + Q_{11}J_1(\omega r)Y_1(\omega r) \tag{6}$$

where Q_{ij} are polynomials of r with unknown coefficients. These coefficients are calculated as solutions of overdetermined systems of linear algebraic equations (as in [7] and [8], where the same approach was used for describing the long periodic waves on a slope). In the cases considered, the compatibility of these overdetermined systems appears to be a matter of luck, and we see no reason why the luck should continue. Nevertheless, we take the liberty of making an "experimental" conjecture that the terms derived are only the lowest terms of a three dimensional exact solution to the surface wave equations (8) - (11) over a horizontal bottom. The exact three-dimensional solutions are not known at the moment, but we present an "intermediate" object (solutions of orders ε) which, we hope, describes more accurately the behavior of the long periodic waves as compared with the Lamb linear solution.

2 Basic equations

The aim of this section is to remind briefly the derivation of shallow water equation in the form suitable for us and to fix notations. We follow mainly to book of Mei [6]. Non-dimensional coordinates are used as follows:

$$x = \frac{x'}{l_0'},\ y = \frac{y'}{l_0'},\ z = \frac{z'}{h_0'},\ \ t = \frac{g^{\frac{1}{2}} h_0'^{\frac{1}{2}}}{l_0'} t',\ \eta = \frac{\eta'}{a_0'},\ \varphi = \frac{h_0'}{a_0' l_0' g^{\frac{1}{2}} h_0'^{\frac{1}{2}}} \varphi',\ h = \frac{h'}{h_0'} \quad (7)$$

where the prime denotes physical variables, and a_0', l_0', and h_0' denote the characteristic wave amplitude, depth, and wavelength, respectively; η' is the surface elevation; g is the acceleration of gravity; x and y are the horizontal coordinate; z is the vertical coordinate; and t is the time. The scaled governing equation and the boundary conditions for the irrotational wave problem read

$$\varphi_{xx} + \varphi_{yy} + \mu^{-2}\varphi_{zz} = 0, \quad -1 < z < \varepsilon\eta(x,y,t) \quad (8)$$

$$\eta_t + \varepsilon\varphi_x\eta_x + \varepsilon\varphi_y\eta_y - \mu^{-2}\varphi_z = 0, \quad z = \varepsilon\eta(x,y,t) \quad , \quad (9)$$

$$\varphi_t + \eta + \frac{1}{2}\varepsilon\left(\varphi_x^2 + \varphi_y^2\right) + \varepsilon\mu^{-2}\varphi_z^2 = 0, \quad z = \varepsilon\eta(x,y,t) \quad , \quad (10)$$

$$\varphi_z = 0, \quad z = -1 \quad (11)$$

where ε and μ are the measures of nonlinearity and frequency dispersion defined by

$$\varepsilon = a_0'/h_0', \qquad \mu = h_0'/l_0'. \quad (12)$$

The potential $\varphi(x,y,z,t)$ is expanded in powers of vertical coordinate

$$\varphi(x,y,z,t) = \sum_{m=0}^{\infty} (z+1)^m F_m(x,y,t) \quad (13)$$

We use ∇ to denote the horizontal gradient $(\partial/\partial x, \partial/\partial y)$. Substituting (13) into (8) and equating to zero the coefficients of each power of $z+1$, we have

$$F_{m+2} = \frac{-\mu^2\nabla^2 F_m}{(m+2)(m+1)} \quad (14)$$

The boundary condition at the bottom (11) gives

$$F_1 = 0 \tag{15}$$

Denoting $f(x, y, t) \equiv F_0(x, y, t)$ and expanding all expressions in powers of μ , we obtain the first terms of φ

$$\varphi = f - \frac{1}{2!}\mu^2(z+1)^2\nabla^2 f + O\left(\mu^4\right) \tag{16}$$

Expression (16) satisfies (8) and (11). Substitution of (16) into (9) and (10) gives the Boussinesq-type equations for two functions: potential at bottom $f(x, y, t)$ and surface elevation $\eta(x, y, t)$:

$$\eta_t + \varepsilon\nabla\eta \cdot \nabla f + (1+\varepsilon\eta)\nabla^2 f = 0 \tag{17}$$

$$\eta + f_t + \frac{1}{2}\varepsilon(\nabla f)^2 = 0 \tag{18}$$

The last two equations are equivalent to equations (1.16) and (1.17) of chapter 11 of [4].

Expressing the surface elevation $\eta(x, y, t)$ in terms of $f(x, y, t)$ and its derivatives, we have the single equation of type (1) for the function $f(x, y, t)$.

