The exact region of stability for MacCormack scheme *

Hoon Hong
Research Institute for Symbolic Computation
Johannes Kepler University
A-4040 Linz, Austria
e-mail: hhong@risc.uni-linz.ac.at

Abstract

Let the two dimensional scalar avection equation be given by $\frac{\partial u}{\partial t} = \hat{a}\frac{\partial u}{\partial x} + \hat{b}\frac{\partial u}{\partial y}$. We prove that the stability region of the MacCormack scheme for this equation is *exactly* given by

$$\left(\hat{a}\frac{\Delta_t}{\Delta_x}\right)^{\frac{2}{3}} + \left(\hat{b}\frac{\Delta_t}{\Delta_y}\right)^{\frac{2}{3}} \leq 1$$

where Δ_t, Δ_x and Δ_y are the grid distances. It is interesting to note that the stability region is identical to the one for Lax-Wendroff scheme proved by Turkel.

1 Introduction

Stability analysis is one of the most fundamental tasks in devising reliable numerical methods for solving partial differential equations. In this paper, we investigate the stability of MacCormack's scheme [8] which is widely used in solving hyperbolic partial differential equations, especially in aerodynamics. In particular we consider the following scalar two dimensional avection equation:

$$\frac{\partial u}{\partial t} = \hat{a}\frac{\partial u}{\partial x} + \hat{b}\frac{\partial u}{\partial y}.$$

In [12] Wendroff derived the following sufficient condition for the stability region:

$$a^2b^2 + a^2 + b^2 \leq 1/8.$$

where a and b are respectively $\hat{a}\frac{\Delta_t}{\Delta_x}$ and $\hat{b}\frac{\Delta_t}{\Delta_y}$, where again Δ_t, Δ_x and Δ_y are the grid distances. As it was remarked [12], numerical sampling shows that it is not a necessary condition. The main result of this paper is the derivation of the following *necessary and sufficient* condition:

$$a^{\frac{2}{3}} + b^{\frac{2}{3}} \leq 1.$$

Figure 1 compares the region obtained by Wendroff and the one obtained here.

It is interesting to note that the exact stability region of MacCormack scheme derived here is identical to that for Lax-Wendroff scheme proved by Turkel [11]. In fact, one will notice in the subsequent sections that the underlying proof techniques are similar.

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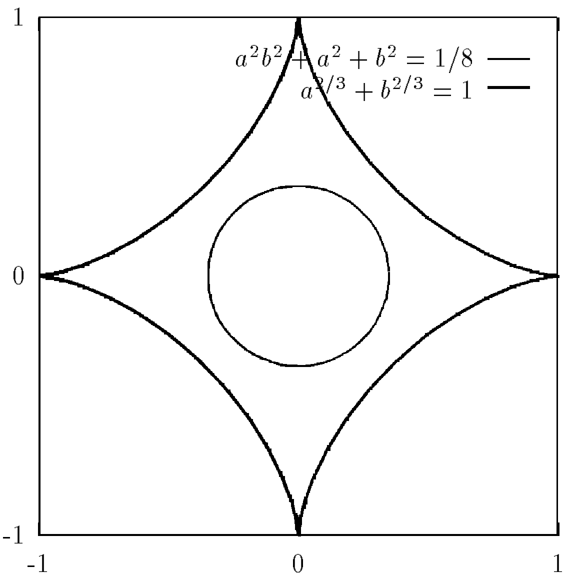

Figure 1: Stability Regions

The structure of the paper is as follows. In the next section, we will give a precise description of the stability problem and the exact solution for MacCormack scheme. In the following section, we will give a proof.

2 Stability of MacCormack Scheme

In this section, we recall the definitions of the Lax-Wendroff scheme and the MacCormack scheme. Then we give a precise statement of the stability problem for the MacCormack scheme and the solution for it.

