Technical Report

A Basis for Multivariate Hermite Interpolation †

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Multivariate Hermite interpolation is a problem of finding a polynomial, given its values and the values of its partial derivatives on several points. In solving this problem, a fundamental task is to find an interpolation basis (a basis in which the coefficients of the linear combination are exactly the given values). A basis for univariate case has been found long ago. This paper provides a basis for multivariate case (under certain non-degeneracy condition).

1. Introduction

Interpolation is a problem of finding a polynomial, given its values (and the values of its derivatives) on several points. It is a fundamental problem in numerical analysis (Werner and Schaback, 1979), in computer algebra (Zippel, 1993), in the study of residue class rings arising in polynomial ideal theory (Moeller, 1976; Pedersen *et al.*, 1993).

One way to tackle interpolation problems is to find an interpolation basis (a basis in which the coefficients of the linear combination are exactly the given values). For the univariate case without derivatives, Lagrange interpolation formula gives a basis. For the univariate case with derivatives, a basis has been also found (Berezin and Zhidkov, 1973).

In this paper, we provide a basis for the most general case: multivariate with partial derivatives (under certain non-degeneracy condition). The degenerate case arises only when the interpolation points share coordinate values, that is, some points are aligned along the coordinate axes. It will be an interesting study to cover the degenerate cases also.

The structure of the paper is as follows. Section 2 states precisely the problem of finding an interpolation base. Section 3 propose an interpolation basis. Section 4 proves that it is indeed an interpolation basis.

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2. Problem Statement

We begin by fixing some notations/conventions that will be used throughout this paper.

NOTATION 2.1.

$\mathbf{0}$	the n-tuple of zeros, $(0, \ldots, 0)$.
$\mathbf{e}$	an n-tuple of exponents, $(e_1, \ldots, e_n), e_i \in \mathbb{N}$.
$\lvert\mathbf{e}\rvert$	the size, $e_1 + \cdots + e_n$.
$\mathbf{e}!$	the factorial, $e_1! \cdots e_n!$.
$\mathbf{e} - \mathbf{e}'$	the difference, $(e_1 - e'_1, \ldots, e_n - e'_n)$.
$\mathbf{e} \leq \mathbf{e}'$	less or equal, $e_i \leq e'_i$ for all i.
$\mathbf{x}$	an n-tuple of variables, $(x_1, \ldots, x_n)$.
$\mathbf{x}^{\mathbf{e}}$	the power product, $x_1^{e_1} \cdots x_n^{e_n}$.
K	a field of characteristic zero.
P	an element of $K[\mathbf{x}]$.
$P^{(\mathbf{e})}$	the partial derivative, $\frac{\partial^{e_1+\cdots+e_n}}{\partial x_1^{e_1} \cdots \partial x_n^{e_n}} P$.

The Multivariate Hermite Interpolation problem is stated as follows:

Given: $\mathbf{a}_i \in K^n$, $i = 1, \ldots, m$, distinct
$b_{ij} \in K$, $i = 1, \ldots, m$, $j = 1, \ldots, \mu_i$.
$\mathbf{e}_{ij} \in \mathbb{N}^n$, $i = 1, \ldots, m$, $j = 1, \ldots, \mu_i$.

Find: $P \in K[\mathbf{x}]$ such that $P^{(\mathbf{e}_{ij})}(\mathbf{a}_i) = b_{ij}$.

This problem naturally motivates us to define the notion of "interpolation basis".

DEFINITION 2.1. (INTERPOLATION BASIS) *An interpolation basis of* $\mathbf{a}_i$ *and* $\mathbf{e}_{ij}$ *is a collection of polynomials* w_{ij} *in* $K[\mathbf{x}]$ *such that*

$$w_{ij}^{(\mathbf{e}_{i'j'})}(\mathbf{a}_{i'}) = \delta_{ii'}\delta_{jj'}. \blacksquare$$

Once we have found an interpolation basis w_{ij}, the Hermite interpolation problem can be immediately solved, since we can trivially set

$$P = \sum_{i=1}^{m} \sum_{j=1}^{\mu_i} b_{ij} w_{ij}.$$

Now we are ready to state the problem tackled in this paper.

