RISC-Linz
Research Institute for Symbolic Computation
Johannes Kepler University
A-4040 Linz, Austria, Europe

An Introduction to the Symbolic Integration of Elementary Functions

Andreas Neubacher

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Abstract

This paper is intended as an introduction to the theory of symbolic integration. First a review of integral calculus is given. Then differential fields are presented as an algebraic framework for representing elementary functions and an algorithm for computing indefinite integrals of such functions is discussed in detail. Finally, a partial implementation of the algorithm is described, and in conclusion some remarks on the experiences gained by doing this work are given.

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Introduction

1.1 Contents

This paper is intended as an introduction to the theory of symbolic integration, i.e. the computation of indefinite integrals. It should give the reader a clear understanding of how this problem originally appearing in calculus can be reformulated and solved by algebraic means.

To reach this goal, we begin with a review of integral calculus. Then the theory of differential fields is presented in a (hopefully) natural and intuitive way. After a description of how indefinite integrals fit into this algebraic framework there is a detailed exposition of an algorithm for computing such integrals. A description of a partial implementation of this algorithm and some notes on the insights gained by doing this work conclude the main part of this paper. It is followed by an appendix containing examples, an appendix giving some algorithms required for but not directly related to integration, and the bibliography.

To more clearly state what the reader can expect from this paper, we will point out its restrictions:

- What this paper does not give is a complete introduction to differential algebra. Only the parts which are needed for integration are covered here. The intention was to clearly explain the relation between the definition of integration in calculus and the definition of integration in differential algebra. The reader who would like to study differential algebra in its own right is referred to Kaplansky's book [17], which is intended as an introduction to this area.

- Another point to note is that the algorithm presented in this paper is not the most general one. Here tutorial considerations were given priority over practical applicability. The transcendental case treated here requires only knowledge of elementary algebra up to field extensions. The more general mixed algebraic case is deemed to be beyond an introductory text (see e.g. Bronstein's work in [2]). Nevertheless, a firm intuitive understanding of the material covered here will aid the interested reader in discovering more advanced topics.

- As a consequence of this latter point, the partial implementation based on the algorithm presented in this paper is not usable for practical applications. Rather, it served to point out problems omitted in most of the literature and helped make this paper more oriented towards computational requirements.

Despite all these deficiencies, the author believes that this paper will serve as a helpful introduction to the subject of symbolic integration for readers who intend to do research in this area. Readers whose interest stems from their need to compute indefinite integrals are expected to gain an additional insight not available through methods known from calculus.

Finally, regular users of computer algebra systems might get an understanding of the reasons why their system is unable to solve certain integrals and be able to see whether the integrals are the problem or the system.

1.2 Structure

Chapter 2 ("Integration in Calculus") gives a review of the basics of integral calculus. As the reader is expected to be familiar with this material, it is intended mostly for clarifying terms and to make this paper more self-contained. It is recommended to at least skim over this chapter with emphasis on Sections 2.5 and 2.7.

In Chapter 3 ("An Algebraic Approach") derivatives and integrals are introduced in an algebraic context. Readers who have already encountered the notions of differential fields, elementary functions, etc. are again advised to at least browse through this chapter (especially Sections 3.6, 3.7.1, and 3.8) for getting familiar with the notations used in this paper. It should also be noted that many references to Chapter 2 are given to stress the relation to integral calculus. The author has found that this aspect is seemingly taken for granted and thus omitted in much of the literature on differential algebra.

Chapter 4 ("An Outline of the Algorithm") uses the notions introduced previously for describing the ideas behind an algorithm for computing indefinite integrals. It also gives an overview of its structure and thus is intended to enable the reader to properly place the subalgorithms appearing in later chapters.

Chapters 5 ("How to Integrate Polynomials"), 6 ("How to Solve the Risch Differential Equation"), and 7 ("How to Integrate Proper Fractions") then give a detailed exposition of the various parts of the algorithm, with emphasis both on the mathematical and the computational aspects.

Chapter 8 ("Notes on the Implementation") contains a brief description of a partial implementation of the algorithm done by the author.

Finally, Chapter 9 ("Conclusion") points out some omitted topics and lists some of the enlightening insights the author gained from working on this paper and from implementing the algorithm presented here.

Appendix A ("Examples") contains several examples which should help the reader's understanding of the ideas presented in this paper, and Appendix B ("Some Results from Computer Algebra") gives several subalgorithms which are not directly related to integration and therefore were not included in the main chapters on the algorithm.

1.3 Peculiarities

There are some points concerning the style and the structure of this paper of which the reader should be aware:

- The author had the feeling that having examples in the main chapters would interfere with the natural flow of thought. Therefore almost all examples were put into Appendix A. Each section of this appendix is given the number and heading of the corresponding chapter.

- Another point to note is that this paper is not written in the classic "definition – theorem – proof" structure. As there is a strong emphasis on computational aspects, the author believes that separating the ideas into various theorems would have interfered with presenting the formal parts in an intuitive manner and relating them to the pseudo code descriptions of the algorithm.

  Still it is understood that care was taken to keep everything correct. For a more formal exposition of the material covered here the reader is referred to the research papers listed in the bibliography.

- At the beginning of each chapter there is a section titled "Checkpoint", which gives a review of the previous chapter's contents and an overview of the material covered in the current chapter. Thus it serves as an introduction and should help the reader's orientation.

- Paragraphs marked as

  Notation: ...

  $\square$

  should be read carefully. The notations introduced there will be used without further references later in the paper and may make things incomprehensible if the reader is not aware of them.

Chapter 2

Integration in Calculus

2.1 Checkpoint

Although the reader is assumed to have a basic knowledge of calculus and algebra, we deem it helpful to give a short overview on how integrals and derivatives are defined in calculus.

The reason for this is that being used to the elementary techniques of integration as taught in school the reader may find it hard to become familiar with the algebraic approach. Therefore, we will first show where the problem of integration is rooted, present the methods found in calculus, point out their deficiencies and thus motivate taking an alternative approach, which will be described in the next chapter.

Note that the following exposition is rather brief as it should only refresh the memory of the reader and clarify terms. If not otherwise specified, numbers are elements of the set of real numbers $\mathbf{R}$ and all functions range over $\mathbf{R}$. For further details (and especially proofs) we recommend consulting an introductory book on calculus such as [16] or [20], for example.

2.2 The Definite Integral

The theory of integration originally stems from the *area problem*:

> Given a continuous function $f(x)$ over a finite interval $[a, b]$, compute the area of the region between the graph of f and the x-axis.

The classic way to solve this problem is to divide the interval into smaller sub-intervals and approximate the area between the graph and the x-axis by the sum of the areas of rectangles over these sub-intervals. By making the sub-intervals smaller and smaller, the approximation can be made arbitrarily good.

This idea is formalized by the following definition:

Definition 2.1 (Definite Integral) *Given a closed interval $[a, b]$ and a function f which is continuous over this interval, let P be a* partition *of $[a, b]$, i.e. $P = \{x_0, x_1, \ldots, x_n\}$ with $x_i \in [a, b]$, $x_0 = a$, and $x_n = b$. Let $x_i^* \in [x_{i-1}, x_i]$ arbitrary but fixed for $i = 1, \ldots, n$. Furthermore, define $||P|| := \max\{x_i - x_{i-1} \mid i = 1, \ldots, n\}$.*

Then the definite integral of f from a to b *is defined as follows:*

$$\int_a^b f(x)\,dx := \lim_{||P|| \to 0} f(x_1^*)(x_1 - x_0) + \cdots + f(x_n^*)(x_n - x_{n-1}).$$

The sum inside the limit is called a Riemann sum[1]*.*

[1] An equivalent definition using the *Darboux method* of upper and lower sums is given in [20].

Note that the terms of the sum are the areas of rectangles of heights $f(x_i^*)$ over the intervals $[x_{i-1}, x_i]$ when $f(x_i^*)$ is positive. So this definition solves the area problem for positive f and generalizes it to arbitrary continuous functions.

Still, evaluating the limit appearing in the definition is not practical in most cases so that other ways of computing definite integrals were devised.

2.3 The Derivative

Before we can do some more work on integrals, we first have to consider the notion of derivatives, which is closely related. Derivatives are the solution to another classic problem in calculus, the *tangent problem*:

> Given a real number a and a function $f(x)$, compute the slope of the tangent to the graph of $f(x)$ at the point $(a, f(a))$, if a unique tangent exists in this point.

Here the classic solution is to approximate the tangent by the secant through $(a, f(a))$ and a point $(a+h, f(a+h))$. As h goes towards zero, the slope of the secant approaches the slope of the tangent.

Again this idea is formalized in a definition.

Definition 2.2 (Derivative) *Given a function $f(x)$, the* derivative $f'(x)$ of f *is defined as follows:*

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

The domain of $f'(x)$ is the set of all x for which $f(x)$ is defined and the limit exists.

Sometimes the symbols $\frac{df}{dx}$ or $\dot{f}$ are used instead of f'. The computation of derivatives is called differentiation*. A function for which a derivative exists is called* differentiable.

Note that $f'(a)$ is a solution to the tangent problem, i.e. $f'(a)$ is the slope of the tangent to the graph of $f(x)$ at the point $(a, f(a))$.

Using the above definition, several properties of derivatives can be proved.

Lemma 2.1 *Let c be a constant, f and g differentiable functions. Then the following holds:*

1. *If $f \equiv c$, then $f' \equiv 0$, and if $f \equiv \mathrm{id}$, then $f' \equiv 1$.*
2. $(cf)' = cf'$. *(Linearity I)*
3. $(f+g)' = f' + g'$. *(Linearity II)*
4. $(fg)' = f'g + fg'$. *(Product Rule)*
5. $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$. *(Quotient Rule)*
6. $(f \circ g)' = g'(f' \circ g)$. *(Chain Rule)*
7. $(f^r)' = rf'f^{r-1}$ *for* $r \in \mathbf{Q}$.

The connection between derivatives and definite integrals which is not really obvious at first sight will be given in the following section.

2.4 The Indefinite Integral

Let us begin by considering the inverse operation to differentiation.

Definition 2.3 (Indefinite Integral) *Let f be defined on the interval $[a, b]$. If there exists a function F such that F is continuous on $[a, b]$ and*

$$F'(x) = f(x) \text{ for } x \in [a, b],$$

then F is called an indefinite integral of f on $[a, b]$ *and we write*

$$F(x) = \int f(x)\, dx.$$

Sometimes the terms antiderivative *or* primitive *are used instead of* indefinite integral. *The computation of integrals is called* integration. *A function for which an integral exists is called* integrable.

Note that indefinite integrals are not unique: When $F = \int f(x)\, dx$ and c is a constant, then we also have $F + c = \int f(x)\, dx$, because $(F + c)' = F' + c' = F' = f$. When we talk about $\int f\, dx$, we always take it as an arbitrary but fixed representative of the set of functions F with $F' = f$, or in other words $\{F \mid F' = f\} = \{\int f\, dx + c \mid c \in \mathbf{R}\}$.

The following theorem gives the relation between definite and indefinite integrals.

Theorem 2.1 (Fundamental Theorem of Integral Calculus) *Let f be continuous on $[a, b]$. If F is an indefinite integral of f on $[a, b]$, then*

$$\int_a^b f(x)\, dx = F(b) - F(a).$$

There we have it: If we can compute indefinite integrals, then computing definite integrals becomes trivial.

Before checking if finding an antiderivative is really easier than evaluating the limit from Definition 2.1 let us discuss the kinds of functions we want to integrate.

2.5 Elementary Functions

Up to now, we only talked about rational functions and fractional powers. But in many practical applications also transcendental functions such as *log, exp, sin, cos*, etc. appear. We will now briefly discuss these functions.

There are many equivalent definitions of *log* and *exp*. One way is to set

$$\begin{aligned} \log(x) &:= \sum_{i=0}^{\infty} (-1)^i \frac{(x-1)^{i+1}}{i+1}, \text{ and} \\ \exp(x) &:= \sum_{i=0}^{\infty} \frac{x^i}{i!}. \end{aligned}$$

This is the definition of the natural logarithm and the exponential function by their power series. Grossman in [16] defines the natural logarithm as the inverse function to exponentiation and uses the symbol *ln*, while Salas in [20] defines logarithms to be functions f with $f(xy) = f(x) + f(y)$.

Suffice it to say that, independent of the definition, the following properties should hold:

- $\log'(x) = \frac{1}{x}$ and $\int \log(x)\, dx = x(\log(x) - 1)$.
- $\exp'(x) = \exp(x)$ and $\int \exp(x)\, dx = \exp(x)$.

- $\log(fg) = \log(f) + \log(g)$ and $\log(f^g) = g\log(f)$.
- $\exp(f+g) = \exp(f)\exp(g)$ and $\exp(fg) = (\exp(f))^g$.
- $\log(\exp(x)) = \exp(\log(x)) = x$.

We will not elaborate on the trigonometric and hyperbolic functions, as these can be defined by the exponential functions, e.g.

$$\begin{aligned} \sin(x) &:= \frac{\exp(ix) - \exp(-ix)}{2i} \quad \text{and} \\ \cos(x) &:= \frac{\exp(ix) + \exp(-ix)}{2}, \end{aligned}$$

where $i := \sqrt{-1}$. By using this trick of switching to complex numbers, one can make do with logarithmic and exponential functions[2].

Taking everything together, we now have some idea of the kind of functions we will call *elementary*: Functions built by the application of addition, multiplication, (fractional) powers, and the logarithmic and exponential (and thus trigonometric and hyperbolic) functions.

2.6 Techniques of Integration

Now we will try to get an idea of the many methods used for computing indefinite integrals of the functions described in the previous section. Again, we refer the reader to the literature for details and a more exhaustive coverage of the problem. First, let us list some simple properties, which are analogous to the ones given in Lemma 2.1.

Lemma 2.2 *Let c be a constant, f and g integrable functions. Then the following holds:*

1. *If $f \equiv 1$, $\int f\,dx = x$.*
2. *If c is a constant, $\int cf\,dx = c\int f\,dx$.* *(Linearity I)*
3. *$\int f + g\,dx = \int f\,dx + \int g\,dx$.* *(Linearity II)*
4. *$\int fg'\,dx = fg - \int f'g\,dx$.* *(Integration by Parts)*
5. *$\int x^r\,dx = \frac{1}{r+1}x^{r+1}$ for $r \in \mathbf{Q}/\{-1\}$.*
6. *There is no useful analogon to the quotient rule.*
7. *$\int (f\circ g)g'\,dx = F\circ g$, where $F = \int f\,dx$.* *(Integration by Substitution)*

Among the numerous heuristics based on these properties are integration by partial fractions, tricks for finding "good" factors f and g for the integration by parts, and tricks for finding a "good" function g (depending on the structure of f) for the integration by substitution.

The goal is to have each application of these techniques yield ever simpler integrals until finally one can find the solutions by looking them up in a table of integrals.

[2] It should be noted that in most books on calculus trigonometric and hyperbolic functions are defined independently of *exp*, and in most tables of integrals they are covered separately.

2.7 The Problem with Heuristics

Now that we know several simplification techniques, computing indefinite integrals still seems to be more of an art than of a craft. Or, as Grossman writes in Chapter 8 of his book [16] on calculus:

> *The aim of this chapter is to learn to recognize which functions can be integrated in the sense of actually finding a recognizable indefinite integral. There are many techniques which have evolved over a period of decades [...]. More than any other chapter in the book, this one consists primarily of a collection of tricks. [...]. There is no short, easy way to learn these techniques and it will be necessary to do great numbers of problems.*
>
> *After having completed [...] this chapter, it will be apparent that there are many integrals for which none of our techniques will yield an answer.*

This quotation points out two major problems:

1. Not all indefinite integrals of elementary functions can be expressed as elementary functions[3].

2. There is no algorithm telling us when to apply which technique that is guaranteed to compute a solution (when it exists) in a finite number of steps. This means that one has to rely mostly on experience and intuition.

Taking 1 and 2 together we see that when we do not succeed in computing an indefinite integral, we never know whether we are just too stupid for finding a solution or whether such a solution does not exist at all.

It remains an open question whether devising an algorithm based entirely on the direct application of the classic heuristics of integral calculus is possible. What would have to be done is finding a set of detailed simplification rules for a given set of functions such that the above mentioned problems do not appear.

The remainder of this paper is devoted to showing that there are ways of solving the problem of indefinite integration algorithmically (at least for a restricted set of functions). The approach taken sacrifices the techniques well suited to human intuition for the guarantee that a solution is found or the non-existence of a solution is proven in a finite number of steps.

[3] As an example assume that we only know rational functions, but have never heard of logarithms. Then we are unable to compute $\int \frac{1}{x}\,dx$, because the solution of this integral is $\log(x)$, which cannot be expressed as a finite rational function.

Chapter 3

An Algebraic Approach

3.1 Checkpoint

In the previous chapter we have seen where the problem of indefinite integration comes from. We have also hinted that the methods of calculus have not yet been developed far enough for yielding an algorithm.

In this chapter we will describe an approach which does lead to an algorithm for computing indefinite integrals, namely the theory of *differential algebra*.

While this area is seemingly self-contained and independent of calculus, the ideas presented here are originally an abstraction of the techniques covered in the previous chapter. Unfortunately, the literature on differential algebra either ignores this fact or assumes that the reader is already aware of it. This sometimes makes applying the intuition gained during the study of calculus hard for people new to the area. We will try to amend this situation.

3.2 From Calculus to Algebra

In calculus the definitions of the definite integral and of the derivative are based upon limits, which are not easy to handle computationally. Nevertheless, the properties of derivatives listed in Lemma 2.1 do not refer to limits at all. Furthermore, the definition of the indefinite integral and the properties listed in Lemma 2.2 do not make any direct use of limits, either.

When computing indefinite integrals, one normally applies their properties without going back to the original definitions. This process of abstraction can be taken even further: By defining derivatives and indefinite integrals only by their "essential properties" we should get a clearer picture of how integration can be handled.

This method of getting rid of superfluous ballast is fundamental to mathematics: The abstraction from computing with numbers, matrices, polynomials, etc. to computing in domains such as rings or fields is a well known example of its application. The author assumes that the reader is aware of the relation between *polynomial functions* over the reals, which are taken as continuous functions from $\mathbf{R}$ to $\mathbf{R}$, and *polynomials*, i.e. elements of the abstract algebraic domain *polynomial ring*. In the remainder of this paper, we will mostly deal with elements of the polynomial ring over the rationals, $\mathbf{Q}[x]$, and of its quotient field $\mathbf{Q}(x)$. Readers who do not feel familiar with these concepts are referred to introductory books on algebra such as [19].

3.3 Differential Fields

The following definition contains the "essential properties" of differentiation.

Definition 3.1 (Derivation) *Let D be a ring (field).*

- *The operator* $(.)' : D \to D$ *is called a* derivation *iff*
  - $\forall u, v \in D : (u+v)' = u' + v'$, *and*
  - $\forall u, v \in D : (uv)' = uv' + u'v$.
- D *together with* $(.)'$ *is called a* differential ring (field).
- $C_D := \{c \in D | c' = 0\}$ *is the* ring (field) of constants of D.

Notation: To avoid confusion between Definition 2.2 and this definition, we will always use the term *derivative* with the symbol $\frac{d.}{dx}$ for the former and the term *derivation* with the symbol $(.)'$ for the latter.

□

In Kaplansky's introductory book on differential algebra [17] the above definition is taken as a starting point which is not discussed any further. As we want to use differential algebra as a tool for solving indefinite integrals, we have to show that derivation operators are in some sense related to derivatives as defined in the previous chapter.

The properties (Linearity II) and (Product Rule) of the derivative as given in Lemma 2.1 are exactly what is required for $\frac{d.}{dx}$ being a derivation, therefore it is clear that everything we prove for derivations also holds for derivatives. We still need to check whether Definition 3.1 is strong enough so that everything we need for integration can be deduced from it.

Before we do this, we will give some properties of the derivation analogous to those listed in Lemma 2.1.

Notation: From now on, let D be a differential field of characteristic 0 with derivation $(.)'$ as defined above[1].

□

Lemma 3.1 *Let f and g be non-zero elements of D and let c be an element of C_D. Then the following holds:*

(i) $1' = 0$, *i.e.* 1 *is always a constant.*

(ii) $(cf)' = cf'$. *(Linearity I)*

(iii) $(f+g)' = f' + g'$. *(Linearity II)*

(iv) $(fg)' = f'g + fg'$. *(Product Rule)*

(v) $(\frac{f}{g})' = \frac{f'g - fg'}{g^2}$. *(Quotient Rule)*

(vi) There is no analogon to the chain rule, as we did not define the operator "$\circ$" for elements of D.

(vii) $(f^n)' = nf'f^{n-1}$ *for all integers n.*

Proof:

(i) $1' = (1 \cdot 1)' = 1' + 1'$, therefore $1' = 0$.

(ii) $(cf)' = c'f + cf' = cf'$ by definition of C_D.

(iii) By definition.

[1] Actually we are interested in the field of elementary functions over the real or complex numbers, but we will stay a bit more general for the time being. We could stay even more general by also considering differential rings, but as we will not have much use for these, we will omit them.

(iv) By definition.

(v) $0 = 1' = (g \cdot \frac{1}{g})' = g(\frac{1}{g})' + g'\frac{1}{g}$, therefore $(\frac{1}{g})' = \frac{-g'}{g^2}$ and $(f \cdot \frac{1}{g})' = \frac{f'g-fg'}{g^2}$.

(vii) Use induction for positive n, then apply (v) for negative n. Note that this is not as general as the corresponding property of Lemma 2.1 because we have no way of defining fractional powers of elements of D, yet.

□

From this lemma we see that computing with derivatives seems to be quite similar to computing with derivations. We will now formalize this relation for rational functions, but before giving the theorem we need to clarify a bit of notation:

In calculus, the x appearing in the notation $\frac{df}{dx}$ is a symbol without any meaning. (We mentioned in Definition 2.2 that e.g. the notation $\dot{f}$ is sometimes used instead.) When x is used in calculus to denote a function, it is — implicitly — defined to be the identity function. In the context of differential fields we will use x in a different sense.

Notation: Given a differential field D, x denotes an arbitrary but fixed element of D with x transcendental over C_D and $x' = 1$, if such elements exist.

□

From Lemma 3.1 we see that $x' = 1$ implies $(x + c)' = 1$ for all constants $c \in D$. This means that we can regard x as a representative of the set of elements of D whose derivation is 1.

The following theorem basically says that the homomorphism from rational functions over $\mathbf{Q}$ to $\mathbf{Q}(x)$ extends to the derivative/derivation. Therefore it does not matter whether we work in an algebraic or an analytic context.

We have to be content with having this result for rational functions only until we are ready to introduce *log*, *exp*, and fractional powers based on Definition 3.1.

Theorem 3.1 *Let $D = \mathbf{Q}(x)$ be a differential field with derivation $(.)'$ such that $C_D = \mathbf{Q}$ and x is as defined above. Furthermore, let $\frac{d}{dx}$ be the derivative according to Definition 2.2, and let H be the homomorphism from rational functions over $\mathbf{Q}$ to $\mathbf{Q}(x)$. Then we have*

$$\begin{aligned} H(\frac{df}{dx}) &= H(f)' \quad \textit{for all rational functions } f, \textit{ and} \\ \frac{d}{dx}H^{-1}(\bar{f}) &= H^{-1}(\bar{f}') \quad \textit{for all } \bar{f} \in D. \end{aligned}$$

Proof: For proving the first equality we will use induction over the structure of rational functions. These are built from constants and x (interpreted as a function, namely the identity) by addition, multiplication, and taking quotients. For the induction base we have

- $H(\frac{dc}{dx}) = 0 = H(c)'$ for $c \in \mathbf{Q}$, and
- $H(\frac{dx}{dx}) = 1 = H(x)'$.

For the induction step note that under the induction hypothesis of $H(\frac{dg}{dx}) = H(g)'$ and $H(\frac{dh}{dx}) = H(h)'$ for rational functions g, h over $\mathbf{Q}$ we have

- $H(\frac{d}{dx}(g + h)) = H(\frac{dg}{dx}) + H(\frac{dh}{dx}) = H(g)' + H(h)' = H(g + h)'$,
- $H(\frac{d}{dx}(gh)) = H(\frac{dg}{dx}h + g\frac{dh}{dx}) = H(g)'H(h) + H(g)H(h)' = H(gh)'$, and
- $H(\frac{d}{dx}(\frac{g}{h})) = H(\frac{1}{h^2}(\frac{dg}{dx}h - g\frac{dh}{dx})) = \frac{1}{H(h)^2}(H(g)'h - gH(h)') = H(\frac{g}{h})'$.

From this follows that $H(\frac{df}{dx}) = H(f)'$ for all rational functions f over $\mathbf{Q}$.

The second equality can be proven analogously.

□

From now on we will be bit sloppy and omit the operator H when there should be no danger of confusion.

3.4 Differential Field Extensions

In the previous section we have seen that differential fields can be related to rational functions in a straightforward way. Now we will go on and present methods for introducing algebraic and transcendental functions.

The idea is as follows: As we only have the algebraic tools of fields, extension fields, etc. at our disposal, we cannot directly use fractional powers as we did in calculus. We also want to make do without limits, so we cannot define log and exp by their power series as we did in Chapter 2. What we can and will do is to add a new element to a given differential field and define it such that it shows the properties which are required of a certain algebraic or transcendental function.

We will start by adapting the notion of field extensions to differential fields.

Definition 3.2 (Differential Extension Field) *Let D_1 and D_2 be differential fields with derivations $(.)'_1$ and $(.)'_2$. D_2 is called a* differential extension field *of D_1 (Symbol: $D_1 \subseteq D_2$) iff*

- *D_2 is an extension field of D_1, and*
- *$\forall u \in D_1 : (u)'_1 = (u)'_2$.*

As a sideremark let us mention that according to [10] (which in turn refers to [18] for a proof) for every differential field D there exists a *universal extension field U* with the property that every finitely generated extension field of D can be embedded in U, the embedding leaving D fixed. We will need this result for Theorem 3.2.

Notation: From now on, let θ be the element of a differential extension field of D. $D(\theta)$ denotes the smallest differential extension field of D containing θ.

□

Now we will see how we can employ this mechanism for introducing analogons to the elementary functions as defined in Section 2.5.

3.5 Log and Exp

We will deal with algebraic functions and especially fractional powers later on and first see what we can do about *log* and *exp*.

Our goal still is indefinite integration, which is defined to be the inverse operation to differentiation as we have seen in Chapter 2. Therefore it seems reasonable that for our purposes it should suffice to simply specify how the algebraic counterparts to transcendental functions are treated by the derivation operator. When this is the same as computing the derivatives of these functions we can transfer results depending only on these differential properties safely from calculus to algebra and back.

The following definition introduces *log* and *exp* in this way.

Definition 3.3 (Log and Exp) *θ is called ...*

- logarithmic over D, *iff $\exists u \in D : \theta' = \frac{u'}{u}$. In this case we will write $\theta = \log(u)$.*

- exponential over D, *iff* $\exists u \in D : \theta' = u'\theta$. *In this case we will write* $\theta = \exp(u)$.

Note that the expression $\theta = \log(u)$ may be a bit misleading. For example, when we know that $\theta' = \frac{1}{x}$ then the choice of $u = x$ (i.e. $\theta = \log(x)$) is the most obvious one. Actually we may choose $u = cx$ (i.e. $\theta = \log(cx)$) for any constant c. The same goes for $\exp(u)$ with $x + c$ instead of cx.

The important thing here is that the uniqueness of u is not guaranteed. This is caused by the fact that when working in an algebraic context, we do not consider "real" functions but only elements of fields with certain differential properties.

Fortunately, for indefinite integration these ambiguities are not much of a problem. If we have given a function f containing e.g. a term $\exp(u)$ as defined in calculus we know that in order to be able to represent f algebraically we have to take a differential field containing u and extend it by θ_1 with the property $\theta_1' = u'\theta_1$. If f also contains $\exp(u + c)$ with c being a constant, then we can represent this by introducing a θ_2 with $\theta_2' = u'\theta_2$ and noting that $\frac{\theta_1}{\theta_2}$ is a constant representing $\exp(c)$. All this will be explained in more detail later in this chapter.

