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Abstract

The goal of this paper is to estimate in a theoretical way the reduction of the number of cells of, and the overall number of neighbouring cells in, a grid when some cells in a uniform grid are replaced by a larger cell in an overlaying non-uniform grid. We assume that the grids are represented by a 2^k-tree and the replacement of cells preserves the representation.

1 Introduction.

Considering the use of grids in the Robot Motion Planning Problem (e.g. [Barraquand & Latombe 91], [Conte, Longhi & Zulli 93], [Van Geem 94], [Van Geem 95], [Tournassoud & Faverjon 88]), we are interested in finding out how much better it is to use a non-uniform grid instead of a uniform grid. This paper is addressing this question.

In Section 2 some definitions are given, needed throughout this paper. Then we discuss the number of cells in uniform and non-uniform grids in Sections 3 and 4. In Section 5 we give a first example. Section 6 consideres some properties of quantities introduced in the preceding sections and relations between the introduced parameters are studied. In Section 7 we consider for the parameters some values which are corresponding to some practical applications carried out over the last few years at the Research Institute for Symbolic Computation (as also discussed in [Van Geem 95]). Finally in Section 8 we consider a particular family of polygons in 2-dimensional space. We compute the number of cells in non-uniform grids modelling these polygons and compare for some particular polygons these amounts with the number of cells needed in a uniform grid.

2 Preparations.

In the following we will use some terminology which we will define here. We will consider both some uniform and non-uniform grids on a space of dimension $k \geq 1$.

A grid consists of k-dimensional cubes. We only consider grids that can be represented by a 2^k-tree. These grids can be constructed in the following way. We start with an initial k-dimensional cube. This cube can be subdivided in 2^k smaller cubes of equal size. Each of these can be subdivided again in 2^k smaller cubes of equal size, and so on. Each node of the corresponding 2^k-tree represents a cube. The initial cube is represented by the root of the tree, the children of node N represent the cubes of smaller size in which the cube corresponding to node N was subdivided. The leaves of the tree correspond to cubes that were not subdivided. We call these cubes cells. When all cells are of equal size, we call the grid a uniform grid. In the other case we call the grid a non-uniform grid.

When a cube is represented by a node on height s in the 2^k-tree, we call the cube a cube of size s. If the cube is a cell, we call it a cell of size s.

Cells C_1 and C_2 in a uniform grid are called neighbours when C_1 has a face in common with C_2. Cells C_1 and C_2 in a non-uniform grid are called neighbours when a face of C_1 coincides partially or completely with a face of C_2.

We assume a coordinate system with axes parallel to the edges of the initial cube. In each dimension we consider an orientation along the coordinate axis. We now say that for each dimension there are two directions, considering the orientation on the corresponding axis. In this sense, we also say that a cell has a neighbour in a particular direction.

3 Uniform Grid.

We consider the k-dimensional uniform grid to be represented by a 2^k-tree and denote h for the height of the tree and d for the number of cells along an axis ($d = 2^{h-1}$). A straightforward computation gives us the following results. The number of cells in the grid is equal to d^k, the number of neighbours of a cell in the grid is $2k$, and the total number of neighbours is $2kd^k$.

4 Non-Uniform Grid.

We still denote k for the dimension of the grid. We denote h for the height of the tree. By h is also expressed the smallest size of a cell in the non-uniform grid. An upper bound for the number of cells in the non-uniform grid is given by $(2^{h-1})^k$.

4.1 Number of Cells.

Assume that we construct the non-uniform grid by subdividing stepwise the k-dimensional cubes, so that in step s we only subdivide cubes of size s in cubes of size $s+1$. We denote the number of subdivisions during step s of cubes of size s in cubes of size $s+1$ by σ_s $(s = 1, ..., h-1)$.

Then the number of cells of size $s+1$, denoted by A_{s+1}, is equal to

$$A_{s+1} = \sigma_s 2^k - \sigma_{s+1}, \text{ where } \sigma_h = 0,$$

since by subdividing a cube of size s one cell (considering the 2^k-tree as it was before executing step s) of size s is replaced by 2^k cells (considering the 2^k-tree as it is after executing step s) of size $s+1$ and in the next step σ_{s+1} of those will be subdivided further. (When $\sigma_1 = 0$, we have $A_1 = 1$, when $\sigma_1 = 1$, we have $A_1 = 0$.)