This equation can be rewritten in polar coordinates (r, θ) as follows:

$$\begin{aligned} &\frac{1}{r^2}f_{\theta\theta} + \frac{1}{r}f_r + f_{rr} - f_{tt} + \alpha\varepsilon\left(-\frac{2}{r^2}f_\theta f_{\theta t} - 2f_r f_{rt}\right) \\ &\quad +\beta\varepsilon\left(-\frac{1}{r^2}f_t f_{\theta\theta} - \frac{1}{r}f_t f_r - f_t f_{rr}\right) = 0 \end{aligned} \tag{19}$$

3 Periodic problem

We suppose that the solution is periodic in time and can be expanded in Fourier series in an area excluding a neighborhood of axis of symmetry and infinity. Making some additional scaling, we can assume that frequency ω is equal to 1 without loss of generality.

$$\begin{aligned} f\left(r,\theta,t\right) &= S_{00}^1\left(r,\theta)\right)\sin\left(t\right) + C_{00}^1(r,\theta)\cos\left(t\right) \\ &+S_{10}^2(r,\theta)\varepsilon\ \sin\left(2t\right) + C_{10}^2(r,\theta)\varepsilon\ \cos\left(2t\right) + ... \end{aligned} \tag{20}$$

In the zero-order, we have the following linear problems for $S_{00}^1(r,\theta)$ and $C_{00}^1(r,\theta)$

$$S_{00}^1 + \frac{1}{r^2}S_{00\theta\theta}^1 + \frac{1}{r}S_{00r}^1 + S_{00rr}^1 = 0 \tag{21}$$

$$C_{00}^1 + \frac{1}{r^2}C_{00\theta\theta}^1 + \frac{1}{r}C_{00r}^1 + C_{00rr}^1 = 0 \tag{22}$$

Their general solution expressed in polar coordinates can be presented as a series:

$$
\begin{aligned}
&\alpha_0^J J_0(r) + \alpha_0^Y Y_0(r) \\
&+ \left(\alpha_{1S}^J J_1(r) + \alpha_{1S}^Y Y_1(r)\right)\sin\theta + \left(\alpha_{1C}^J J_1(r) + \alpha_{1C}^Y Y_1(r)\right)\cos\theta + ... \\
&+ \left(\alpha_{nS}^J J_n(r) + \alpha_{nS}^Y Y_n(r)\right)\sin n\theta + \left(\alpha_{nC}^J J_n(r) + \alpha_{nC}^Y Y_n(r)\right)\cos n\theta + ... \qquad (23)
\end{aligned}
$$

where α_{nC}^J and α_{nC}^Y are arbitrary constant coefficients providing the convergency. We concentrate our attention on the first two terms of this series (zero and the first harmonics in θ)

Let us denote by $S = S(r)$ and $C = C(r)$ two arbitrary solutions of Bessel equation

$$rZ_{rr} + Z_r + rZ = 0, \tag{24}$$

Functions $S(r)$, $C(r)$ and their derivatives $S'(r)$, $C'(r)$ can be expressed in terms of Bessel functions as follows:

$$S(r) = a_{11}J_0(r) + a_{12}Y_0(r), \tag{25}$$
$$S'(r) = -a_{11}J_1(r) - a_{12}Y_1(r) \tag{26}$$
$$C(r) = a_{21}J_0(r) + a_{22}Y_0(r), \tag{27}$$
$$C'(r) = -a_{21}J_1(r) - a_{22}Y_1(r) \tag{28}$$

Then the first two terms of (23) can be written in the form:

$$S_{00}^1(r,\theta) = A_S S - (A_{SSS}S' + A_{SSC}C')\sin\theta - (A_{SCS}S' + A_{SCC}C')\cos\theta \tag{29}$$
$$C_{00}^1(r,\theta) = A_C C - (A_{CSS}S' + A_{CSC}C')\sin\theta - (A_{CCS}S' + A_{CCC}C')\cos\theta \tag{30}$$

where A_X and A_{XYZ} are arbitrary constant coefficients (We can assume $A_S = 1$ and $A_C = 1$ almost without loss of generality).

Lamb [4] considers two particular cases of (29), (30):

(i)

$$S_{00}^1(r,\theta) = \varepsilon J_0(r) \tag{31}$$
$$C_{00}^1(r,\theta) = -\varepsilon Y_0(r) \tag{32}$$

($A_X = 0,\quad A_{XYZ} = 0,\quad a_{12} = -\varepsilon,\quad a_{21} = \varepsilon,\quad a_{11} = a_{22} = 0$). The solution describes axisymmetric wave motion with periodic source located in the center of polar coordinates system.

(ii)

$$S_{00}^1(r,\theta) = 0 \tag{33}$$
$$C_{00}^1(r,\theta) = -\varepsilon J_1(r)\cos\theta \tag{34}$$

($A_S = 0$, $A_C = 0$, all A_{XYZ} are equal to zero except for $A_{CCC} = -\varepsilon$). The solution describes the simplest case of regular unsymmetrical wave motion in a circular basin.

The first solution is used in §§ 191-195 of book of Lamb [4] for describing axisymmetric wave motion with periodic source located in the center of polar coordinates system. The second one is used there for describing the simplest case of regular unsymmetrical wave motion in a circular basin. The aim of subsequent consideration is to give next-order corrections to these linear solutions.