Let T_x and T_y be the translation operators respectively defined by

$$T_x u(x,y,t) = u(x+\Delta_x, y, t), \quad T_y u(x,y,t) = u(x, y+\Delta_y, t).$$

Let

$$\begin{aligned} L &= I + \frac{1}{2}a(T_x - T_x^{-1}) + \frac{1}{2}b(T_y - T_y^{-1}) \\ &\quad + \frac{1}{2}a^2(T_x - 2I + T_x^{-1}) + \frac{1}{2}b^2(T_y - 2I + T_y^{-1}) \\ &\quad + \frac{1}{4}ab(T_x - T_x^{-1})(T_y - T_y^{-1}). \end{aligned}$$

Then the Lax-Wendroff [6] scheme is given by

$$u(x,y,t+\Delta_t) \approx Lu(x,y,t).$$

Let

$$\begin{aligned}
G_{++} &= I + a(T_x - I) + b(T_y - I),\\
G_{--} &= I + a(I - T_x^{-1}) + b(I - T_y^{-1}),\\
G_{-+} &= I + a(I - T_x^{-1}) + b(T_y - I),\\
G_{+-} &= I + a(T_x - I) + b(I - T_y^{-1}),\\
M_1 &= \frac{1}{2}(I + G_{++}G_{--}),\\
M_2 &= \frac{1}{2}(I + G_{-+}G_{+-}),\\
M &= M_2 M_1.
\end{aligned}$$

Then the MacCormack scheme [8] is given by

$$u(x,y,t+2\Delta_t) \approx Mu(x,y,t).$$

As observed in [12], the two operators M and L are related by

$$M = L^2 - Q^2$$

where

$$Q = \frac{1}{4}ab(T_x - 2I + T_x^{-1})(T_y - 2I + T_y^{-1}).$$

In order to obtain the amplifications for the Lax-Wendroff scheme and the MacCormack scheme, one replaces T_x and T_y with $e^{i\xi}$ and $e^{i\eta}$ respectively, obtaining

$$\begin{aligned}
r &= 1 - a^2(1-\cos\xi) - b^2(1-\cos\eta) - ab\sin\xi\sin\eta\\
s &= a\sin\xi + b\sin\eta\\
q &= ab(1-\cos\xi)(1-\cos\eta)\\
l &= r + is\\
m &= l^2 - q^2
\end{aligned}$$

where l and m are respectively the amplifications of the Lax-Wendroff and the MacCormack scheme.

Now the stability problem for the MacCormack scheme is to find a necessary and sufficient condition on a and b such that for all real values of ξ and η the absolute value of m is not greater than 1. More formally, the problem is to find a quantifier-*free* formula $F(a,b)$ which is equivalent to the quantified formula

$$\forall(\xi,\eta)\in\mathbb{R}^2 \;\; |m|^2 \le 1.$$

Let us observe that we can remove the trigonometric functions from the above formula by carrying out the usual parameterization with $u_1 = \sin\xi$, $v_1 = \cos\xi$, $u_2 = \sin\eta$ and $v_2 = \cos\eta$, along with the condition $u_1^2 + v_1^2 = u_2^2 + v_2^2 = 1$. The resulting formula (involving only algebraic operations) belongs to the language of the first order theory of real closed fields.

It is well known that the quantifier elimination in that theory can be carried out algorithmically [10]. Further, there have been various breakthroughs in devising more efficient algorithms [1, 4, 3, 5, 2, 9]. There also exists an implementation [4].

Thus, in principle, the stability problems can be solved completely automatically by a computer program. In fact, there have been interesting and important progress in this direction as reported in [7] where Liska and Steinberg succeeded in deriving the stability conditions for several problems by using the quantifier elimination program **qepcad** [4].

But, in spite of these various progresses, the complete automation for the MacCormack scheme is not yet feasible due to the huge requirement of computing times. Thus, the result reported in this paper is based on "human-brain" proof. But it must be emphasized that during conjecturing the main result, computer played essential role, as will be reported in some other paper.