The total number A of cells in a non-uniform grid corresponding to the tuple $(\sigma_1, ..., \sigma_{h-1})$, is equal to

$$A = 1 + (2^k - 1)\sum_{s=1}^{h-1} \sigma_s.$$

4.2 Acceptable Grids.

Not every tuple $(\sigma_1, ..., \sigma_{h-1})$ of $h-1$ natural numbers corresponds to a grid. A necessary and sufficient condition for such a tuple to represent a grid is:

$$\sigma_1 \in \{0, 1\},$$

$$\sigma_{s+1} \leq 2^k \sigma_s \text{ (for } s = 1, ...h-2).$$

Denote the minimum desired amount of cells in the non-uniform grid by D. Considering a tuple $(\sigma_1, ..., \sigma_{h-1})$, σ_t (for $t = 2, ..., h-2$) must at least be equal to $\lceil \frac{\sigma_{t+1}}{2^k} \rceil$ in order to end up with the minimum desired amount of cells after $h-1$ steps. More cells than the minimum desired amount can be created, so σ_t can have all natural numbers in the interval

$$[\lceil\frac{\sigma_{t+1}}{2^k}\rceil, \sigma_{t-1}2^k] \text{ for } t = 2, ..., h-2$$

as its value. Obviously, if the desired amount of cells is strictly more than 1, then $h \geq 1$ and $\sigma_1 = 1$. The value for σ_{h-1} is $\lceil\frac{D}{2^k}\rceil$, since we want a grid with at least D cells of smallest size, but with not more than the absolutely necessary amount of cells of smallest size.

When for some t the interval $[\lceil\frac{\sigma_{t+1}}{2^k}\rceil, \sigma_{t-1}2^k]$ is empty, the tuple $(\sigma_1,...,\sigma_{h-1})$ is not acceptable, since then the inequation $\sigma_{t+1} \leq 2^k\sigma_t$ does not hold, which was the necessary and sufficient condition of above.

4.3 Number of Cubes.

The number C_s of cubes constructed in the first s steps is equal to

$$C_1 = 1 + \sigma_1(2^k - 1),$$
$$C_s = C_{s-1} - \sigma_s + 2^k\sigma_s \text{ (for } s = 2, ..., h-1).$$

The number of cells that are not subdivided in step s is equal to $\sigma_{s-1}2^k - \sigma_s$. The number of cells of size s in the non-uniform grid is equal to $C_{s-1} - \sigma_s$.

5 An Example.

Let $k = 3, h = 5$. Assume that the number of cells of smallest size in the non-uniform grid has to be 400. We look for all tuples $(\sigma_1, ..., \sigma_4)$, using the notation of above.

We observe that $\sigma_1 = 1$ and $\sigma_4 = 50$ must hold. Several values for σ_2 and σ_3 are possible here. For σ_2 all values between 1 and 8 are acceptable. For a given σ_2, σ_3 can have all values between 7 and U_{σ_2}, where $U_{\sigma_2} = \sigma_2 2^k$.

The maximal number of cells in the non-uniform grid is 862, which occurs for the highest acceptable values for σ_2 and σ_3, namely $\sigma_2 = 8$, $\sigma_3 = 64$.

6 Some Properties and Numbers.

6.1 Realisable Precision.

Here we assume that k and h are given fixed values. We assume that D can be written as $D = 2^{k(h-1)-p}$. We address ourselves to the question what the upper and lower bounds for p are.

Obviously the lower bound for p is 0, to which corresponds the uniform grid with $2^{k(h-1)}$ cells.

The grid corresponding to the upper bound of p is the non-uniform grid with minimal amount of cells. This grid corresponds to the $h-1$-tuple $(\sigma_1, ..., \sigma_{h-1}) = (1, ..., 1)$. The minimal value for D therefore is equal to

$$D_{\text{min}} = \sigma_{h-1}2^k = 2^k.$$

Thus, p cannot be larger than $k(h-2)$.

So, p must be an integer between 1 and $k(h-2)$. When D is given, we can compute p from the formula: $p = k(h-1) - \log_2 D$.