4 Calculation of $S^2_{10}\,(r,\theta)$

$S^2_{10}(r)$ is presented as a sum of four components:

$$S^2_{10}(r) = -\frac{1}{2}\alpha S_{grad2} + \frac{1}{2}\alpha C_{grad2} + \frac{1}{2}\beta S_{mod2} - \frac{1}{2}\beta C_{mod2} \tag{35}$$

where

$$\begin{aligned}
S_{grad2} = {} & \frac{1}{2}rSS' + \left(A^2_{SCC} + A^2_{SSC}\right)\left(\frac{1}{4}rC^2 + \frac{1}{4}rCC' - \frac{1}{4}C'^2\right) \\
& + \left(A^2_{SCS} + A^2_{SSS}\right)\left(\frac{1}{4}rS^2 + \frac{1}{4}rSS' - \frac{1}{4}S'^2\right) \\
& + \left(A_{SCC}A_{SCS} + A_{SSC}A_{SSS}\right)\left(\frac{1}{2}CS + \frac{1}{4}C'S + \frac{1}{4}CS' - \frac{1}{2}C'S'\right) \\
& + \begin{pmatrix} A_{SSC}\left(-\frac{1}{4}rCS + \frac{3}{8}C'S + \frac{1}{8}CS' + \frac{1}{4}rC'S'\right) \\ +A_{SSS}(-\frac{1}{4}rS^2 + \frac{1}{2}SS' + \frac{1}{4}rS'^2) \end{pmatrix}\sin\theta \\
& + \begin{pmatrix} A_{SCC}\left(-\frac{1}{4}rCS + \frac{3}{8}C'S + \frac{1}{8}CS' + \frac{1}{4}rC'S'\right) \\ +A_{SCS}(-\frac{1}{4}rS^2 + \frac{1}{2}SS' + \frac{1}{4}rS'^2) \end{pmatrix}\cos\theta \\
& + \begin{pmatrix} \frac{1}{4}A_{SSC}A_{SCC}CC' + \frac{1}{4}A_{SCS}A_{SSS}SS' \\ +\frac{1}{4}\left(A_{SCS}A_{SSC} + A_{SCC}A_{SSC}\right)(CS' + C'S) \end{pmatrix}\sin 2\theta \\
& + \begin{pmatrix} \frac{1}{8}\left(A^2_{SCC} - A^2_{SSS}\right)rCC' + \frac{1}{8}\left(A^2_{SCS} - A^2_{SSS}\right)SS' \\ +\frac{1}{4}\left(A_{SCC}A_{SCS} - A_{SSC}A_{SSS}\right)r(CS' + C'S) \end{pmatrix}\cos 2\theta
\end{aligned} \tag{36}$$

$$\begin{aligned}
C_{grad2} = {} & \frac{1}{2}rCC' + \left(A^2_{CCC} + A^2_{CSC}\right)\left(\frac{1}{4}rC^2 + \frac{1}{4}rCC' - \frac{1}{4}C'^2\right) \\
& + \left(A^2_{CCS} + A^2_{CSS}\right)\left(\frac{1}{4}rS^2 + \frac{1}{4}rSS' - \frac{1}{4}S'^2\right) \\
& + \left(A_{CCC}A_{CCS} + A_{CSC}A_{CSS}\right)\left(\frac{1}{2}CS + \frac{1}{4}C'S + \frac{1}{4}CS' - \frac{1}{2}C'S'\right) \\
& + \begin{pmatrix} 2A_{CSC}\left(-\frac{1}{4}rC^2 + \frac{1}{2}rCC' + \frac{1}{4}C'^2\right) \\ +2A_{SSS}(-\frac{1}{4}rCS + \frac{1}{8}C'S + \frac{3}{8}CS' + \frac{1}{4}rC'S') \end{pmatrix}\sin\theta
\end{aligned} \tag{37}$$

$$+\begin{pmatrix} 2A_{CCC}\left(-\frac{1}{4}rC^2+\frac{1}{2}rCC'+\frac{1}{4}C'^2\right) \\ +2A_{CCS}(-\frac{1}{4}rCS+\frac{1}{8}C'S+\frac{3}{8}CS'+\frac{1}{4}rC'S') \end{pmatrix}\cos\theta$$
$$+\begin{pmatrix} \frac{1}{4}A_{CCC}A_{CSC}CC'+\frac{1}{4}A_{CCS}A_{CSS}SS' \\ +\frac{1}{4}\left(A_{CCS}A_{CSC}+A_{CCC}A_{CSS}\right)(CS'+C'S) \end{pmatrix}\sin 2\theta$$
$$+\begin{pmatrix} \frac{1}{8}\left(A_{CCC}^2-A_{CSC}^2\right)rCC'+\frac{1}{8}\left(A_{CCS}^2-A_{CSS}^2\right)SS' \\ +\frac{1}{4}\left(A_{CCC}A_{CCS}-A_{CSC}A_{CSS}\right)r(CS'+C'S) \end{pmatrix}\cos 2\theta$$