Now we state the main result of this paper, that we will prove in the next section.

Theorem 1 (Stability of MacCormack Scheme)

$$\forall(\xi,\eta)\in\mathbb{R}^2 \;\; |m|^2 \le 1 \quad\Longleftrightarrow\quad a^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1. \;\blacksquare$$

3 Proof

In this section, we give a proof of the theorem stated above. The proof will consists of proofs of four lemmas. The dependency among the lemmas is as follows. The first lemma is used for proving the second lemma. The second lemma is used for proving the third and the fourth lemma. The third lemma proves that the proposed stability condition is necessary. The fourth lemma proves that it is also sufficient.

Slightly different variations of the first two lemmas were proved by Turkel [11]. Thus, the reader who is familiar with [11] might want to skip reading the proofs of the two lemmas. We decided to include the complete proof here because our lemmas are slight different from Turkel's in that ours do not have absolute value function in the formulation, and thus require careful adjustments in the proofs.

Lemma 1 (Univariate quantifier elimination) *Let*

$$P(x) = (ax+b)^4 - a^2x^4 - b^2.$$

Then we have

$$\forall x\in\mathbb{R} \;\; P(x)\le 0 \quad\Longleftrightarrow\quad a^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1. \;\blacksquare$$

Proof. The proof will be by case analysis. First three cases take care of the "degenerate" situations.

Case: $a = 0$

$$\begin{aligned}
\text{L.H.S.} &\Longleftrightarrow \forall x\in\mathbb{R}\;\; b^4 - b^2 \le 0\\
&\Longleftrightarrow b^4 - b^2 \le 0\\
&\Longleftrightarrow b^2(b^2-1) \le 0\\
&\Longleftrightarrow b^2 \le 1\\
&\Longleftrightarrow 0^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1\\
&\Longleftrightarrow \text{R.H.S.}
\end{aligned}$$

Case: $b = 0$

$$\begin{aligned}
\text{L.H.S.} &\Longleftrightarrow \forall x\in\mathbb{R}\;\; a^4x^4 - a^2x^4 \le 0\\
&\Longleftrightarrow \forall x\in\mathbb{R}\;\; a^2(a^2-1)x^4 \le 0
\end{aligned}$$

$$
\begin{aligned}
&\iff a^2(a^2-1) \le 0 \\
&\iff a^2 \le 1 \\
&\iff a^{\frac{2}{3}} + 0^{\frac{2}{3}} \le 1 \\
&\iff \text{R.H.S.}
\end{aligned}
$$

Case: $a = \pm 1$

$$
\begin{aligned}
\text{L.H.S.} &\iff \forall x \in \mathbb{R} \quad \pm 4bx^3 + 6b^2x^2 \pm 4b^3x + b^4 - b^2 \le 0 \\
&\iff b = 0 \\
&\iff (\pm 1)^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1 \\
&\iff \text{R.H.S.}
\end{aligned}
$$

Case: Otherwise, that is, $a \neq 0, a \neq \pm 1, b \neq 0$.

Note that $P(x) = (a^4 - a^2)x^4 + \cdots$. Since $(a^4 - a^2)$ is not zero, P is of degree 4 in x. Thus we have

$$\text{L.H.S} \iff \text{C1} \ \wedge \ \text{C2}$$

where

$$
\begin{aligned}
\text{C1} &: \ a^4 - a^2 \le 0 \\
\text{C2} &: \ \forall x \in \mathbb{R} \ [\ P'(x) = 0 \implies P(x) \le 0\ ].
\end{aligned}
$$

We see immediately

$$
\begin{aligned}
\text{C1} &\iff a^2 \le 1 \\
&\iff a^2 < 1
\end{aligned}
$$

because $a \neq \pm 1$. Let us now eliminate the quantifier from C2.