6.2 Acceptable Height.

Here we assume that both k and D are given. Since

$$2^{k(h-2)} < D \leq 2^{k(h-1)}$$

must hold for minimal h, we get

$$1 + \frac{\log_2 D}{k} \leq h_{\text{min}} < 2 + \frac{\log_2 D}{k}.$$

Thus the minimal h for which D can be realised is then $h_{\text{min}} = \lceil 1 + \frac{\log_2 D}{k} \rceil$.

An upper bound for h does not exist. Indeed, if $(\sigma_1, ..., \sigma_{h-1})$ is a tuple for which D is realised, then the tuple $(1, \sigma_1, ..., \sigma_{h-1})$ also realises D and is larger.

6.3 Grids With Many Cells.

From the paragraph on acceptable grids it is clear that the highest amount H of cells in a k-dimensional non-uniform grid with at least D cells of smallest size and corresponding to a 2^k-tree with given height h is reached for the highest acceptable values for $(\sigma_2, ..., \sigma_{h-2})$. Note that H is the upper bound for the number of cells in a non-uniform grid with D cells of smallest size. This is a smaller upper bound than $(2^{h-1})^k$.

Denote $\tau_{h-1} = \lceil \frac{D}{2^k} \rceil$, and for $t = 1, ..., h-2$: $\tau_t = \lceil \frac{\tau_{t+1}}{2^k} \rceil$. Then, providing that $D \leq (2^{h-1})^k$, we have

$$\tau_t \leq (2^k)^{t-2}.$$

Thus the highest value for σ_t (for $t = 2, ..., h-2$) is $(2^k)^{t-2}$. Therefore,

$$H = 1 + (2^k - 1)1 + (2^k - 1)(\sum_{s=2}^{h-2}(2^k)^{s-2}) + (2^k - 1)\lceil\frac{D}{2^k}\rceil$$

$$= 1 + (2^k - 1)(1 + \lceil\frac{D}{2^k}\rceil) + ((2^k)^{h-3} - 1).$$

When we assume that $D = 2^{k(h-1)-p}$, then straithforward rewriting gives us the following result:

$$H = 2^k - 1 + 2^{k(h-3)} - 2^{k(h-1)-p} - 2^{k(h-2)-p}.$$

6.4 Grids With Few Cells.

The fewest number L of cells in a non-uniform grid with D cells of smallest size is equal to

$$L = 1 + (\sum_{s=2}^{h-2}\lceil\frac{\tau_{s+1}}{2^k}\rceil)(2^k - 1),$$

where $\tau_{h-1} = \lceil\frac{D}{2^k}\rceil$ and $\tau_s = \lceil\frac{\tau_{s+1}}{2^k}\rceil$ for $s = h-2, h-3, ..., 3$.

We now compute the value for L for fixed h and for $D = 2^{k(h-1)-p}$. We now have

$$\tau_{h-1} = \lceil 2^{(h-2)k-p}\rceil = \max(1, 2^{(h-2)k-p}).$$

Assuming that $(h-2)k - p > 1$,

$$\tau_{h-2} = \lceil 2^{(h-3)k-p}\rceil = \max(1, 2^{(h-3)k-p}).$$

As long as $(h-t)k - p > 1$ holds, we get

$$\tau_{h-t} = \lceil 2^{(h-t-1)k-p}\rceil = \max(1, 2^{(h-t-1)k-p}),$$

for $s > 1$. In general, $\tau_s = \max(1, 2^{(s-1)k-p})$. Define q such that $q < h - \frac{p+1}{k}$ and $q + 1 \geq h - \frac{p+1}{k}$. This gives us

$$L = 1 + (\sum_{s=2}^{h-(q+1)} 1 + \lceil 2^{(h-q)k-p}\rceil + \sum_{s=h-q+1}^{h-1} 2^{(s-1)k-p})(2^k - 1)$$

$$= 1 + (h - q - 2 + \lceil 2^{(h-q)k-p}\rceil)(2^k - 1) + \frac{(2^k)^{h-q+1}((2^k)^{q-1} - 1)}{2^{k+p}}.$$

6.5 Fixed Number of Cells.

In contrast to the previous, in this subsection we assume the total amount of cells A in the non-uniform grid to be given. We also assume that k and p (and thus $D = 2^{k(h-1)-p}$) are given. We now look for acceptable values for h. Observe that from the formula for H, we now can deduce the minimal acceptable value for h if we assume that A is equal to H (i.e. the cells of smallest size are spread as widely as possible over the grid). Analoguously, from the formula for L we can deduce the maximal acceptable value for h, assuming that $A = L$ (corresponding to the most densely distribution of the cells of smallest size). We do get an upper bound for h here, since the total amount A of cells in the grid is given.