As with the derivation operator, we have to make sure that computationally the algebraic versions of log and exp follow the same rules as their counterparts from calculus.

Lemma 3.2 *Let* $u, v, \log(u), \exp(u)$, *etc. be elements of a differential field* D, $c \in C_D$, n *an integer. Then we have*

(i) $\log(uv) = \log(u) + \log(v) + c$.

(ii) $\log(u^n) = n\log(u) + c$.

(iii) $\exp(u + v) = \exp(u)\exp(v)$.

(iv) $\exp(nu) = \exp(u)^n$.

(v) $\exp(\log(u)) = u$.

(vi) $\log(\exp(u)) = u$.

Note that in most cases we will take $c = 0$ when applying this lemma.

Proof: As already mentioned, log and exp are only defined by the structure of their derivatives, which we will use to prove the equalities given above.

(i) Let $\theta = \log(u) + \log(v) + c$. Then $\theta' = \frac{u'}{u} + \frac{v'}{v} = \frac{u'v+uv'}{uv} = \frac{(uv)'}{uv}$, and therefore $\theta = \log(uv)$.

Now let $\theta = \log(uv)$. Then $(\theta - \log(u) - \log(v))' = 0$, and therefore $\theta - \log(u) - \log(v) = c$ for some $c \in C_D$.

(ii) For positive n proved by induction over n, using (i). Analogous proof for negative n.

(iii) Let $\theta = \exp(u)\exp(v)$. Then $\theta' = u'\exp(u)\exp(v) + v'\exp(u)\exp(v) = (u + v)'\theta$, and therefore $\theta = \exp(u + v)$.

(iv) For positive n proved by induction over n, using (iii). Analogous proof for negative n.

(v) For arbitrary but fixed $u \in D$ we have $u' = u'\frac{u}{u} = \frac{u'}{u}u = \log(u)'u$, and therefore $u = \exp(\log(u))$.

(vi) For arbitrary but fixed $u \in D$ we have $u' = u'\frac{\exp(u)}{\exp(u)} = \frac{u'\exp(u)}{\exp(u)} = \frac{\exp(u)'}{\exp(u)}$, and therefore $u = \log(\exp(u))$.

□

Now we are ready to talk about algebraic functions. Here some distinctions have to be made:

- Algebraic functions in their most general form (i.e. functions f implicitly defined by $g_n f^n + \ldots + g_1 f + g_0 = 0$ where the g_i are elementary functions) normally do not appear in integration problems from calculus. Therefore we will not consider them, although they would fit into the framework presented here: Given $u_i \in D$ we could extend to $D(\theta)$ with θ such that $u_n\theta^n + \ldots + u_1\theta + u_0 = 0$.
- Fractional powers $f = g^{\frac{m}{n}}$ (i.e. functions f such that $f^n = g^m$) with $m, n \in \mathbf{Z}$, which certainly are interesting for calculus, can be introduced by using *log* and *exp*: If $\theta = \exp(\frac{m}{n}\log(u))$ then according to Lemma 3.2 we have $\theta^n = u^m$, i.e. $\theta = u^{\frac{m}{n}}$.

Actually, these remarks were only added for completeness, because in the next section we will exclude algebraic functions from the domain of the algorithm presented in this paper.

For concluding this section on logarithms and exponentials we will extend Theorem 3.1 to elementary functions as defined in Section 2.5.

First note that the homomorphism H from rational functions over $\mathbf{Q}$ to $\mathbf{Q}(x)$ canonically extends to a homomorphism from elementary functions to the subfield of a universal extension field of $\mathbf{Q}(x)$ which contains only exponential and logarithmic extensions by

- $H(\log(f)) := \theta$ with $\theta' := \frac{H(f)'}{H(f)}$, and
- $H(\exp(f)) := \theta$ with $\theta' := H(f)'\theta$.

We need to restrict the universal extension field because otherwise H would not be a bijection.

Theorem 3.2 *Let $\bar{U}$ be a subfield of a universal extension field of $\mathbf{Q}(x)$ containing only logarithmic and exponential extensions with derivation $(.)'$. Let $\frac{d}{dx}$ be the derivative according to Definition 2.2. Then we have*

$$\begin{aligned} H(\frac{df}{dx}) &= H(f)' \quad \textit{for all elementary functions } f, \textit{ and} \\ \frac{d}{dx}H^{-1}(\bar{f}) &= H^{-1}(\bar{f}') \quad \textit{for all } \bar{f} \in U. \end{aligned}$$

Proof: Since H treats logarithmic and exponential terms in a way conforming to Definition 3.3 we can proceed by induction over the structure of elementary functions similar to Theorem 3.1, taking the rational functions for the induction base.

□

Again we will simply write f instead of $H(f)$ when there is no danger of confusion. Furthermore, from now on we will sometimes call elements of differential fields "functions", although this term does not have any special meaning in such a context.

To make Theorems 3.1 and 3.2 a bit simpler we only considered the case where the constant field was $\mathbf{Q}$. This may often suffice, but the proofs do not depend on the structure of the constant field so that everything also holds when the constant field is an extension field of $\mathbf{Q}$ (e.g. $\mathbf{R}$ or a field $\mathbf{Q}(\alpha_1, \ldots, \alpha_n)$ with α_i algebraic or transcendental over $\mathbf{Q}$).

3.6 Transcendental Elementary Differential Fields

The problem of integration (as defined in Section 3.8) was completely solved[2] for the first time by R. H. Risch in 1969 (see [7]). Only recently has research for an integration algorithm

[2] For elements of differential fields with only transcendental extensions.

for mixed transcendental and algebraic fields shown success (see [2, 4, 13]). As already mentioned in the introduction, these results are deemed to be beyond the scope of this paper. Therefore we will present an algorithm only for the class of functions Risch used. It is interesting enough to be dealt with while the algorithm given here (which is different from Risch's) should be simple enough to serve as an introductory example.

Let us begin by considering the building blocks of these functions. First we will define the notion of elementary functions, which corresponds to the one given in Section 2.5.

Definition 3.4 (Elementary Function Field) *Let $C(x)$ be a differential field, where C is a field of constants explicitly finitely generated over $\mathbf{Q}$, and x is transcendental over C with $x' = 1$.*

Let $D_0 := C(x)$, $\theta_0 := x$.

For $m \in \mathbf{N}$ and $1 \leq i \leq m$, let $D_i := D_{i-1}(\theta_i)$ be a differential extension field of D_{i-1}, where θ_i is algebraic, logarithmic, or exponential over D_{i-1}.

We will then call D_m an elementary function field *and its elements* elementary functions*.*

Note that once again we do not work with the full generality of differential algebra: Requiring $\theta_0' = 1$ is not necessary for the general notion of elementary differential fields, but it is needed for mapping elementary functions to differential fields, as shown in Theorems 3.1 and 3.2.

For the integration algorithm presented here to work, we need to be even more restrictive. As we have seen in the previous sections, we can make do with rational functions together with *log* and *exp*. Algebraic extensions would cause troubles, so whenever we extend a differential field D by some θ we will require θ to be transcendental over D. In addition to that, we need to have control over the constant field, i.e. it must not happen that extending by θ implicitly introduces some new constants. Example A.3.5 shows the necessity for these requirements which are covered by the next definition.

Definition 3.5 (Transcendental Elementary Monomial) *θ is called a* transcendental elementary monomial *over D iff*

- *$\theta' = 1$ or θ is logarithmic or θ is exponential over D, and*
- *θ is transcendental over D, and*
- *$C_D = C_{D(\theta)}$.*

In the literature a host of similar but not completely identical definitions of building blocks of elementary functions can be found. They are called *monomial* in [5, 7], *regular monomial* in [9, 10], *elementary monomial* in [2], and *elementary generator* in [15]. For the special, restricted application given in this paper we found it necessary to introduce yet another definition.

Based on this we can now describe the special type of differential field we will be dealing with in the remainder of this paper.

Definition 3.6 (Transcendental Elementary Differential Field) *Let $C(x)$ be a differential field, where C is a field of constants explicitly finitely generated over $\mathbf{Q}$, and x is transcendental over C with $x' = 1$.*

Let $D_0 := C(x)$, $\theta_0 := x$.

For $m \in \mathbf{N}$ and $1 \leq i \leq m$, let $D_i := D_{i-1}(\theta_i)$ be a differential extension field of D_{i-1}, where θ_i is a transcendental elementary monomial over D_{i-1} with $\theta_i' \neq 1$.

We will then call D_m a transcendental elementary differential field *and its elements* transcendental elementary functions*. Depending on the situation, x will be called* variable of differentiation *or* variable of integration*.*

From now on we will only consider transcendental elementary differential fields, if not mentioned otherwise. Although this reduces the set of functions we can handle, we gain in that the algorithms for the transcendental case are *much* simpler.

Before defining integrals in the context of transcendental elementary differential fields, we will give some results on elements of such fields which we will need later in this paper.

3.7 Additional Theory

3.7.1 Notation

Notation: From now on, let

- D_m and D be transcendental elementary differential fields,
- $\bar{D}_m$ and $\bar{D}$ be transcendental elementary differential fields built up from the same monomials as D_m and D, respectively, but whose fields of constants $C_{\bar{D}_m}$ and $C_{\bar{D}}$ are the algebraic closures of C_{D_m} and C_D,
- θ be a transcendental elementary monomial over D,
- $D[\theta]$ denote the ring of polynomials in θ with coefficients in D,
- $D(\theta)$ denote the quotient field of $D[\theta]$,
- $D\langle\theta\rangle$ denote $D[\theta, \theta^{-1}]$ when θ is exponential, and $D[\theta]$ otherwise[3]. We will call elements of this domain *generalized polynomials.*

□

Throughout this paper we will interpret an element $f \in D_m$ as a univariate rational function, i.e. as an element of $D(\theta)$ where $D = D_{m-1}$ and $\theta = \theta_m$. We will denote parts of f as follows[4]:

Notation: Let $f = \frac{f_{\bar{N}}}{f_{\bar{D}}} = f_P + f_F$, where

- $f_{\bar{N}} \in D[\theta]$, the *numerator* of f.
- $f_{\bar{D}} \in D[\theta]$, the *denominator* of f with $\gcd(f_{\bar{N}}, f_{\bar{D}}) = 1$ and $\mathrm{ldcf}(f_{\bar{D}}) = 1$.
- $f_P = f_{P+} + f_0 + f_{P-} \in D\langle\theta\rangle$, the *polynomial part* of f.
    - $f_{P+} = \sum_{i=1}^{\delta_f} f_i\theta^i$, the *positive polynomial part* of f, where $0 \leq \delta_f$, $f_i \in D$, and $f_{\delta_f} \neq 0$ when $0 < \delta_f$.
    - $f_{P-} = \sum_{i=\gamma_f}^{-1} f_i\theta^i$, the *negative polynomial part* of f, where $\gamma_f \leq 0$, $f_i \in D$, $f_{\gamma_f} \neq 0$ when $\gamma_f < 0$, and $\gamma_f = 0$ always holds when θ is not exponential.
    - $f_0 \in D$. We will refer to this as the *θ-free part* of f. This is not the same as the *constant term* of f (an element of C_{D_m}), which denotes the term with degree 0 when f is written as a sum of power products of $\theta_0, \ldots, \theta_m$.

      The θ-free part and the constant term are the same only when $\theta = \theta_0 = x$, because then $D = D_0 = C_D$.
- $f_F = \frac{f_N}{f_D} \in D(\theta)$, the *fractional part* of f.
    - $f_N \in D[\theta]$, the *numerator of the fractional part* of f.

[3] This latter notation is according to Bronstein in [1].

[4] This notation closely follows the one used by Davenport in [5].

- $f_D \in D[\theta]$, the *denominator of the fractional part* of f with $\deg(f_N) < \deg(f_D)$, $\gcd(f_N, f_D) = 1$, and $\mathrm{ldcf}(f_D) = 1$. When θ is exponential, θ must not divide f_D.

Sometimes (esp. in Section 6.5) we will also use the expressions f_L ("leading quotient") and f_T ("trailing quotient"), which we define as follows: Let $f_F = \frac{p_r\theta^r+\ldots+p_0}{q_s\theta^s+\ldots+q_0}$, where $r = \deg(f_N)$ and $s = \deg(f_D)$. Then

- $f_L := \frac{p_r}{q_s}$ if $r = s-1$, $f_L := 0$ otherwise.
- $f_T := \frac{p_0}{q_0}$ if $q_0 \neq 0$, $f_T := 0$ otherwise.

When $f = f_F$ (i.e. when $f_P = 0$), f is called a *proper fraction*. If f is a proper fraction and f_D is squarefree, f is called a *normal fraction*.

□

Note that δ_f and γ_f are interpreted as operators giving the most positive and most negative powers of θ in f *when f is known*. In the case where f is *not* known, we will interpret δ_f and γ_f as unknowns for which the conditions given above must hold. This allows us to compute them independently of f.

3.7.2 Lemmas

The following lemma gives the relation between the degree of a polynomial and that of its derivation.

Lemma 3.3 *Let $f \in D\langle\theta\rangle$. Then*

- $\delta_{f'} = \delta_f$ *if θ is logarithmic and $f_{\delta_f} \notin C_D$ or if θ is exponential.*
- $\delta_{f'} = \delta_f - 1$ *otherwise.*
- $\gamma_{f'} = \gamma_f$.

Proof: We have
$$f = f_{\delta_f}\theta^{\delta_f} + \ldots + f_{\gamma_f}\theta^{\gamma_f}$$
with $\gamma_f = 0$ if θ is not exponential. From this follows that
$$f' = (f_{\delta_f})'\theta^{\delta_f} + f_{\delta_f}\delta_f\theta'\theta^{\delta_f-1} + \ldots + (f_{\gamma_f})'\theta^{\gamma_f} + f_{\gamma_f}\gamma_f\theta'\theta^{\gamma_f-1}.$$

In the base case $\theta' = 1$, $(f_{\delta_f})' = 0$, and $\gamma_f = 0$. Therefore,
$$f' = f_{\delta_f}\delta_f\theta^{\delta_f-1} + \ldots + f_1,$$
and thus $\delta_{f'} = \delta_f - 1$ and $\gamma_{f'} = \gamma_f$.

If θ is logarithmic we have $\theta' \in D$. Furthermore, θ is transcendental over D so that we get $\deg(\theta'\theta^{\delta_f-1}) = \delta_f - 1$. Therefore, if $f_{\delta_f} \in C_D$, then $(f_{\delta_f})' = 0$ and thus $\delta_{f'} = \delta_f - 1$. Otherwise, $(f_{\delta_f})' \neq 0$ and thus $\delta_{f'} = \delta_f$. By definition we have $\gamma_f = 0$, so that $\gamma_{f'} = \gamma_f$.

If θ is exponential, then $\theta' = u'\theta$ for some $u \in D$ and therefore we have
$$f' = ((f_{\delta_f})' + f_{\delta_f}\delta_f u')\theta^{\delta_f} + \ldots + ((f_{\gamma_f})' + f_{\gamma_f}\gamma_f u')\theta^{\gamma_f},$$
and thus $\delta_{f'} = \delta_f$ and $\gamma_{f'} = \gamma_f$.

□

3.7.3 A Structure Theorem

An important question concerning transcendental elementary monomials and transcendental elementary differential fields still remains open: How can one check whether some θ is a transcendental elementary monomial over a given transcendental elementary differential field D?

The following theorem essentially says that the relations between *log* and *exp* given in Lemma 3.2 are sufficient for testing whether we have a transcendental elementary monomial.

Theorem 3.3 (Structure Theorem) *Let D_m be a transcendental elementary differential field, $L := \{i \in \mathbf{N} \mid 1 \leq i \leq m, \theta_i = \log(u_i), u_i \in D_{i-1}\}$, $E := \{i \in \mathbf{N} \mid 1 \leq i \leq m, \theta_i = \exp(u_i), u_i \in D_{i-1}\}$, $u \in D_m$.*

- $\log(u)$ *is not a transcendental elementary monomial over D_m iff*

$$u = c \prod_{i \in L} u_i^{r_i} \prod_{i \in E} \theta_i^{r_i},$$

where $c \in C_{D_m}$ and $r_i \in \mathbf{Q}$ for $1 \leq i \leq m$.

- $\exp(u)$ *is not a transcendental elementary monomial over D_m iff*

$$u = c + \sum_{i \in L} r_i \theta_i + \sum_{i \in E} r_i u_i,$$

where $c \in C_{D_m}$ and $r_i \in \mathbf{Q}$ for $1 \leq i \leq m$.

For a discussion of this result in a more general context see [11].

Proof: Corollaries 3.2 and 3.3 of [11] show that the corresponding equation holds when $\log(u)$ and $\exp(u)$ are no transcendental elementary monomials over D_m.

Now assume that

$$u = c \prod_{i \in L} u_i^{r_i} \prod_{i \in E} \theta_i^{r_i}.$$

We take the logarithm of both sides and applying the transformation rules listed in Lemma 3.2 yields

$$\log(u) = \log(c) + \sum_{i \in L} r_i \theta_i + \sum_{i \in E} r_i u_i,$$

i.e. either $\log(u)$ is algebraic over D_m (when $\log(c) \in C_{D_m}$) or the constant fields of D_m and $D_m(\log(u))$ differ (when $\log(c) \notin C_{D_m}$).

If we have

$$u = c + \sum_{i \in L} r_i \theta_i + \sum_{i \in E} r_i u_i$$

we can take the exponential of both sides and applying the transformation rules listed in Lemma 3.2 yields

$$\exp(u) = \exp(c) \prod_{i \in L} u_i^{r_i} \prod_{i \in E} \theta_i^{r_i},$$

i.e. either $\exp(u)$ is algebraic over D_m (when $\exp(c) \in C_{D_m}$) or the constant fields of D_m and $D_m(\exp(u))$ differ (when $\exp(c) \notin C_{D_m}$).

□

Now that we have covered derivation and explained what kind of elementary functions intend to integrate, we can finally turn towards an algebraic definition of integration.

3.8 The Problem of Integration in Finite Terms

We borrowed[5] the title for this section from Risch's fundamental paper on integration [7] and we will reformulate part (a) of its Main Theorem for specifying the problem. But first we have to define what "integral" in the algebraic sense means. Indefinite integrals in calculus are defined as the inverse operation to differentiation, and we already have the derivation operator as an algebraic counterpart to derivatives, so the following definition is rather straightforward.

Definition 3.7 (Integral) *Let f be an element of a differential field D, F an element of some extension field of D.*
F is called an integral of f *(Symbol: $F = \int f$), iff $F' = f$.*

Note that according to this definition integrals are not unique: if F is an integral of f, then for every constant c obviously $F + c$ is an integral of f, too.

Furthermore, this definition is based upon the definition of differential fields as given in Section 3.3, which is too abstract to be directly related to the notions of calculus. For example, we cannot talk about something like a "variable of integration" here.

With only this definition at hand, computing integrals is trivial: given $f \in D$, we can simply extend D to $D(F)$, where the differential operator of the extension field is *defined* to be such that $F' = f$. Still, this does not give us an explicit elementary function, which is what we are really interested in[6].

We will now refine (or rather restrict) the above definition. This is similar to going from differential fields to transcendental elementary fields.

Definition 3.8 (Elementary Integral) *Let $m, n \in \mathbf{N}_0$, $m \leq n$, D_m and D_n elementary function fields with $D_m \subseteq D_n$, $f \in D_m$, $F \in D_n$.*

- *F is called an* elementary integral of f *(Symbol: $F = \int f\,dx$), iff $F' = f$.*
- *x is called the* variable of integration.
- *When $F = \int f\,dx$, then the constant term of F is called the* constant of integration.

If there is no danger of confusion, we will simply use the term *integral* for elementary integrals.

If we need to assure uniqueness of the integral, the representative for $\int f\,dx$ will be chosen to be the F for which $F' = f$ and whose constant term is equal to zero.

We know from the last chapter that we can relate the elements of the differential fields involved to their counterparts in calculus, and we have a distinct "variable of integration" as known from calculus. We have also seen that the algebraic notion of derivation is compatible with derivatives from calculus, so that we can expect that the same goes for integration. But note that this definition is much more restrictive than the definition of indefinite integrals in Section 2.4 because we are only considering elementary functions.

Adding the condition that the field containing the integral must be an elementary function field has some drawbacks: such a field might not exist and if it does, finding the integral is not trivial any more.

The following theorem[7] lays the foundation for the development of an algorithm to test whether an integral exists and, when it does exist, compute it.

[5] "Lesser artists borrow, great artists steal." — this is only a diploma thesis, after all ...

[6] But remember that this is exactly the way how we introduced *log* and *exp*. There we accepted this method because these two functions are much better "understood" in calculus than most other "unsolvable" integrals.

[7] We could not get the original reference [6] and the versions given in [7] and [10] slightly differ, so we present only a sketch containing the main point of the theorem.

Theorem 3.4 (Liouville 1833) *Let* $f \in D_m$.
If $\int f\,dx$ *exists, then*

$$\int f\,dx = v_0 + \sum_{i=1}^{k} c_i \log(v_i),$$

where $k \in \mathbf{N}_0$, $v_0 \in D_m$, $v_i \in \bar{D}_m$ *and* $c_i \in C_{\bar{D}_m}$ $(i = 1, \ldots, k)$.

Basically, this theorem says that if the integral of an elementary function f again is an elementary function, then it consists of a part which is in the original field, and a part which consists of terms which are logarithmic over the original field extended by some algebraic constants.

This finally leads to a formulation of the actual problem. Following Risch, we will state it as a theorem[8] of which most of the remainder of this paper will be the proof.

Theorem 3.5 (The Problem of Integration in Finite Terms) *Let* $m \in \mathbf{N}_0$, D_m *a transcendental elementary differential field with* C_{D_m} *explicitly finitely generated over* $\mathbf{Q}$, $f \in D_m$.
Then one can determine in a finite number of steps whether there are $k \in \mathbf{N}_0$, $v_0 \in D_m$, $v_i \in \bar{D}_m$ *and* $c_i \in C_{\bar{D}_m}$ $(i = 1, \ldots, k)$ *such that*

$$f = [v_0 + \sum_{i=1}^{k} c_i \log(v_i)]'.$$

If they do exist they can be computed in a finite number of steps.

Although Liouville's theorem also holds for elementary functions, Theorem 3.5 only formulates the problem of computing elementary integrals of transcendental elementary functions, i.e. the mixed algebraic case is not considered. An algorithm for this more general problem can be found in [2].

Now that we have given a framework in which to work and that we have defined the problem we want to solve we are ready to describe the solution: an algorithm for computing integrals and the fields containing them.

[8] This is basically part (a) of the Main Theorem of [7], adapted to the notations used in this paper.

Chapter 4

An Outline of the Algorithm

4.1 Checkpoint

In the previous chapters we set up the theoretical background and we defined the problem we intend to solve. Now we are prepared to start talking about an algorithm for computing elementary integrals.

In this chapter we will describe the basic structure of the algorithm, give some additional theory, and then show how the subalgorithms presented in the following three chapters fit into the big picture.

4.2 Recursion and Divide-and-Conquer

We will solve the problem of integration in finite terms by induction (recursion) over the tower of extensions of the transcendental elementary differential field D_m containing the function f to be integrated.

- The induction (recursion) base is the case $m = 0$. In this case, D_{m-1} is a field of constants, θ_m is equal to x, and we know that $\int f\,dx$ is always an elementary integral.
- The induction (recursion) step for $m > 0$ consists of reducing the problem of integrating f to the problem of integrating elements of D_{m-1}. This latter problem can then be solved by applying the integration algorithm recursively.

In every single recursion step we will use a divide-and-conquer approach for simplifying the integrals to be computed. Here the idea is to decompose the function f into a polynomial part f_P and a fractional part f_F (as described in Section 3.6), and to integrate these parts separately.

4.3 Decomposition

Before formulating the algorithm in more detail, we first have to address a problem caused by restricting ourselves to elementary integrals.

While being true in the general case, the equality $\int u + v\,dx = \int u\,dx + \int v\,dx$ need not hold if we require the integrals to be elements of elementary function fields. It might be possible that although there are no elementary function fields containing $\int u\,dx$ and $\int v\,dx$ there is cancellation in the sum, and thus $\int u + v\,dx$ could be an elementary integral.

We will show now that this cannot happen. We base this on the analysis done in Chapter 5 of [15], although our version of the proof is of a slightly different structure.

Lemma 4.1 (Decomposition) *Let $f \in D_m$. Then*

f has an elementary integral
if and only if
both f_P and f_F have elementary integrals.

Furthermore, if f has an elementary integral then

$$\int f\,dx = \int f_P\,dx + \int f_F\,dx.$$

This means that we can safely use the divide-and-conquer approach mentioned in the previous section. From the proof, we will also get some information on the structure of the integrals.

Proof: First assume that $\int f_P\,dx$ and $\int f_F\,dx$ are elementary. We have to show that there exists an elementary function F such that $F' = f$. If we let $F = \int f_P\,dx + \int f_F\,dx$ then F is elementary and we have $F' = f_P + f_F = f$, i.e. $F = \int f\,dx$.

Now assume that $\int f\,dx$ is a elementary integral. From Theorem 3.4 we know that

$$\int f\,dx = v + \sum_{i=1}^{l} c_i \log(v_i)$$

with $l \in \mathbf{N}_0$, $v \in D_m$, $v_i \in \bar{D}_m$ and $c_i \in C_{\bar{D}_m}$ $(i = 1, \ldots, l)$.

We can rewrite this as

$$f = v' + \sum_{i=1}^{l} c_i \frac{v_i'}{v_i}.$$

Decomposing into polynomial and fractional parts yields

$$f_P + f_F = (v_P + v_F)' + \sum_{i=k+1}^{l} c_i \frac{v_i'}{v_i} + \sum_{i=1}^{k} c_i \frac{v_i'}{v_i}.$$

Note that we can assume that the v_i are monic polynomials because $\frac{v'}{v} = \frac{a'}{a} + \frac{b'}{b}$ if $v = ab$ and $\frac{v'}{v} = \frac{a'}{a} - \frac{b'}{b}$ if $v = \frac{a}{b}$. Furthermore, we indexed the v_i such that $v_1, \ldots, v_k$ are elements of $D_{m-1}\langle\theta_m\rangle$ whose degrees w.r.t. θ_m are greater than 0, and $v_{k+1}, \ldots, v_l$ are elements of D_{m-1}. Reorganizing the terms on the right side leads to

$$f_P + f_F = \underbrace{v_P' + \sum_{i=k+1}^{l} c_i \frac{v_i'}{v_i}}_{A} + \underbrace{v_F' + \sum_{i=1}^{k} c_i \frac{v_i'}{v_i}}_{B}.$$

Now we will distinguish cases depending on the type of θ_m:

$\theta_m = x$: Here we know that v_P' is a polynomial and all terms of the sum in part A are equal to 0, because the v_i appearing there must be constants. Furthermore, v_F' is a proper fraction, as are the terms appearing in the sum in part B. Therefore, we have

$$\begin{aligned} f_P &= v_P', \\ f_F &= v_F' + \sum_{i=1}^{k} c_i \frac{v_i'}{v_i}, \end{aligned}$$

and integrating gives us

$$\begin{aligned} \int f_P\,dx &= v_P, \\ \int f_F\,dx &= v_F + \sum_{i=1}^{k} c_i \log(v_i). \end{aligned}$$

$\theta_m = \log(u)$**:** This case is similar to the previous one. v'_P again is a polynomial and the sum in part A is an element of D_{m-1} and thus must be part of f_0. v'_F is a proper rational function as are the terms appearing in the sum in part B. Therefore, we have

$$\begin{aligned} f_P &= v'_P + \sum_{i=k+1}^{l} c_i \frac{v'_i}{v_i}, \\ f_F &= v'_F + \sum_{i=1}^{k} c_i \frac{v'_i}{v_i}, \end{aligned}$$

and integrating gives us

$$\begin{aligned} \int f_P \, dx &= v_P + \sum_{i=k+1}^{l} c_i \log(v_i), \\ \int f_F \, dx &= v_F + \sum_{i=1}^{k} c_i \log(v_i). \end{aligned}$$

$\theta_m = \exp(u)$**:** This case is a bit different. Although part A again is a polynomial (more precisely an element of $D_{m-1}\langle\theta_m\rangle$), and v'_F is a proper rational function, the terms of the sum appearing in part B need not be proper rational functions.