$$
\begin{aligned}
& \text{C2} \\
\iff & \forall x \in \mathbb{R} \ [\, 4a(ax+b)^3 - 4a^2x^3 = 0 \implies P(x) \le 0\,] \\
\iff & \forall x \in \mathbb{R} \ [\, (ax+b)^3 - ax^3 = 0 \implies P(x) \le 0\,] \\
\iff & \forall x \in \mathbb{R} \ [\, ax + b - a^{\frac{1}{3}}x = 0 \implies P(x) \le 0\,] \\
\iff & \forall x \in \mathbb{R} \ [\, x = \frac{b}{a^{\frac{1}{3}} - a} \implies P(x) \le 0\,] \\
\iff & P\left(\frac{b}{a^{\frac{1}{3}} - a}\right) \le 0.
\end{aligned}
$$

Finally we obtained a quantifier-free formula. Let us simplify this formula.

$$
\begin{aligned}
& P\left(\frac{b}{a^{\frac{1}{3}} - a}\right) \le 0 \\
\iff & \left(a\frac{b}{a^{\frac{1}{3}} - a} + b\right)^4 - a^2\left(\frac{b}{a^{\frac{1}{3}} - a}\right)^4 - b^2 \le 0 \\
\iff & \left(\frac{a^{\frac{1}{3}}b}{a^{\frac{1}{3}} - a}\right)^4 - a^2\left(\frac{b}{a^{\frac{1}{3}} - a}\right)^4 - b^2 \le 0 \\
\iff & \left(a^{\frac{1}{3}}b\right)^4 - a^2b^4 - b^2\left(a^{\frac{1}{3}} - a\right)^4 \le 0 \\
\iff & a^{\frac{4}{3}}b^2 - a^2b^2 - \left(a^{\frac{1}{3}} - a\right)^4 \le 0 \\
\iff & \left(a^{\frac{1}{3}} - a\right)\left(ab^2 - \left(a^{\frac{1}{3}} - a\right)^3\right) \le 0 \\
\iff & \left(a^{\frac{1}{3}} - a\right)a\left(b^2 - \left(1 - a^{\frac{2}{3}}\right)^3\right) \le 0 \\
\iff & (a - a^3)\,a\left(b^{\frac{2}{3}} - \left(1 - a^{\frac{2}{3}}\right)\right) \le 0 \\
\iff & (1 - a^2)\left(a^{\frac{2}{3}} + b^{\frac{2}{3}} - 1\right) \le 0.
\end{aligned}
$$

In order to link the second and the third steps from the bottom, we have used the elementary fact that $(u^3 - v^3)$ has the same sign as $(u - v)$.

Now putting together, we finally obtain

$$
\begin{aligned}
\text{L.H.S.} &\iff \text{C1} \ \wedge \ \text{C2} \\
&\iff a^2 < 1 \ \wedge \ (1 - a^2)\left(a^{\frac{2}{3}} + b^{\frac{2}{3}} - 1\right) \le 0 \\
&\iff a^2 < 1 \ \wedge \ \left(a^{\frac{2}{3}} + b^{\frac{2}{3}} - 1\right) \le 0 \\
&\iff a^{\frac{2}{3}} + b^{\frac{2}{3}} - 1 \le 0 \\
&\iff \text{R.H.S.}
\end{aligned}
$$

This completes the proof of the lemma. ∎

Remark: This univariate quantifier elimination problem is small enough to be carried out by a computer program. In fact, we have tried the quantifier elimination program **qepcad** [4]. Given the input

$$\forall x \ \ (ax + b)^4 - a^2x^4 - b^2 \le 0,$$

the program, in about 5 seconds on a DEC station, outputs the quantifier-free formula:

$$a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 \le 1.$$

By suitable rewriting, one see that it is equivalent to

$$a^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1. \ \blacksquare$$

Remark: It is interesting to note that the polynomial shown above

$$a^6 + 3a^4b^2 + 3a^2b^4 + b^6 - 3a^4 + 21a^2b^2 - 3b^4 + 3a^2 + 3b^2 - 1$$

is the discriminant of the input polynomial

$$P(x) = (ax + b)^4 - a^2x^4 - b^2$$

with respect to x upto some trivial factor. Thus, for this particular case, the discriminant computation was sufficient. It is because P' has only one real root, say α. In that case, it is well known that $P(\alpha)$ is the discriminant of $P(x)$ upto some constant multiple. But it is not true in general when P' has more than one real root. ∎