$$S_{mod2}=\begin{pmatrix} \frac{1}{2}rSS'+\frac{1}{2}\left(A_{SCC}^2+A_{SSC}^2\right)\left(-\frac{1}{2}C^2-\frac{1}{2}rCC'\right) \\ +\frac{1}{2}\left(A_{SCS}^2+A_{SSS}^2\right)\left(-\frac{1}{2}S^2-\frac{1}{2}rSS'\right) \\ +\left(A_{CCC}A_{CCS}+A_{CSC}A_{CSS}\right)\left(-\frac{1}{2}CS-\frac{1}{4}rC'S-\frac{1}{4}rCS'\right) \end{pmatrix} \tag{38}$$
$$+\begin{pmatrix} -2A_{CSS}\left(-\frac{1}{4}rCS+\frac{3}{8}C'S+\frac{1}{8}CS'+\frac{1}{4}rC'S'\right) \\ -2A_{CSC}(-\frac{1}{4}rC^2+\frac{1}{4}rC'^2) \end{pmatrix}\sin\theta$$
$$+\begin{pmatrix} -2A_{SCC}\left(-\frac{1}{4}rCS-\frac{1}{8}C'S+\frac{1}{8}CS'+\frac{1}{4}rC'S'\right) \\ -2A_{SCS}(-\frac{1}{4}rS^2+\frac{1}{4}rS'^2) \end{pmatrix}\cos\theta$$
$$+\begin{pmatrix} A_{SCC}A_{SSC}(-\frac{1}{2}rCC'-\frac{1}{2}C'^2) \\ +A_{SCS}A_{SSS}\left(-\frac{1}{2}rSS'-\frac{1}{2}S'^2\right) \\ +\left(A_{SCS}A_{SSC}+A_{SCC}A_{SSS}\right)\left(-\frac{1}{4}rC'S-\frac{1}{4}rC'S-\frac{1}{2}C'S'\right) \end{pmatrix}\sin 2\theta$$
$$+\begin{pmatrix} \frac{1}{2}\left(A_{SCC}^2-A_{SSC}^2\right)\left(-\frac{1}{2}rCC'-\frac{1}{2}C'^2\right) \\ +\frac{1}{2}\left(A_{SCS}^2-A_{SSS}^2\right)\left(-\frac{1}{2}rSS'-\frac{1}{2}S'^2\right) \\ +\left(A_{SCC}A_{SCS}-A_{SSC}A_{SSS}\right)\left(-\frac{1}{4}rC'S-\frac{1}{4}rC'S-\frac{1}{2}C'S'\right) \end{pmatrix}\cos 2\theta$$

$$C_{mod2}=\begin{pmatrix} \frac{1}{4}rCC'+\frac{1}{2}\left(A_{CCC}^2+A_{CSC}^2\right)\left(-\frac{1}{2}C^2-\frac{1}{2}rCC'\right) \\ +\frac{1}{2}\left(A_{CCS}^2+A_{CSS}^2\right)\left(-\frac{1}{2}S^2-\frac{1}{2}rSS'\right) \\ +\left(A_{CCC}A_{CCS}+A_{CSC}A_{CSS}\right)\left(-\frac{1}{2}CS-\frac{1}{4}rC'S-\frac{1}{4}rCS'\right) \end{pmatrix} \tag{39}$$
$$+\begin{pmatrix} -2A_{CSC}\left(-\frac{1}{4}rCS-\frac{1}{8}C'S+\frac{1}{8}CS'+\frac{1}{4}rC'S'\right) \\ -2A_{CSS}(-\frac{1}{4}rC^2+\frac{1}{4}rC'^2) \end{pmatrix}\sin\theta$$
$$+\begin{pmatrix} -2A_{CCS}\left(-\frac{1}{4}rCS-\frac{1}{8}C'S+\frac{1}{8}CS'+\frac{1}{4}rC'S'\right) \\ -2A_{CCC}(-\frac{1}{4}rC^2+\frac{1}{4}rC'^2) \end{pmatrix}\cos\theta$$
$$+\begin{pmatrix} \frac{1}{2}\left(A_{CCC}^2-A_{CSC}^2\right)\left(-\frac{1}{2}rCC'-\frac{1}{2}C'^2\right) \\ +\frac{1}{2}\left(A_{CCS}^2-A_{CSS}^2\right)\left(-\frac{1}{2}rSS'-\frac{1}{2}S'^2\right) \\ +\left(A_{SCS}A_{SSC}+A_{SCC}A_{SSS}\right)\left(-\frac{1}{4}rC'S-\frac{1}{4}rC'S-\frac{1}{2}C'S'\right) \end{pmatrix}\sin 2\theta$$
$$+\begin{pmatrix} \frac{1}{2}\left(A_{SCC}^2-A_{SSC}^2\right)\left(-\frac{1}{2}rCC'-\frac{1}{2}C'^2\right) \\ +\frac{1}{2}\left(A_{SCS}^2-A_{SSS}^2\right)\left(-\frac{1}{2}rSS'-\frac{1}{2}S'^2\right) \\ +\left(A_{SCC}A_{SCS}-A_{SSC}A_{SSS}\right)\left(-\frac{1}{4}rC'S-\frac{1}{4}rC'S-\frac{1}{2}C'S'\right) \end{pmatrix}\cos 2\theta$$