The reason for this is that $\deg(v'_i) = \deg(v_i)$ (for $i = 1, \ldots, k$) and thus $\frac{v'_i}{v_i}$ is not a proper fraction. Let us have a closer look at these terms: The leading term of v'_i is $(\theta_m^{n_i})' = n_i u' \theta_m^{n_i}$, with n_i being the degree of v_i. Therefore, $\frac{v'_i}{v_i} = n_i u' +$ *(a proper fraction)*. This means that $\frac{v'_i}{v_i} - n_i u'$ is a proper fraction and we have

$$\begin{aligned} f_P &= v'_P + \sum_{i=k+1}^{l} c_i \frac{v'_i}{v_i} + u' \sum_{i=1}^{k} c_i n_i, \\ f_F &= v'_F + \sum_{i=1}^{k} c_i \frac{v'_i}{v_i} - u' \sum_{i=1}^{k} c_i n_i. \end{aligned}$$

Integrating then yields

$$\begin{aligned} \int f_P \, dx &= v_P + \sum_{i=k+1}^{l} c_i \log(v_i) + u \sum_{i=1}^{k} c_i n_i, \\ \int f_F \, dx &= v_F + \sum_{i=1}^{k} c_i \log(v_i) - u \sum_{i=1}^{k} c_i n_i. \end{aligned}$$

From this we see that if f has an elementary integral, then f_P and f_F also have elementary integrals and $\int f \, dx = \int f_P \, dx + \int f_F \, dx$.

□

4.4 An Outline of the Algorithm

Now we know that the divide-and-conquer approach will work and we can describe the overall structure of an algorithm for solving the problem of integration in finite terms.

Given a transcendental elementary differential field D_m and an element $f \in D_m$, we will proceed as follows:

(i) First we decompose f into $f_P + f_F$. In the base case and the logarithmic case, this can be done by a simple division. In the exponential case, we also have to guarantee that f_D is not divisible by θ_m. For this we may need to do a partial fraction decomposition as described in Section B.3.3.

As we have seen in the previous section, we can now integrate f_P and f_F independently of each other.

(ii) $\int f_P \, dx$ is computed by `integrate_poly`, which is described in Chapter 5. Depending on the type of the transcendental elementary monomial θ_m we will distinguish between three cases:

Base case: This is trivial, as we have $\int cx^n \, dx = \frac{c}{n+1}x^{n+1}$ for constant c.

Logarithmic case: Here the coefficients are elements of D_{m-1} which is not a constant field, so we have to call the integration algorithm recursively.

Exponential case: Here the main problem is finding solutions of so-called Risch differential equations $y' + fy = g$ for y. This is done by the algorithm `solve_rde` which we describe in Chapter 6.

(iii) The fractional part f_F is integrated by the algorithm `integrate_frac`, which is described in Chapter 7. Here all three cases are handled in the same way:

- For computing the part of the integral which is a proper fraction we will use Hermite's method. This requires doing a partial fraction decomposition of f_F.
- The logarithmic terms then are computed using Rothstein's method as described in [10]. Here we need to compute resultants and to factor polynomials over $\bar{D}_m$.

If $\theta_m = \exp(u)$ we have to subtract $u \sum_{i=1}^{k} c_i n_i$ from the result as we have seen in the proof of Lemma 4.1.

(iv) If either one of $\int f_P \, dx$ and $\int f_F \, dx$ is not an elementary integral, then $\int f \, dx$ is not an elementary integral and we cannot give a solution. Otherwise, we compute the elementary function field containing both integrals and return $\int f_P \, dx + \int f_F \, dx$ as the solution.

To assure uniqueness of the result we will always guarantee that the computed integrals do not contain any constant terms, i.e. the constant of integration is always 0.

$(\bar{D}_n,\, I_f) \leftarrow$ `integrate`$(D_m,\, f)$

Integrate an element of a transcendental elementary differential field.

Inputs:

D_m : a transcendental elementary differential field, $m \in \mathbf{N}_0$.

f : the function to be integrated, $f \in D_m$.

Outputs:

$\bar{D}_n$: a transcendental elementary differential field, $n \in \mathbf{N}_0$, $m \leq n$.

I_f : the integral $\int f\, dx \in \bar{D}_n$.

If no solution exists, the output is undefined and the message "no solution" is returned.

Begin

Step 1: *Split f into a polynomial and a fractional part.*

Compute f_P and f_F.

Step 2: *Integrate the polynomial part.*

$(\bar{D}_{n_1},\, I_{f_P}) \leftarrow$ `integrate_poly`$(D_m,\, f_P)$

Step 3: *Integrate the fractional part.*

$(\bar{D}_{n_2},\, I_{f_F}) \leftarrow$ `integrate_frac`$(D_m,\, f_F)$

Step 4: *Prepare the output.*

if the integrals in Steps 2 and 3 exist **then**
  Let $\bar{D}_n$ be an elementary function field containing both $\bar{D}_{n_1}$ and $\bar{D}_{n_2}$.
  $I_f \leftarrow I_{f_P} + I_{f_F}$
  return
else
  return "no solution"
endif

End

Algorithm 1: `integrate`

Chapter 5

How to Integrate Polynomials

5.1 Checkpoint

In this chapter we will describe how generalized polynomials[1] are integrated. The resulting algorithm `integrate_poly` is called in Step 2 of the algorithm `integrate`.

As explained in Chapter 4, we can apply the main integration algorithm recursively to elements of the coefficient domain D, so that it is sufficient to reduce the problem of integrating elements of $D\langle\theta\rangle$ to integrating elements of D.

Depending on the type of the transcendental elementary monomial θ we will distinguish three cases: the base case with $\theta' = 1$, the logarithmic case with $\theta' = \log(u)$ for some $u \in D$, and the exponential case with $\theta' = \exp(u)$ for some $u \in D$.

5.2 The Base Case

Given a transcendental elementary differential field D which consists only of constants (i.e. $D = C_D$), a transcendental elementary monomial $\theta = x$, and $p = \sum_{i=0}^{\delta_p} p_i\theta^i \in D[\theta]$, we know from the proof of Lemma 4.1 that $\int p\,dx$ is also a polynomial. Lemma 3.3 gives us a bound on its degree and we get

$$\int \sum_{i=0}^{\delta_p} p_i\theta^i\,dx = \sum_{i=0}^{\delta_p+1} q_i\theta^i.$$

After differentiation we have

$$\sum_{i=0}^{\delta_p} p_i\theta^i = \sum_{i=1}^{\delta_p+1} iq_i\theta^{i-1},$$

and comparing coefficients of like powers of θ we get

$$q_i = \frac{p_{i-1}}{i} \quad \text{for } i = 1, \ldots, \delta_p + 1.$$

Thus the solution is

$$\int p\,dx = \sum_{i=1}^{\delta_p+1} \frac{p_{i-1}}{i}\theta^i + q_0,$$

where q_0 is the constant of integration.

[1] I.e. elements of $D\langle\theta\rangle$ — see Section 3.7.1 for a definition of $D\langle\theta\rangle$, δ_p, γ_p, etc.

5.3 The Logarithmic Case

Given a transcendental elementary differential field D, a transcendental elementary monomial $\theta = \log(u)$, and $p = \sum_{i=0}^{\delta_p} p_i\theta^i \in D[\theta]$, we know from Lemma 4.1 that $\int p\,dx$ consists of a polynomial $q \in D[\theta]$ and some logarithmic terms. We get a bound on the degree of q from Lemma 3.3 and we see that

$$\int \sum_{i=0}^{\delta_p} p_i\theta^i\,dx = \sum_{i=0}^{\delta_p+1} q_i\theta^i + \sum_{j=1}^{h} d_j \log(v_j),$$

where $d_j \in C_{\bar{D}}$, $v_j \in D$, $q_i \in D$, and q_{δ_p+1} must be a constant (possibly equal to zero). We differentiate, getting

$$\sum_{i=0}^{\delta_p} p_i\theta^i = \sum_{i=0}^{\delta_p+1}(q_i'\theta^i + iq_i\frac{u'}{u}\theta^{i-1}) + \sum_{j=1}^{h} d_j\frac{v_j'}{v_j},$$

and after reorganizing the terms on the right side to make comparing coefficients of like powers of θ easier we have

$$\begin{aligned}\sum_{i=0}^{\delta_p} p_i\theta^i \;&=\; q'_{\delta_p+1}\theta^{\delta_p+1} + \\ &\quad + \sum_{i=1}^{\delta_p} q_i'\theta^i + (i+1)q_{i+1}\frac{u'}{u}\theta^i + \\ &\quad + q_0' + q_1\frac{u'}{u} + \sum_{j=1}^{h} d_j\frac{v_j'}{v_j}.\end{aligned}$$

From this we get the following equations for the coefficients:

$$0 = q'_{\delta_p+1} \tag{5.1}$$

$$p_i = q_i' + (i+1)q_{i+1}\frac{u'}{u} \quad \text{for } i = 1,\ldots,\delta_p \tag{5.2}$$

$$p_0 = q_0' + q_1\frac{u'}{u} + \sum_{j=1}^{h} d_j\frac{v_j'}{v_j} \tag{5.3}$$

Equation 5.1 only means that q_{δ_p+1} is constant. If we consider Equation 5.2 with $i = \delta_p$ and integrate, we have

$$\int p_{\delta_p}\,dx = q_{\delta_p} + (\delta_p+1)q_{\delta_p+1}\theta,$$

which seems to indicate a possible way of computing q_{δ_p+1} and q_{δ_p} by applying the integration algorithm recursively to p_{δ_p}.

The problem is that integration gives us a solution which is unique only up to constant terms. This means that what we actually get from the algorithm is

$$\int p_{\delta_p}\,dx = \bar{q}_{\delta_p} + c_{\delta_p+1}\theta,$$

where $\bar{q}_{\delta_p}$ is the part of q_{δ_p} which does not contain any constant terms and c_{δ_p+1} is some constant of which we know that it is equal to $(\delta_p+1)q_{\delta_p+1}$. We do not get the — unique — constant term of q_{δ_p} (which we will denote by $\bar{c}_{\delta_p}$), because it takes the part of the constant of integration.

Other values of i can be handled in the same way: by splitting q_i into a non-constant and a constant part Equation 5.2 becomes

$$p_i = \bar{q}_i' + \bar{c}_i' + (i+1)(\bar{q}_{i+1} + \bar{c}_{i+1})\frac{u'}{u}$$

for $i = 1, \ldots, \delta_p - 1$. Assuming $\bar{q}_{i+1}$ has already been computed from a previous equation we can move it to the left side and have

$$p_i - (i+1)\bar{q}_{i+1}\frac{u'}{u} = \bar{q}_i' + \bar{c}_i' + (i+1)\bar{c}_{i+1}\frac{u'}{u}.$$

Applying the integration algorithm now yields

$$\int p_i - (i+1)\bar{q}_{i+1}\frac{u'}{u}\,dx = \bar{q}_i + c_{i+1}\theta,$$

where again $\bar{q}_i$ and c_{i+1} are uniquely determined and we have $\bar{c}_{i+1} = \frac{c_{i+1}}{i+1}$. We cannot compute $\bar{c}_i$ in this step because it is the constant of integration.

From Equation 5.3 (for the θ-free parts) we get

$$p_0 = \bar{q}_0' + \bar{c}_0' + (\bar{q}_1 + \bar{c}_1)\frac{u'}{u} + \sum_{j=1}^{h} d_j \frac{v_j'}{v_j}$$

and after integrating as above we have

$$\int p_0 - \bar{q}_1\frac{u'}{u}\,dx = \bar{q}_0 + \bar{c}_1\theta + \sum_{j=1}^{h} d_j \log(v_j).$$

Here the integral is unique up to a constant $\bar{c}_0$, which is the constant of integration for the whole integral $\int p\,dx$. The sum $\sum_{j=1}^{h} d_j \log(v_j)$ contains all the new logarithms which may appear according to Lemma 4.1.

From the above discussion we see that processing the equations from the one containing p_{δ_p} to the one with p_0 allows us to successively compute $\bar{q}_{\delta_p}, \ldots, \bar{q}_0$ and $c_{\delta_p+1}, \ldots, c_1$. This gives us

$$\begin{aligned} q_{\delta_p+1} &= \frac{c_{\delta_p+1}}{\delta_p+1}, \\ q_i &= \bar{q}_i + \frac{c_i}{i} \quad \text{for } i = 1, \ldots, \delta_p, \text{ and} \\ q_0 &= \bar{q}_0. \end{aligned}$$

By this q is uniquely determined except for the constant of integration. Together with the sum of logarithms possibly appearing in the integral of p_0 we have the complete solution.

If any of the integrals appearing above are not elementary integrals or do not have the desired form (i.e. they must be elements of $D[\theta]$ except for the integral containing p_0), then $\int p\,dx$ is not a elementary integral.

5.4 The Exponential Case

Given a transcendental elementary differential field D, a transcendental elementary monomial $\theta = \exp(u)$ and $p = \sum_{i=\gamma_p}^{\delta_p} p_i\theta^i \in D\langle\theta\rangle$, we again apply Lemmas 4.1 and 3.3, getting

$$\int \sum_{i=\gamma_p}^{\delta_p} p_i\theta^i\,dx = \sum_{i=\gamma_p}^{\delta_p} q_i\theta^i + \sum_{j=1}^{h} d_j \log(v_j),$$

where $v_j \in D$, and $q_i \in D$. Note that the term $u \sum_{i=1}^{k} c_i n_i$, which appears in the proof of Lemma 4.1, goes into q_0.

Differentiation yields

$$\sum_{i=\gamma_p}^{\delta_p} p_i \theta^i = \sum_{i=\gamma_p}^{\delta_p} (q_i' \theta^i + q_i i u' \theta^i) + \sum_{j=1}^{h} d_j \frac{v_j'}{v_j}.$$

We reorganize the terms on the right side, getting

$$\begin{aligned} \sum_{i=\gamma_p}^{\delta_p} p_i \theta^i &= \sum_{i=\gamma_p}^{-1} q_i' \theta^i + q_i i u' \theta^i + \\ &\quad + q_0' + \sum_{j=1}^{h} d_j \frac{v_j'}{v_j} + \\ &\quad + \sum_{i=1}^{\delta_p} q_i' \theta^i + q_i i u' \theta^i, \end{aligned}$$

and as in the previous cases we set up equations for coefficients of like powers of θ:

$$p_i = q_i' + q_i i u' \quad \text{for } i = 1, \ldots, \delta_p \tag{5.4}$$

$$p_0 = q_0' + \sum_{j=1}^{h} d_j \frac{v_j'}{v_j} \tag{5.5}$$

$$p_i = q_i' + q_i i u' \text{ for } i = \gamma_p, \ldots, -1 \tag{5.6}$$

Integrating both sides of Equation 5.5 shows that q_0, the logarithmic terms, and the constant of integration for the whole integral come from $\int p_0 \, dx$.

The equations for all other q_i are of the form $g = y' + fy$, the so-called *Risch differential equation*. We will describe an algorithm for the solution of such equations in Chapter 6.

If $\int p_0 \, dx$ is not an elementary integral or if the solution to at least one of the Risch differential equations cannot be computed, then $\int p \, dx$ is not an elementary integral.

5.5 Algorithms

$$(\bar{D}_n,\, I_p) \leftarrow \texttt{integrate\_poly}(D_m,\, p)$$

Integrate a univariate polynomial over a transcendental elementary differential field.

Inputs:

D_m : a transcendental elementary differential field.

p : $\in D_{m-1}\langle\theta_m\rangle$, the polynomial to be integrated.

Outputs:

$\bar{D}_n$: a transcendental elementary differential field with $\bar{D}_n = \bar{D}_m(\theta_{m+1}, \ldots, \theta_n)$, where $n \geq m$ and $\theta_{m+1}, \ldots, \theta_n$ are logarithmic over $\bar{D}_{m-1}$.

I_p : the integral $\int p\, dx \in \bar{D}_n$.

If no solution exists, the output is undefined and the message "no solution" is returned.

Begin
Case $\theta_m = x$**:**
  $\bar{D}_n \leftarrow D_m$
  $I_p \leftarrow$ `integrate_poly_base`$(D_m,\, p)$
  return
Case $\theta_m = \log(u)$**:**
  $(\bar{D}_n,\, I_p) \leftarrow$ `integrate_poly_log`$(D_m,\, p)$
  return
Case $\theta_m = \exp(u)$**:**
  $(\bar{D}_n,\, I_p) \leftarrow$ `integrate_poly_exp`$(D_m,\, p)$
  return
End

Algorithm 2: `integrate_poly`

$I_p \leftarrow$ `integrate_poly_base`$(D,\, p)$

Integrate a univariate polynomial over a transcendental elementary differential field — base case.

Inputs:
D : a field of constants.
p : $\in D[x]$, the polynomial to be integrated.

Outputs:
I_p : the integral $\int p\, dx \in D[x]$.

Begin
$I_p \leftarrow \sum_{i=1}^{\delta_p+1} \frac{p_{i-1}}{i}\theta^i$
return
End

Algorithm 3: `integrate_poly_base`

$$(\bar{D}_n,\, I_p) \leftarrow \texttt{integrate\_poly\_log}(D_m,\, p)$$

Integrate a univariate polynomial over a transcendental elementary differential field — logarithmic case.

Inputs:

D_m : a transcendental elementary differential field with $\theta_m = \log(u)$ for some $u \in D_{m-1}$.

p : $\in D_{m-1}[\theta_m]$, the polynomial to be integrated.

Outputs:

$\bar{D}_n$: a transcendental elementary differential field with $\bar{D}_n = \bar{D}_m(\theta_{m+1}, \ldots, \theta_n)$, where $n \geq m$ and $\theta_{m+1}, \ldots, \theta_n$ are logarithmic over $\bar{D}_{m-1}$.

I_p : the integral $\int p\, dx \in \bar{D}_n$.

If no solution exists, the output is undefined and the message "no solution" is returned.

Begin

Step 1: *Integrate recursively.*

$(D_I,\, I) \leftarrow \texttt{integrate}(D_{m-1},\, p_{\delta_p})$
Let $\bar{q}_{\delta_p}$ and c_{δ_p+1} such that $I = c_{\delta_p+1}\theta_m + \bar{q}_{\delta_p}$, where c_{δ_p+1} is constant and $\bar{q}_{\delta_p} \in D_{m-1}$.
for $i = \delta_p - 1$ **downto** 1 **do**
  $(D_I,\, I) \leftarrow \texttt{integrate}(D_{m-1},\, p_i - (i+1)\bar{q}_{i+1}\frac{u'}{u})$
  Let $\bar{q}_i$ and c_{i+1} such that $I = c_{i+1}\theta_m + \bar{q}_i$, where c_{i+1} is constant and $\bar{q}_i \in D_{m-1}$.
endfor
$(D_n,\, I) \leftarrow \texttt{integrate}(D_{m-1},\, p_0 - \bar{q}_1\frac{u'}{u})$
Let $\bar{q}_0$, c_1 and $\sum_{j=1}^{k} d_j \log(v_j)$ such that $I = c_1\theta_m + \bar{q}_0 + \sum_{j=1}^{k} d_j \log(v_j)$, where $c_1 \in C_{D_m}$, $\bar{q}_0 \in D_{m-1}$, $d_j \in C_{\bar{D}_m}$, and $v_j \in \bar{D}_{m-1}$.
if at least one of the results is "no solution" or one of the D_I is not equal to D_m or one of the integrals does not have the desired structure **then**
  return "no solution"
endif

Step 2: *Compute the solution.*

$q_0 \leftarrow \bar{q}_0$
for $i = 1$ **to** δ_p **do**
  $q_i \leftarrow \bar{q}_i + \frac{c_i}{i}$
endfor
$q_{\delta_p+1} \leftarrow \frac{c_{\delta_p+1}}{\delta_p+1}$
$I_p \leftarrow \sum_{i=0}^{\delta_p+1} q_i\theta_m^i + \sum_{j=1}^{k} d_j \log(v_j)$
return

End

Algorithm 4: `integrate_poly_log`

$(\bar{D}_n, I_p) \leftarrow$ `integrate_poly_exp`(D_m, p)

Integrate a univariate polynomial over a transcendental elementary differential field — exponential case.

Inputs:

D_m : a transcendental elementary differential field with $\theta_m = \exp(u)$ for some $u \in D_{m-1}$.

p : $\in D_{m-1}[\theta_m, \theta_m^{-1}]$, the polynomial to be integrated.

Outputs:

$\bar{D}_n$: a transcendental elementary differential field with $\bar{D}_n = \bar{D}_m(\theta_{m+1}, \ldots, \theta_n)$, where $n \geq m$ and $\theta_{m+1}, \ldots, \theta_n$ are logarithmic over $\bar{D}_{m-1}$.

I_p : the integral $\int p\,dx \in \bar{D}_n$.

If no solution exists, the output is undefined and the message "no solution" is returned.

Begin

Step 1: *Solve the Risch differential equations for the positive powers.*

for $i = 1$ **to** δ_p **do**
  $q_i \leftarrow$ `solve_rde`(D_{m-1}, iu', p_i)
endfor
if at least one of the results is "no solution" **then return** "no solution"

Step 2: *Integrate recursively.*

$(\bar{D}_n, q_0) \leftarrow$ `integrate`(D_{m-1}, p_0)
if the integral cannot be computed **then return** "no solution"

Step 3: *Solve the Risch differential equations for the negative powers.*

for $i = \gamma_p$ **to** -1 **do**
  $q_i \leftarrow$ `solve_rde`(D_{m-1}, iu', p_i)
endfor
if at least one of the results is "no solution" **then return** "no solution"

Step 4: *Compute the solution.*

$I_p \leftarrow \sum_{i=\gamma_p}^{\delta_p} q_i \theta_m^i$
return

End

Algorithm 5: `integrate_poly_exp`

Chapter 6

How to Solve the Risch Differential Equation

6.1 Checkpoint

This chapter deals with computing the solutions of Risch differential equations, i.e. equations of the form $y' + fy = g$ with given f and g. For our purpose of doing integration algorithmically we need to solve such equations in algorithm `integrate_poly_exp`, so we will restrict ourselves to equations over transcendental elementary differential fields where f is of the form given in Section 5.4.

There are various approaches for solving this problem. The first one to realize its significance to integration and devise an algorithm was Robert H. Risch (in [7]). More recent solutions are presented in [5] by James H. Davenport and in [1] by Manuel Bronstein. The latter bases his work on an algorithm by Michael Rothstein given in [9].

The algorithm presented in this chapter is based on [1] and [5]. In Section 6.4 Bronstein's one-step reduction is used to compute the denominator of the solution, while Section 6.5 shows how to compute the numerator using Davenport's approach. We chose the respective algorithms because they have a clear structure and should be fairly easy to understand, thus fitting the tutorial purposes of this paper.

Unfortunately, there is one major drawback: Davenport's approach imposes several additional restrictions on f, which are explained in Section 6.5.5. Therefore the algorithm presented here is not as general as e.g. the one given in [1].

6.2 Overview

Let us first state the problem more clearly: Given a transcendental elementary differential field $D(\theta)$ and $f, g \in D(\theta)$ where f is *weakly normalized*, find an element y of $D(\theta)$ for which

$$y' + fy = g$$

holds.

The condition of weak normalization will be defined in Section 6.3. Meanwhile, let us just note that it does not restrict the set of functions f for the purposes of integration.

For computing y we will proceed as follows:

(i) Compute the denominator w of y.

(ii) Remove the denominator from the equation. This is done by modifying f and g in such a way that the resulting equation $v' + \bar{f}v = \bar{g}$ is a Risch differential equation such that v is guaranteed to be a polynomial, i.e. an element of $D\langle\theta\rangle$.

(iii) Solve the new equation by recursion over the structure of the differential field, distinguishing between the base case ($\theta = x$) of the recursion, a logarithmic case ($\theta = \log(u)$) and an exponential case ($\theta = \exp(u)$). Each case will be handled according to the following scheme:

- Check whether conditions hold which enable us to directly compute v or to conclude that a solution does not exist. If this is the case then we are finished.
- Compute $v_{\delta_v}\theta^{\delta_v}$, the leading term of v w.r.t. θ, from the leading terms of $\bar{f}$ and $\bar{g}$. If this is not possible, then no solution exists.
- Remove the leading term from the equation by substituting $v \leftarrow \hat{v} + v_{\delta_v}\theta^{\delta_v}$. This gives an equation $\hat{v}' + \bar{f}\hat{v} = \hat{\bar{g}}$ with $\hat{\bar{g}} = \bar{g} - (v_{\delta_v}\theta^{\delta_v})' - \bar{f}v_{\delta_v}\theta^{\delta_v}$ and $\delta_{\hat{\bar{g}}} < \delta_{\bar{g}}$.
- The leading term of $\hat{v}$ now is the next term of v and we can iterate until all terms with positive exponents have been computed or until we see that no solution exists.

In the exponential case, the same scheme can be used for computing the terms with negative exponents (starting from the leading term of v w.r.t. θ^{-1}).

The remaining equation then either yields v_0 or it is unsolvable. In the latter case, the original equation has no solution, either.

(iv) If v and w could both be computed, we have $y = \frac{v}{w}$ as the solution to the original equation, otherwise, there is no solution.

6.3 Weak Normalization

For being able to compute exact bounds on the degree of y we must ensure that f fulfills certain conditions.

Definition 6.1 (Weak Normalization) *Let $f \in D(\theta)$. f is called* weakly normalized w.r.t. θ *if and only if the following three conditions hold:*

(a) $\int f\,dx$ is an elementary integral.

(b) $\int f\,dx$ must not be of the form $h + \sum_{i=1}^{k} n_i \log(p_i)$, where h is in some extension field of $D(\theta)$, $n_i \in \mathbf{N}$, and $p_i \in D[\theta]$ with $\delta_{p_i} > 0$.

In other words, logarithmic terms depending on θ must not appear linearly with positive integer coefficients.

(c) If $\int f\,dx$ is of the form $h + \sum_{i=1}^{k} r_i \log(p_i)$, where h is in some extension field of $D(\theta)$, $r_i \in \mathbf{Q}$, and $p_i \in D[\theta]$ with $\delta_{p_i} > 0$, then $\sum_{i=1}^{k} r_i\delta_{p_i}$ must not be a negative integer.

Conditions (a) and (b) appear both in Davenport's [5] and Bronstein's [1]. They simplify computing the denominator of y. Condition (c) is not given in [1], but it is required for Davenport's method of computing the numerator of y.

We will now show that we can always weakly normalize a given f when we have to solve Risch differential equations for computing elementary integrals.

(a) As we have seen in Section 5.4, the Risch differential equations to be solved in the integration algorithm are of a special form. We always have $f = nu'$ for some integer n and some element u of a transcendental elementary differential field. Therefore, $\int f\,dx$ always is an elementary integral.

When we need to solve Risch differential equations recursively, then the f appearing in the subproblem also always has an integral which is an element of the transcendental elementary differential field containing u.

From this follows that Condition (a) is always fulfilled in the context of integration.

(b) Assume that $\int f\,dx = h + \sum_{i=1}^{k} n_i \log(p_i)$, where h is in some extension field of $D(\theta)$, $n_i \in \mathbf{N}$, and $p_i \in D[\theta]$ with $\delta_{p_i} > 0$.

Let $p := \prod_{i=1}^{k} p_i^{-n_i}$ and substitute $y \leftarrow p\bar{y}$. Then

$$\begin{aligned} y' + fy = g \quad &\Leftrightarrow \quad p\bar{y}' + p'\bar{y} + fp\bar{y} = g \\ &\Leftrightarrow \quad \bar{y}' + \underbrace{\left(\frac{p'}{p} + f\right)}_{\bar{f}} \bar{y} = p^{-1}g \end{aligned}$$

and $\bar{f}$ fulfills Condition (b), because $\int \bar{f}\,dx = \log(p) + \int f\,dx = h$, i.e. the logarithmic terms with positive integer coefficients have been removed.