Lemma 2 (Bivariate quantifier elimination) *Let*

$$P(x, y) = (ax + by)^4 - a^2x^4 - b^2y^4.$$

Then we have

$$\forall (x, y) \in \mathbb{R}^2 \ \ P(x, y) \le 0 \iff a^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1. \ \blacksquare$$

Proof.

$$
\begin{aligned}
& \text{L.H.S.} \\
\iff & [\ \forall x \in \mathbb{R} \ \ P(x, 0) \le 0\ ] \ \wedge \\
& [\ \forall x \in \mathbb{R} \forall y \in \mathbb{R} - \{0\} \ \ P(x, y) \le 0\ ] \\
\iff & [\ \forall x \in \mathbb{R} \ \ (a^4 - a^2)x^4 \le 0\ ] \ \wedge \\
& [\ \forall x \in \mathbb{R} \forall y \in \mathbb{R} - \{0\} \\
& y^4\left(\left(a\frac{x}{y} + b\right)^4 - a^2\left(\frac{x}{y}\right)^4 - b^2\right) \le 0\ ]
\end{aligned}
$$

$$
\begin{aligned}
&\iff a^2 \le 1 \ \wedge \\
&\qquad \forall z \in \mathbb{R} \ \ (az+b)^4 - a^2 z^4 - b^2 \le 0 \\
&\iff a^2 \le 1 \ \wedge \ a^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1 \quad \text{(by Lemma 1)} \\
&\iff a^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1 \\
&\iff \text{R.H.S.}
\end{aligned}
$$

This completes the proof of the lemma. ■

Lemma 3 (Necessary condition)

$$\forall(\xi,\eta) \in \mathbb{R}^2 \ \ |m|^2 \le 1 \quad \Longrightarrow \quad a^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1. \ ■$$

Proof. The proof will be by case analysis. The first three cases will take care of "degenerate" situations.

Case: $a = 0$ and $b = 0$

We see that $r = 1, s = 0, q = 0$. Thus, $|m|^2 = 1$. Hence we have

$$
\begin{aligned}
\text{L.H.S.} &\iff \forall(\xi,\eta) \in \mathbb{R}^2 \ \ 1 \le 1 \\
&\iff 0^{\frac{2}{3}} + 0^{\frac{2}{3}} \le 1 \\
&\iff \text{R.H.S}
\end{aligned}
$$

Case: $a = 0$ and $b^2 = 1$

We see that $r = \cos\eta, s = \pm\sin\eta, q = 0$. Thus, $|m|^2 = 1$. Hence we have

$$
\begin{aligned}
\text{L.H.S.} &\iff \forall(\xi,\eta) \in \mathbb{R}^2 \ \ 1 \le 1 \\
&\iff 0^{\frac{2}{3}} + (\pm 1)^{\frac{2}{3}} \le 1 \\
&\iff \text{R.H.S}
\end{aligned}
$$

Case: $b = 0$ and $a^2 = 1$

We see that $r = \cos\xi, s = \pm\sin\xi, q = 0$. Thus, $|m|^2 = 1$. Hence we have

$$
\begin{aligned}
\text{L.H.S.} &\iff \forall(\xi,\eta) \in \mathbb{R}^2 \ \ 1 \le 1 \\
&\iff (\pm 1)^{\frac{2}{3}} + 0^{\frac{2}{3}} \le 1 \\
&\iff \text{R.H.S}
\end{aligned}
$$

