Implicitization of Curves Parameterized by Generalized Trigonometric Polynomials. *

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Abstract

Consider a plane curve given parametrically by a generalized trigonometric polynomial, that is, $x + iy = \sum_{k=1}^{n} a_k e^{ik\theta}$. In this paper, we obtain an implicitization of the curve, that is, an equation in x and y which captures all the points on the curve and, if any, only finitely many more points.

1 Introduction

Generalized trigonometric polynomials, $\sum_{k=1}^{n} a_k e^{ik\theta}$, $a_k, \theta \in \mathbb{R}$, are interesting and important mathematical objects. They have a very natural interpretation as the superpositions of rotating vectors (often called phasers) on a plane, having applications in numerous areas such as linear differential equations, Fourier analysis, almost periodic functions, representation of groups, electrical circuit analysis, caustics in fracture mechanics, etc.

Consider the trajectory of a generalized trigonometric polynomial on the complex plane. It defines a curve given by

$$x + iy = \sum_{k=1}^{n} a_k e^{ik\theta}.$$

Written by components, we obtain

$$\begin{aligned} x &= \sum_{k=1}^{n} a_k \cos k\theta \\ y &= \sum_{k=1}^{n} a_k \sin k\theta. \end{aligned}$$

Thus, we can view it as a real plane curve given parametrically in θ.

In this paper, we study the implicitization of this curve, that is, we would like to find an equation in x and y that captures all the points on the curve (and if any, only finitely many more points).

One could tackle the problem as follows: (1) Rewrite $\cos k\theta$ and $\sin k\theta$ as polynomials in $\cos\theta$ and $\sin\theta$. (2) Parameterize $\cos\theta$ and $\sin\theta$ by the usual rational parameterization of a circle, obtaining a rational parameterization of the curve. (3) Implicitize the rational parameterization by using general methods in computational algebraic geometry such as Buchberger's Groebner basis method [1, 2, 7, 11, 8], Collins' cylindrical algebraic decomposition method [5, 9, 6], Ritt-Wu's characteristic set method [12], etc.

However one can often devise a more efficient method for a particular problem class by taking advantage of its special structure. This is what we do in this paper. By utilizing the particular structure of trigonometric polynomial, we devised a much simpler and computationally cheaper method.

In essence, we show that the resultant of the following two polynomials with respect to z gives an implicitization of the curve:

$$-(x + iy) + \sum_{k=1}^{n} a_k z^k$$

$$-(x - iy) z^n + \sum_{k=1}^{n} a_k z^{n-k}.$$

There is no magic about this form, since the first polynomial can be obtained by simply naming $e^{i\theta}$ with z, and the second polynomial can be obtained by conjugating the first one and multiplying out the common denominator. Further, it is straightforward to show that the resultant captures all the points on the curve.

Non-triviality lied in showing that it captures only finitely many more points, if any (or equivalently

*This research was done in the framework of the European project POSSO.

showing that it is irreducible). Thus, the proof of this fact is the main contribution of this paper. We also show that the resultant is a *real* polynomial, even though its definition involves complex numbers.

The structure of the paper is as follows: In Section 2, we review several classical results on resultants which will be used heavily in Section 4. In Section 3, we state more precisely the implicitization method informally described above. In Section 4, we prove that the method indeed gives implicitization. In Section 5, we give a small example arising from the theory of caustics in fracture mechanics.

2 Review of Resultants

In this section, we review some well known facts about resultants. For a nice summary on the subject, see [4] where one can also find an efficient method for computing resultants.

Consider two polynomials $A, B \in \mathbb{C}[x_1, \ldots, x_\nu]$ such that

$$\begin{aligned} A &= \sum_{k=0}^{m} A_k(x_1, \ldots, x_{\nu-1}) x_\nu^k \\ B &= \sum_{k=0}^{n} B_k(x_1, \ldots, x_{\nu-1}) x_\nu^k \end{aligned}$$

where A_m and B_n are non-zero polynomials and $m, n \geq 0$.

Definition 1 (Resultant) *The resultant of A and B with respect to x_ν, written as $\mathrm{res}_{x_\nu}(A, B)$, is the determinant of the (Sylvester) matrix:*

$$\begin{bmatrix} A_m & A_{m-1} & \cdots & A_0 & & & \\ & A_m & A_{m-1} & \cdots & A_0 & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & A_m & A_{m-1} & \cdots & A_0 \\ B_n & B_{n-1} & \cdots & B_0 & & & \\ & B_n & B_{n-1} & \cdots & B_0 & & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & B_n & B_{n-1} & \cdots & B_0 \end{bmatrix}$$

in which there are n rows of A coefficients, m rows of B coefficients, and all elements not shown are zero. ■

The following classical theorems will be used in the subsequent sections.