We thus get from the formula for H:

$$h_{\min} = 3 + \frac{1}{k}\log_2 \frac{A + 1 - 2^k}{1 - 2^{-2k-p} - 2^{-k-p}}.$$

7 Reasonable D, k, and h

In practical examples k typically might be large, say $k = 6, 7, 8, \ldots$. We assume h usually is equal to $2, 3, 4$ or 5. Assuming that D can be written as $D = 2^{kh-p}$, then typically p will be 2 or 3, although acceptible values are up to $k(h-1)$.

Assuming that the amount of cells in the non-uniform ($k =$)6-dimensional grid is ($A =$)15000, we get from the formula above for $p = 2$ and for $p = 3$, $h_{\min} = 6$. For $k = 7$ and $p = 2$ or 3, $h_{\min} = 5$.

8 A Particular Family of Polygons.

In this section, we consider the 2-dimensional case. We study the amount of cells needed for modelling a particular kind of polygons.

The family of polygons that we consider here, have the following characteristics. The edges are all parallel to the sides of the cells of the grid and they are all of equal length. The length of the edges is equal to the length of a cell of size $h-2$, where h is the height of the quadtree corresponding to the grid. We assume that the grid is not larger than necessary for modelling the polygon. The polygon is concave and the number of edges is equal to $16j-4$, where $j = 1, 2, \ldots$. In Figure 1 the polygons for $j = 1$ and $j = 2$ are drawn. Here, the polygons are drawn in the uniform grid.

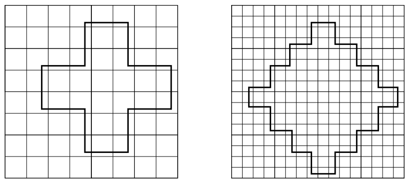

Figure 1: Polygons: Cases $j = 1$ and $j = 2$.

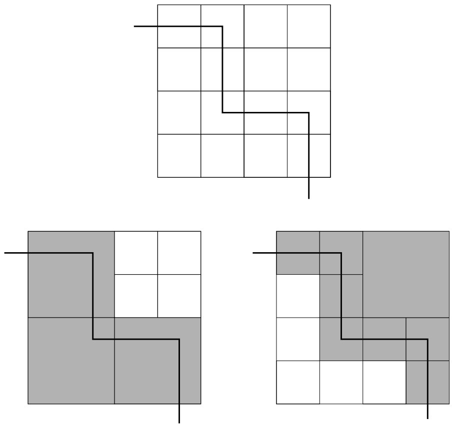

Figure 2: Building Block of Type One.

This family of polygons can be constructed with the building blocks drawn in Figures 2 and 3. These building blocks can be glued together only in some particular combinations. In order to build a staircase, we can only glue two identical building blocks together. For this, an additional vertex has to be inserted, as demonstrated in Figure 4. In order to glue two staircases together, only type one and two, and type two and three can be combined, as demonstrated in Figure 5.

In order to construct the 12-gon ($j = 1$) of the family, we have to glue building block one to building block two (rotated over $\frac{\pi}{2}$), followed by building block three (rotated over π) and building block two (its mirror image). In general, for a $(16j - 4)$-gon of the family, we need a staircase of j copies of building block one, followed by a staircase of j copies of building block

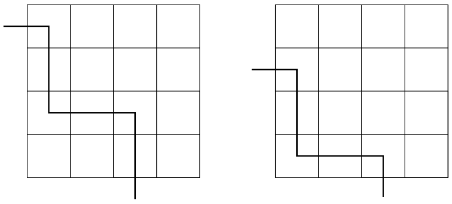

Figure 3: Building Blocks of Types Two and Three.

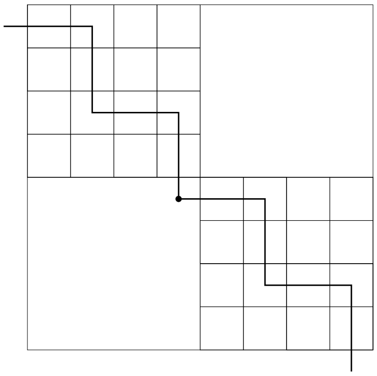

Figure 4: Staircase.

two (rotated over $\frac{\pi}{2}$), followed by a staircase of j copies of building block three (rotated over π) and connected by a staircase of j copies of building block two (its mirror image).