(c) Assume that $\int f\,dx = h + \sum_{i=1}^{k} r_i \log(p_i)$, where h is in some extension field of $D(\theta)$, $r_i \in \mathbf{Q}$, and $p_i \in D[\theta]$ with $\delta_{p_i} > 0$. Furthermore, assume that $\sum_{i=1}^{k} r_i\delta_{p_i} < 0$.

Let $n_i \in \mathbf{N}$ such that $n_i = 0$ for $r_i > 0$ and $n_i - 1 < -r_i \leq n_i$ for $r_i < 0$. If we let $p := \prod_{i=1}^{k} p_i^{n_i}$ and again substitute $y \leftarrow p\bar{y}$ then the resulting equation for $\bar{y}$ is the same as in the previous case, but this time we have $\int \bar{f}\,dx = \log(p) + \int f\,dx = h + \sum_{i=1}^{k} (r_i + n_i)\log(p_i)$.

From the definition of n_i we see that $r_i + n_i$ is either 0 or a positive fraction. Thus, $\bar{f}$ fulfills Condition (c), because we have $\sum_{i=1}^{k} (r_i + n_i)\delta_{p_i} \geq 0$.

This shows that if f is not weakly normalized at the outset, we can apply the necessary substitutions, solve for $\bar{y}$, and then substitute back to get the solution for the original equation. For a more intuitive explanation of this mechanism consult the examples given in Section A.6.3.

The following lemma, which is essentially the same[1] as Davenport's Lemma 2.1 in [5], gives the consequence of Condition (c) which is the reason for introducing it and which will be used in the computation of the numerator.

Lemma 6.1 *Let f be weakly normalized w.r.t. θ with $\theta = x$ or θ logarithmic. Then there do not exist $h \in D$ and a negative integer N such that $\int f_L\,dx = h + N\theta$.*

Proof: See Lemma 2.1 in [5].

□

6.4 How to Compute the Denominator

Given a transcendental elementary differential field $D(\theta)$ and $f, g \in D$ we want to compute the denominator $y_{\bar{D}}$ of the solution y of the Risch differential equation $y' + fy = g$. Bronstein's Theorem 1 in [1] gives the following formula:

Let $h = \gcd(f_{\bar{D}}, g_{\bar{D}})$. Then

$$y_{\bar{D}} = \frac{\gcd(g_{\bar{D}}, \frac{dg_{\bar{D}}}{d\theta})}{\gcd(h, \frac{dh}{d\theta})}.$$

If $g_{\bar{D}}$ does not divide $f_{\bar{D}} y_{\bar{D}}^2$ then the equation has no solution in $D(\theta)$.

Proof: See the proofs of Theorem 1 and Corollary 1 in [1].

□

[1] Due to notational differences we write f_L instead of f_∞ for the "leading quotient" of f as defined in Section 3.7.1. Furthermore, we do not formulate the lemma for "primitive" monomials θ, but for $\theta = x$ or θ logarithmic.

Note that here the expression $\frac{d}{d\theta}$ is not related to the derivation operator $(.)'$. It is simply used as a shorthand notation with the definition

$$\frac{df}{d\theta} := \sum_{i=\gamma_f}^{\delta_f} i f_i \theta^{i-1}$$

for $f \in D\langle\theta\rangle$.

Now we want to remove the denominator from the equation. At this point we have

$$\left(\frac{y_{\bar{N}}}{y_{\bar{D}}}\right)' + f\frac{y_{\bar{N}}}{y_{\bar{D}}} = g$$

where f, g, and $y_{\bar{D}}$ are known. We need to find an equivalent Risch differential equation for $y_{\bar{N}}$.

Expanding the derivation yields

$$\frac{y'_{\bar{N}} y_{\bar{D}} - y_{\bar{N}} y'_{\bar{D}}}{y^2_{\bar{D}}} + f\frac{y_{\bar{N}}}{y_{\bar{D}}} = g$$

and multiplying by $y_{\bar{D}}$ we get

$$y'_{\bar{N}} + (f - \frac{y'_{\bar{D}}}{y_{\bar{D}}})y_{\bar{N}} = g y_{\bar{D}},$$

which is a Risch differential equation.

6.5 How to Compute the Numerator

Now we have given a transcendental elementary differential field D, an elementary monomial θ over D, and f and g such that the solution of $y' + fy = g$ must be an element of $D\langle\theta\rangle$.

We will compute y in a term by term fashion, starting with the leading term. To do this, we have to be able to find an exact bound on the highest power of θ in y, which we denote by δ_y. In the exponential case we also need an exact bound on the negative powers of θ, which we denote by γ_y.

These bounds can only be computed when certain conditions are satisfied. This gives rise to the notion of *cancellation*, which will be explained in the following section.

6.5.1 Cancellation

In the equations below we expand $y' + fy = g$, writing "..." for those terms which we are not interested in.

If we consider the most significant terms with respect to θ we get

$$y'_{\delta_y}\theta^{\delta_y} + \ldots + f_{\delta_f} y_{\delta_y}\theta^{\delta_f+\delta_y} + \ldots = g_{\delta_g}\theta^{\delta_g} + \ldots \tag{6.1}$$

if $\theta = x$ or $\theta = \log(u)$ and

$$y'_{\delta_y}\theta^{\delta_y} + y_{\delta_y}\delta_y u'\theta^{\delta_y} + \ldots + f_{\delta_f} y_{\delta_y}\theta^{\delta_f+\delta_y} + \ldots = g_{\delta_g}\theta^{\delta_g} + \ldots \tag{6.2}$$

if $\theta = \exp(u)$. In the latter case we also need to consider the least significant terms with respect to θ, i.e.

$$\begin{aligned}\ldots + y'_{\gamma_y}\theta^{\gamma_y} + y_{\gamma_y}\gamma_y u'\theta^{\gamma_y} + \ldots +&\\ + f_T y_{\gamma_y}\theta^{\gamma_y} + \ldots + f_{\gamma_f} y_{\gamma_y}\theta^{\gamma_f+\gamma_y} &= \ldots + g_{\delta_g}\theta^{\delta_g},\end{aligned} \tag{6.3}$$

where f_T is the *trailing quotient* of f as defined in Section 3.7.1.

If $\delta_f \geq 1$ then we see from Equations 6.1 and 6.2 that the term with the highest power of θ is $f_{\delta_f} y_{\delta_y} \theta^{\delta_f+\delta_y}$ and therefore we must have $\delta_y = \delta_g - \delta_f$ and $y_{\delta_y} = \frac{g_{\delta_g}}{f_{\delta_f}}$. Similarly we see from Equation 6.3 that if $\gamma_f \leq -1$ then the term with the lowest (i.e. most negative) power of θ is $f_{\gamma_f} y_{\gamma_y} \theta^{\gamma_f+\gamma_y}$ and therefore we have $\gamma_y = \gamma_g - \gamma_f$ and $y_{\gamma_y} = \frac{g_{\gamma_g}}{f_{\gamma_f}}$.

If $\delta_f = 0$ or $\gamma_f = 0$, things are not so simple. For example, if we have $\delta_f = 0$ and $f_0 \neq 0$ in the non-exponential case, then the leading terms on the left side of equation 6.1 are $y'_{\delta_y}\theta^{\delta_y} + f_0 y_{\delta_y}\theta^{\delta_y}$ and thus the coefficient of the highest power of θ is

$$y'_{\delta_y} + f_0 y_{\delta_y}.$$

Here we can distinguish between two cases:

- If there does not exist a v such that $v' + f_0 v = 0$, i.e. if f_0 is not of the form $-\frac{v'}{v}$, then equation 6.1 only holds when $\delta_y = \delta_g$ and $y'_{\delta_y} + f_0 y_{\delta_y} = g_{\delta_g}$.
- If there is a v such that $v' + f_0 v = 0$ (the so-called *cancellation*) then with $y_{\delta_y} = v$ equation 6.1 holds for any $\delta_y > \delta_g$.

Weak normalization of f lets us avoid most of these situations. In some cases we can also find substitutions which transform the original equation into one where cancellation cannot occur. Unfortunately, this is not possible under certain conditions. That is the reason why the algorithm presented here only works when additional restrictions are imposed upon f. These are explained in detail in Section 6.5.5.

6.5.2 The Base Case

Here we are in the situation where $D = C_D$ and $\theta' = 1$. The leading terms of y and y' are

$$\begin{aligned} y &= y_{\delta_y}\theta^{\delta_y} + y_{\delta_y-1}\theta^{\delta_y-1} + \ldots \\ y' &= y_{\delta_y}\delta_y\theta^{\delta_y-1} + y_{\delta_y-1}(\delta_y-1)\theta^{\delta_y-2} + \ldots \end{aligned}$$

For the purpose of computing δ_y and y_{δ_y} we distinguish between the following cases, depending on the degree of f:

$f_P \neq 0$:

Comparing the leading terms w.r.t. θ we get the equation

$$f_{\delta_f} y_{\delta_y}\theta^{\delta_f+\delta_y} + \ldots = g_{\delta_g}\theta^{\delta_g} + \ldots$$

We can compute y by observing that

- If $g = 0$ then $y = 0$.
- If $g \neq 0$ and $\delta_g < \delta_f$ then the equation has no solution.
- Otherwise, we have $\delta_y = \delta_g - \delta_f$ and $y_{\delta_y} = \frac{g_{\delta_g}}{f_{\delta_f}}$. In this case we can substitute and iterate as described in Section 6.2.

$f_P = 0, \deg(f_N) = \deg(f_D) - 1$:

Comparing the leading terms w.r.t. θ we get the equation

$$(\delta_y + f_L)y_{\delta_y}\theta^{\delta_y-1} + \ldots = g_{\delta_g}\theta^{\delta_g} + \ldots,$$

where f_L is the *leading quotient* of f as defined in Section 3.7.1. Here we may have cancellation by $\delta_y + f_L = 0$. Condition (c) of weak normalization guarantees that f_L never is a negative integer (see Lemma 6.1) and we can compute y_P by observing that

- If $g = 0$ then $y = 0$.
- If $g_P = 0$, then $y_{P+} = 0$ and $y_0 = \frac{g}{f}$. If y_0 is not a constant, the equation has no solution.
- If $g_P \neq 0$, we have $\delta_y = \delta_g + 1$ and $y_{\delta_y} = \frac{g_{\delta_g}}{\delta_y + f_L}$. In this case we can again substitute and iterate as described in Section 6.2.

$f_P = 0, \deg(f_N) < \deg(f_D) - 1$:

This case is handled like the previous one, with the exception that $f_L = 0$.

6.5.3 The Logarithmic Case

In this case $\theta = \log(u)$ for some $u \in D$. The leading terms of y and y' are

$$\begin{aligned} y &= y_{\delta_y}\theta^{\delta_y} + y_{\delta_y - 1}\theta^{\delta_y - 1} + \ldots \\ y' &= y'_{\delta_y}\theta^{\delta_y} + (y'_{\delta_y - 1} + y_{\delta_y}\delta_y\theta')\theta^{\delta_y - 1} + \ldots \end{aligned}$$

Again we distinguish between several cases depending on the degree of f.

$f_{P+} \neq 0$: Comparing the leading terms w.r.t. θ we get the equation

$$f_{\delta_f} y_{\delta_y} \theta^{\delta_f + \delta_y} + \ldots = g_{\delta_g}\theta^{\delta_g} + \ldots$$

We can compute y by observing that

- If $g = 0$ then $y = 0$.
- If $g \neq 0$ and $\delta_g < \delta_f$ then the equation has no solution.
- Otherwise, we have $\delta_y = \delta_g - \delta_f$ and $y_{\delta_y} = \frac{g_{\delta_g}}{f_{\delta_f}}$. In this case we can substitute and iterate as described in Section 6.2.

$f_P = 0, \deg(f_N) = \deg(f_D) - 1$: Comparing the leading terms w.r.t. θ we get the equation

$$y'_{\delta_y}\theta^{\delta_y} + \ldots = g_{\delta_g}\theta^{\delta_g} + \ldots$$

Cancellation happens when $y'_{\delta_y} = 0$, i.e. when y_{δ_y} is a constant. In this case, the equation for the next higher powers of θ in y is

$$y'_{\delta_y - 1}\theta^{\delta_y - 1} + y_{\delta_y}\delta_y\theta'\theta^{\delta_y - 1} + f_L y_{\delta_y}\theta^{\delta_y} + \ldots = g_{\delta_g}\theta^{\delta_g} + \ldots$$

Here, cancellation would mean that

$$y_{\delta_y}\delta_y\theta' + y'_{\delta_y - 1} + f_L y_{\delta_y} = 0,$$

or equivalently (after integrating and solving for $y_{\delta_y - 1}$)

$$y_{\delta_y - 1} = -y_{\delta_y}\left(\delta_y\theta + \int f_L\, dx\right) + c,$$

where c is a constant of integration. As in the similar situation appearing in the base case, Condition (c) of weak normalization guarantees that $\delta_y\theta + \int f_L\, dx \neq 0$ (again see Lemma 6.1). Therefore cancellation could only happen when $y_{\delta_y - 1}$ contained θ, which is a contradiction to θ being transcendental over D and $y_{\delta_y - 1}$ being an element of D.

Therefore we see that cancellation is only possible for the leading term of y and only if it is constant. If we let $\delta = \delta_g + 1$ then we must have $y'_\delta = 0$, i.e. y_δ must be a constant (possibly equal to zero) which we will call $\bar{c}_\delta$. From this follows that the equation for the coefficients of θ^{δ_g} is

$$\bar{c}_\delta \delta\theta' + y'_{\delta-1} + f_L \bar{c}_\delta = g_{\delta_g}.$$

Now we use a method similar to the one applied in Section 5.3 and integrate, getting

$$\bar{c}_\delta \delta\theta + \bar{y}_{\delta-1} + \bar{c}_{\delta-1} + \bar{c}_\delta \int f_L \, dx = \int g_{\delta_g} \, dx,$$

where $\bar{y}_{\delta-1}$ is the non-constant part of $y_{\delta-1}$ and $\bar{c}_{\delta-1}$ is the constant part and also the constant of integration.

We can now compute y in the following way:

- If $g = 0$ then $y = 0$.
- If $g_P = 0$, then $y_{P+} = 0$ and $y_0 = \frac{g}{f}$. If y_0 is not a constant, the equation has no solution.
- Let $\delta = \delta_g + 1$, $d_1\theta + h_1 + c_1 = \int g_{\delta_g} \, dx$ and $d_2\theta + h_2 + c_2 = \int f_L \, dx$, where c_1 and c_2 are the constants of integration.
- If the integrals cannot be computed or if they contain logarithmic terms not in $D(\theta)$ or if d_1 and d_2 are not constant or if $d_1 \neq 0$ and $\delta + d_2 = 0$ then the equation has no solution.
- If $d_1 \neq 0$ then $\delta_y = \delta$, $\bar{c}_{\delta_y} = \frac{d_1}{\delta_y + d_2}$, and $\bar{y}_{\delta_y - 1} = h_1 - \bar{c}_{\delta_y} h_2$. After substituting δ_g is reduced by at least one and we get a new equation for which the above analysis still holds.
- Otherwise, $\delta_y = \delta - 1$ and $\bar{y}_{\delta_y} = h_1$. After substituting δ_g is reduced by at least one and we get a new equation for which the above analysis still holds, so that we can iterate until y has been computed or the unsolvability of the equation is shown.

$f_P = 0, \deg(f_N) < \deg(f_D) - 1$: This case is handled like the previous one, with the exception that $f_L = 0$.

$f_{P+} = 0, f_0 \neq 0, \exp(\int f_0 \, dx) \notin D$: Comparing the leading terms w.r.t. θ we get the equation

$$y'_{\delta_y}\theta^{\delta_y} + f_0 y_{\delta_y}\theta^{\delta_y} + \ldots = g_{\delta_g}\theta^{\delta_g} + \ldots$$

Cancellation would mean that $y'_{\delta_y} + f_0 y_{\delta_y} = 0$. The solution for this differential equation is $y_{\delta_y} = \exp(-\int f_0 \, dx)$. Together with the condition above this would imply that $y_{\delta_y} \notin D$, a contradiction to y being an element of $D[\theta]$.

Knowing that cancellation is impossible we can compute y in the following way:

- If $g = 0$ then $y = 0$.
- If $g_P = 0$, then $y_{P+} = 0$ and $y_0 = \frac{g}{f}$. If y_0 is not in D (i.e. if it contains θ), the equation has no solution.
- Let $\delta = \delta_g$ and solve $y'_\delta + f_0 y_\delta = g_{\delta_g}$ recursively over D.
- If y_δ cannot be computed then the original equation has no solution.
- Otherwise, $\delta_y = \delta$ and $y_{\delta_y} = y_\delta$. After substituting δ_g is reduced by at least one and we get a new equation for which the above analysis still holds, so that we can iterate until y has been computed or the unsolvability of the equation is shown.

$f_{P+} = 0, f_0 \neq 0, \exp(\int f_0\, dx) \in D$: Comparing the leading terms w.r.t. θ we get the same equation as in the previous case, namely

$$y'_{\delta_y}\theta^{\delta_y} + f_0 y_{\delta_y}\theta^{\delta_y} + \ldots = g_{\delta_g}\theta^{\delta_g} + \ldots$$

Here we may have $y'_{\delta_y} + f_0 y_{\delta_y} = 0$ because $y_{\delta_y} = \exp(-\int f_0\, dx)$ is possible. We can circumvent this by substituting $y \leftarrow \bar{y}\exp(-\int f_0\, dx)$. This gives us

$$\bar{y}' + (f - f_0)\bar{y} = g\exp(\textstyle\int f_0\, dx)$$

and equivalently

$$\bar{y}' + f_F\bar{y} = g\exp(\textstyle\int f_0\, dx).$$

In this latter equation f_P is equal to zero so $\bar{y}$ can be computed by applying one of the previous cases.

The procedure therefore looks as follows:

- Let $\bar{g} = g\exp(\int f_0\, dx)$.
- Solve $\bar{y}' + f_F\bar{y} = \bar{g}$ by applying one of the previous cases.
- If $\bar{y}$ cannot be computed then the original equation has no solution.
- Otherwise, $y = \bar{y}\exp(-\int f_0\, dx)$ and we are finished.

6.5.4 The Exponential Case

This case is different from the previous two in that we have $f_P, g_P, y \in D\langle\theta\rangle$. Therefore we must consider both δ_y and γ_y, which leads to the four cases listed below:

$\delta_f > 0 > \gamma_f$: Comparing the leading terms w.r.t. θ and θ^{-1} we get the equations

$$\begin{aligned} f_{\delta_f} y_{\delta_y}\theta^{\delta_f+\delta_y} + \ldots &= g_{\delta_g}\theta^{\delta_g} + \ldots \\ \ldots + f_{\gamma_f} y_{\gamma_y}\theta^{\gamma_f+\gamma_y} &= \ldots + g_{\gamma_g}\theta^{\gamma_g} \end{aligned}$$

We see that cancellation is impossible and therefore we can compute y_{P+} as follows:

- Let $\delta = \delta_g - \delta_f$ and $\gamma = \gamma_g - \gamma_f$.
- If $g = 0$ or $\delta \leq 0$ then $\delta_y = 0$ and $y_{P+} = 0$.
- If $\delta < \gamma$ then the equation has no solution.
- Otherwise, $\delta_y = \delta$, $y_{\delta_y} = \frac{g_{\delta_g}}{f_{\delta_f}}$ and we can substitute and iterate as described in Section 6.2.

Computing y_{P-} can be done analogously:

- Let $\delta = \delta_g - \delta_f$ and $\gamma = \gamma_g - \gamma_f$.
- If $g = 0$ or $0 \leq \gamma$ then $\gamma_y = 0$ and $y_{P-} = 0$.
- If $\delta < \gamma$ then the equation has no solution.
- Otherwise, $\gamma_y = \gamma$, $y_{\gamma_y} = \frac{g_{\gamma_g}}{f_{\gamma_f}}$ and we can substitute and iterate as described in Section 6.2.

Now only y_0 remains to be computed. We consider the equation remaining after the substitutions done above and get the following:

- Let $\delta = \delta_g - \delta_f$ and $\gamma = \gamma_g - \gamma_f$.
- If $g = 0$ or $\text{sign}(\delta) = \text{sign}(\gamma) \neq 0$ then $y_0 = 0$.

- If $\delta = \gamma = 0$ we must have $\frac{g_{\delta_g}}{f_{\delta_f}} = \frac{g_{\gamma_g}}{f_{\gamma_f}}$, which is the value of y_0.
- Otherwise the equation has no solution.

$\delta_f = 0 > \gamma_f$: Comparing the leading terms w.r.t. θ and θ^{-1} we get the equations

$$\begin{aligned} (y'_{\delta_y} + \delta_y u' y_{\delta_y} + f_0 y_{\delta_y})\theta^{\delta_y} + \ldots &= g_{\delta_g}\theta^{\delta_g} + \ldots \\ \ldots + f_{\gamma_f} y_{\gamma_y}\theta^{\gamma_f+\gamma_y} &= \ldots + g_{\gamma_g}\theta^{\gamma_g} \end{aligned}$$

We see that we might have cancellation in the first equation. This happens when $f_0 = -\delta_y u' - \frac{y'_{\delta_y}}{y_{\delta_y}}$. Integrating gives us the following criterion:

We check whether $\int f_0\, dx$ can be expressed in the form $-Mu - \log(v)$ where v is in D and M is a non-negative integer with $M > \delta_g$. When this is the case we substitute $y \leftarrow \bar{y}\theta^M v$. We have $\theta^M v = \exp(-\int f_0\, dx)$ and therefore get the equation $\bar{y}' + (-f_0 + f)\bar{y} = g\frac{1}{\theta^M v}$ for $\bar{y}$, which is free of cancellation.

With cancellation safely avoided we can proceed as in the previous case and first compute y_{P+}:

- Let $\delta = \delta_g$ and $\gamma = \gamma_g - \gamma_f$.
- If $g = 0$ or $\delta = 0$ then $\delta_y = 0$ and $y_{P+} = 0$.
- If $\delta < \gamma$ then the equation has no solution.
- Otherwise, $\delta_y = \delta$ and y_{δ_y} is the unique solution to the equation

$$y'_{\delta_y} + (\delta_y u' + f_0)y_{\delta_y} = g_{\delta_g}$$

  which is a Risch differential equation over D. If such a y_{δ_y} does not exist the original equation has no solution. Otherwise, we again substitute and iterate as described in Section 6.2.

y_{P-} is computed as follows:

- Let $\delta = \delta_g$ and $\gamma = \gamma_g - \gamma_f$.
- If $g = 0$ or $0 \leq \gamma$ then $\gamma_y = 0$ and $y_{P-} = 0$.
- Otherwise, $\gamma_y = \gamma$, $y_{\gamma_y} = \frac{g_{\gamma_g}}{f_{\gamma_f}}$ and we can substitute and iterate as described in Section 6.2.

Now only y_0 remains to be computed. We consider the equation remaining after the substitutions done above and get the following:

- Let $\delta = \delta_g$ and $\gamma = \gamma_g - \gamma_f$.
- If $g = 0$ or $\mathrm{sign}(\delta) = \mathrm{sign}(\gamma) \neq 0$ then $y_0 = 0$.
- If $\delta = \gamma = 0$ then $y_0 = \frac{g_{\gamma_g}}{f_{\gamma_f}}$ and we must have $y'_0 + f y_0 = g$.
- Otherwise the equation has no solution.

$\delta_f > 0 = \gamma_f$: Comparing the leading terms w.r.t. θ and θ^{-1} we get the equations

$$\begin{aligned} f_{\delta_f} y_{\delta_y}\theta^{\delta_f+\delta_y} + \ldots &= g_{\delta_g}\theta^{\delta_g} + \ldots \\ \ldots + (y'_{\gamma_y} + \gamma_y u' y_{\gamma_y} + f_0 y_{\gamma_y} + f_T y_{\gamma_y})\theta^{\gamma_y} &= \ldots + g_{\gamma_g}\theta^{\gamma_g} \end{aligned}$$

We see that we may have cancellation in the second equation. This happens when $f_0 + f_T = -\gamma_y u' - \frac{y'_{\gamma_y}}{y_{\gamma_y}}$. Integrating gives us the following criterion:

We check whether $\int f_0 + f_T\,dx$ can be expressed in the form $Nu - \log(w)$ where w is in D and N is a positive integer with $N > -\gamma_g$. When this is the case we substitute $y \leftarrow \bar{y}\frac{w}{\theta^N}$. We have $\frac{w}{\theta^N} = \exp(-\int f_0 + f_T\,dx)$ and therefore get the equation $\bar{y}' + (-f_0 - f_T + f)\bar{y} = g\frac{\theta^N}{w}$ for $\bar{y}$, which is free of cancellation.

With cancellation safely avoided we can proceed as in the previous case and first compute y_{P+}:

- Let $\delta = \delta_g - \delta_f$ and $\gamma = \gamma_g$.
- If $g = 0$ or $\delta \leq 0$ then $\delta_y = 0$ and $y_{P+} = 0$.
- Otherwise, $\delta_y = \delta$, $y_{\delta_y} = \frac{g_{\delta_g}}{f_{\delta_f}}$ and we can substitute and iterate as described in Section 6.2.

y_{P-} is computed as follows:

- Let $\delta = \delta_g - \delta_f$ and $\gamma = \gamma_g$.
- If $g = 0$ or $\gamma = 0$ then $\gamma_y = 0$ and $y_{P-} = 0$.
- If $\delta < \gamma$ then the equation has no solution.
- Otherwise, $\gamma_y = \gamma$ and y_{γ_y} is the unique solution to the equation

  $$y'_{\gamma_y} + (\gamma_y u' + f_0 + f_T)y_{\gamma_y} = g_{\gamma_g}$$

  which is a Risch differential equation over D. If such a y_{γ_y} does not exist the original equation has no solution. Otherwise, we again substitute and iterate as described in Section 6.2.

Now only y_0 remains to be computed. We consider the equation remaining after the substitutions done above and get the following:

- Let $\delta = \delta_g - \delta_f$ and $\gamma = \gamma_g$.
- If $g = 0$ or $\operatorname{sign}(\delta) = \operatorname{sign}(\gamma) \neq 0$ then $y_0 = 0$.
- If $\delta = \gamma = 0$ then $y_0 = \frac{g_{\delta_g}}{f_{\delta_f}}$ and we must have $y_0' + fy_0 = g$.
- Otherwise the equation has no solution.

$\delta_f = 0 = \gamma_f$: This case can only be solved when several restrictions are imposed upon f. These will be explained in detail in the following section.