Theorem 1 (Projection of solution) *Let $C = \mathrm{res}_{x_\nu}(A, B) \in \mathbb{C}[x_1, \ldots, x_{\nu-1}]$. Let $(p_1, \ldots, p_\nu) \in \mathbb{C}^\nu$ be such that*

$$\begin{aligned} A(p_1, \ldots, p_\nu) &= 0 \\ B(p_1, \ldots, p_\nu) &= 0 \end{aligned}$$

Then we have

$$C(p_1, \ldots, p_{\nu-1}) = 0. \blacksquare$$

Theorem 2 (Extension of solution) *Let $C = \mathrm{res}_{x_\nu}(A, B) \in \mathbb{C}[x_1, \ldots, x_{\nu-1}]$. Let $(p_1, \ldots, p_{\nu-1}) \in \mathbb{C}^{\nu-1}$ be such that*

$$C(p_1, \ldots, p_{\nu-1}) = 0.$$

Then one of the followings holds:

(a) *$A_m(p_1, \ldots, p_{\nu-1}) = 0$ and $B_n(p_1, \ldots, p_{\nu-1}) = 0$.*

(b) *$A_i(p_1, \ldots, p_{\nu-1}) = 0$ for $m \geq i \geq 0$.*

(c) *$B_i(p_1, \ldots, p_{\nu-1}) = 0$ for $n \geq i \geq 0$.*

(d) *There exists a $p_\nu \in \mathbb{C}$ such that*

$$\begin{aligned} A(p_1, \ldots, p_\nu) &= 0 \\ B(p_1, \ldots, p_\nu) &= 0. \blacksquare \end{aligned}$$

Theorem 3 (Commutativity of evaluation) *Let $(p_1, \ldots, p_{\nu-1}) \in \mathbb{C}^{\nu-1}$ be such that*

$$\begin{aligned} 0 &\neq A_m(p_1, \ldots, p_{\nu-1}) \\ 0 &\neq B_n(p_1, \ldots, p_{\nu-1}). \end{aligned}$$

Let

$$\begin{aligned} A^* &= A(p_1, \ldots, p_{\nu-1}, x_\nu) \\ B^* &= B(p_1, \ldots, p_{\nu-1}, x_\nu). \end{aligned}$$

Let

$$\begin{aligned} C &= \mathrm{res}_{x_\nu}(A, B) \\ C^* &= \mathrm{res}_{x_\nu}(A^*, B^*). \end{aligned}$$

Then we have

$$C^* = C(p_1, \ldots, p_{\nu-1}). \blacksquare$$

3 Implicitization

Consider a curve given parametrically by

$$\begin{aligned} x &= \sum_{k=1}^{n} a_k \cos k\theta \qquad (1) \\ y &= \sum_{k=1}^{n} a_k \sin k\theta. \end{aligned}$$

where $a_1, \ldots, a_n, \theta \in \mathbb{R}$ and $a_n \neq 0$.

Let $F, G \in \mathbb{C}[x, y, z]$ be defined by

$$\begin{aligned} F &= -(x + iy) + \sum_{k=1}^{n} a_k z^k \\ G &= -(x - iy) z^n + \sum_{k=1}^{n} a_k z^{n-k}. \end{aligned}$$

Let $H \in \mathbb{C}[x, y]$ be defined by

$$H = \mathrm{res}_z(F, G),$$

that is, H is the determinant of the matrix:

$$\begin{bmatrix} a_n & a_{n-1} & \cdots & a_1 & a_0 & & & & \\ & a_n & a_{n-1} & \cdots & a_1 & a_0 & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & a_n & a_{n-1} & \cdots & a_1 & a_0 & \\ \overline{a_0} & a_1 & \cdots & a_{n-1} & a_n & & & & \\ & \overline{a_0} & a_1 & \cdots & a_{n-1} & a_n & & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & & \\ & & & \overline{a_0} & a_1 & \cdots & a_{n-1} & a_n & \end{bmatrix}$$

where $a_0 = -(x + iy)$ and $\overline{a_0} = -(x - iy)$, where the upper bar stands for complex conjugation.