Case: Otherwise

By expanding $|m|^2$ with respect to ξ and η around $(0,0)$, we obtain

$$|m|^2 = 1 + \frac{1}{2}\left((a\xi + b\eta)^4 - a^2\xi^4 - b^2\eta^4\right) + O(\xi^6 + \eta^6).$$

Let

$$P(\xi,\eta) = (a\xi + b\eta)^4 - a^2\xi^4 - b^2\eta^4.$$

One can easily verify that P is a non-zero polynomial (*i.e.* does not vanish identically).[1] Thus, the behaviour of $|m|^2$ near the origin is determined by that of P. Hence we have

$$
\begin{aligned}
\text{L.H.S.} &\Longrightarrow \forall(\xi,\eta) \in \mathbb{R}^2 \ \ P(\xi,\eta) \le 0 \\
&\iff a^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1 \quad \text{(by Lemma 2)} \\
&\iff \text{R.H.S}
\end{aligned}
$$

This completes the proof of the lemma. ■

[1] In fact, the first three cases were introduced in order to filter out the situation when P is a zero polynomial.

Lemma 4 (Sufficient condition)

$$\forall(\xi,\eta) \in \mathbb{R}^2 \ \ |m|^2 \le 1 \quad \Longleftarrow \quad a^{\frac{2}{3}} + b^{\frac{2}{3}} \le 1. \ ■$$

Proof. Let us first note

$$
\begin{aligned}
|m|^2 &= |l^2 - q^2|^2 \\
&= (r + is)^2 - q^2|^2 \\
&= |r^2 - s^2 - q^2 + 2rsi|^2 \\
&= (r^2 - s^2 - q^2)^2 + 4r^2s^2 \\
&= r^4 + s^4 + q^4 - 2r^2s^2 - 2r^2q^2 + 2s^2q^2 + 4r^2s^2 \\
&= r^4 + s^4 + q^4 + 2r^2s^2 + 2r^2q^2 + 2s^2q^2 - 4r^2q^2 \\
&= (r^2 + s^2 + q^2)^2 - 4r^2q^2.
\end{aligned}
$$

Thus we have

$$
\begin{aligned}
&\forall(\xi,\eta) \in \mathbb{R}^2 \ \ |m|^2 \le 1 \\
\iff &\forall(\xi,\eta) \in \mathbb{R}^2 \ \ (r^2 + s^2 + q^2)^2 - 4r^2q^2 \le 1 \\
\Longleftarrow &\forall(\xi,\eta) \in \mathbb{R}^2 \ \ (r^2 + s^2 + q^2)^2 \le 1 \\
\iff &\forall(\xi,\eta) \in \mathbb{R}^2 \ \ r^2 + s^2 + q^2 \le 1. \qquad (1)
\end{aligned}
$$

A digression to tell an inside story. Above, I took a "bold" step of ignoring the term $4r^2q^2$. It is hard to believe that the proof will go through without the term, but it turns out to be so. In fact, I have spent several months, in vain, searching for proofs which take account of the term because I was preconditioned to the idea that the term must play an essential role. Only after drawing lots of pictures using plotting programs and inspecting lots of computation traces using computer algebra systems, one morning, I "accidently" hit upon the intuition that the term might not play an important role for stability. Then I could reach, after only few more hours of work, the end of the proof, to my joy! The end of the digression.