6.5.5 Restrictions of Davenport's Approach

As we have seen in the previous sections, Davenport's method works fine when we can make sure that cancellation does not occur. Now we will consider the case where $\delta_f = 0$ and $\gamma_f = 0$, i.e. $f = f_0 + f_F$. The leading terms w.r.t. θ and θ^{-1} give rise to the equations

$$(y'_{\delta_y} + \delta_y u' y_{\delta_y} + f_0 y_{\delta_y})\theta^{\delta_y} + \ldots \quad = \quad g_{\delta_g}\theta^{\delta_g} + \ldots \tag{6.4}$$

$$\ldots + (y'_{\gamma_y} + \gamma_y u' y_{\gamma_y} + f_0 y_{\gamma_y} + f_T y_{\gamma_y})\theta^{\gamma_y} \quad = \quad \ldots + g_{\gamma_g}\theta^{\gamma_g} \tag{6.5}$$

Cancellation occurs when

there exist an $M > \delta_g$ and a $v \in D$ such that
$$f_0 = -Mu' - \frac{v'}{v} \tag{6.6}$$

or when

there exist an $N > -\gamma_g$ and a $w \in D$ such that
$$f_0 + f_T = Nu' - \frac{w'}{w}. \tag{6.7}$$

Equivalent criteria are

$$\text{there exist an } M > \delta_g \text{ and a } v \in D \text{ such that} \quad \int f_0\,dx = -Mu - \log(v) \tag{6.8}$$

and

$$\text{there exist an } N > -\gamma_g \text{ and a } w \in D \text{ such that} \quad \int f_0 + f_T\,dx = Nu - \log(w). \tag{6.9}$$

When we had cancellation previously it could never appear simultaneously in y_{δ_y} and y_{γ_y} and therefore we could avoid it by a substitution. These substitutions also work here if only one of (6.8) and (6.9) hold. If both equations hold things are not so easy:

If we substitute $y \leftarrow \bar{y}v\theta^M$ (i.e. $y \leftarrow \bar{y}\exp(-\int f_0\,dx)$), we get the equation

$$\bar{y}' + \underbrace{(Mu' + \frac{v'}{v} + f)}_{\bar{f}}\,\bar{y} = \underbrace{g\frac{1}{v\theta^M}}_{\bar{g}}.$$

Here $\bar{f}_0 = Mu' + \frac{v'}{v} + f_0 = 0$, so that we cannot have cancellation in $\bar{y}_{\delta_{\bar{y}}}$. The problem is that we have

$$\begin{aligned}\int \bar{f}_0 + \bar{f}_T\,dx &= Mu + \log(v) + \int f_0 + f_T\,dx \\ &= (M+N)u - \log(\frac{w}{v})\end{aligned}$$

and therefore cancellation occurs in $\bar{y}_{\gamma_{\bar{y}}}$. We get a similar problem with cancellation in $\bar{y}_{\delta_{\bar{y}}}$ when we substitute $y \leftarrow \bar{y}\frac{w}{\theta^N}$.

Still, if we want to stick to Davenport's approach we only have two possibilities:

- We can try to compute y_{δ_y} or y_{γ_y} directly. Unfortunately, this cannot be done without changing the scheme we were following in the previous steps:

  From (6.4) and (6.8) we know that the leading term of y w.r.t. θ is of the form $cv\theta^M$ where c is a constant which depends on the other terms in y. Such dependencies cannot be unriddled using the methods presented here.

  The same goes for the leading term of y w.r.t. θ^{-1}.

- Alternatively, we can try to find a substitution $y \leftarrow a\bar{y}$ with $a \in D(\theta)$ such that the resulting equation

  $$\bar{y}' + (\frac{a'}{a} + f)\bar{y} = \frac{g}{a}$$

  is free of cancellation. If we let $h = \frac{a'}{a}$, then this means that $\int h_0 + f_0\,dx$ and $\int h_0 + f_0 + h_T + f_T\,dx$ must not be of the form given in equations (6.8) and (6.9).

  We will now show that such an a does not exist: Assume a is a polynomial $\sum_{i=0}^k a_i\theta^i$. We have $\theta = \exp(u)$, so

  $$\begin{aligned} h &= \frac{\sum_{i=0}^k (a_i' + a_i i u')\theta^i}{a} \\ &= \underbrace{\frac{a_k'}{a_k} + ku'}_{h_0} + \frac{\sum_{i=0}^{k-1}(a_i' + a_i i u' - a_i h_0)\theta^i}{a}\end{aligned}$$

  From this follows that $\int h_0\,dx = ku + \log(a_k)$ and $\int h_0 + h_T\,dx = \int \frac{a_0'}{a_0}\,dx = \log(a_0)$ — exactly the form we want to avoid.

When $a \in D(\theta)$, i.e. $a = \frac{a_{\bar{N}}}{a_{\bar{D}}}$ with $a_{\bar{N}}, a_{\bar{D}} \in D[\theta]$, then $\frac{a'}{a} = \frac{a'_{\bar{N}}}{a_{\bar{N}}} - \frac{a'_{\bar{D}}}{a_{\bar{D}}}$ and we are in the same situation as above.

In the proof of Lemma 6.5 in the original paper [5] Davenport wrongly assumes that equation 6.8 is sufficient for avoiding cancellation. Bronstein mentions this problem in [1]. We used his method of considering the trailing quotient f_T (in Bronstein's notation $\frac{B(0)}{A(0)}$) for introducing equation 6.9 and thus imposing sufficient restrictions upon f.

Naturally, y *can* be computed for arbitrary (weakly normalized) f, e.g. by using Bronstein's version of Rothstein's algorithm (as given in [1]). Here the equation is transformed so that no negative powers of θ remain, then δ_y is computed and the polynomial equation $f_D y' + f_N y = f_D g$ is solved for y.

The same could be done to avoid the problems described in this section, but the schemes of the algorithms are so different that they could not be merged without introducing unnecessary complications. For being able to compute solutions to Risch differential equations without having to impose restrictions upon f it might be best to just take some algorithm not using Davenport's approach.

Again we want to stress the fact that we chose the algorithm presented here because its structure appeared to be simpler than that of e.g. Rothstein's algorithm and we decided to trade generality off against understandability.

6.6 Algorithms

$$y \leftarrow \texttt{solve\_rde}(D_m,\, f,\, g)$$

Solve a Risch differential equation over a transcendental elementary differential field.

Inputs:
D_m : a transcendental elementary differential field, $m \geq 0$.
$f,\, g$: $\in D_m$.

Outputs:
y : $\in D_m$, the solution to the Risch differential equation $y' + fy = g$.

Begin

Step 1: *Compute the denominator of the solution.*
$h \leftarrow \gcd(f_{\bar{D}}, g_{\bar{D}})$
$w \leftarrow \frac{\gcd(g_{\bar{D}}, \frac{dg_{\bar{D}}}{d\theta_m})}{\gcd(h_{\bar{D}}, \frac{dh_{\bar{D}}}{d\theta_m})}$

Step 2: *Test whether the denominator is valid.*
if $\mathrm{rem}(f_{\bar{D}} w^2, g_{\bar{D}}) \neq 0$ **then return** "no solution"

Step 3: *Set up the equation for the numerator of the solution.*
$\bar{f} \leftarrow f - \frac{w'}{w}$
$\bar{g} \leftarrow gw$

Step 4: *Compute the numerator of the solution.*
Case $\theta_m = x$**:**
$v \leftarrow \texttt{solve\_rde\_base}(D_m, \bar{f}, \bar{g})$
Case $\theta_m = \log(u)$**:**
$v \leftarrow \texttt{solve\_rde\_log}(D_m, \bar{f}, \bar{g})$
Case $\theta_m = \exp(u)$**:**
$v \leftarrow \texttt{solve\_rde\_exp}(D_m, \bar{f}, \bar{g})$
if the result is "no solution" **then return** "no solution"

Step 5: *Prepare the output.*
$y \leftarrow \frac{v}{w}$
return y

End

Algorithm 6: `solve_rde`

$y \leftarrow$ `solve_rde_base`$(D,\, f,\, g)$

Solve a Risch differential equation — base case.

Inputs:

D : a differential field $C(x)$, where C is a field of constants.

$f,\, g :\in D$, such that the solution y of $y' + fy = g$ is in $C[x]$.

Outputs:

$y :\in C[x]$, the solution to the Risch differential equation $y' + fy = g$.

Begin

Step 1: *Initialize.*

$y \leftarrow 0$

Step 2: *Case distinction.*

Case $f_P \neq 0$**:**
  while $g \neq 0$ **do**
    if $\delta_g < \delta_f$ **then return** "no solution"
    $\delta \leftarrow \delta_g - \delta_f$
    $\tilde{y} \leftarrow \frac{g_{\delta_g}}{f_{\delta_f}}$
    $y \leftarrow y + \tilde{y} x^{\delta}$
    $g \leftarrow g - \tilde{y} \delta x^{\delta - 1} - f \tilde{y} x^{\delta}$
  endwhile
  return y

Case $f_P = 0, \deg(f_N) = \deg(f_D) - 1$**:**
  while $g \neq 0$ **do**
    if $g_P = 0$ **then**
      if $\frac{g}{f}$ is constant **then return** $y + \frac{g}{f}$
      else return "no solution"
    endif
    $\delta \leftarrow \delta_g + 1$
    $\tilde{y} \leftarrow \frac{g_{\delta_g}}{\delta + f_L}$
    $y \leftarrow y + \tilde{y} x^{\delta}$
    $g \leftarrow g - \tilde{y} \delta x^{\delta - 1} - f \tilde{y} x^{\delta}$
  endwhile
  return y

Case $f_P = 0, \deg(f_N) < \deg(f_D) - 1$**:**
  while $g \neq 0$ **do**
    if $g_P = 0$ **then**
      if $\frac{g}{f}$ is constant **then return** $y + \frac{g}{f}$
      else return "no solution"
    endif
    $\delta \leftarrow \delta_g + 1$

$\tilde{y} \leftarrow \frac{g_{\delta_g}}{\delta}$
$y \leftarrow y + \tilde{y}x^{\delta}$
$g \leftarrow g - \tilde{y}\delta x^{\delta-1} - f\tilde{y}x^{\delta}$
endwhile
return y
End

Algorithm 7: `solve_rde_base`

$y \leftarrow$ `solve_rde_log`(D_m, f, g)

Solve a Risch differential equation — logarithmic case.

Inputs:

D_m : a transcendental elementary differential field with $\theta_m = \log(u)$ for some $u \in D_{m-1}$.

f, g : $\in D_m$, such that the solution y of $y' + fy = g$ is in $D_{m-1}[\theta_m]$.

Outputs:

y : $\in D_{m-1}[\theta_m]$, the solution to the Risch differential equation $y' + fy = g$.

Begin

Step 1: *Initialize.*

$D \leftarrow D_{m-1}$
$\theta \leftarrow \theta_m$
$y \leftarrow 0$

Step 2: *Case distinction.*

Case $f_{P+} \neq 0$**:**
  while $g \neq 0$ **do**
    if $\delta_g < \delta_f$ **then return** "no solution"
    $\delta \leftarrow \delta_g - \delta_f$
    $\tilde{y} \leftarrow \frac{g_{\delta_g}}{f_{\delta_f}}$
    $y \leftarrow y + \tilde{y}\theta^\delta$
    $g \leftarrow g - \tilde{y}'\theta^\delta - \tilde{y}\delta\frac{u'}{u}\theta^{\delta-1} - f\tilde{y}\theta^\delta$
  endwhile
  return y

Case $f_P = 0, \deg(f_N) = \deg(f_D) - 1$**:**
  $y \leftarrow 0$
  while $g \neq 0$ **do**
    if $g_P = 0$ **then**
      if $\frac{g}{f}$ is constant **then return** $y + \frac{g}{f}$
      else return "no solution"
    endif
    $\delta \leftarrow \delta_g + 1$
    $(D_{I_1}, I_1) \leftarrow$ `integrate`(D, g_{δ_g})
    $(D_{I_2}, I_2) \leftarrow$ `integrate`(D, f_L)
    if at least one result is "no solution" or $D_{I_1} \neq D_m$ or $D_{I_2} \neq D_m$ **then**
      return "no solution"
    endif
    Let d_1 and h_1 s.th. $I_1 = d_1\theta + h_1$, where d_1 is constant and $h_1 \in D$.
    Let d_2 and h_2 s.th. $I_2 = d_2\theta + h_2$, where d_2 is constant and $h_2 \in D$.
    if such d_1, h_1, d_2, h_2 do not exist or $d_1 \neq 0$ and $\delta + d_2 = 0$ **then**
      return "no solution"
    endif
    $c \leftarrow \frac{d_1}{\delta + d_2}$
    $\tilde{y} \leftarrow c\theta + h_1 - ch_2$
    $y \leftarrow y + \tilde{y}\theta^{\delta-1}$

$g \leftarrow g - \tilde{y}'\theta^{\delta-1} - \tilde{y}(\delta-1)\frac{u'}{u}\theta^{\delta-2} - f\tilde{y}\theta^{\delta-1}$
endwhile
return y
Case $f_P = 0, \deg(f_N) < \deg(f_D) - 1$**:**
while $g \neq 0$ **do**
if $g_P = 0$ **then**
if $\frac{g}{f}$ is constant **then return** $y + \frac{g}{f}$
else return "no solution"
endif
$\delta \leftarrow \delta_g + 1$
$(D_I, I) \leftarrow$ `integrate`(D, g_{δ_g})
if the result is "no solution" or $D_I \neq D_m$ **then return** "no solution"
Let c and $\bar{y}$ s.th. $I = c\theta + \bar{y}$, where c is constant and $\bar{y} \in D$.
if such C and $\bar{y}$ do not exist **then return** "no solution"
$\tilde{y} \leftarrow \frac{c}{\delta}\theta + \bar{y}$
$y \leftarrow y + \tilde{y}\theta^{\delta-1}$
$g \leftarrow g - \tilde{y}'\theta^{\delta-1} - \tilde{y}(\delta-1)\theta^{\delta-2} - f\tilde{y}\theta^{\delta-1}$
endwhile
return y
Case $f_{P+} = 0, f_0 \neq 0, \exp(\int f_0\, dx) \notin D$**:**
while $g \neq 0$ **do**
if $g_P = 0$ **then**
if $\frac{g}{f}$ does not contain θ **then return** $y + \frac{g}{f}$
else return "no solution"
endif
$\delta \leftarrow \delta_g$
$\tilde{y} \leftarrow$ `solve_rde`(D, f_0, g_{δ_g})
if the result is "no solution" **then return** "no solution"
$y \leftarrow y + \tilde{y}\theta^{\delta}$
$g \leftarrow g - \tilde{y}'\theta^{\delta} - \tilde{y}\delta\frac{u'}{u}\theta^{\delta-2} - f\tilde{y}\theta^{\delta}$
endwhile
return y
Case $f_{P+} = 0, f_0 \neq 0, \exp(\int f_0\, dx) \in D$**:**
$\bar{y} \leftarrow$ `solve_rde_log`$(D, f_F, g\exp(\int f_0\, dx))$
if the result is "no solution" **then return** "no solution"
$y \leftarrow \bar{y}\exp(-\int f_0\, dx)$
return y
End

Algorithm 8: `solve_rde_log`

$y \leftarrow \texttt{solve\_rde\_exp}(D_m, f, g)$

Solve a Risch differential equation — exponential case.

Inputs:

D_m : a transcendental elementary differential field with $\theta_m = \exp(u)$ for some $u \in D_{m-1}$.

$f, g : \in D_m$, such that the solution y of $y' + fy = g$ is in $D_{m-1}\langle\theta_m\rangle$.

Outputs:

$y : \in D_{m-1}\langle\theta_m\rangle$, the solution to the Risch differential equation $y' + fy = g$.

Begin

Step 1: *Initialize.*

$D \leftarrow D_{m-1}$
$\theta \leftarrow \theta_m$
$y \leftarrow 0$

Step 2: *Case distinction.*

Case $\delta_f > 0 > \gamma_f$**:**

[*Compute* y_{P+}.]
$\delta \leftarrow \delta_g - \delta_f$
$\gamma \leftarrow \gamma_g - \gamma_f$
while $g \neq 0$ and $\delta > 0$ **do**
  if $\delta < \gamma$ **then return** "no solution"
  $\tilde{y} \leftarrow \frac{g_{\delta_g}}{f_{\delta_f}}$
  $y \leftarrow y + \tilde{y}\theta^{\delta}$
  $g \leftarrow g - \tilde{y}'\theta^{\delta} - \tilde{y}\delta u'\theta^{\delta} - f\tilde{y}\theta^{\delta}$
  $\delta \leftarrow \delta_g - \delta_f$
  $\gamma \leftarrow \gamma_g - \gamma_f$
endwhile

[*Compute* y_{P-}.]
$\delta \leftarrow \delta_g - \delta_f$
$\gamma \leftarrow \gamma_g - \gamma_f$
while $g \neq 0$ and $0 > \gamma$ **do**
  if $\delta < \gamma$ **then return** "no solution"
  $\tilde{y} \leftarrow \frac{g_{\gamma_g}}{f_{\gamma_f}}$
  $y \leftarrow y + \tilde{y}\theta^{\gamma}$
  $g \leftarrow g - \tilde{y}'\theta^{\gamma} - \tilde{y}\gamma u'\theta^{\gamma} - f\tilde{y}\theta^{\gamma}$
  $\delta \leftarrow \delta_g - \delta_f$
  $\gamma \leftarrow \gamma_g - \gamma_f$
endwhile

[*Compute* y_0.]
$\delta \leftarrow \delta_g - \delta_f$
$\gamma \leftarrow \gamma_g - \gamma_f$
if $g \neq 0$ and $\delta = \gamma = 0$ **then**
  if $\frac{g_{\delta_g}}{f_{\delta_f}} \neq \frac{g_{\gamma_g}}{f_{\gamma_f}}$ **then return** "no solution"
  $\tilde{y} \leftarrow \frac{g_{\gamma_g}}{f_{\gamma_f}}$

$y \leftarrow y + \tilde{y}$
else if $g \neq 0$ and $\text{sign}(\delta) \neq \text{sign}(\gamma)$ **then**
return "no solution"
endif
return y
Case $\delta_f = 0 > \gamma_f$**:**
[*Check for cancellation.*]
Find an integer $M > \delta_g$ and a $v \in D$ such that $\int f_0 \, dx = -Mu - \log(v)$.
if such M and v do not exist **then**
$M \leftarrow 0$
$v \leftarrow 1.$
endif
[*Shift.*]
$\bar{f} \leftarrow Mu' + \frac{v'}{v} + f$
$\bar{g} \leftarrow g\frac{1}{\theta^M v}$
[*Compute* $\bar{y}_{P+}$.]
$\delta \leftarrow \delta_{\bar{g}}$
$\gamma \leftarrow \gamma_{\bar{g}} - \gamma_{\bar{f}}$
while $\bar{g} \neq 0$ and $\delta \neq 0$ **do**
if $\delta < \gamma$ **then return** "no solution"
$\tilde{y} \leftarrow$ `solve_rde`$(D, \delta u' + \bar{f}_0, \bar{g}_{\delta_{\bar{g}}})$
if the result is "no solution" **then return** "no solution"
$\bar{y} \leftarrow \bar{y} + \tilde{y}\theta^\delta$
$\bar{g} \leftarrow \bar{g} - \tilde{y}'\theta^\delta - \tilde{y}\delta u'\theta^\delta - \bar{f}\tilde{y}\theta^\delta$
$\delta \leftarrow \delta_{\bar{g}}$
$\gamma \leftarrow \gamma_{\bar{g}} - \gamma_{\bar{f}}$
endwhile
[*Compute* $\bar{y}_{P-}$.]
$\gamma \leftarrow \gamma_{\bar{g}} - \gamma_{\bar{f}}$
while $\bar{g} \neq 0$ and $0 > \gamma$ **do**
$\tilde{y} \leftarrow \frac{\bar{g}_{\gamma_{\bar{g}}}}{\bar{f}_{\gamma_{\bar{f}}}}$
$\bar{y} \leftarrow \bar{y} + \tilde{y}\theta^\gamma$
$\bar{g} \leftarrow \bar{g} - \tilde{y}'\theta^\gamma - \tilde{y}\gamma u'\theta^\gamma - \bar{f}\tilde{y}\theta^\gamma$
$\gamma \leftarrow \gamma_{\bar{g}} - \gamma_{\bar{f}}$
endwhile
[*Compute* $\bar{y}_0$.]
$\delta \leftarrow \delta_{\bar{g}}$
$\gamma \leftarrow \gamma_{\bar{g}} - \gamma_{\bar{f}}$
if $\bar{g} \neq 0$ and $\delta = \gamma = 0$ **then**
$\tilde{y} \leftarrow \frac{\bar{g}_{\gamma_{\bar{g}}}}{\bar{f}_{\gamma_{\bar{f}}}}$
if $\tilde{y}' + \bar{f}\tilde{y} \neq \bar{g}$ **then return** "no solution"
$\bar{y} \leftarrow \bar{y} + \tilde{y}$
else if $\bar{g} \neq 0$ and $\text{sign}(\delta) \neq \text{sign}(\gamma)$ **then**
return "no solution"
endif
[*Shift back.*]
$y \leftarrow \bar{y}\theta^M v$
return y
Case $\delta_f > 0 = \gamma_f$**:**
[*Check for cancellation.*]
Find an integer $N > -\gamma_g$ and a $w \in D$ such that $\int f_0 + f_T \, dx = Nu - \log(w)$.

if such N and w do not exist **then**
  $N \leftarrow 0$
  $w \leftarrow 1.$
endif
[*Shift.*]
$\bar{f} \leftarrow -Nu' + \frac{w'}{w} + f$
$\bar{g} \leftarrow g\frac{\theta^N}{w}$
[*Compute* $\bar{y}_{P+}$.]
$\delta \leftarrow \delta_{\bar{g}} - \delta_{\bar{f}}$
while $\bar{g} \neq 0$ and $\delta > 0$ **do**
  $\tilde{y} \leftarrow \frac{\bar{g}_{\delta_{\bar{g}}}}{\bar{f}_{\delta_{\bar{f}}}}$
  $\bar{y} \leftarrow \bar{y} + \tilde{y}\theta^\delta$
  $\bar{g} \leftarrow \bar{g} - \tilde{y}'\theta^\delta - \tilde{y}\delta u'\theta^\delta - \bar{f}\tilde{y}\theta^\delta$
  $\delta \leftarrow \delta_{\bar{g}} - \delta_{\bar{f}}$
endwhile
[*Compute* $\bar{y}_{P-}$.]
$\delta \leftarrow \delta_{\bar{g}} - \delta_{\bar{f}}$
$\gamma \leftarrow \gamma_{\bar{g}}$
while $\bar{g} \neq 0$ and $\gamma \neq 0$ **do**
  if $\delta < \gamma$ **then return** "no solution"
  $\tilde{y} \leftarrow$ `solve_rde`$(D,\, \gamma u' + \bar{f}_0 + \bar{f}_T,\, \bar{g}_{\gamma_{\bar{g}}})$
  if the result is "no solution" **then return** "no solution"
  $\bar{y} \leftarrow \bar{y} + \tilde{y}\theta^\gamma$
  $\bar{g} \leftarrow \bar{g} - \tilde{y}'\theta^\gamma - \tilde{y}\gamma u'\theta^\gamma - \bar{f}\tilde{y}\theta^\gamma$
  $\delta \leftarrow \delta_{\bar{g}} - \delta_{\bar{f}}$
  $\gamma \leftarrow \gamma_{\bar{g}}$
endwhile
[*Compute* $\bar{y}_0$.]
$\delta \leftarrow \delta_{\bar{g}} - \delta_{\bar{f}}$
$\gamma \leftarrow \gamma_{\bar{g}}$
if $\bar{g} \neq 0$ and $\delta = \gamma = 0$ **then**
  $\tilde{y} \leftarrow \frac{\bar{g}_{\delta_{\bar{g}}}}{\bar{f}_{\delta_{\bar{f}}}}$
  if $\tilde{y}' + \bar{f}\tilde{y} \neq \bar{g}$ **then return** "no solution"
  $\bar{y} \leftarrow \bar{y} + \tilde{y}$
else if $\bar{g} \neq 0$ and $\operatorname{sign}(\delta) \neq \operatorname{sign}(\gamma)$ **then**
  return "no solution"
endif
[*Shift back.*]
$y \leftarrow \bar{y}\frac{w}{\theta^N}$
return y

Case $\delta_f = 0 = \gamma_f$:

[*Check for cancellation.*]
Find integers $M \geq 0$ and $N \geq 0$ and $v, w \in D$ such that $\int f_0\, dx = -Mu - \log(v)$ and $\int f_0 + f_T\, dx = Nu - \log(w)$.
if such M, v, N, and w exist and either $\delta_g < M$ and $-M - N < \gamma_g$, or $\delta_g < M + N$ and $-N < \gamma_g$ **then**
  return "Cancellation"
endif
if M and v exist with $\delta_g < M$ **then**
  $S \leftarrow M$

$s \leftarrow v.$
else if N and w exist with $-N < \gamma_g$ **then**
$S \leftarrow -N$
$s \leftarrow w.$
else
$S \leftarrow 0$
$s \leftarrow 1.$
endif
[*Shift.*]
$\bar{f} \leftarrow Su' + \frac{s'}{s} + f$
$\bar{g} \leftarrow g \frac{1}{\theta^S s}$
[*Compute* $\bar{y}_{P+}$.]
while $\bar{g} \neq 0$ and $\delta_{\bar{g}} \neq 0$ **do**
$\tilde{y} \leftarrow$ `solve_rde`$(D, \delta_{\bar{g}} u' + \bar{f}_0, \bar{g}_{\delta_{\bar{g}}})$
if the result is "no solution" **then return** "no solution"
$\bar{y} \leftarrow \bar{y} + \tilde{y}\theta^{\delta_{\bar{g}}}$
$\bar{g} \leftarrow \bar{g} - \tilde{y}'\theta^{\delta_{\bar{g}}} - \tilde{y}\delta_{\bar{g}} u'\theta^{\delta_{\bar{g}}} - \bar{f}\tilde{y}\theta^{\delta_{\bar{g}}}$
endwhile
[*Compute* $\bar{y}_{P-}$.]
while $\bar{g} \neq 0$ and $\gamma_{\bar{g}} \neq 0$ **do**
$\tilde{y} \leftarrow$ `solve_rde`$(D, \gamma_{\bar{g}} u' + \bar{f}_0 + \bar{f}_T, \bar{g}_{\gamma_{\bar{g}}})$
if the result is "no solution" **then return** "no solution"
$\bar{y} \leftarrow \bar{y} + \tilde{y}\theta^{\gamma_{\bar{g}}}$
$\bar{g} \leftarrow \bar{g} - \tilde{y}'\theta^{\gamma_{\bar{g}}} - \tilde{y}\gamma_{\bar{g}} u'\theta^{\gamma_{\bar{g}}} - \bar{f}\tilde{y}\theta^{\gamma_{\bar{g}}}$
endwhile
[*Compute* $\bar{y}_0$.]
if $g \neq 0$ **then**
$\tilde{y} \leftarrow \frac{\bar{g}_F}{\bar{f}_F}$
if $\tilde{y} \notin D$ or $\tilde{y}' + \bar{f}\tilde{y} \neq \bar{g}$ **then return** "no solution"
$\bar{y} \leftarrow \bar{y} + \tilde{y}$
endif
[*Shift back.*]
$y \leftarrow \bar{y}\theta^S s$
return y
End

Algorithm 9: `solve_rde_exp`

Chapter 7

How to Integrate Proper Fractions

7.1 Checkpoint

In this chapter we will describe how proper fractions are integrated. The resulting algorithm `integrate_frac` is called in Step 3 of algorithm `integrate`.

As explained in Chapter 4, we have given a proper fraction $f \in D(\theta)$, and we need to find a proper fraction $g \in D(\theta)$, $k \in \mathbf{N}_0$, $v_i \in \bar{D}(\theta)$, and $c_i \in C_{\bar{D}}$ $(i = 1, \ldots, k)$, such that

$$\int f\, dx = g + \sum_{i=1}^{k} c_i \log(v_i) + d.$$

If $\theta = \exp(u)$ then $d = -u \sum_{i=1}^{k} c_i \deg(v_i)$, otherwise $d = 0$.

Computing the rational part g can be done using squarefree factorization and partial fraction decomposition, while computing the logarithmic part requires complete factorization and therefore might introduce additional algebraic constants and logarithmic monomials.

7.2 How to Compute the Rational Part

Given a proper fraction $f \in D(\theta)$ we want to find a proper fraction $g \in D(\theta)$ and a normal fraction $h \in D(\theta)$ such that

$$\int f\, dx = g + \int h\, dx.$$

We will solve this problem using Hermite's algorithm as given in e.g. [15]. Then the logarithmic part can be computed from $\int h\, dx$.

First we compute a squarefree factorization of f_D, i.e. squarefree and relatively prime polynomials $q_1, \ldots, q_l \in D[\theta]$ such that $f_D = \prod_{i=1}^{l} q_i^i$. In Section B.2 we explain how this can be done.

Now we can do a partial fraction decomposition of f using the method described in Section B.3.3, i.e. we can compute $p_1, \ldots, p_l \in D[\theta]$ such that

$$f = \sum_{i=1}^{l} \frac{p_i}{q_i^i}$$

where the terms on the right again are proper fractions. This means that we can now handle each of the $\frac{p_i}{q_i^i}$ separately.