5 Calculation of $C_{10}^2(r,\theta)$

$C_{10}^2(r)$ is presented as a sum of two components:

$$C_{10}^2(r)=-\frac{1}{2}\alpha\Pi_{grad}+\frac{1}{2}\beta\Pi_{mod2} \tag{40}$$

where

$$
\begin{aligned}
\Pi_{grad} = {} & \begin{pmatrix}
-\frac{1}{2}CS - \frac{1}{4}rC'S - \frac{1}{4}rCS' \\
+\left(A_{CCC}A_{SCC} + A_{CSC}A_{SSC}\right)\left(\frac{1}{4}C^2 + \frac{1}{4}rCC' - \frac{1}{4}C'^2\right) \\
+\left(A_{CCS}A_{SCS} + A_{CSS}A_{SSS}\right)\left(\frac{1}{4}S^2 + \frac{1}{4}rSS' - \frac{1}{4}S'^2\right) \\
+\frac{1}{2}\begin{pmatrix} A_{CCS}A_{SCC} + A_{CCC}A_{SCS} \\ +A_{CSS}A_{SSC} + A_{CSC}A_{SSS} \end{pmatrix}\left(\frac{1}{2}CS + \frac{1}{4}rC'S + \frac{1}{4}rCS' - \frac{1}{2}C'S'\right)
\end{pmatrix} \\
& + \begin{pmatrix}
A_{SSC}\left(-\frac{1}{4}rC^2 + \frac{1}{2}CC' + \frac{1}{4}rC'^2\right) + A_{CSS}(-\frac{1}{4}rS^2 + \frac{1}{2}SS' + \frac{1}{4}rS'^2) \\
+A_{SSS}(-\frac{1}{4}rCS + \frac{1}{8}C'S + \frac{3}{8}CS' + \frac{1}{4}rC'S') \\
+A_{CSC}(-\frac{1}{4}rCS + \frac{3}{8}C'S + \frac{1}{8}CS' + \frac{1}{4}rC'S')
\end{pmatrix}\sin\theta \\
& + \begin{pmatrix}
A_{SCC}\left(-\frac{1}{4}rC^2 + \frac{1}{2}CC' + \frac{1}{4}rC'^2\right) + A_{CCS}(-\frac{1}{4}rS^2 + \frac{1}{2}SS' + \frac{1}{4}rS'^2) \\
+A_{CCC}(-\frac{1}{4}rCS + \frac{3}{8}C'S + \frac{1}{8}CS' + \frac{1}{4}rC'S') \\
+A_{SCS}(-\frac{1}{4}rCS + \frac{1}{8}C'S + \frac{3}{8}CS' + \frac{1}{4}rC'S')
\end{pmatrix}\cos\theta \\
& + \begin{pmatrix}
\left(A_{CSC}A_{SCC} + A_{CCC}A_{SSC}\right)\frac{1}{4}rCC' \\
+\left(A_{CSS}A_{SCS} + A_{CCS}A_{SSS}\right)\frac{1}{4}rSS' \\
+\frac{1}{2}r\begin{pmatrix} A_{CSS}A_{SCC} + A_{CSC}A_{SCS} \\ +A_{CCS}A_{SSC} + A_{CCC}A_{SSS} \end{pmatrix}(CS' + C'S)
\end{pmatrix}\sin 2\theta \\
& + \begin{pmatrix}
\left(A_{CCC}A_{SCC} - A_{CSC}A_{SSC}\right)\frac{1}{4}rCC' \\
+\left(A_{CCS}A_{SCS} - A_{CSS}A_{SSS}\right)\frac{1}{4}rSS' \\
-\frac{1}{2}r\begin{pmatrix} -A_{CCS}A_{SCC} - A_{CCC}A_{SCS} \\ +A_{CSS}A_{SSC} + A_{CSC}A_{SSS} \end{pmatrix}(CS' + C'S)
\end{pmatrix}\cos 2\theta
\end{aligned} \tag{41}
$$