Since ξ and η always occur as arguments of trigonometric functions, we can restrict their ranges.

$$
\begin{aligned}
&\forall(\xi,\eta) \in \mathbb{R}^2 \ \ r^2 + s^2 + q^2 \le 1 \\
\iff &\forall(\xi,\eta) \in [-\pi,\pi]^2 \ \ r^2 + s^2 + q^2 \le 1. \qquad (2)
\end{aligned}
$$

Here the choice of $[-\pi,\pi]$ instead of $[0,2\pi]$ is intentional as will be seen now. Let

$$\alpha = \sin\frac{\xi}{2}, \quad \beta = \sin\frac{\eta}{2}.$$

Then we have

- $-1 \le \alpha, \ \beta \le 1$
- $0 \le \cos\frac{\xi}{2}, \quad \cos\frac{\eta}{2} \le 1$
- $\cos\frac{\xi}{2} = \sqrt{1-\alpha^2}, \quad \cos\frac{\eta}{2} = \sqrt{1-\beta^2}.$

(If we had chosen $[0,2\pi]$ as the range of ξ and η instead, we could have had difficulty since we could not have determined the sign of $\cos\frac{\xi}{2}$ and $\cos\frac{\eta}{2}$ from α and β.)

We rewrite $r, s,$ and q in terms of α and β, obtaining

$$
\begin{aligned}
r &= 1 - 2a^2\alpha^2 - 2b^2\beta^2 - 4ab\alpha\beta\sqrt{1-\alpha^2}\sqrt{1-\beta^2} \\
s &= 2a\alpha\sqrt{1-\alpha^2} + 2b\beta\sqrt{1-\beta^2} \\
q &= 4ab\alpha^2\beta^2.
\end{aligned}
$$

After some straightforward calculation, we obtain

$$\begin{aligned}
w_1(\alpha,\beta) &= r^2+s^2+q^2 \\
&= 1-4a^2\alpha^4-4b^2\beta^4 \\
&\quad +4a^4\alpha^4+4b^4\beta^4 \\
&\quad +8a^2b^2\alpha^2\beta^2 \\
&\quad +16a^2b^2\alpha^2\beta^2(1-\alpha^2-\beta^2+2\alpha^2\beta^2) \\
&\quad +16a^3b\alpha^3\beta\sqrt{1-\alpha^2}\sqrt{1-\beta^2} \\
&\quad +16ab^3\alpha\beta^3\sqrt{1-\alpha^2}\sqrt{1-\beta^2}.
\end{aligned}$$

Then we have

$$\begin{aligned}
& \forall(\xi,\eta)\in[-\pi,\pi]^2 \;\; r^2+s^2+q^2\le 1 \\
\iff & \forall(\alpha,\beta)\in[-1,1]^2 \;\; w_1(\alpha,\beta)\le 1. \qquad (3)
\end{aligned}$$

Now observe

$$1-\alpha^2-\beta^2+2\alpha^2\beta^2 = 1-\alpha^2(1-\beta^2)-\beta^2(1-\alpha^2)\le 1.$$

Thus we have

$$\begin{aligned}
& \forall(\alpha,\beta)\in[-1,1]^2 \;\; w_1(\alpha,\beta)\le 1 \\
\Longleftarrow & \forall(\alpha,\beta)\in[-1,1]^2 \;\; w_2(\alpha,\beta)\le 1 \qquad (4)
\end{aligned}$$

where $w_2(\alpha,\beta)$ is obtained from $w_1(\alpha,\beta)$ by replacing the term $(1-\alpha^2-\beta^2+2\alpha^2\beta^2)$ with 1, that is,

$$\begin{aligned}
w_2(\alpha,\beta) &= 1-4a^2\alpha^4-4b^2\beta^4 \\
&\quad +4a^4\alpha^4+4b^4\beta^4 \\
&\quad +24a^2b^2\alpha^2\beta^2 \\
&\quad +16a^3b\alpha^3\beta\sqrt{1-\alpha^2}\sqrt{1-\beta^2} \\
&\quad +16ab^3\alpha\beta^3\sqrt{1-\alpha^2}\sqrt{1-\beta^2}.
\end{aligned}$$