We claim that the polynomial H gives an implicitization of the curve defined by the equations (1). More in detail, we claim

Theorem 4 (Implicitization)

- *$H = 0$ captures all the points on the curve.*
- *$H = 0$ captures, if any, only finitely many more points.*
- *H is a real polynomial, that is $H \in \mathbb{R}[x, y]$.*

The next section is devoted in proving the claims.

4 Proof

First we show that H captures all the points on the curve. More precisely, let

$$\begin{aligned} S &= \left\{ (x, y) \in \mathbb{R}^2 \mid (\exists \theta \in \mathbb{R}) \begin{array}{l} x = \sum_{k=1}^n a_k \cos k\theta \\ y = \sum_{k=1}^n a_k \sin k\theta \end{array} \right\} \\ T &= \{ (x, y) \in \mathbb{R}^2 \mid H(x, y) = 0 \}. \end{aligned}$$

Theorem 5 (Containment) $S \subseteq T$. ■

Proof.

Let $(p, q) \in S$. We need to show that $(p, q) \in T$. First note that from the definition of S, there must exist $t \in \mathbb{R}$ such that

$$p = \sum_{k=1}^n a_k \cos kt$$
$$q = \sum_{k=1}^n a_k \sin kt.$$

By using the elementary fact: $\cos t + i \sin t = e^{it}$, we obtain

$$\begin{aligned} p + iq &= \sum_{k=1}^n a_k e^{ikt} \\ p - iq &= \sum_{k=1}^n a_k e^{-ikt}. \end{aligned}$$

By moving the left hand sides to the right and by multiplying e^{int} to the second equation, we obtain

$$\begin{aligned} 0 &= -(p + iq) + \sum_{k=1}^n a_k e^{ikt} \\ 0 &= -(p - iq)e^{int} + \sum_{k=1}^n a_k e^{i(n-k)t}. \end{aligned}$$

Note that the right hand sides are the polynomials F and G evaluated at p, q, e^{it}. Let r denote e^{it}. Then, we have

$$\begin{aligned} F(p, q, r) &= 0 \\ G(p, q, r) &= 0 \end{aligned}$$

By Theorem 1, we immediately have

$$H(p, q) = 0.$$

Thus, $(p, q) \in T$. ■

Now one might wonder whether $S = T$. Unfortunately this is not true in general, as illustrated by the following counter example:

$$\begin{aligned} x &= 5/2 \cos \theta - \cos 2\theta \\ y &= 5/2 \sin \theta - \sin 2\theta. \end{aligned}$$

The associated polynomial H is

$$-\frac{21}{4} - \frac{33}{4}x^2 - \frac{33}{4}y^2 + x^4 + 2x^2y^2 + y^4 + \frac{25}{2}x.$$

A simple calculation shows that the point $(1, 0)$ is on the curve T given by $H = 0$, but it is not on the curve S given by the parametric equations. In fact, in this particular case, we have

$$T - S = \{(1, 0)\}.$$

Observe that the set $\{(1, 0)\}$ is a finite set. Thus one naturally wonders whether $T - S$ is always a finite set. The following theorem gives an affirmative answer.

Theorem 6 (Finite Exceptions) *$T - S$ is a finite set.* ■

Proof. [1]

Let $(p, q) \in T - S$. Since $(p, q) \in T$, we have

$$H(p, q) = 0.$$

[1] From the basic facts of algebraic geometry, one sees that the finite exception property is equivalent to the irreducibility of H. Thus we have initially tried to prove the irreducibility of H, but without success. The proof given here *directly* shows the finite exception property (thus providing an *indirect* proof of the irreducibility of H).

Now we recall Theorem 2. Since $a_n \neq 0$, we have

$$\begin{aligned} F_n(p,q) &= a_n \neq 0 \\ G_0(p,q) &= a_n \neq 0. \end{aligned}$$

Thus only the case *(d)* applies, and we conclude that there exists $r \in \mathbb{C}$ such that

$$\begin{aligned} F(p,q,r) &= 0 \\ G(p,q,r) &= 0. \end{aligned}$$

From the definition of F and G, we have

$$\begin{aligned} p + iq &= \sum_{k=1}^{n} a_k r^k && (2) \\ (p - iq) r^n &= \sum_{k=1}^{n} a_k r^{n-k}. \end{aligned}$$