$$
\begin{aligned}
\Pi_{mod2} = {} & \begin{pmatrix}
\frac{1}{4}r\left(C'S + CS'\right) \\
+\frac{1}{2}\left(A_{CCC}A_{SCC} + A_{CSC}A_{SSC}\right)\left(-\frac{1}{2}C^2 - \frac{1}{2}rCC'\right) \\
+\frac{1}{2}\left(A_{CCS}A_{SCS} + A_{CSS}A_{SSS}\right)\left(-\frac{1}{2}S^2 - \frac{1}{2}rSS'\right) \\
+\begin{pmatrix} A_{CCS}A_{SCC} + A_{CCC}A_{SCS} \\ +A_{CSS}A_{SSC} + A_{CSC}A_{SSS} \end{pmatrix}\left(-\frac{1}{2}CS - \frac{1}{4}rC'S - \frac{1}{4}rCS'\right)
\end{pmatrix} \\
& + \begin{pmatrix}
-A_{CSC}\left(-\frac{1}{4}rCS - \frac{1}{8}C'S + \frac{1}{8}CS' + \frac{1}{4}rC'S'\right) \\
-A_{SSS}\left(-\frac{1}{4}rCS + \frac{1}{8}C'S - \frac{1}{8}CS' + \frac{1}{4}rC'S'\right) \\
-A_{SSC}(-\frac{1}{4}rC^2 + \frac{1}{4}rC'^2) - A_{CSS}(-\frac{1}{4}rS^2 + \frac{1}{4}rS'^2)
\end{pmatrix}\sin\theta \\
& + \begin{pmatrix}
-A_{CCC}\left(-\frac{1}{4}rCS - \frac{1}{8}C'S + \frac{1}{8}CS' + \frac{1}{4}rC'S'\right) \\
-A_{SCS}\left(-\frac{1}{4}rCS + \frac{1}{8}C'S - \frac{1}{8}CS' + \frac{1}{4}rC'S'\right) \\
-A_{SCC}(-\frac{1}{4}rC^2 + \frac{1}{4}rC'^2) - A_{CCS}(-\frac{1}{4}rS^2 + \frac{1}{4}rS'^2)
\end{pmatrix}\cos\theta \\
& + \begin{pmatrix}
\frac{1}{2}\left(A_{CSC}A_{SCC} + A_{CCC}A_{SSC}\right)(-\frac{1}{2}rCC' - \frac{1}{2}C'^2) \\
+\frac{1}{2}\left(A_{CSS}A_{SCS} + A_{CCS}A_{SSS}\right)\left(-\frac{1}{2}rSS' - \frac{1}{2}S'^2\right) \\
+\frac{1}{2}\begin{pmatrix} A_{CSS}A_{SCC} + A_{CSC}A_{SCS} \\ +A_{CCS}A_{SSC} + A_{CCC}A_{SSS} \end{pmatrix}(-\frac{1}{4}rC'S - \frac{1}{4}rC'S - \frac{1}{2}C'S')
\end{pmatrix}\sin 2\theta \\
& + \begin{pmatrix}
\frac{1}{2}\left(A_{CCC}A_{SCC} - A_{CSC}A_{SSC}\right)(-\frac{1}{2}rCC' - \frac{1}{2}C'^2) \\
+\frac{1}{2}\left(A_{CCS}A_{SCS} + A_{CSS}A_{SSS}\right)\left(-\frac{1}{2}rSS' - \frac{1}{2}S'^2\right) \\
+\frac{1}{2}\begin{pmatrix} A_{CCS}A_{SCC} + A_{CCC}A_{SCS} \\ -A_{CSS}A_{SSC} - A_{CSC}A_{SSS} \end{pmatrix}(-\frac{1}{4}rC'S - \frac{1}{4}rC'S - \frac{1}{2}C'S')
\end{pmatrix}\cos 2\theta
\end{aligned} \tag{42}
$$