This allows us to compute the $h-1$-tuple $(\sigma_1, ..., \sigma_{h-1})$ and thus A and D. Indeed, when considering the inside of the obstacle on the left side of the contour in Figures 2 and 3, all building blocks contribute 1 to σ_{h-2}, building block one contributes 0 to σ_{h-1}, building block two contributes 2 to σ_{h-1}, and building block three contributes 3 to σ_{h-1}. (When the inside is on the right hand side of the contour in the figures, then building block one contributes 3 and building block three contributes 0 to σ_{h-1}, all the other contributions stay the same.) Glueing together two building blocks to a staircase in all cases contributes 1 to σ_{h-2}. In the cases where two building blocks of type one or two are glued, there is a contribution of 1 to σ_{h-1},

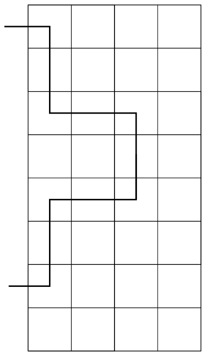

Figure 5: Glueing Two Different Kinds of Building Blocks.

and for glueing two building blocks of type three there is no contribution to σ_{h-1}. Glueing two building blocks of different type does not affect the $h-1$-tuple further. The relation between j and h is given by:

$$j = h - 2.$$

Thus, for the 12-gon we get the tuple $(1,4,7)$, $A = 37$, $D = 28$. For the 28-gon we get the tuple $(1,4,9,15)$, $A = 88$, $D = 60$. And in general, for the $(16j-4)$-gon we get $(1,4,...,4^{j-1},5j-1,8j-1)$, $A = 4^j + 39j - 6$ and $D = 32j - 4$. The uniform grid consists of $U = 4^{j+2}$ cells. The ratio $\frac{A}{U}$ for $j = 1,...,13$ is given in following Table.

j	$\frac{A}{U}$	j	$\frac{A}{U}$
1	0.57812500	8	0.06279182
2	0.34375000	9	0.06258225
3	0.17089844	10	0.06252289
4	0.09912109	11	0.06250630
5	0.07403564	12	0.06250172
6	0.06597900	13	0.06250047
7	0.06351852		

Considering the same family of polynomials, but with longer edges, corresponds to adding another level to the quadtree. If we now suppose that the length of the edges is equal to the length of the cells of size $\hat{h} - 3 - t$, where $\hat{h}$ is the height of the quadtree, then we get a tuple that is t longer than in the previous ($\hat{h} = h + t$), and for which the first part of the tuple is the same a the tuple we had before. In fact, we add σ_{h-1+s}, for $s = 1, ..., t$. Assuming that every vertex of the polygon with n vertices lies in the SW-cube (south-west, the third branch in the quadtree), we get the following formula:

$$\sigma_{h-1+s} = 1 + 2^s \frac{n}{2}.$$

Indeed, for each edge for which we have to refine the grid, on s levels lower than the original one we split 2^s cells along that edge. In the staircase of type one, we get $\frac{\frac{n}{4}-1}{2}$ horizontal and $\frac{\frac{n}{4}-1}{2}$ vertical edges, each for which 2^s cubes have to be split. In the staircase of type two, we have $\frac{\frac{n}{4}+1}{2}$ edges for which 2^s cubes have to be split. The same amount is to be considered for the staircase of type two in mirror image. And in the staircase of type three no split has to be performed. Then for the vertices, there is one convex vertex more for which a cube has to be split on s levels, than there are concave vertices for which we counted a cube double in the discussion of the edges above. So, we get $\sigma_{h-1+s} = 2^s(2\frac{\frac{n}{4}-1}{2} + 2\frac{\frac{n}{4}+1}{2} + 0) + 1$, which can be rewritten in the formula presented earlier.