By multiplying the first equation by $\overline{r}^n$, we obtain

$$\begin{aligned} (p + iq)\overline{r}^n &= \sum_{k=1}^{n} a_k r^k \overline{r}^n \\ (p - iq) r^n &= \sum_{k=1}^{n} a_k r^{n-k}. \end{aligned}$$

Thus, we have

$$\overline{\sum_{k=1}^{n} a_k r^k \overline{r}^n} = (p - iq) r^n = \sum_{k=1}^{n} a_k r^{n-k}.$$

By collecting the both sides into one, we get

$$\begin{aligned} 0 &= \sum_{k=1}^{n} a_k (\overline{r}^k r^n - r^{n-k}) \\ &= \sum_{k=1}^{n} a_k (\overline{r}^k r^k - 1) r^{n-k}. && (3) \end{aligned}$$

Now we claim that $\overline{r}r \neq 1$. Assume otherwise. Then there exists $t \in \mathbb{R}$ such that $r = e^{it}$. From the equation (2), we have

$$p + iq = \sum_{k=1}^{n} a_k e^{ikt}.$$

So, we should have

$$\begin{aligned} p &= \sum_{k=1}^{n} \cos kt \\ q &= \sum_{k=1}^{n} \sin kt. \end{aligned}$$

Thus, $(p,q) \in S$. But this contradicts the initial assumption that $(p,q) \in T - S$. Thus, we have proved the claim that $\overline{r}r \neq 1$.

Since $\overline{r}r \neq 1$, we can divide the equation (3) safely by $\overline{r}r - 1$, obtaining

$$0 = \sum_{k=1}^{n} a_k ((\overline{r}r)^{k-1} + \cdots + \overline{r}r + 1) r^{n-k}.$$

Now from Lemma 1 (which is given right after this proof), we see that only finitely many values of r can satisfy the equation. Thus there are only finitely many values for (p,q). So we have finally proved that $T - S$ is a finite set. ■

Now we state and prove a technical lemma that was needed in the proof of the above theorem.

Lemma 1 (Finite Solution) *Let $a_1, \ldots, a_n \in \mathbb{R}$ and $a_n \neq 0$. Let*

$$P(r) = \sum_{k=1}^{n} a_k ((\overline{r}r)^{k-1} + \cdots + \overline{r}r + 1) r^{n-k}.$$

Then the equation $P(r) = 0$ has only finitely many solutions in $\mathbb{C}$. ■

Proof. The whole proof will essentially consists of proving the following two claims:

C1: There are only finitely many solutions on any circle centered at the origin of the complex plane. Precisely, for all $\rho \in \mathbb{R}$, the set

$$\{r \in \mathbb{C} \mid P(r) = 0, \rho = \overline{r}r\}$$

is finite.

C2: There are only finitely many circles (centered at the origin) containing a solution. Precisely, the set

$$\{\overline{r}r \mid P(r) = 0\}$$

is finite.

From these two, one immediately sees that there can be only finite many solutions for $P(r) = 0$.

Now we begin by proving the claim **C1**. Let $\rho = \overline{r}r$. Then

$$P(r) = \sum_{k=1}^{n} a_k (\rho^{k-1} + \cdots + 1) r^{n-k}. \quad (4)$$

Note that the trailing coefficient of P is

$$a_n(\rho^{n-1} + \cdots + 1).$$

This is non-zero since $a_n \neq 0$ and $\rho \geq 0$. So for every value of ρ, $P(r)$ is a non-zero polynomial in r, and thus it has only finitely many solutions. Hence we have proved the claim **C1**.

Next we will prove the claim **C2**. Conjugating the equation $P(r) = 0$, we obtain

$$0 = \sum_{k=1}^{n} a_k(\rho^{k-1} + \cdots + 1)\overline{r}^{n-k}.$$

Multiplying by r^{n-1}, we obtain

$$\begin{aligned} 0 &= \sum_{k=1}^{n} a_k(\rho^{k-1} + \cdots + 1)\overline{r}^{n-k} r^{n-1}. \\ &= \sum_{k=1}^{n} a_k(\rho^{k-1} + \cdots + 1)\rho^{n-k} r^{k-1}. \qquad (5) \end{aligned}$$

Summarizing, we have obtained two equations:

$$\begin{aligned} 0 &= A(\rho, r) = \sum_{k=1}^{n} a_k(\rho^{k-1} + \cdots + 1)\rho^{n-k} r^{k-1}. \\ 0 &= B(\rho, r) = \sum_{k=1}^{n} a_k(\rho^{k-1} + \cdots + 1) r^{n-k}. \end{aligned}$$

Now we will show that the resultant of the two polynomials A, B with respect to r is a non-zero polynomial in ρ. We will do so by showing that it is non-zero when evaluated at $\rho = 0$. For this, we need to be cautious about the possibility that some of a_k's might be zero. Thus let us assume, without losing any generality, that

$$a_1 = a_2 = \cdots = a_{\mu-1} = 0, a_\mu \neq 0$$

for some μ such that $1 \leq \mu \leq n$.