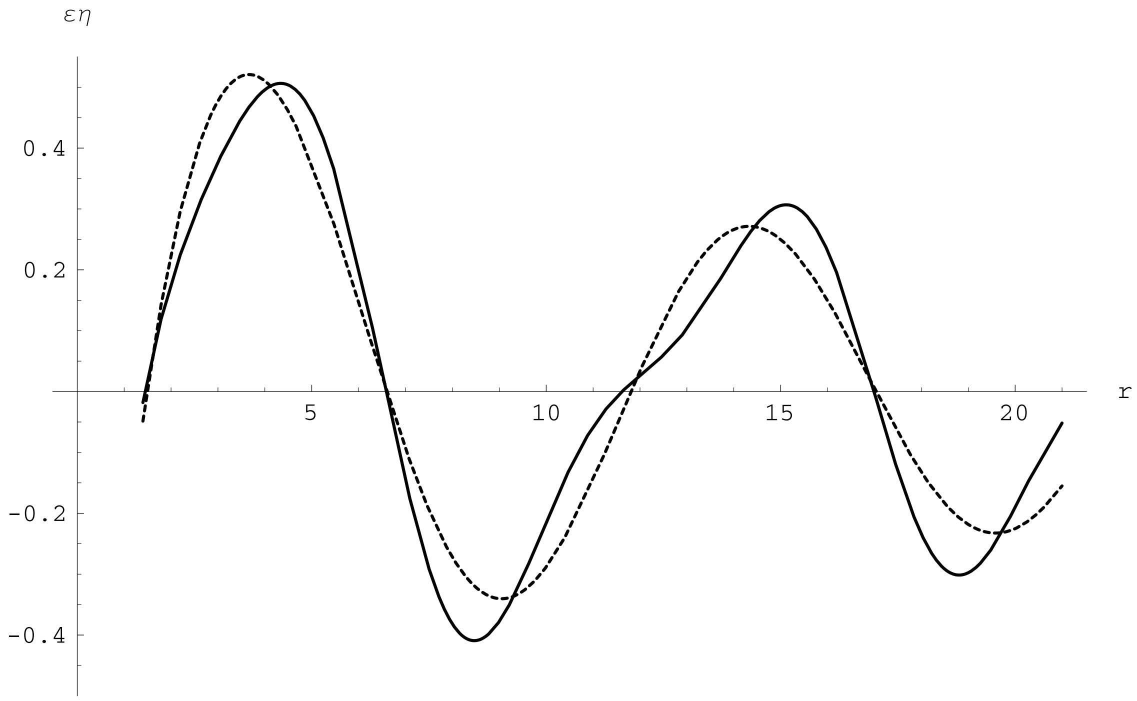

Fig. 1. Axisymmetric progressive waves. Dependence of surface elevation η on radius r. The solid line is (ε)-order solution, and the dashed line is zero-order solution. Face slope is more steep and rear slope is more gentle as compared with zero order Lamb solution. $\varepsilon = 0.1, \qquad t = \pi$.

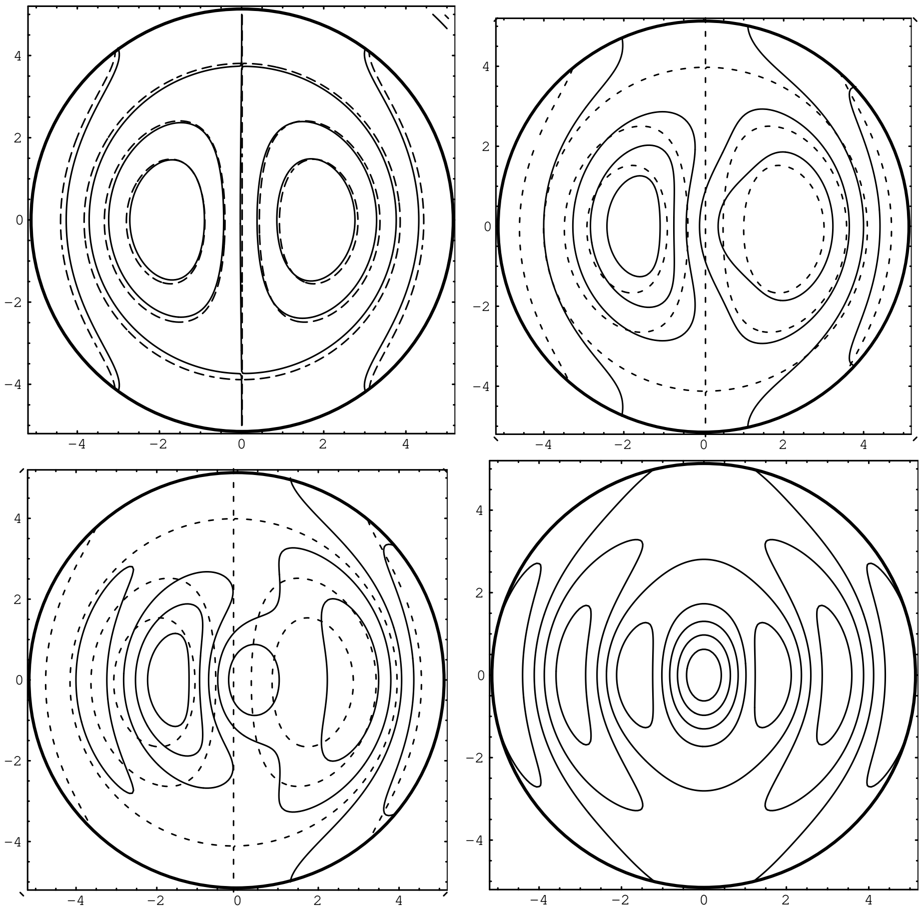

Fig. 2. Contours of unsymmetric waves (case ii) for $t = \frac{3}{4}\pi, \quad \frac{5}{6}\pi, \quad \frac{11}{12}\pi, \quad \pi$. The solid lines are ($\varepsilon$)-order solutions, and the dashed lines are zero-order solutions. $\varepsilon = 0.2$.

6 Examples

We give here explicit corrections for Lamb's linear solutions.

Case i.