For each $\frac{p}{q^j}$ we have $\gcd(q, q') = 1$ (because q is squarefree), so we can find $a, b \in D[\theta]$ with $aq' + bq = \frac{p}{j-1}$ using the method described in Section B.3.2. Then we have

$$\begin{aligned}
\int \frac{p}{q^j}\,dx &= \int \frac{(j-1)aq'}{q^j} + \frac{(j-1)b}{q^{j-1}}\,dx \\
&= \int -\left(\frac{a}{q^{j-1}}\right)' + \frac{a'}{q^{j-1}} + \frac{(j-1)b}{q^{j-1}}\,dx \\
&= -\frac{a}{q^{j-1}} + \int \frac{a' + (j-1)b}{q^{j-1}}\,dx,
\end{aligned}$$

using the equality $\left(\frac{a}{q^{j-1}}\right)' = \frac{a'}{q^{j-1}} - \frac{(j-1)aq'}{q^j}$ for the step from the first line to the second.

The fractions in the last line are proper and the exponent of q has been reduced by 1, so we can repeat this computation with the integral appearing in the last line until we are left with only q in the denominator. After having done this for all $\frac{p_i}{q_i^i}$ we can collect the terms outside the integrals into g and the terms inside the integrals into h.

7.3 How to Compute the Logarithmic Part

It remains to show how normal fractions can be integrated. That is, given a proper fraction $h \in D(\theta)$ with $h_{\bar{D}}$ squarefree, we want to find a $k \in \mathbf{N}_0$, $v_i \in \bar{D}(\theta)$, and $c_i \in C_{\bar{D}}$ $(i = 1, \ldots, k)$ such that

$$\int h\,dx = \sum_{i=1}^{k} c_i \log(v_i) + d$$

where $d = -u \sum_{i=1}^{k} c_i \deg(v_i)$ if $\theta = \exp(u)$, and $d = 0$, otherwise. Furthermore, the properties of logarithms allow us to require the v_i to be squarefree, pairwise relatively prime, and that $c_i \neq c_j$ for $i \neq j$.

Nearly identical methods for solving this problem with introducing only a minimal number of new algebraic constants were first given by M. Rothstein in [9] and by B. M. Trager in [12]. We base the following exposition on [10] and [15].

First let us check whether $\int h\,dx$ really can only consist of a logarithmic part: If $\int h\,dx$ were of the form $g + \sum c_i \log(v_i)$ with g a proper fraction, then we would have $h = g' + \sum c_i \frac{v_i'}{v_i} = \frac{g_{\bar{N}}'}{g_{\bar{D}}} - \frac{g_{\bar{N}} g_{\bar{D}}'}{g_{\bar{D}}^2} + \sum c_i \frac{v_i'}{v_i}$ and $h_{\bar{D}} = g_{\bar{D}}^2$ would not be squarefree.

Now we will consider the non-exponential case, i.e. $d = 0$ and

$$h = \sum_{i=1}^{k} c_i \frac{v_i'}{v_i}.$$

From this we can conclude that

$$h_{\bar{D}} = \prod_{i=1}^{k} v_i.$$

If we let $u_i := \frac{h_{\bar{D}}}{v_i} = \prod_{\substack{j=1 \\ j \neq i}}^{k} v_j$ we have $h_{\bar{D}}' = \sum_{i=1}^{k} v_i' u_i$ and $\frac{v_i'}{v_i} = \frac{v_i' u_i}{v_i u_i} = \frac{v_i' u_i}{h_{\bar{D}}}$. From this follows

$$\begin{aligned}
h_{\bar{N}} &= h_{\bar{D}} \sum_{i=1}^{k} c_i \frac{v_i'}{v_i} \\
&= h_{\bar{D}} \sum_{i=1}^{k} c_i \frac{v_i' u_i}{h_{\bar{D}}} \\
&= \sum_{i=1}^{k} c_i v_i' u_i.
\end{aligned}$$

Now we will show how we can compute $v_{\hat{\imath}}$ for $1 \leq \hat{\imath} \leq k$. We have

$$\begin{aligned} v_{\hat{\imath}} &= \gcd(h_{\bar{N}} - \sum_{i=1}^{k} c_i v_i' u_i, v_{\hat{\imath}}) \\ &= \gcd(h_{\bar{N}} - c_{\hat{\imath}} v_{\hat{\imath}}' u_{\hat{\imath}}, v_{\hat{\imath}}) \\ &= \gcd(h_{\bar{N}} - c_{\hat{\imath}} \sum_{i=1}^{k} v_i' u_i, v_{\hat{\imath}}) \\ &= \gcd(h_{\bar{N}} - c_{\hat{\imath}} h_{\bar{D}}', v_{\hat{\imath}}) \end{aligned}$$

and for $\hat{\jmath} \neq \hat{\imath}$ with $1 \leq \hat{\jmath} \leq k$ we get

$$\begin{aligned} \gcd(h_{\bar{N}} - c_{\hat{\imath}} h_{\bar{D}}', v_{\hat{\jmath}}) &= \gcd(\sum_{i=1}^{k} (c_i - c_{\hat{\imath}}) v_i' u_i, v_{\hat{\jmath}}) \\ &= \gcd((c_{\hat{\jmath}} - c_{\hat{\imath}}) v_{\hat{\jmath}}' u_{\hat{\jmath}}, v_{\hat{\jmath}}) \\ &= 1. \end{aligned}$$

We see that only $v_{\hat{\imath}}$ divides $h_{\bar{N}} - c_{\hat{\imath}} h_{\bar{D}}'$ and therefore we have

$$\gcd(h_{\bar{N}} - c_{\hat{\imath}} h_{\bar{D}}', h_{\bar{D}}) = v_{\hat{\imath}}.$$

This means that we can compute $v_{\hat{\imath}}$ if we know $c_{\hat{\imath}}$.

The c_i are exactly those values α for which $\gcd(h_{\bar{N}} - \alpha h_{\bar{D}}', h_{\bar{D}}) \neq 1$. From the properties of resultants given in Section B.4 we know that $\operatorname{res}(p, q) = 0$ if and only if $\gcd(p, q) \neq 1$, so the c_i must be the distinct roots (w.r.t. α) of $\operatorname{res}_\theta(h_{\bar{N}} - \alpha h_{\bar{D}}', h_{\bar{D}})$. (Note that we have $h_{\bar{N}} - \alpha h_{\bar{D}}' \in (D[\alpha])[\theta]$ so that the resultant is an element of $D[\alpha]$.) If any of the c_i is not a constant, then $\int h\,dx$ is not elementary.

If $\theta = \exp(u)$, we have

$$h = \sum_{i=1}^{k} c_i \frac{v_i'}{v_i} - u' \sum_{i=1}^{k} c_i \deg(v_i)$$

and analogously to the previous case

$$h_{\bar{N}} = \sum_{i=1}^{k} c_i v_i' u_i - h_{\bar{D}} u' \sum_{i=1}^{k} c_i \deg(v_i).$$

From $h_{\bar{D}} = \prod_{i=1}^{k} v_i$ follows that

$$\gcd(h_{\bar{N}} - \sum_{i=1}^{k} c_i v_i' u_i - h_{\bar{D}} u' \sum_{i=1}^{k} c_i \deg(v_i), v_{\hat{\imath}}) = \gcd(h_{\bar{N}} - \sum_{i=1}^{k} c_i v_i' u_i, v_{\hat{\imath}}).$$

Therefore the equations for the non-exponential case still hold and we can handle the exponential case analogously, with the only difference that in the end $u \sum_{i=1}^{k} c_i \deg(v_i)$ must be subtracted.

7.4 Algorithms

$(\hat{D},\, I) \leftarrow$ `integrate_frac`$(D(\theta),\, f)$

Integrate a proper fraction.

Inputs:

$D(\theta)$: a differential field.

f : $\in D(\theta)$, a proper fraction.

Outputs:

$\hat{D}$: a differential field $\bar{D}(\theta, \log(v_1), \ldots, \log(v_k))$, where $v_i \in \bar{D}$ and $\bar{D}$ is equal to D extended by some algebraic constants.

I : $\in \bar{D}(\theta)$ such that $I = \int f\, dx$.

Begin

Step 1: *Compute the squarefree factorization.*

$(q_1, \ldots, q_l) \leftarrow$ `squarefree`(f_D)

Step 2: *Compute the rational part.*

$(g,\, h) \leftarrow$ `integrate_partial`$(f_N,\, f_D,\, (q_1, \ldots, q_l))$

Step 3: *Compute the logarithmic part.*

$(\bar{D}(\theta),\, d,\, c_1,\, v_1,\, \ldots,\, c_k,\, v_k) \leftarrow$ `integrate_normal_frac`$(D(\theta),\, h)$

Step 4: *Return.*

return $(\bar{D}(\theta, \log(v_1), \ldots, \log(v_k)),\, g + d + \sum_{i=1}^{k} c_i \log(v_i))$

End

Algorithm 10: `integrate_frac`

$(g,\, h) \leftarrow$ `integrate_partial`$(p,\, \mathrm{q},\, (q_j,\, \ldots,\, q_l))$

Compute the rational part of the integral of a proper fraction.

Inputs:
$p, q\, : \in D[\theta]$, such that $\frac{p}{q}$ is a proper fraction

$q_i\, : \in D[\theta]$, squarefree, such that $q = \prod_{i=j}^{l} q_i^i$.

Outputs:
$g,\, h\, : \in D(\theta)$, g a proper fraction, h a normal fraction, with $\int \frac{p}{q}\, dx = g + \int h\, dx$.

Begin

Step 1: *Recursion base.*

if $j = l$ **then**
$(g,\, h) \leftarrow$ `integrate_squarefree`$(p,\, q_l,\, l)$
return $(g,\, h)$
endif

Step 2: *Partial fraction decomposition.*

$(p_j,\, \bar{p}) \leftarrow$ `solve_spe`$(p,\, \prod_{i=j+1}^{l} q_i^i,\, q_j^j)$

Step 3: *Compute the rational part for a single term.*

$(g_j,\, h_j) \leftarrow$ `integrate_squarefree`$(p_j,\, q_j,\, j)$

Step 4: *Recursion.*

$(\bar{g},\, \bar{h}) \leftarrow$ `integrate_partial`$(\bar{p},\, \prod_{i=j+1}^{l} q_i^i,\, (q_{j+1}, \ldots, q_l))$
return $(\bar{g} + g_j,\, \bar{h} + h_j)$

End

Algorithm 11: `integrate_partial`

$(g, h) \leftarrow \texttt{integrate\_squarefree}(p, q, q_i, i)$

Compute the rational part of the integral of a proper fraction whose denominator contains a single squarefree factor.

Inputs:
p, q : $\in D[\theta]$, such that $\frac{p}{q}$ is a proper fraction.
q_i : $\in D[\theta]$, squarefree.
i : $\in \mathbf{N}$, such that $q_i^i = q$.

Outputs:
g, h : $\in D(\theta)$, g a proper fraction, h a normal fraction, with $\int \frac{p}{q}\, dx = g + \int h\, dx$.

Begin

Step 1: *Recursion base.*
if $i = 1$ **then return** $(0, \frac{p}{q})$

Step 2: *Decomposition.*
$(a, b) \leftarrow \texttt{solve\_spe}(\frac{p}{i-1}, q_i', q_i)$

Step 3: *Recursion.*
$(\bar{g}, h) \leftarrow \texttt{integrate\_squarefree}(a' + (i-1)b, \frac{q}{q_i}, q_i, i-1)$
return $(\bar{g} - \frac{a}{q_i^{i-1}}, h)$

End

Algorithm 12: `integrate_squarefree`

$$(\bar{D}(\theta),\, d,\, c_1,\, v_1,\, \ldots,\, c_k,\, v_k) \leftarrow \texttt{integrate\_normal\_frac}(D(\theta),\, h)$$

Integrate a normal fraction.

Inputs:
$D(\theta)$: a differential field.
h : $\in D(\theta)$, a normal fraction.

Outputs:
$\bar{D}(\theta)$: $D(\theta)$ extended by some algebraic constants.
d : $\in D$, $d = -u \sum_{i=1}^{k} c_i \deg(v_i)$ if $\theta = \exp(u)$, otherwise $d = 0$.
c_i : $\in C_{\bar{D}}$.
v_i : $\in \bar{D}(\theta)$.

c_i, v_i, and d are such that $\int h\, dx = d + \sum_{i=1}^{k} c_i \log(v_i)$.

Begin

Step 1: *Add new variable.*
Represent $a \leftarrow h_N - \alpha h'_D$ and $b = h_D$ as elements of $(D[\alpha])[\theta]$, where α is a new variable.

Step 2: *Compute the resultant.*
$r \leftarrow \texttt{resultant}((D[\alpha])[\theta],\, a,\, b)$

Step 3: *Split r into linear factors.*
Compute the roots $c_1, \ldots, c_k$ of $r \in D[\alpha]$ w.r.t. α.
if any of the c_i is not constant **then return** "no solution"

Step 4: *Compute the arguments of the logarithms.*
for $i \leftarrow 1$ **to** k **do**
  $v_i \leftarrow \gcd(h_N - c_i h'_D, h_D)$
endfor

Step 5: *Compute d.*
if $\theta = \exp(u)$ **then** $d \leftarrow -u \sum_{i=1}^{k} c_i \deg(v_i)$
else $d \leftarrow 0$
return $(\bar{D}(\theta),\, c_1,\, v_1,\, \ldots,\, c_k,\, v_k,\, d)$

End

Algorithm 13: `integrate_normal_frac`

Chapter 8

Notes on the Implementation

8.1 Checkpoint

In the previous chapters the theory for an algorithm for computing elementary integrals of transcendental elementary functions is presented. In this chapter an overview of the partial implementation done by the author is given.

8.2 The Software Environment

While the author was working on this paper he was also doing maintenance for SACLIB, a freely distributable computer algebra library written in `C` (see [21]). Therefore `C` was an obvious choice as the implementation language, with arbitrary precision integer and polynomial arithmetic and low-level input and output being done by calls to SACLIB functions.

The package was developed under `AmigaDos` and then ported to `Unix`. It should run on any system where SACLIB is installed.

8.3 What Has Been Done

The implementation only allows computing elementary integrals of elements of a differential field $\mathbf{Q}(x, \theta)$ with θ exponential over $\mathbf{Q}(x)$. A further restriction is that in the integral of a normal fraction (see Section 7.3) no new algebraic constants may appear.

If an integral $\int f\,dx$ cannot be computed, the integration function returns a partial result I and a g such that $\int f\,dx = I + \int g\,dx$. Furthermore, it gives the reason for failure, which may be that

- not all parts of the algorithm have been implemented, or
- new algebraic constants are required, or
- the given function does not fulfill the restrictions described in Section 6.5.5, or
- there is no elementary integral.

In a complete implementation, only the last item should cause the integration function to return a partial result.

The functions provided by the package can either be accessed directly through `C` function calls or through a *very* simple interactive shell.

8.4 Implemented Algorithms

Not all the algorithms given in this paper have been implemented.

- The following algorithms have been implemented as described in this paper: `integrate`, `integrate_poly`, `integrate_poly_base`, `integrate_poly_exp`, `solve_rde`, `solve_rde_base`, `integrate_frac`, `integrate_partial`, `integrate_squarefree`, `squarefree`, `resultant`, `solve_spe`, and `eea`.
- The following algorithms have been implemented only in prototype versions returning default values: `integrate_poly_log`, `solve_rde_log`, `solve_rde_exp`.
- The algorithm `integrate_normal_frac` has been implemented only in a restricted version because SACLIB does not provide functions for complete factorization. Implementing e.g. the algorithm presented in [12] would have taken too much time, so the SACLIB function for factoring multivariate polynomials over $\mathbf{Z}$ is used. If the polynomial splits into linear factors over $\mathbf{Z}$, algebraic extensions are not necessary. Factors of the form $ax^n + b = 0$ with a, b, $n \in \mathbf{Z}$, and $\sqrt[n]{-\frac{b}{a}} \in \mathbf{Q}$ can be handled, too. If factors of a different form appear, only a partial result is returned.

In addition to the algorithms described in this paper, a host of functions for elementary computations (construction and selection, arithmetic, predicates, etc.) with elements of differential rings and fields had to be written. All in all, the package consists of about 140 kB of `C` source code in about 120 files (using SACLIB's "one function per file" convention).

8.5 Data Representation

As `C` does not provide operator overloading for user written functions and as SACLIB does not support generic functions, working with data of a complex structure is not really comfortable in the environment described above.

Fortunately, computations with elements of transcendental elementary differential fields are similar to computations with elements of a rational function field in several variables: The differential properties are only required for differentiation and integration, and algebraic relations between monomials are not allowed. Therefore the implementation could be built upon functions for computations with multivariate polynomials over the integers as provided by SACLIB.

- Elements of a differential ring $R_m := \mathbf{Z}[x, \theta_1, \ldots, \theta_m]$ are represented by $(m+1)$-variate polynomials over $\mathbf{Z}$ in the SACLIB sparse recursive representation.
- Elements of a differential field $D_m := \mathbf{Q}(x, \theta_1, \ldots, \theta_m)$ are represented by a normalized numerator/denominator pair of elements of the differential ring R_m in the representation described above.
- ◦ The differential properties (or *domain data*) of R_m and D_m are stored in a separate data structure, namely a list containing a type flag (for the types x, *logarithmic monomial*, and *exponential monomial*) together with some type specific information for each monomial.

  The design of the data structures allows further extensions of the package by introducing additional types. There is (very restricted) support for transcendental constants, for instance.

As a consequence of keeping the domain data apart from the elements of the corresponding domains, functions such as the derivation require an additional parameter: the domain data must be given together with the elements to be operated on.

To keep the package consistent, all functions operating over elements of differential rings and fields take the domain data as an additional parameter, even if it is not really needed (such as in multiplication).

8.6 Availability and Maintenance

It has to be noted that the implementation in its present form is too restricted to be of any practical use. For this reason we do not give any timings or source code listings.

As explained in Section 9.3 there are no plans for further extension of the package, so maintenance will be restricted to the correction of bugs and possibly minor modifications.

For interested readers, the package is available upon request from the author[1].

[1] Send `Internet` e-mail to `aneubach@risc.uni-linz.ac.at` or mail to Andreas Neubacher, Research Institute for Symbolic Computation, Johannes Kepler University, 4040 Linz, Austria.

Chapter 9

Conclusion

9.1 Checkpoint

In this paper the author has tried to present the complete theory for an algorithm for the computation of elementary integrals.

After a summary of how integration is defined in calculus, the algebraic notion of differential fields is introduced and the theory necessary for clearly defining the problem of integration is presented. This forms the foundation for a detailed exposition of the algorithm. A brief overview of the work done for implementing the algorithm then finishes the main part of this paper.

The following section will make the reader aware of some points not covered in this paper and afterwards some concluding remarks on the insights gained by doing an implementation are given.

9.2 Omitted Topics

- As shown in Section 6.5.5, the algorithm given here suffers from the need for imposing certain restrictions on the functions to be integrated. We mentioned that there are other approaches which avoid these problems, but did not give any descriptions of these alternative algorithms. The reader is referred to [1], [7], [9], and [10] for further details.

- We also noted the fact that the set of functions which can be integrated by the algorithms presented here and in the papers cited above is rather restricted as it does not contain mixed algebraic functions. This problem is extensively covered in [2] and [13].

  An alternative problem is addressed in [3], where an algorithm for solving the base case of Risch differential equations in terms of so-called special incomplete Γ functions (e.g. the error function) is presented.

- The top level algorithm `integrate` takes an element of a transcendental elementary differential field as the function to be integrated. In general, the user wants to simply enter some function in the usual notation, and then expects the program to give a result. This leads to the following problem:

  > Given a function f and its representation $\bar{f}$ as an element an elementary function field, decide whether there exists a transcendental elementary differential field containing an element $\hat{f}$ which is also a representation for f. If such a transcendental elementary differential field exists, compute it.

The Structure Theorem given in Section 3.7.3 may be a first step towards a solution but some additional work needs to be done.

9.3 Insights Gained

As already mentioned there are currently no plans for extending the implementation to full functionality. The main reason for this is that the author has found that developing computer algebra software operating over domains of a complex structure in plain `C` is definitely *not* a good idea.

- The approach of keeping the domain data in a seperate variable introduced a lot of hard-to-find bugs (mostly from oversight).
- On the other hand, coupling the domain data with the elements would have introduced additional bookkeeping overhead especially in low level functions.
- Furthermore, the lack of operator overloading and generic functions makes future extensions complicated and programming uncomfortable.
- Another aspect is that many operations in computer algebra are more naturally formulated recursively instead of iteratively, so it might have been possible to formulate the algorithms more elegantly in e.g. a functional language.

This leads to the obvious conclusion that a programming language suitable for computer algebra should give full support for high-level user defined types (hiding all the necessary bookkeeping), perhaps be functional (with powerful iterative constructs), provide support for program development (such as modules), and still compile to fast and efficient executables.

Unfortunately, it seems that no current programming language or computer algebra system fulfills all of these requirements: either they miss some features or the execution speed is not satisfactory ...

Although the implementation does not have much use as a tool for computing integrals, it was of great value in at least one respect: it helped in improving this paper.

- Working on the theory with the intention of somehow implementing everything forces one to consider additional aspects of the problems at hand. One tends to organize things such that they can be transformed into "implementable algorithms[1]" in a natural way. This paper shows a close relation between the theory and the pseudocode description of the algorithms.
- After a first version of the paper with all the algorithms in pseudocode was finished, implementation was begun. Due to technical problems only realized by turning things into actual code, some of the algorithms had to be restructured, parameters had to be changed, etc. These changes are reflected in the paper so that the structure of the pseudocode algorithms is identical to that of the final source code.
- "To err is human", so some of the algorithms contained errors in their first versions. Several bugs were detected during test runs of the implementation and then removed from the pseudocode versions. The algorithms which are listed in Section 8.4 as having been fully implemented are quite well tested. Unfortunately, this does not mean that they are guaranteed to be free of errors, but rather that they have a greater chance of being correct than the *other* pseudocode algorithms, which have not been "reverse engineered" from actual, tested source code.

[1] Not everything which is called an "algorithm" in the literature can be turned into an executable program without first being completely redesigned. The author had to do substantial work on some of the literature until the shape of an algorithm emerged.

These experiences cause the author to conclude that theoretical work gains a lot if it goes through the process of implementation, testing, and error correction. Even if the resulting programs are not designed to be fast or efficient, they will help improve the theory and algorithm descriptions. These improvements should then help others turning the algorithms into fast, efficient software, as the author hopes he has done with this paper.

Appendix A

Examples

A.1 Checkpoint

The following sections contain examples which should illustrate the concepts and algorithms presented in the previous chapters.

This chapter is organized in such a way that each section corresponds to a chapter and each subsection corresponds to a section in one of these chapters. For easier reference, sections and subsections are numbered according to the corresponding chapters and sections, with the prefix "A" added. Therefore, as there are no examples for e.g. Section 3.2, the corresponding subsection A.3.2 is missing.

A.2 Integration in Calculus

A.2.6 Techniques of Integration

The *integration by parts* is just one of many standard techniques which can be found in any book on calculus. We will describe it here using the notation $\dot{f}$ to indicate the derivative of a function f.

Example A.2.1 (Integration by Parts) *From the product rule of differentiation* $\dot{(uv)} = \dot{u}v + u\dot{v}$ *we obtain* $uv = \int \dot{u}v + \int u\dot{v}$, *which is equivalent to*

$$\int u\dot{v} = uv - \int \dot{u}v.$$

□

A.2.7 The Problem with Heuristics

To illustrate in which sense the well known heuristics for integration are lacking we give Example 4 from Chapter 8 of Grossman's book on calculus [16].

Example A.2.2 *For computing $\int x^3 \exp(x^2)\,dx$ we might try to apply integration by parts. The following table lists all possible choices of u and $\dot v$.*

u	$\dot v$	$\dot u$	v	$uv - \int v\dot u$
$x^3e^{x^2}$	1	$(2x^4+3x^2)e^{x^2}$	x	$x^4e^{x^2} - \int(2x^5+3x^3)e^{x^2}\,dx$
$x^2e^{x^2}$	x	$(2x^3+2x)e^{x^2}$	$\frac{x^2}{2}$	$\frac{x^4}{2}e^{x^2} - \int(x^5+x^3)e^{x^2}\,dx$
xe^{x^2}	x^2	$(2x^2+1)e^{x^2}$	$\frac{x^3}{3}$	$\frac{x^4}{3}e^{x^2} - \frac{1}{3}\int(2x^5+x^3)e^{x^2}\,dx$
e^{x^2}	x^3	$2xe^{x^2}$	$\frac{x^4}{4}$	$\frac{x^4}{4}e^{x^2} - \frac{1}{2}\int x^5e^{x^2}\,dx$
x^3	e^{x^2}	$3x^2$	$\int e^{x^2}\,dx = ?$	?
x^2	xe^{x^2}	$2x$	$\frac{1}{2}e^{x^2}$	$\frac{x^2}{2}e^{x^2} - \int xe^{x^2}\,dx$
x	$x^2e^{x^2}$	1	$\int x^2e^{x^2}\,dx = ?$	?

Only one of these choices, namely $u = x^2$ and $\dot v = x\exp(x^2)$ proves useful, yielding the result

$$\begin{aligned}\int x^3\exp(x^2)\,dx &= \frac{x^2}{2}\exp(x^2) - \int x\exp(x^2)\,dx \\ &= \frac{1}{2}(x^2-1)\exp(x^2).\end{aligned}$$

All other combinations only increase the complexity of the integrals to be computed. (In Example A.6.4 we show how integrals of the form $\int x^n\exp(x^m)\,dx$ can be computed using the algorithm presented in this paper.)

□

This example shows that in addition to the method of integration by parts, one also would need rules for choosing "good" combinations of u and $\dot v$, which in turn might require a measure for the "complexity" of an integral or something similar. Without this, one must rely on experience, intuition, and trial-and-error.

A.3 An Algebraic Approach

A.3.3 Differential Fields

In Section 3.3 we concentrated on the differential field $\mathbf{Q}(x)$ because rational functions form the basis of the fields handled by the integration algorithm described in this paper. Nevertheless, there are also other fields (rings) which fulfill Definition 3.1.

Example A.3.1 *Take the ring of infinitely differentiable functions on the real numbers. We know from Lemma 2.1 that (Linearity II) and (Product Rule) hold for the derivative according to Definition 2.2. Therefore, if we define the derivation operator $f' := \frac{df}{dx}$ we have a differential ring.*

□

Note that in this example we had to take infinitely differentiable functions to ensure closedness w.r.t. differentiation. Furthermore, this ring is not of characteristic 0, because there may exist functions $f \not\equiv 0$ and $g \not\equiv 0$ with $fg \equiv 0$.

A.3.4 Differential Field Extensions

Now we will see what we can do with field extensions.

Example A.3.2 *Let $D = \mathbf{Q}(x)$ with the usual derivative. We can extend this to $D(\theta)$ by defining how the derivation operates on θ. Some possibilities are:*

$\theta' = 1$: *This means that* $\theta' = x'$, *i.e.* $(\theta - x)' = 0$ *and thus* $\theta - x \in C_D$. *So we have* $\theta = x + c$, *where* c *is some constant.*

$\theta' = \frac{1}{x}$: *From Section 2.5 we know that* $\log(cx)' = \frac{1}{x}$ *so we could interpret* θ *as* $\log(cx)$, *where* c *is some constant.*

$\theta' = \theta$: *Here we have the equality* $\exp(x+c)' = \exp(x+c)$ *from Section 2.5 and therefore could interpret* θ *as* $\exp(x+c)$, *where* c *is some constant.*

$\theta' = \frac{1}{2\theta}$: *This looks like the derivative of a square root, namely* $((x+c)^{\frac{1}{2}})' = \frac{1}{2}(x+c)^{-\frac{1}{2}}$, *so we could interpret* θ *as* $\sqrt{x+c}$, *where* c *is some constant.*

□

A.3.5 Log and Exp

The following example should give some intuition on how to get from an elementary function to a differential field representing it.