Now we will apply Theorem 3 by evaluating ρ to be 0. Let $L_A, L_B \in \mathbb{C}[\rho]$ be the leading coefficients of A and B in r respectively. Then we have

$$\begin{aligned} L_A &= a_n(\rho^{n-1} + \cdots + 1) \\ L_B &= a_\mu(\rho^{\mu-1} + \cdots + 1) \end{aligned}$$

Since $a_n \neq 0$ and $a_\mu \neq 0$, we see that

$$\begin{aligned} 0 &\neq L_A(0) \\ 0 &\neq L_B(0). \end{aligned}$$

Let

$$\begin{aligned} A^* &= A(0, r) \\ B^* &= B(0, r). \end{aligned}$$

Let

$$\begin{aligned} C &= \mathrm{res}_r(A, B) \\ C^* &= \mathrm{res}_r(A^*, B^*). \end{aligned}$$

By Theorem 3, we have

$$C^* = C(0). \qquad (6)$$

Now we will show that C^* is non-zero (and thus that C is a non-zero polynomial). By the definition of resultant, we see that C^* is the determinant of the matrix:

$$\begin{bmatrix} a_n & & & & & & & \\ & a_n & & & & & & \\ & & \cdots & & & & & \\ & & & a_n & & & & \\ a_\mu & \cdots & \cdots & a_{n-1} & a_n & & & \\ & a_\mu & \cdots & \cdots & a_{n-1} & a_n & & \\ & & \cdots & \cdots & \cdots & \cdots & \cdots & \\ & & & a_\mu & \cdots & \cdots & a_{n-1} & a_n \end{bmatrix}$$

Note that it is a lower triangular matrix, and thus its determinant is the product of the diagonal elements, that is, a certain power of a_n in this case. Since $a_n \neq 0$, the determinat C^* is also non-zero.

Recalling the relation (6), we conclude that C is a non-zero polynomial. Therefore, there can only finitely many values for ρ. Thus, we have proved the claim **C2**. ■

Now, we show that the polynomial H has only real coefficients, that is, $H \in \mathbb{R}[x, y]$, in spite of the fact that its definition involves imiginary numer unit i. In fact, we will prove a more general result.

Let F and G be two polynomials in $\mathbb{C}[x_1, \ldots, x_\nu]$ such that

$$\begin{aligned} F &= \sum_{k=0}^{n} F_k x_\nu^k \\ G &= \sum_{k=0}^{n} \overline{F_k} x_\nu^{n-k} \end{aligned}$$

where $F_k \in \mathbb{C}[x_1, \ldots, x_{\nu-1}]$ and $\overline{F_k}$ is the complex conjugate of F_k (in the sense that the coefficients of F_k are replaced by their complex conjugates). Let $H = \mathrm{res}_{x_\nu}(F, G)$.

Note that in this paper, we are interested only in a particular case where $\nu = 3, x_1 = x, x_2 = y, x_3 = z$ and

$$F_k = \begin{cases} a_k & \text{for } k \geq 1 \\ -(x + iy) & \text{for } k = 0. \end{cases}$$

But the following result holds in general.

Theorem 7 (Real) $H \in \mathbb{R}[x_1, \ldots, x_{\nu-1}]$. ■

Proof. We will show that $H = \overline{H}$. From the definition of resultant, we have

$$H = \det \begin{bmatrix} F_n & \cdots & \cdots & F_0 & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & F_n & \cdots & \cdots & F_0 \\ \overline{F_0} & \cdots & \cdots & \overline{F_n} & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & \overline{F_0} & \cdots & \cdots & \overline{F_n} \end{bmatrix}$$