Counting that out for various values for j and t, we get the results of the next Table.

j	t	number cells uniform grid	number cells non-uniform grid	Non-uni/uni * 100 Non-uni/uni * 100
1	1	256	184	71.875000
1	2	1024	475	46.386719
1	3	4096	1054	25.732422
1	4	16384	2209	13.482666
1	5	65536	4516	6.890869
1	6	262144	9127	3.481674
1	7	1048576	18346	1.749611
2	1	1024	763	74.511719
2	2	4096	2110	51.513672
2	3	16384	4801	29.302979
2	4	65536	10180	15.533447
2	5	262144	20935	7.986069
2	6	1048576	42442	4.047585
2	7	4194304	85453	2.037358

j	t	number cells uniform grid	number cells non-uniform grid	Non-uni/uni * 100 Non-uni/uni * 100
3	1	4096	2290	55.908203
3	2	16384	6517	39.776611
3	3	65536	14968	22.839355
3	4	262144	31867	12.156296
3	5	1048576	65662	6.262016
3	6	4194304	133249	3.176904
3	7	16777216	268420	1.599908
4	1	16384	6169	37.652588
4	2	65536	17692	26.995850
4	3	262144	40735	15.539169
4	4	1048576	86818	8.279610
4	5	4194304	178981	4.267240
4	6	16777216	363304	2.165461
4	7	67108864	731947	1.090686
5	1	65536	15808	24.121094
5	2	262144	44995	17.164230
5	3	1048576	103366	9.857750
5	4	4194304	220105	5.247712
5	5	16777216	453580	2.703547
5	6	67108864	920527	1.371692
5	7	268435456	1854418	0.690825
6	1	262144	39655	15.127182
6	2	1048576	110314	10.520363
6	3	4194304	251628	5.999303
6	4	16777216	534256	3.184414
6	5	67108864	1099507	1.638393
6	6	268435456	2230006	0.830742
6	7	1073741824	4491001	0.418257
7	1	1048576	99598	9.498405
7	2	4194304	265489	6.329751
7	3	16777216	597268	3.559995
7	4	67108864	1260823	1.878773
7	5	268435456	2587930	0.964079

j	t	number cells uniform grid	number cells non-uniform grid	Non-uni/uni * 100 Non-uni/uni * 100
8	1	4194304	256309	6.110883
8	2	16777216	637240	3.798246
8	3	67108864	1399099	2.084820
8	4	268435456	2922814	1.088833
8	5	1073741824	5970241	0.556022
9	1	16777216	692572	4.128051
9	2	67108864	1552735	2.313755
9	3	268435456	3273058	1.219309
9	4	1073741824	6713701	0.625262

We observe that the number of cells in the uniform grid grow rapidly, while for the non-uniform grid the relative growth of the number of cells is less. When we assume that the total amount of cells in the grid is fixed, for increasing j the highest possible t is decreased. However, with the same amount of cells the non-uniform grid can be much more precise in modelling the polygons than the uniform grid can. If we assume that the amount of cells that can be stored is about 30000 at most, we see that with increasing j the highest possible t decreases. However, the highest possible value for t for non-uniform grids is considerably larger than the highest possible value for t for the uniform grid. This is demonstrated by the next Table.

j	Non-Uniform Grid, t	Uniform Grid, t
1	7	4
2	5	3
3	4	2
4	2	1
5	1	-

9 Conclusion.

In this paper we have addressed ourselves to the question how many cells a non-uniform grid can have when the representation of the k-dimensional grid is done using a 2^k-tree.

For this, we introduced several parameters and quantities:

- We found bounds on the acceptable amounts of subdivision in each level of the tree.
- For the most widely spread distribution of cells of smallest size over the grid, for given dimension k and given height h of the tree, we found a formula for the total amount H of cells in the grid. This is an upper bound for the amount of cells. Similarly, we found a lower bound L, the total amount of cells in the grid when the cells of smallest size are most densily distributed (all positioned in one branch of the tree) over the grid.
- We looked for a formula giving us the height a tree must have in order to model a grid with given amount of cells of smallest size and given total amount of cells.

We also considered some particular values for the parameters corresponding to some practical applications carried out over the last years at the Research Institute for Symbolic Computation, in the fields of geometric modelling and obstacle avoidance.

Finally we studied the amount of cells needed in a non-uniform grid for modelling a particular family of polygons in 2 dimensions. We compared this amount with the number of cells in a uniform grid and observed a considerable reduction of the total number of cells needed when using a non-uniform grid.

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