We will be a bit sloppy by avoiding to make a distinction between elementary functions and elements of the differential fields used to represent them. If we have a function f and a representation u in some differential field, then formally we would have to write $H(f) = u$. As there should not be any danger of confusion, we will use the shorter notation $f = u$.

Example A.3.3

$f = 7x + \exp(x^2)$: *We can represent* f *as an element of* $\mathbf{Q}(x, \theta)$ *with* θ *such that* $\theta' = 2x\theta$. *From this we get* $f = 7x + \theta$.

$f = \log(x) + \log(\log(x))$: *Here we need to extend twice, getting* $\mathbf{Q}(x, \theta_1, \theta_2)$ *with* $\theta_1 = \log(x)$ *(i.e.* $\theta_1' = \frac{1}{x}$*) and* $\theta_2 = \log(\theta_1)$ *(i.e.* $\theta_2' = \frac{\theta_1'}{\theta_1} = \frac{1}{x\theta_1}$*). This gives us the representation* $f = \theta_1 + \theta_2$.

$f = \sqrt{x}$: *As explained in Section 3.5 we can write this fractional power as* $\exp(\frac{1}{2}\log(x))$. *Using the differential field* $\mathbf{Q}(x, \theta_1, \theta_2)$ *with* $\theta_1 = \log(x)$ *and* $\theta_2 = \exp(\frac{1}{2}\theta_1)$ *we have the representation* $f = \theta_2$.

Note that one can prove $\theta_2^2 = x$ *using Lemma 3.2.*

$f = \sin(x)$: *Again we can use an exponential element for representation. Take the field* $\mathbf{Q}(i)(x, \theta)$, *where* i *is a constant with the property* $i^2 = -1$ *and let* $\theta = \exp(ix)$. *Then, using the equality given in Section 2.5, we have the representation* $f = \frac{1}{2i}\theta - \frac{1}{2i}\theta^{-1}$.

□

It is not always possible to find a unique differential field for a given elementary function. The problems given below show some of the ambiguities we may encounter.

Example A.3.4

$f = \exp(x) + \exp(3x)$: *Here* $D = \mathbf{Q}(x)$ *can be extended to* $D(\theta_1, \theta_2)$ *with* $\theta_1 = \exp(x)$, $\theta_2 = \exp(3x)$ *and therefore we have* $f = \theta_1 + \theta_2$. *A singly extended field* $D(\theta)$ *would also suffice:* $\theta = \exp(x)$ *yields* $f = \theta + \theta^3$ *and alternatively* $\theta = \exp(3x)$ *yields* $f = \theta^{\frac{1}{3}} + \theta$.

$f = \log(x) + \log(x^5)$: *Taking* $D = \mathbf{Q}(x)$ *we can extend to* $D(\theta_1, \theta_2)$ *with* $\theta_1 = \log(x)$ *and* $\theta_2 = \log(x^5)$ *so that we have the representation* $f = \theta_1 + \theta_2$. *Alternatively we can use the field* $D(\theta)$ *with either* $\theta = \log(x)$ *and* $f = \theta + 5\theta = 6\theta$, *or* $\theta = \log(x^5)$ *and* $f = \frac{6}{5}\theta$.

$f = \exp(\frac{1}{2}\log(x) + x)$: *Again we have more than one way of representing* f. *If we define* $\theta_1 = \log(x)$ *and* $\theta_2 = \exp(\frac{1}{2}\theta_1 + x)$, *we get* $f = \theta_2$. *But we can also take* $\theta_2 = \exp(\frac{1}{2}\theta_1)$ *and* $\theta_3 = \exp(x)$, *getting* $f = \theta_2\theta_3$, *i.e.* $f = \sqrt{x}\exp(x)$.

□

A.3.6 Transcendental Elementary Differential Fields

The definition of transcendental elementary monomials may need some clarification.

Example A.3.5 *Let $D = \mathbf{Q}(x, \log(x))$. We want to extend this differential field by some θ.*

- *Assume that we do not require θ to be transcendental over D, e.g. $\theta = \exp(\frac{1}{2}\log(x))$ so that the polynomial $p = \theta^2 - x$ is equal to zero.*

  In this case, equalities essential to the algorithm such as

  > *For all $a, b \in D(\theta)$ with $\deg(a) \neq \deg(b)$ we have $\deg(a+b) = \max\{\deg(a), \deg(b)\}$, where $\deg$ is computed w.r.t. θ.*

  do not hold anymore: e.g. if $a = \theta^4$ and $b = -2x\theta^2 + x + 5$, then $a + b = p^2 + 5 = 5$.

- *In the case of constants, something similar may happen. If we have $\theta = \log(2x)$, then $(\theta - \log(x))' = \frac{1}{x} - \frac{1}{x} = 0$, i.e. $\theta - \log(x)$ is a constant (which we could call $\log(2)$).*

  The property

  > *If θ is not exponential and the leading coefficient of $a \in D(\theta)$ is constant, then $\deg(a') = \deg(a) - 1$, where $\deg$ is computed w.r.t. θ.*

  is used several times in the algorithm, but here it does not hold: e.g. if $a = \theta^2 - 2\log(x)\theta + \log(x)^2 + x$, then this means that $a = (\theta - \log(x))^2 + x$ and therefore $a' = 1$.

□

If we restrict ourselves to integrating elements of transcendental elementary differential fields some of the ambiguities from Example A.3.4 disappear.

Example A.3.6

$f = \exp(x) + \exp(3x)$: *Of the fields given in Example A.3.4 the only transcendental elementary differential field is $D(\theta)$ with $D = \mathbf{Q}(x)$ and $\theta = \exp(x)$, because $\exp(3x)$ is not transcendental over $D(\exp(x))$ and $\exp(x)$ is not transcendental over $D(\exp(3x))$.*

$f = \log(x) + \log(x^5)$: *$\log(x)$ and $\log(x^5)$ are algebraically dependent, but we can use $\mathbf{Q}(x, \theta)$ with either $\theta = \log(x)$ or $\theta = \log(x^5)$ for representing f as 6θ or $\frac{6}{5}\theta$, respectively.*

$f = \exp(\frac{1}{2}\log(x) + x)$: *$\exp(\frac{1}{2}\log(x))$ is not transcendental over $D = \mathbf{Q}(x)$ so we can only take $D(\theta_1, \theta_2)$ with $\theta_1 = \log(x)$ and $\theta_2 = \exp(\frac{1}{2}\theta_1 + x)$ for representing f.*

This shows some of the problems we have with algebraic functions: We can deal with $f = \sqrt{x}\exp(x)$ if we write it as $\exp(\frac{1}{2}\log(x) + x)$, but we cannot handle something like $\exp(\frac{1}{2}\log(x))$ (i.e. $\sqrt{x}$) directly.

□

A.3.7 Additional Theory

Now we will give some examples for the notation introduced in Section 3.7.1.

Example A.3.7

- *Let*
$$f = 5x^3 + 2,$$
$D = \mathbf{Q}$, and $\theta = x$. f obviously is a polynomial with
$$\begin{aligned} f_{P+} &= 5\theta^3, \\ f_0 &= 2, \text{ and} \\ f_{P-} &= 0. \end{aligned}$$
Furthermore, $\delta_f = 3$ and $\gamma_f = 0$.

- *Let*
$$f = \frac{3x}{(x+1)^2},$$
$D = \mathbf{Q}$, and $\theta = x$. f is a proper fraction with
$$\begin{aligned} f_{\bar{N}} &= 3x, \text{ and} \\ f_{\bar{D}} &= (x+1)^2. \end{aligned}$$

- *Let*
$$f = \frac{1}{x^2 - 1},$$
$D = \mathbf{Q}$, and $\theta = x$. f is a normal fraction with
$$\begin{aligned} f_{\bar{N}} &= 1, \text{ and} \\ f_{\bar{D}} &= x^2 - 1. \end{aligned}$$
f is normal because its denominator splits into $(x-1)(x+1)$ and therefore is squarefree.

- *Let*
$$f = \frac{5x^5 - 5x^3 + 2x^2 - 1}{x^2 - 1},$$
$D = \mathbf{Q}$, and $\theta = x$. f is an elementary function with
$$\begin{aligned} f_P &= 5x^3 + 2 \quad \text{and} \\ f_F &= \frac{1}{x^2 - 1}. \end{aligned}$$
Furthermore, $\delta_f = 3$ and $\gamma_f = 0$.

- *Let*
$$\begin{aligned} f = {} & \frac{\log(x)\log(x^3 + \log(x))^2 + (2\log(x)^2 - 10x)\log(x^3 + \log(x))}{2x\log(x)} + \\ & + x\log(x) + 7, \end{aligned}$$
$D = \mathbf{Q}(x, \theta_1)$ with $\theta_1 = \log(x)$, and $\theta = \log(x^3 + \theta_1)$. Here we have $f_F = 0$ and f_P splits into
$$\begin{aligned} f_{P+} &= \frac{1}{2x}\theta^2 + \frac{\theta_1^2 - 5x}{x\theta_1}\theta, \\ f_0 &= x\theta_1 + 7, \text{ and} \\ f_{P-} &= 0, \end{aligned}$$
with $\delta_f = 2$ and $\gamma_f = 0$. Note that the θ-free part f_0 is not identical to the constant term, which is 7.

- *Let*

$$f = x\exp(x)^2 + 4\log(x)\exp(x) + \frac{x}{\log(x)} + 5exp(x)^{-1} + 9\log(x)^2\exp(x)^{-4}.$$

If we have $D = \mathbf{Q}(x, \theta_1)$*,* $\theta_1 = \log(x)$*, and* $\theta = \exp(x)$ *we get* $f_F = 0$*,*

$$\begin{aligned} f_{P+} &= x\theta^2 + 4\theta_1\theta, \\ f_0 &= \frac{x}{\theta_1}, \\ f_{P-} &= 5\theta^{-1} + 9\theta_1^2\theta^{-4}, \end{aligned}$$

and $\delta_f = 2$ *and* $\gamma_f = -4$*.*

We could also have $D = \mathbf{Q}(x, \theta_1)$*,* $\theta_1 = \exp(x)$*, and* $\theta = \log(x)$*. This would lead to*

$$\begin{aligned} f_{P+} &= 9\theta_1^{-4}\theta^2 + \left(4\theta_1 + \frac{1}{x}\right)\theta, \\ f_0 &= x\theta_1^2 + 5\theta_1^{-1}, \\ f_{P-} &= 0, \text{ and} \\ f_F &= \frac{x}{\theta}, \end{aligned}$$

with $\delta_f = 2$ *and* $\gamma_f = 0$*.*

- *Now, let* $D = \mathbf{Q}(x)$*,* $\theta = \exp(x)$ *and*

$$f = \frac{2\theta^5 + 6\theta^3 + 2x\theta^2 + 3x}{\theta^2(\theta^2 + 3)}.$$

This is an elementary function with

$$\begin{aligned} f_{P+} &= 2\theta, \\ f_0 &= 0, \\ f_{P-} &= x\theta^{-2}, \text{ and} \\ f_F &= \frac{x}{\theta^2 + 3}. \end{aligned}$$

Here we have $\delta_f = 1$*,* $\gamma_f = -2$*, and* f_F *is a normal fraction.*

□

A.3.8 The Problem of Integration in Finite Terms

First, let us consider the consequences of Definition 3.7 (Integral).

Example A.3.8 *Let* $D = \mathbf{Q}(x, \exp(x^2))$*. The problem of computing* $\int \exp(x^2)$ *could be solved by extending* D *to* $D(F)$ *with* $F = \int \exp(x^2)$*. We might call this new element* error function*. We do not have any explicit elementary function related to this element, but only know its differential property* $F' = \exp(x^2)$*.*

□

Definition 3.8 (Elementary Integral) does not allow such an easy way out.

Example A.3.9 *As we see from Example A.6.4 (with* $m = 2$ *and* $n = 0$*) there is no solution to the Risch differential equation* $y' + 2xy = 1$*, and therefore there is no elementary function field containing an element* F *with* $F' = \exp(x^2)$*. This means that* $\exp(x^2)$ *has no elementary integral.*

□

A.4 An Outline of the Algorithm

A.4.3 Decomposition

Below we give a trivial example showing that $\int u + v\,dx = \int u\,dx + \int v\,dx$ need not always hold.

Example A.4.1 *Let* $u = x + \exp(x^2)$ *and* $v = x - \exp(x^2)$. *Neither* $\int u\,dx$ *nor* $\int v\,dx$ *is elementary, but* $\int u + v\,dx = \int 2x\,dx$ *is.*

□

Note that this example does not contradict Lemma 4.1 because splitting $2x$ into $x + \exp(x^2)$ and $x - \exp(x^2)$ is not equivalent to splitting it into a polynomial and a fractional part.

A.4.4 An Outline of the Algorithm

The structure of the top level function `integrate` is not too complicated, but nevertheless we give an example so that all levels of the algorithm are covered here.

Example A.4.2 *Let* $f \in \mathbf{Q}(x)$ *with*

$$f = \frac{5x^5 - 5x^3 + 2x^2 - 1}{x^2 - 1}.$$

Dividing $f_{\bar{N}}$ *by* $f_{\bar{D}}$ *yields*

$$\begin{aligned} f_P &= 5x^3 + 2 \quad \textit{and} \\ f_F &= \frac{1}{x^2 - 1}. \end{aligned}$$

We integrate these separately in Examples A.5.1 and A.7.3, getting the solution

$$\begin{aligned} \int f\,dx &= \int 5x^3 + 2\,dx + \int \frac{1}{x^2 - 1}\,dx \\ &= \frac{5}{4}x^4 + 2x + \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x+1). \end{aligned}$$

□

A.5 How to Integrate Polynomials

A.5.2 The Base Case

The following example is given for completeness only — it should be clear how polynomials are integrated in the base case.

Example A.5.1 *Let* $p \in \mathbf{Q}(x)$ *with*

$$p = 5x^3 + 2.$$

From the formula given in Section 5.2 we can immediately compute the integral

$$\int p\,dx = \frac{5}{4}x^4 + 2x.$$

□

A.5.3 The Logarithmic Case

In the following three examples we will integrate logarithmic polynomials by applying the scheme described in Section 5.3.

Example A.5.2 *Let $p \in D(\theta)$ with $D = \mathbf{Q}(x)$, $\theta = \log(x)$, and*

$$p = (4x+1)\theta^2 - 3\theta + 1 + \frac{1}{x^2-1}.$$

Following the algorithm, we compute

$$\begin{aligned}
\int p_2\,dx &= \int 4x+1\,dx &&= \underbrace{2x^2+x}_{\bar{q}_2},\\
\int p_1 - 2\bar{q}_2\frac{x'}{x}\,dx &= \int -4x-5\,dx &&= \underbrace{-2x^2-5x}_{\bar{q}_1},\\
\int p_0 - \bar{q}_1\frac{x'}{x}\,dx &= \int \frac{1}{x^2-1}+2x+6\,dx &&= \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x+1) +\\
& && \quad + \underbrace{x^2+6x}_{\bar{q}_0},
\end{aligned}$$

and $c_1 = c_2 = c_3 = 0$, so that finally we have

$$\int p\,dx = (2x^2+x)\theta^2 - (2x^2+5x)\theta + (x^2+6x) + \frac{1}{2}\log(x-1) - \frac{1}{2}\log(x+1).$$

In Example A.7.3 we show how the new logarithmic terms appearing in the integral of p_0 are computed.

□

In the previous example, all constants were equal to zero. We will now show what happens when this is not the case.

Example A.5.3 *Let $p \in D(\theta)$ with $D = \mathbf{Q}(x)$, $\theta = \log(x)$, and*

$$p = \frac{4x^2+5}{x}\theta^2 + \frac{3}{x}.$$

Here we get

$$\begin{aligned}
\int p_2\,dx &= \int 4x + \frac{5}{x}\,dx &&= \underbrace{2x^2}_{\bar{q}_2} + \underbrace{5}_{c_3}\theta,\\
\int p_1 - 2\bar{q}_2\frac{x'}{x}\,dx &= \int -4x\,dx &&= \underbrace{-2x^2}_{\bar{q}_1} + \underbrace{0}_{c_2}\theta,\\
\int p_0 - \bar{q}_1\frac{x'}{x}\,dx &= \int \frac{1}{x} + 2x\,dx &&= \underbrace{x^2}_{\bar{q}_0} + \underbrace{3}_{c_1}\theta,
\end{aligned}$$

and therefore

$$\int p\,dx = \frac{5}{3}\theta^3 + 2x^2\theta^2 - (-2x^2+3)\theta + x^2.$$

□

Finally, let's see how we can detect that there is no elementary integral for a given p.

Example A.5.4 *Let $p \in D(\theta)$ with $D = \mathbf{Q}(x)$, $\theta = \log(x)$, and*

$$p = \frac{1}{x+1}\theta.$$

The first integral to be computed yields

$$\int p_1\,dx = \int \frac{1}{x+1}\,dx = \log(x+1).$$

This cannot be written in the form $\bar{q}_1 + c_2\theta$ because $\log(x+1)$ is transcendental over $D(\theta)$. Therefore, $\int p\,dx$ is not an elementary integral.

□

A.5.4 The Exponential Case

The relation between polynomials in the exponential case and Risch differential equations should be clear from the description given in Section 5.4. Therefore the following example rather concentrates on another point, namely how to compute

$$\int \sin(x)^2\,dx = \frac{x}{2} - \frac{1}{2}\sin(x)\cos(x)$$

using the methods presented in this paper. As we have already shown in Example A.3.3 we can use the equalities

$$\begin{aligned} \sin(x) &= \frac{1}{2i}\exp(ix) - \frac{1}{2i}\exp(-ix), \text{ and} \\ \cos(x) &= \frac{1}{2}\exp(ix) + \frac{1}{2}\exp(-ix) \end{aligned}$$

for representing sin and cos as elements of a transcendental elementary differential field.

Example A.5.5 *Let $p \in D(\theta)$ with $D = \mathbf{Q}(i)(x)$, i a constant such that $i^2 = -1$, $\theta = \exp(ix)$, and*

$$p = -\frac{1}{4}\theta^2 + \frac{1}{2} - \frac{1}{4}\theta^{-2}.$$

We know that

$$\int p\,dx = q_2\theta^2 + q_0 + q_{-2}\theta^{-2},$$

where the q_i must fulfill the equations

$$\begin{aligned} q_2' + 2iq_2 &= -\frac{1}{4}, \\ q_0 &= \int \frac{1}{2}\,dx, \text{ and} \\ q_{-2}' - 2iq_{-2} &= -\frac{1}{4}. \end{aligned}$$

Solving the Risch differential equations and computing the integral we get

$$\begin{aligned} q_2 &= -\frac{1}{8i}, \\ q_0 &= \frac{x}{2}, \text{ and} \\ q_{-2} &= \frac{1}{8i}, \end{aligned}$$

so that finally we have

$$\int p\,dx = \frac{x}{2} - \frac{1}{8i}(\theta^2 - \theta^{-2}).$$

□

Note that the result derived in this example can be interpreted as $\frac{x}{2} - \frac{1}{2}\sin(x)\cos(x)$ according to the equalities given above.

A.6 How to Solve the Risch Differential Equation

A.6.3 Weak Normalization

Now we will try to clarify the effect weak normalization has on a given integration problem.

Example A.6.1 *Let $f, g \in \mathbf{Q}(x)$ with*

$$\begin{aligned} f &= 2x + \frac{3}{x} \quad \textit{and} \\ g &= \frac{5}{x^2}. \end{aligned}$$

Before trying to solve the Risch differential equation $y' + fy = g$ we have to check whether f is weakly normalized. From

$$\int f\, dx = x^2 + 3\log(x)$$

we see that Condition (b) is violated. This situation can be amended as follows: Let $p = x^{-3}$ and substitute $y \leftarrow p\bar{y}$. Furthermore, let

$$\begin{aligned} \bar{f} &= \frac{p'}{p} + f &&= 2x, \quad \textit{and} \\ \bar{g} &= p^{-1}g &&= 5x. \end{aligned}$$

Then $y' + fy = g$ is equivalent to $\bar{y}' + \bar{f}\bar{y} = \bar{g}$. The latter equation has the solution $\bar{y} = \frac{5}{2}$, which is computed in Example A.6.4, and from $y = p\bar{y}$ we get

$$y = \frac{5}{2x^3},$$

the solution to the original equation.

□

The mechanism employed in the previous example becomes more intuitive when it is interpreted as follows: The original equation relates to the integral

$$\int \frac{5}{x^2} \exp(x^2 + 3\log(x))\, dx.$$

Instead of computing this integral (i.e. solving the original equation) directly, we use the equality

$$\exp(x^2 + 3\log(x)) = x^3 \exp(x^2)$$

and compute

$$\int 5x \exp(x^2)\, dx.$$

Therefore, the effect of weak normalization is the reduction of expressions of the form $\exp(n\log(u))$ to the form u^n.

If n is a rational number and Condition (c) is violated, things are a bit more complicated. The following example covers such a situation.

Example A.6.2 *Let $f,\ g \in \mathbf{Q}(x)$ with*

$$\begin{aligned} f &= \frac{1}{2x} - \frac{3}{2(x+1)} \quad \text{and} \\ g &= \frac{1}{2}(x+1)(4x+3). \end{aligned}$$

Again we see that f is not weakly normalized because

$$\int f\,dx = \frac{1}{2}\log(x) - \frac{3}{2}\log(x+1),$$

i.e. $r_1 = \frac{1}{2}$, $p_1 = x$, $r_2 = -\frac{3}{2}$, $p_2 = x+1$ in the notation of Section 6.3, and by $\sum_{i=1}^{2} r_i \delta_{p_i} = -1$ Condition (c) is violated.

Here we let $n_1 = 0$ and $n_2 = 2$, getting $p = (x+1)^2$. Substituting as in the previous example we have

$$\begin{aligned} \bar{f} &= \frac{p'}{p} + f &&= \frac{2x+1}{2x(x+1)}, \quad \text{and} \\ \bar{g} &= p^{-1} g &&= 2 - \frac{1}{2(x+1)}. \end{aligned}$$

In Example A.6.5 the resulting equation $\bar{y}' + \bar{f}\bar{y} = \bar{g}$ is solved with $\bar{y} = x$ and we have

$$y = p\bar{y} = x(x+1)^2$$

as the solution to the original equation.

□

Here, weak normalization results in replacing the integral

$$\int \frac{1}{2}(x+1)(4x+3)\exp(\frac{1}{2}\log(x) - \frac{3}{2}\log(x+1))\,dx$$

by

$$\int \left(2 - \frac{1}{2(x+1)}\right) \exp(\frac{1}{2}\log(x) + \frac{1}{2}\log(x+1))\,dx,$$

applying the equality

$$\exp(\frac{1}{2}\log(x) - \frac{3}{2}\log(x+1)) = \frac{1}{(x+1)^2}\exp(\frac{1}{2}\log(x) + \frac{1}{2}\log(x+1)).$$

A.6.4 How to Compute the Denominator

The formula given in Section 6.4 is not too complicated so a single example should suffice here.

Example A.6.3 *Let $f,\ g \in D(\theta)$ with $D = \mathbf{Q}(x)$, $\theta = \log(x)$,*

$$\begin{aligned} f &= \frac{4\theta+1}{x\theta(\theta+1)}, \quad \text{and} \\ g &= \frac{3\theta^2+8\theta+3}{x\theta(\theta+1)^3}. \end{aligned}$$

For solving the Risch differential equation $y' + fy = g$ we first compute the denominator $y_{\bar{D}}$ of the solution. According to the formula given in Section 6.4 we compute

$$\begin{aligned} h &= \gcd(f_{\bar{D}}, g_{\bar{D}}) \;=\; \gcd(x\theta(\theta+1), x\theta(\theta+1)^3) \;=\; x\theta(\theta+1), \\ \frac{dg_{\bar{D}}}{d\theta} &= x(\theta+1)^2(4\theta+1), \quad \text{and} \\ \frac{dh}{d\theta} &= x(2\theta+1). \end{aligned}$$

From this we get

$$y_{\bar{D}} = \frac{\gcd(g_{\bar{D}}, \frac{dg_{\bar{D}}}{d\theta})}{\gcd(h, \frac{dh}{d\theta})} = \frac{x(\theta+1)^2}{x} = (\theta+1)^2.$$

Now we need to set up the equation for the numerator $y_{\bar{N}}$ *of the solution. If we let*

$$\begin{aligned} \bar{f} &= f - \frac{y'_{\bar{D}}}{y_{\bar{D}}} = \frac{4\theta+1}{x\theta(\theta+1)} - \frac{2}{x(\theta+1)} = \frac{2\theta+1}{x\theta(\theta+1)} \quad \text{and} \\ \bar{g} &= g y_{\bar{D}} = \frac{3\theta^2+8\theta+3}{x\theta(\theta+1)}, \end{aligned}$$

then the numerator must fulfill the Risch differential equation

$$y'_{\bar{D}} + \bar{f} y_{\bar{D}} = \bar{g}.$$

The solution for this equation (which we compute in Example A.6.6) is $\theta+3$*, so that we get*

$$y = \frac{\theta+3}{\theta+1}$$

as the solution for the original equation.

□

A.6.5 How to Compute the Numerator

The Base Case

Here we will address the integration problem raised in Example A.2.2.

Example A.6.4 *The integral*

$$\int x^n \exp(x^m)\, dx$$

directly leads to the Risch differential equation

$$y' + \underbrace{mx^{m-1}}_{f} y = \underbrace{x^n}_{g}.$$

Here are in the base case with $f_P \neq 0$*. Therefore, if* $n < m-1$ *the equation has no solution (e.g.* $\int \exp(x^2)\, dx$ *is not an elementary integral).*

Otherwise, we have $\delta_y = n - (m-1)$ *and* $y_{\delta_y} = \frac{1}{m}$*. Substituting* $y \leftarrow \frac{1}{m}x^{n-m+1} + \bar{y}$ *yields the equation*

$$\bar{y}' + \underbrace{mx^{m-1}}_{\bar{f}} \bar{y} = \underbrace{-\frac{n-m+1}{m}x^{n-m}}_{\bar{g}}.$$

We can continue in this manner until either $\bar{g} = 0$*, in which case we have a solution, or* $\delta_{\bar{g}} < m-1$*, in which case there is no solution.*

□

The following problem, which appears in Example A.6.2 is an example for the base case with $f_P = 0$ and $\deg(f_N) = \deg(f_D) - 1$.

Example A.6.5 *Let* $f, g \in \mathbf{Q}(x)$ *with*

$$\begin{aligned} f &= \frac{2x+1}{2x(x+1)} \quad \text{and} \\ g &= 2 - \frac{1}{2(x+1)}. \end{aligned}$$

We have $g_P = g_0 = 2 \neq 0$ *and therefore*

$$\delta_y = \delta_g + 1 = 1.$$

Furthermore, we have $f_L = \frac{2}{2}$ *and thus*

$$y_{\delta_y} = \frac{g_0}{\delta_y + f_L} = 1.$$

This far we know that $y = x + \bar{y}$ *for some* $\bar{y}$*. Substitution yields*

$$y' + fy = g \Leftrightarrow \bar{y}' + f\bar{y} = 0,$$

so we see that $\bar{y} = 0$ *and we get the solution*

$$y = x.$$

□

The Logarithmic Case

Now we will consider a similar situation in the logarithmic case, which we encountered as a subproblem in Example A.6.3.