Let $w_3(\alpha,\beta)$ be obtained from $w_2(\alpha,\beta)$ by dropping the term $\sqrt{1-\alpha^2}\sqrt{1-\beta^2}$, that is,

$$\begin{aligned}
w_3(\alpha,\beta) &= 1-4a^2\alpha^4-4b^2\beta^4 \\
&\quad +4a^4\alpha^4+4b^4\beta^4 \\
&\quad +24a^2b^2\alpha^2\beta^2 \\
&\quad +16a^3b\alpha^3\beta \\
&\quad +16ab^3\alpha\beta^3.
\end{aligned}$$

Now we claim that

$$\begin{aligned}
& \forall(\alpha,\beta)\in[-1,1]^2 \;\; w_2(\alpha,\beta)\le 1 \\
\Longleftarrow & \forall(\alpha,\beta)\in[-1,1]^2 \;\; w_3(\alpha,\beta)\le 1 \qquad (5)
\end{aligned}$$

In order to prove this claim, let us assume that

$$\forall(\alpha,\beta)\in[-1,1]^2 \;\; w_3(\alpha,\beta)\le 1.$$

Let $(\alpha^*,\beta^*)\in[-1,1]^2$ be arbitrary but fixed. we need to show that

$$w_2(\alpha^*,\beta^*)\le 1.$$

We will do so by case analysis.

Case: $ab\alpha^*\beta^*\ge 0$

Clearly, we have

$$w_2(\alpha^*,\beta^*)\le w_3(\alpha^*,\beta^*).$$

From the hypothesis, we also have

$$w_3(\alpha^*,\beta^*)\le 1.$$

Thus, we have

$$w_2(\alpha^*,\beta^*)\le 1.$$

Case: $ab\alpha^*\beta^*<0$

Clearly, we have

$$w_2(\alpha^*,\beta^*)\le w_2(\alpha^*,-\beta^*)\le w_3(\alpha^*,-\beta^*).$$

From the hypothesis, we also have

$$w_3(\alpha^*,-\beta^*)\le 1.$$

Thus, we have

$$w_2(\alpha^*,\beta^*)\le 1.$$

This completes the proof of the claim that we can drop the radical terms.

Now note that

$$\begin{aligned}
& \forall(\alpha,\beta)\in[-1,1]^2 \;\; w_3(\alpha,\beta)\le 1 \\
\iff & \forall(\alpha,\beta)\in[-1,1]^2 \\
& -a^2\alpha^4-b^2\beta^4+(a\alpha+b\beta)^4\le 0 \\
\Longleftarrow & \forall(\alpha,\beta)\in\mathbb{R}^2 \;\; -a^2\alpha^4-b^2\beta^4+(a\alpha+b\beta)^4\le 0 \\
\iff & a^{\frac{2}{3}}+b^{\frac{2}{3}}\le 1 \quad \text{(by Lemma 2).} \qquad (6)
\end{aligned}$$

Finally, by applying the transitivity of the logical implication ($\Longleftarrow$) to the implication relations in (1), (2), (3), (4), (5) and (6), we conclude that

$$\forall(\xi,\eta)\in\mathbb{R}^2 \;\; |m|^2\le 1 \quad \Longleftarrow \quad a^{\frac{2}{3}}+b^{\frac{2}{3}}\le 1. \;\blacksquare$$

This completes the proof of the lemma. ■

From Lemmas 3 and 4, Theorem 1 immediately follows.

4 Concluding Remarks

In this paper, we have proved that the stability region of the MacCormack scheme for the two dimensional scalar avection equation is *exactly* given by

$$a^{\frac{2}{3}}+b^{\frac{2}{3}}\le 1.$$

As Turkel [11] has done for the Lax-Wendroff scheme, it seems to be possible to generalize the result of this paper to n-valued functions where a and b are commuting real symmetric matrices. This will be investigated in a subsequent note.

Though it is not yet computationally feasible to automate the whole process by a computer program, some part of the proof could be already carried out by computer programs. Furthermore various conjecturing processes were greatly aided by computer programs. These experience will also be reported in another paper.

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