Assuming that

$$S_{00}^{1}(r,\theta) = S\,(r)\,, \qquad C_{00}^{1}(r,\theta) = C\,(r) \tag{43}$$

We have

$$\begin{aligned} f(r,t) &= S\sin(t) + C\cos(t) \\ &+ \varepsilon\left(\frac{1}{2}\left(S^2 - C^2\right) + \frac{3}{4}r(SS' - CC')\right)\sin(2t) \\ &+ \varepsilon\left(CS + \frac{3}{4}r(SC' + S'C)\right)\cos(2t)\,. \end{aligned} \tag{44}$$

Substituting these expressions into (18), we could derive the following expressions for $\eta(r,\theta,t)$.

$$\eta(r,\theta,t) = \left(-\frac{1}{4}S'^2 - \frac{1}{4}C'^2\right)\varepsilon + C\sin t - S\cos t \tag{45}$$

$$+ \left(2CS + \frac{3r}{2}\left(CS' + C'S\right)\right)\varepsilon\sin(2t) \tag{46}$$

$$+ \left(\left(C^2 - S^2\right) + \frac{3r}{2}\left(CC' - SS'\right) - \frac{1}{4}\left(C'^2 - S'^2\right)\right)\varepsilon\cos(2t) \tag{47}$$

Setting $a_{12} = -1$, $a_{21} = 1$, $a_{11} = a_{22} = 0$, we have $S(r) = -Y_0\,(r)$ and $C(r) = J_0\,(r)$. Then the main approximation for $\eta(r,\theta,t)$ is

$$J_0\,(r)\sin(t) + Y_0\,(r)\cos(t) \tag{48}$$

which is equivalent to

$$\sqrt{\frac{2}{\pi r}}\left(\cos\left(r - \frac{\pi}{4}\right)\sin(t) + \sin\left(r - \frac{\pi}{4}\right)\cos(t)\right) = \sqrt{\frac{1}{2\pi r}}\sin\left(r - \frac{\pi}{4} + t\right) \tag{49}$$

for $r \to +\infty$. So, we can consider this case as progressive wave, whereas the case $a_{11} = 1$, $a_{22} = a_{12} = a_{21} = 0$ is a standing (symmetrical) wave. New solution (solid line) and zero-order Lamb solution (broken line) are shown in Fig. 1.

Case ii.

Assuming that

$$S_{00}^{1}(r,\theta) = 0, \qquad C_{00}^{1}(r,\theta) = C'\,(r)\cos\theta \tag{50}$$

(We suppose in the numerical calculation that if $C = -\varepsilon J_0$)

We have

$$f(r,t) = C' \cos\theta \cos(t) + \varepsilon \left(\frac{3}{8}C^2 + \frac{3}{8}rCC' - \frac{1}{4}C'^2 + \left(\frac{3}{8}rCC' + \frac{1}{8}C'^2 \right) \cos 2\theta \right) \sin(2t) . \tag{51}$$

Substituting this expression into (18), we could derive the following expressions for $\eta(r,\theta,t)$.

$$\eta(r,\theta,t) = \left(-\frac{1}{4}\cos^2\theta C^2 - \frac{1}{2r}\cos^2\theta CC' - \frac{1}{4r^2}C'^2 \right) \varepsilon + C' \cos\theta \sin t + \left(\begin{array}{c} \left(-\frac{1}{4}\cos^2\theta - \frac{3}{4}\right) C^2 - \left(\frac{1}{2r}\cos^2\theta + \frac{3}{2}r\cos^2\theta\right) CC' \\ + \left(-\frac{1}{4r^2} + \frac{3}{4} - \frac{1}{2}\cos^2\theta\right) C'^2 \end{array} \right) \varepsilon \cos(2t) \tag{52}$$

The presented nonlinear corrections alter dramatically the topology of contours of surface waves. In particular, the contours vary in time in contrast with Lamb solution and the surface is never flat. Examples of the contours of waves are shown in Fig. 2.

7 Conclusions

Some periodic solutions with the accuracy of ε to the classical surface wave equations (8) - (11) are presented. The intermediate equations are given for illustrating the method of derivation but the expressions (35) - (42) can be proved by substitution into system (8) - (11) (using expression (16) for the potential). Linear versions of these problems when the terms of zero orders in ε only are retained were subject of classical investigation in the book of Lamb [4] §§ 191-195.

We conjecture that these expressions are only the lowest terms of a certain expanded exact three-dimensional solution to surface wave equations (8) - (11). The number of known exact solutions to the surface wave equations is rather small and there are no three-dimensional ones.

The first solution can be used for describing the water dynamics generated by periodic pressure source.

The second one presents the simplest unsymmetrical water waves in circular basin.

The presented formulas can be useful, in particular, for describing wave dynamics in the problems with boundary conditions at the circumferences.

The derived formulas are obtained by the method of unknown coefficients as solutions of some **overdetermined** systems algebraic linear equations. The reason for their solvability remains obscure in the moment.

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