By flipping the matrix left-right (which can be done by n swapping of columns), we obtain

$$H = \det \begin{bmatrix} & & F_0 & \cdots & \cdots & F_n \\ & \cdots & \cdots & \cdots & \cdots & \\ F_0 & \cdots & \cdots & F_n & & \\ & & \overline{F_n} & \cdots & \cdots & \overline{F_0} \\ & \cdots & \cdots & \cdots & \cdots & \\ \overline{F_n} & \cdots & \cdots & \overline{F_0} & & \end{bmatrix} (-1)^n$$

By flipping the matrix top-down (which can be done by n swapping of rows), we obtain

$$H = \det \begin{bmatrix} \overline{F_n} & \cdots & \cdots & \overline{F_0} & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & \overline{F_n} & \cdots & \cdots & \overline{F_0} \\ F_0 & \cdots & \cdots & F_n & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & F_0 & \cdots & \cdots & F_n \end{bmatrix} (-1)^{2n}$$

Since the conjugate operation commutes with determinant operation, we have

$$H = \overline{\det \begin{bmatrix} F_n & \cdots & \cdots & F_0 & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & F_n & \cdots & \cdots & F_0 \\ \overline{F_0} & \cdots & \cdots & \overline{F_n} & & \\ & \cdots & \cdots & \cdots & \cdots & \\ & & \overline{F_0} & \cdots & \cdots & \overline{F_n} \end{bmatrix}}$$

Thus we finally have

$$H = \overline{H}. \blacksquare$$

5 An Example

In this section, we give a simple example arising from the theory of caustics in the facture mechanics, namely

$$\begin{aligned} x &= \cos 2\theta + 2/3 \cos 3\theta \\ y &= \sin 2\theta + 2/3 \sin 3\theta. \end{aligned}$$

In [10] Ioakimidis and Anastasselou tackled the problem of implicitization this curve by using Groebner basis method [1, 2], obtaining a result in about 5 minutes using the computer algebra system Maple V [3] running on 80386 DX MS-DOS microcomputer at 20 MHz.

Using the method described in the present paper, this can be done by simply computing the determinant of the following matrix:

$$\begin{bmatrix} 2/3 & 1 & & a_0 & & \\ & 2/3 & 1 & & a_0 & \\ & & 2/3 & 1 & & a_0 \\ \overline{a_0} & & 1 & 2/3 & & \\ & \overline{a_0} & & 1 & 2/3 & \\ & & \overline{a_0} & & 1 & 2/3 \end{bmatrix}$$

where $a_0 = -(x + iy)$ and $\overline{a_0} = -(x - iy)$. This is small enough for even hand caculation. However, just for comparision, we have carried out the determinant computation using the same computer algebra system on the same machine. We obtained the same polynomial (upto some constant factor) as in [10]:

$$-\tfrac{80}{729} + \tfrac{8}{9}x - \tfrac{55}{27}x^2 - \tfrac{55}{27}y^2 + \tfrac{10}{3}x^4 + \tfrac{20}{3}x^2y^2 + \tfrac{10}{3}y^4$$
$$-3y^2x^4 - 3x^2y^4 - x^6 - y^6$$

in about 300 milli-seconds.

Now one wonders whether the polynomial captures any points not from the curve. In order to resolve this question, we recall the polynomial from Theorem 1:

$$P(r) = \sum_{k=1}^{n} a_k(\overline{r}r)^{k-1} + \cdots + \overline{r}r + 1)r^{n-k}.$$

If this polynomial does not have any root with non-unit magnitude in $\mathbb{C}$, then we can conclude that the implicitization does not introduce any new points. In fact, it is the case with the current example as we will show now.

For the example, $n = 3, a_1 = 0, a_2 = 1, a_3 = 2/3$. Thus, we need to study the equation

$$0 = 2/3((\overline{r}r)^2 + \overline{r}r + 1) + (\overline{r}r + 1)r. \qquad (7)$$

Since $\overline{r}r$ is a non-negative real number, r must be a real number. Thus, we can drop the conjugation, obtaining

$$0 = 2/3r^4 + r^3 + 2/3r^2 + r + 2/3.$$

It has only one real root -1, but which is of unit magnitude. Thus, the equation (7) does not have a root of non-unit magnitude. Hence we conclude that the implicitization does not introduce any new points.

Acknowledgment: The work reported here was initially motivated by a personal communication and in particular by the paper sent by Nikolaos I. Ioakimidis. The example of caustic equation is taken from his paper (co-authored with Anastasselou). I also thank Joseph Schicho and Bernhard Wall for various interesting discussions on the subject.

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