Example A.6.6 *Let* $f, g \in D(\theta)$ *with* $D = \mathbf{Q}(x)$, $\theta = \log(x)$,

$$\begin{aligned} f &= \frac{2\theta + 1}{x\theta^2 + x\theta}, \text{ and} \\ g &= \frac{3}{x} + \frac{5\theta + 3}{x\theta^2 + x\theta}. \end{aligned}$$

We have $f_P = 0$ *and* $\deg(f_N) = \deg(f_D) - 1$*, so following the algorithm given in Section 6.5.3 we set*

$$\delta = \delta_g + 1 = 1$$

and compute

$$\begin{aligned} \int g_{\delta_g}\, dx &= \int \frac{3}{x}\, dx = 3\theta, \text{ and} \\ \int f_L\, dx &= \int \frac{2}{x}\, dx = 2\theta, \end{aligned}$$

from which follows $d_1 = 3$, $h_1 = 0$, $d_2 = 2$*, and* $h_2 = 0$ *according to the notation used in the algorithm description. We get*

$$\begin{aligned} \delta_y &= \delta &&= 1, \\ \bar{c}_{\delta_y} &= \frac{d_1}{\delta_y + d_2} &&= 1, \text{ and} \\ \bar{y}_{\delta_y - 1} &= h_1 - \bar{c}_{\delta_y} h_2 &&= 0. \end{aligned}$$

and therefore we can conclude that $y = \theta + \bar{y}$*, where the leading coefficient of* $\bar{y}$ *is constant (possibly equal to zero).*

Substituting yields

$$\bar{g} = g - \theta' - f\theta = \frac{3}{x} + \frac{5\theta + 3}{x\theta^2 + x\theta} - \frac{1}{x} - \frac{2\theta^2 + \theta}{x\theta^2 + x\theta} = \frac{6\theta + 3}{x\theta^2 + x\theta}.$$

For the equation

$$\bar{y}' + f\bar{y} = \bar{g}$$

we have $\bar{g}_P = 0$ *so that* $\delta_{\bar{y}} = 0$ *and*

$$\bar{y}_{\delta_{\bar{y}}} = \frac{\bar{g}}{f} = 3.$$

Therefore the overall solution is

$$y = \theta + 3.$$

□

The Exponential Case

In the following example we consider a Risch differential equation problem where we might have cancellation in y_{γ_y}.

Example A.6.7 *Let* f, $g \in D(\theta)$ *with* $D = \mathbf{Q}(x)$, $\theta = \exp(x)$,

$$\begin{aligned} f &= \theta - \frac{1}{x} + \frac{6}{\theta^2+3}, \text{ and} \\ g &= 5\theta^2 + \frac{5x-5}{x}\theta + \frac{30\theta}{\theta^2+3}. \end{aligned}$$

Checking for cancellation we see that

$$\begin{aligned} f_0 &= -\frac{1}{x} \text{ and} \\ f_T &= \frac{6}{3} = 2, \end{aligned}$$

so that we have

$$\int f_0 + f_T \, dx = 2x - \log(x).$$

According to the notation used in Section 6.5.3 we have $N = 2$ *and* $w = x$. *We substitute* $y \leftarrow \bar{y}\frac{x}{\theta^2}$ *and get*

$$\begin{aligned} \bar{f} &= f - f_0 - f_T &&= \theta - 2 + \frac{6}{\theta^2+3}, \text{ and} \\ \bar{g} &= g\frac{\theta^2}{x} &&= \frac{5}{x}\theta^4 + \frac{5x-5}{x^2}\theta^3 + \frac{30}{x}\theta - \frac{90\theta}{x(\theta^2+3)}. \end{aligned}$$

For the equation $\bar{y}' + \bar{f}\bar{y} = \bar{g}$ *we have* $\delta_{\bar{f}} = 1$ *and* $\delta_{\bar{g}} = 4$, *from which we get* $\delta_{\bar{y}} = 3$ *and* $\bar{y}_{\delta_{\bar{y}}} = \frac{\bar{g}_{\delta_{\bar{g}}}}{\bar{f}_{\delta_{\bar{f}}}} = \frac{5}{x}$, *and therefore* $\bar{y} = \frac{5}{x}\theta^3 + \bar{\bar{y}}$.

We substitute again and see that $\bar{\bar{y}}$ *must fulfill* $\bar{\bar{y}}' + \bar{f}\bar{\bar{y}} = 0$. *The solution is* $\bar{\bar{y}} = 0$, *so that* $\bar{y} = \frac{5}{x}\theta^3$ *and finally*

$$y = 5\theta.$$

□

Restrictions of Davenport's Approach

Now we will give a function of the kind described in Section 6.5.5.

Example A.6.8 *Let* $f, g \in D(\theta)$ *with* $D = \mathbf{Q}(x)$, $\theta = \exp(x)$,

$$\begin{aligned} f &= -\frac{1}{x} - 1 + \frac{2}{\theta+1}, \text{ and} \\ g &= x - \frac{2x}{\theta+1}. \end{aligned}$$

We have $f_0 = -\frac{1}{x} - 1$ *and* $f_T = 2$, *so that*

$$\begin{aligned} \int f_0 \, dx &= -x - \log(x), \text{ and} \\ \int f_0 + f_T \, dx &= x - \log(x). \end{aligned}$$

This means that there is cancellation both in y_{δ_y} *and in* y_{γ_y} *so that we can neither compute* δ_y *from* δ_f *and* δ_g *nor* γ_y *from* γ_f *and* γ_g. *Furthermore, there is no substitution which would remove this problem.*

The solution (which cannot be computed using the method presented in Section 6.5) is

$$y = x\theta + x + x\theta^{-1}.$$

□

For getting a better notion of the problems encountered in the previous example consider the following table, where some f, g, and y fulfilling $y' + fy = g$ and their corresponding degrees are listed.

f	g	y	δ_f	δ_g	δ_y	γ_f	γ_g	γ_y
$-\frac{1}{x} - 1 + \frac{2}{\theta+1}$	$2x - \frac{2x}{\theta+1}$	$x\theta$	0	0	1	0	0	0
$-\frac{1}{x} - 1 + \frac{2}{\theta+1}$	$-x + \frac{2x}{\theta+1}$	x	0	0	0	0	0	0
$-\frac{1}{x} - 1 + \frac{2}{\theta+1}$	$0 - \frac{2x}{\theta+1}$	$x\theta^{-1}$	0	0	0	0	0	-1

We see that due to cancellation the degree of y does not depend on the degrees of f and g.

A.7 How to Integrate Proper Fractions

A.7.2 How to Compute the Rational Part

In the following examples we will not give the squarefree factorizations, partial fraction decompositions, etc. in detail. These computations are considered to be elementary and therefore are left to the reader[1].

Example A.7.1 *Let* $f \in \mathbf{Q}(x)$ *with*

$$f = \frac{2x^4 - 4x^2 + 4x + 8}{(2x+3)(x^2-1)^2}.$$

[1]Most of the corresponding algorithms are given in Appendix B.

The squarefree factors of $f_{\bar{D}}$ *are* $2x+3$ *and* x^2-1*. A partial fraction decomposition yields*

$$f = \frac{2}{(x^2-1)^2} + \frac{2}{2x+3}.$$

If we consider the first fraction we have $p = 2$, $q = x^2 - 1$, *and* $j = 2$ *in the notation of Section 7.2. According to the scheme given there we now have to find* a *and* b *such that* $aq' + bq = \frac{p}{j-1}$*. With* $a = x$ *and* $b = -2$ *this equation is fulfilled and we get*

$$\begin{aligned}\int \frac{2}{(x^2-1)^2}\,dx &= -\frac{a}{q^{j-1}} + \int \frac{a' + (j-1)b}{q^{j-1}}\,dx \\ &= -\frac{x}{x^2-1} - \int \frac{1}{x^2-1}\,dx.\end{aligned}$$

The integral of $\frac{1}{x^2-1}$ *is computed in Example A.7.3, the integral of* $\frac{2}{2x+3}$ *is* $\log(2x+3)$*, so that we have*

$$\begin{aligned}\int f\,dx &= \int \frac{2}{(x^2-1)^2}\,dx + \int \frac{2}{2x+3}\,dx \\ &= -\frac{x}{x^2-1} - \int \frac{1}{x^2-1}\,dx + \int \frac{2}{2x+3}\,dx \\ &= -\frac{x}{x^2-1} - \frac{1}{2}\log(x-1) + \frac{1}{2}\log(x+1) + \log(2x+3).\end{aligned}$$

□

The scheme employed in the previous example is not restricted to the functions in $\mathbf{Q}(x)$ as the following example shows.

Example A.7.2 *Let* $f \in D(\theta)$ *with* $D = \mathbf{Q}(x)$, $\theta = \log(x)$, *and*

$$f = \frac{2x(1-\theta)}{(\theta^2 - x^2)^2}.$$

Here we have $p = 2x(1-\theta)$, $q = \theta^2 - x^2$, *and* $j = 2$*. The equation* $aq' + bq = \frac{p}{j-1}$ *is fulfilled by* $a = \theta$ *and* $b = -\frac{2}{x}$*, so we get*

$$\begin{aligned}\int f\,dx &= -\frac{a}{q^{j-1}} + \int \frac{a' + (j-1)b}{q^{j-1}}\,dx \\ &= -\frac{\theta}{\theta^2 - x^2} - \int \frac{1}{x\theta^2 - x^3}\,dx.\end{aligned}$$

As we see from Example A.7.4, $\int \frac{1}{x\theta^2 - x^3}\,dx$ *is not elementary and we have to be satisfied with this partial result.*

□

A.7.3 How to Compute the Logarithmic Part

For the resultant computations and factorizations done in the following examples only the results are given. As the examples are not too complicated it should be clear how these results are derived.

Example A.7.3 *Let* $h \in \mathbf{Q}(x)$ *with*

$$h = \frac{1}{x^2-1}.$$

This is a normal fraction with $h_{\bar{N}} = 1$, $h_{\bar{D}} = x^2 - 1$, *and* $h'_{\bar{D}} = 2x$.

Following the algorithm given in Section 7.3 we first compute the resultant

$$\begin{aligned}
\mathrm{res}_x(h_{\bar{N}} - \alpha h'_{\bar{D}}, h_{\bar{D}}) &= \mathrm{res}_x(1 - 2\alpha x, x^2 - 1) \\
&= \det \begin{pmatrix} -2\alpha & 1 & 0 \\ 0 & -2\alpha & 1 \\ 1 & 0 & -1 \end{pmatrix} \\
&= -4\alpha^2 + 1.
\end{aligned}$$

Factorizing w.r.t. α *yields*

$$-4\alpha^2 + 1 = (-2\alpha + 1)(2\alpha + 1),$$

so that we get the distinct roots $c_1 = \frac{1}{2}$ *and* $c_2 = -\frac{1}{2}$. *Then we have*

$$\begin{aligned}
v_1 &= \gcd(h_{\bar{N}} - c_1 h'_{\bar{D}}, h_{\bar{D}}) \\
&= \gcd(1 - x, x^2 - 1) \\
&= x - 1 \quad \textit{(normalized)}
\end{aligned}$$

and

$$\begin{aligned}
v_2 &= \gcd(h_{\bar{N}} - c_2 h'_{\bar{D}}, h_{\bar{D}}) \\
&= \gcd(1 + x, x^2 - 1) \\
&= x + 1 \quad \textit{(normalized)}.
\end{aligned}$$

So finally we get the solution

$$\int \frac{1}{x^2 - 1}\, dx = \frac{1}{2}\log(x - 1) - \frac{1}{2}\log(x + 1).$$

□

The following example shows what happens when an integral is not elementary.

Example A.7.4 *Let* $h \in D(\theta)$ *with* $D = \mathbf{Q}(x)$, $\theta = \log(x)$, *and*

$$h = \frac{1}{x\theta^2 - x^3}.$$

We have $h_{\bar{N}} = 1$, $h_{\bar{D}} = x\theta^2 - x^3$, *and* $h'_{\bar{D}} = \theta^2 + 2\theta - 3x^2$, *so that*

$$\begin{aligned}
\mathrm{res}(1 - \alpha h'_{\bar{D}}, h_{\bar{D}}) &= \mathrm{res}(-\alpha\theta^2 - 2\alpha\theta + 3x^2\alpha + 1, x\theta^2 - x^3) \\
&= \det \begin{pmatrix} -\alpha & -2\alpha & 3x^2\alpha + 1 & 0 \\ 0 & -\alpha & -2\alpha & 3x^2\alpha + 1 \\ x & 0 & -x^3 & 0 \\ 0 & x & 0 & -x^3 \end{pmatrix} \\
&= x^2\left((2x^2 - 2x)\alpha - 1\right)\left((2x^2 + 2x)\alpha - 1\right).
\end{aligned}$$

We see that the roots w.r.t. α *are* $\frac{1}{2x^2 - 2x}$ *and* $\frac{1}{2x^2 + 2x}$. *These are not constants and therefore* $\int h\, dx$ *is not elementary.*

□

Finally we will give an example covering the exponential case, where an additional term may appear in the computation of the integral of a normal function.

Example A.7.5 *Let* $h \in D(\theta)$ *with* $D = \mathbf{Q}(x)$, $\theta = \exp(x)$, *and*

$$h = \frac{1}{\theta^2 + 3}.$$

Here we have $h_{\bar{N}} = 1$, $h_{\bar{D}} = \theta^2 + 3$, *and* $h'_{\bar{D}} = 2\theta^2$. *This leads to the resultant*

$$\begin{aligned}
\mathrm{res}(1 - \alpha h'_{\bar{D}}, h_{\bar{D}}) &= \mathrm{res}(-2\alpha\theta^2 + 1, \theta^2 + 3) \\
&= \det \begin{pmatrix} -2\alpha & 0 & 1 & 0 \\ 0 & -2\alpha & 0 & 1 \\ 1 & 0 & 3 & 0 \\ 0 & 1 & 0 & 3 \end{pmatrix} \\
&= (6\alpha + 1)^2.
\end{aligned}$$

The only root (not counting multiplicities) is $-\frac{1}{6}$ *and we get*

$$\begin{aligned}
\gcd(h_{\bar{N}} - \alpha h'_{\bar{D}}, h_{\bar{D}}) &= \gcd(1 + \frac{1}{3}\theta^2, \theta^2 + 3) \\
&= \theta^2 + 3.
\end{aligned}$$

As θ *is exponential we must subtract* $u \sum_i c_i \deg(v_i)$. *In this example we have* $u = x$, $c_1 = -\frac{1}{6}$, *and* $v_1 = \theta^2 + 3$, *so that the final result is*

$$\int h \, dx = \frac{x}{3} - \frac{1}{6} \log(\theta^2 + 3).$$

□

Appendix B

Some Results from Computer Algebra

B.1 Checkpoint

In previous chapters we did not explain how to do squarefree factorization, partial fraction decomposition, etc. These problems are not special to integration but also appear in other areas of computer algebra. For readers who are not familiar with the methods of symbolic computation this appendix contains a brief description of how these problems can be solved.

The material here is mainly based on the appendix "Algebraic background" in [15] and various articles in [14].

B.2 Squarefree Factorization

Definition B.1 *A polynomial $p \in D[\theta]$ is called* squarefree *when it has no factors of multiplicity greater than 1.*

Given a monic polynomial $p \in D[\theta]$ with $\deg(p) \geq 1$, we want to find a factorization

$$p = \prod_{i=1}^{n} p_i^i,$$

with $p_i \in D[\theta]$, $\gcd(p_i, p_j) = 1$ for $i \neq j$, each p_i is squarefree, and n is the greatest integer with $p_n \neq 1$ (note that either $p_i = 1$ or $\deg(p_i) \geq 1$).

The advantage of this factorization is that we do not need to introduce new algebraic terms while it is sufficient for various applications such as the integration of proper fractions.

Before we show how we can compute the squarefree factors, we prove a lemma which will play an essential part in the algorithm.

Lemma B.1 *Let $p \in D[\theta]$ with $\deg(p) \geq 1$. When θ is exponential we also require $\gcd(p, \theta) = 1$. Let $\prod_{i=1}^{n} p_i^i$ be a squarefree factorization of p. Then we have*

$$\gcd(p, p') = \prod_{i=2}^{n} p_i^{i-1}.$$

Proof:

(i) Assume that p is squarefree. We have to show that $\gcd(p, p') = 1$. Let $n := \deg(p)$. If $p = \prod_{i=1}^{n} l_i$ is a linear factorization of p, we have

$$p' = \sum_{i=1}^{n} l_i' \prod_{\substack{j=1\\ j\neq i}}^{n} l_j.$$

The l_i are relatively prime and $\gcd(l_i, l_i') = 1$ (in the exponential case θ is required not to divide p), so each l_i divides all the terms of the sum but one. From this follows that p and p' must be relatively prime.

(ii) Now assume that p is not squarefree and we have a squarefree factorization $p = \prod_{i=1}^{n} p_i^i$. This time we have

$$\begin{aligned} p' &= \sum_{i=1}^{n} i p_i' p_i^{i-1} \prod_{\substack{j=1\\ j\neq i}}^{n} p_j^j \\ &= (\sum_{i=1}^{n} i p_i' \prod_{\substack{j=1\\ j\neq i}}^{n} p_j) \prod_{i=1}^{n} p_i^{i-1}. \end{aligned}$$

The p_i are squarefree so by (i) we have $\gcd(p_i, p_i') = 1$, therefore the sum in the second line is not divisible by any factor of p and we have $\gcd(p, p') = \prod_{i=2}^{n} p_i^{i-1}$.

□

Using Lemma B.1 we will now show how one can compute squarefree factorizations. Let $q := \gcd(p, p')$. Then

$$\begin{aligned} \frac{p}{q} &= \prod_{i=1}^{n} p_i, \\ \gcd(\frac{p}{q}, q) &= \prod_{i=2}^{n} p_i, \end{aligned}$$

and thus

$$p_1 = \frac{p}{q} \frac{1}{\gcd(\frac{p}{q}, q)}.$$

Now q contains the squarefree factors of p with multiplicities reduced by 1, so we can iterate until we have found all squarefree factors of p.

These computations only use division and derivatives so we always have $p_i \in D[\theta]$, i.e. there are no algebraic extensions necessary.

B.3 The Extended Euclidean Algorithm and Some Applications

B.3.1 The Extended Euclidean Algorithm

Given polynomials $p, q \in D[\theta]$, the classical Euclidean algorithm gives a method for computing $\gcd(p, q)$. It is based on the fact that $\gcd(p, q) = \gcd(q, \mathrm{rem}(p, q))$, where we assume $\deg(p) \geq \deg(q)$. Starting with $r_1 = p$ and $r_2 = q$, we can compute a sequence of remainders $r_{i+1} = \mathrm{rem}(r_{i-1}, r_i)$ until we have $r_{n+1} = 0$. Then $r_n = \gcd(p, q)$.

This scheme can be modified such that factors $a, b \in D[\theta]$ are computed, with $\deg(a) < \deg(q)$, $\deg(b) < deg(p)$, and

$$ap + bq = \gcd(p, q).$$

The idea here is the following: From

$$\begin{aligned} a_{i-1}p + b_{i-1}q &= r_{i-1} \\ a_i p + b_i q &= r_i \end{aligned}$$

we want to get

$$a_{i+1}p + b_{i+1}q = \mathrm{rem}(r_{i-1}, r_i).$$

As we know that $\mathrm{rem}(r_{i-1}, r_i) = r_{i-1} - \mathrm{quot}(r_{i-1}, r_i)r_i$, we have $a_{i+1} = a_{i-1} - \mathrm{quot}(r_{i-1}, r_i)a_i$ and $b_{i+1} = b_{i-1} - \mathrm{quot}(r_{i-1}, r_i)b_i$. As initial values we take $a_1 = 1$, $b_1 = 0$, $a_2 = 0$, and $b_2 = 1$.

To assure uniqueness of the gcd we will normalize it to a monic polynomial and adjust a and b accordingly.

B.3.2 A Special Polynomial Equation

In Section 7.2 we need to solve polynomial equations of the following kind:

Given $p, q, r \in D[\theta]$ with $\gcd(p, q) = 1$, we want to find $a, b \in D[\theta]$ such that

$$r = ap + bq.$$

As we have seen in the previous section, by applying the extended Euclidean algorithm we can find $\bar{a}$ and $\bar{b}$ such that $1 = \bar{a}p + \bar{b}q$. We multiply by r, getting $r = r\bar{a}p + r\bar{b}q$. Now there are two cases:

- If $\deg(p) + \deg(q) \leq \deg(r)$ then we have $a = r\bar{a}$ and $b = r\bar{b}$ with $\deg(a) < \deg(r) + \deg(q)$ and $\deg(b) < \deg(r) + \deg(p)$.

- If $\deg(p) + \deg(q) > \deg(r)$ then we can set $a = \mathrm{rem}(r\bar{a}, q)$ and $b = \mathrm{rem}(r\bar{b}, p)$, getting $\deg(a) < \deg(q)$ and $\deg(b) < \deg(p)$.

B.3.3 Partial Fraction Decomposition

Given a proper fraction $\frac{r}{pq} \in D(\theta)$ with p and q relatively prime, we want to find $a, b \in D[\theta]$ such that

$$\frac{r}{pq} = \frac{a}{q} + \frac{b}{p}$$

and $\frac{a}{q}$ and $\frac{b}{p}$ are again proper fractions.

Multiplying by pq yields $r = ap + bq$ and we can compute a and b by applying the method described in the previous section.

B.4 Resultants

This section gives only a brief overview containing a definition and some properties of resultants. For proofs and some further details see e.g. the section "Resultant Calculus" of the article "Computing in Algebraic Extensions" by R. Loos in [14] or the appendix "Algebraic background" in [15].

Definition B.2 *Let* $p, q \in D[\theta]$ *with* $p = \sum_{i=0}^{m} p_i \theta^i$ *and* $q = \sum_{i=0}^{n} q_i \theta^i$. *The* Sylvester matrix of p and q *is the matrix*

$$S = \underbrace{\begin{pmatrix} p_m & p_{m-1} & \cdots & p_0 & & & \\ & p_m & p_{m-1} & \cdots & p_0 & & \\ & & \ddots & \ddots & \cdots & \ddots & \\ & & & p_m & p_{m-1} & \cdots & p_0 \\ q_n & q_{n-1} & \cdots & q_0 & & & \\ & q_n & q_{n-1} & \cdots & q_0 & & \\ & & \ddots & \ddots & \cdots & \ddots & \\ & & & q_n & q_{n-1} & \cdots & q_0 \end{pmatrix}}_{m+n \text{ columns}} \begin{array}{l} \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} n \text{ rows} \\ \left.\vphantom{\begin{array}{c}a\\a\\a\\a\end{array}}\right\} m \text{ rows} \end{array}$$

The resultant of p and q *(w.r.t.* θ*) is defined to be the determinant of* S*, in symbolic notation*

$$\mathrm{res}_\theta(p, q) = \det(S).$$

We will omit the θ *when there is no danger of confusion.*

In Section 7.3 we use a fundamental property of resultants given in the following lemma.

Lemma B.2 *Let* $p, q \in D[\theta]$, p *and* q *non-zero. Then*

$$\mathrm{res}(p, q) = 0 \textit{ if and only if } \gcd(p, q) = 1$$

The next lemma gives some properties which suggest an algorithm similar to the Euclidean algorithm for computing gcds.

Lemma B.3 *Let* $d \in D$, $p, q, r \in D[\theta]$, $r = \mathrm{rem}(q, p)$, $l = \deg(p)$, $m = \deg(q)$, $n = \deg(r)$. *Then*

(i) $\mathrm{res}(d, q) = d^m$,

(ii) $\mathrm{res}(p, q) = (-1)^{lm} \mathrm{res}(q, p)$,

(iii) $\mathrm{res}(p, q) = p_l^{m-n} \mathrm{res}(p, r)$.

There exist algorithms for computing resultants which are more sophisticated than the one given here, but as this problem is not special to integration we chose a straightforward approach and refer the reader to the corresponding literature for further information.

B.5 Complete Factorization

In Section 7.3 we need to split a polynomial into linear factors. The required theory and algorithms are deemed to be outside the scope of this paper, therefore we refer the reader to Trager's paper [12]. There the problem of complete factorization is covered in the context of an algorithm for integrating (normal) rational functions which is quite similar to the one given in Section 7.3.

B.6 Algorithms

$$(p_1, \ldots, p_n) \leftarrow \texttt{squarefree}(p)$$

Compute the squarefree factorization of a polynomial.

Inputs:
p : a polynomial in $D[\theta]$.

Outputs:
$p_1, \ldots, p_n$: the squarefree factors of p.

Begin

Step 1: *Recursion base.*

$q \leftarrow \gcd(p, p')$
if $\deg(q) = 0$ **then**
  $p_1 \leftarrow p$
  return
endif

Step 2: *Compute the first factor.*

$p_1 \leftarrow \frac{p}{q} \frac{1}{\gcd(\frac{p}{q}, q)}$
$q \leftarrow \gcd(p_1, p_1')$

Step 3: *Recursion.*

$(p_2, \ldots, p_n) \leftarrow \texttt{squarefree}(q)$
return $(p_1, p_2, \ldots, p_n)$

End

Algorithm 14: `squarefree`

$(g,\ a,\ b) \leftarrow \texttt{eea}(p,\ q)$

Compute the gcd of two polynomials using the extended Euclidean algorithm.

Inputs:
$p,\ q :\in D[\theta]$.

Outputs:
$g :\in D[\theta]$, primitive, the gcd of p and q.
$a,\ b :\in D[\theta]$ such that $\deg(a) < \deg(q)$, $\deg(b) < \deg(p)$, and $g = ap + bq$.

Begin

Step 1: *Initialize.*
if $\deg(p) < \deg(q)$ **then**
$t \leftarrow p,\ p \leftarrow q,\ q \leftarrow t$
endif
$a_p \leftarrow 1,\ b_p \leftarrow 0$
$a_q \leftarrow 0,\ b_q \leftarrow 1$

Step 2: *Loop.*
while $q \neq 0$ **do**
$t \leftarrow a_q,\ a_q \leftarrow a_p - \mathrm{quot}(p, q)a_q,\ a_p \leftarrow t$
$t \leftarrow b_q,\ b_q \leftarrow b_p - \mathrm{quot}(p, q)b_q,\ b_p \leftarrow t$
$t \leftarrow q,\ q \leftarrow p - \mathrm{quot}(p, q)q,\ p \leftarrow t$
endwhile

Step 3: *Normalize.*
$g \leftarrow \mathrm{primitive\_part}(p)$
$a \leftarrow \frac{a_p}{\mathrm{content}(p)}$
$b \leftarrow \frac{b_p}{\mathrm{content}(p)}$
return $(g,\ a,\ b)$

End

Algorithm 15: `eea`

$$(a,\, b) \leftarrow \texttt{solve\_spe}(r,\, p,\, q)$$

Solve a special kind of polynomial equation.

Inputs:

$r :\in D[\theta]$.

$p,\, q :\in D[\theta]$ with $\gcd(p, q) = 1$.

Outputs:

$a,\, b :\in D[\theta]$ such that $r = ap + bq$. If $\deg(r) < \deg(p) + \deg(q)$ then $\deg(a) < \deg(q)$ and $\deg(b) < \deg(p)$.

Begin

Step 1: *Apply the extended Euclidean algorithm.*

$(1,\, \bar{a},\, \bar{b}) \leftarrow \texttt{eea}(p,\, q)$

Step 2: *Reduce degrees.*

if $\deg(r) < \deg(p) + \deg(q)$ **then**
  $a \leftarrow \mathrm{rem}(r\bar{a}, q)$
  $b \leftarrow \mathrm{rem}(r\bar{b}, p)$
else
  $a \leftarrow r\bar{a}$
  $b \leftarrow r\bar{b}$
endif
return $(a,\, b)$

End

Algorithm 16: `solve_spe`

$r \leftarrow$ `resultant`$(D[\theta],\, p,\, q)$

Compute the resultant.

Inputs:
$D[\theta]$: a polynomial ring over D.
$p,\, q$: $\in D[\theta]$.

Outputs:
r : $\in D$, the resultant of p and q w.r.t. θ.

Begin

Step 1: *Initialize.*
$l \leftarrow \deg(p)$
$m \leftarrow \deg(q)$

Step 2: *Swap?*
if $l > m$ **then return** $(-1)^{lm}$ `resultant`$(D[\theta],\, q,\, p)$

Step 3: *Recursion base.*
if $l = 0$ **then return** p_0^m

Step 4: *Recursion.*
$r \leftarrow \operatorname{rem}(q, p)$
$n \leftarrow \deg(r)$
return p_l^{m-n} `resultant`$(D[\theta],\, p,\, r)$

End

Algorithm 17: `resultant`

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