Problem: Find an "explicit" expression for an interpolation basis of $\mathbf{a}_i$ and $\mathbf{e}_{ij}$.

3. Solution

This section gives an explicit expression of an interpolation basis. The main idea is to generalize the known formula for the univariate case (Berezin and Zhidkov, 1973) to the multivariate case.
Some notations are needed:

NOTATION 3.1.

- $\Omega_i = \prod_{1\leq k\leq m, k\neq i}(\mathbf{x}-\mathbf{a}_k)^{\mathbf{h}_k}$ where $\mathbf{h}_k \in \mathbb{N}^n$ is such that
  (a) $\forall j \in [1,\ldots,\mu_k]$ $\mathbf{h}_k \not\leq \mathbf{e}_{kj}$ and
  (b) $\Omega_i(\mathbf{a}_i) \neq 0$.
- $\mathbf{E}_{ij} = \{\mathbf{e} : \exists j' \in [1,\ldots,\mu_i]\ \mathbf{e} \leq \mathbf{e}_{ij'} - \mathbf{e}_{ij}\}$.

THEOREM 3.1. (INTERPOLATION BASIS) *The following polynomials w_{ij} form an interpolation basis:*

$$w_{ij} = \frac{(\mathbf{x}-\mathbf{a}_i)^{\mathbf{e}_{ij}}}{\mathbf{e}_{ij}!}\Omega_i \sum_{\mathbf{e}\in\mathbf{E}_{ij}} (\frac{1}{\Omega_i})^{(\mathbf{e})}(\mathbf{a}_i)\frac{(\mathbf{x}-\mathbf{a}_i)^{\mathbf{e}}}{\mathbf{e}!}. \blacksquare$$

Remark: Note that the exponent tuples $\mathbf{h}_i$ might not exist. As an example, consider the case:

$$\begin{aligned}
&n = 2, m = 3,\\
&a_1 = (1,1), a_2 = (1,2), a_3 = (2,1),\\
&\mu_1 = \mu_2 = \mu_3 = 1,\\
&e_{11} = e_{21} = e_{31} = (0,0).
\end{aligned}$$

In this case, one can verify that it is not possible to satisfy the conditions on $\mathbf{h}_1$. The main difficulty comes from the fact that the x_1 coordinates of a_1 and a_2 are the same and likewise the x_2 coordinates of a_1 and a_3 are the same. In general, such a "degenerate " alignment of the points a_i's is the cause for the failure. But sometimes one can still find $\mathbf{h}_i$ even in such degenerate cases. Further, when none of the points a_i share coordinate values, one can alway find $\mathbf{h}_i$'s.
Remark: Note that the exponent tuples $\mathbf{h}_i$ are not uniquely determined (when exist), and depending on which $\mathbf{h}_i$ are chosen, we get different bases. In order to make the choice of $\mathbf{h}_i$ unique, one may define a linear ordering on exponent tuples and choose the smallest one.

4. Proof

We will begin with a few easy lemmas that will be used in the main proof. The following lemma is well known.

LEMMA 4.1. (PARTIAL DERIVATIVE OF PRODUCT)

$$\frac{(PQ)^{(\mathbf{f})}}{\mathbf{f}!} = \sum_{\mathbf{f}=\mathbf{f}'+\mathbf{f}''} \frac{P^{(\mathbf{f}')}}{\mathbf{f}'!}\frac{Q^{(\mathbf{f}'')}}{\mathbf{f}''!}. \blacksquare$$

LEMMA 4.2. (PARTIAL DERIVATIVE OF POWER PRODUCT)

$$\left((\mathbf{x}-\mathbf{a})^{\mathbf{e}}\right)^{(\mathbf{f})}(\mathbf{a}) = \mathbf{e}!\ \delta_{\mathbf{ef}}. \ \blacksquare$$

PROOF. We only need to observe:

$$\begin{aligned}\left((\mathbf{x}-\mathbf{a})^{\mathbf{e}}\right)^{(\mathbf{f})}(\mathbf{a}) &= \prod_{i=1}^{n} \frac{\partial^{f_i}(x_i-a_i)^{e_i}}{\partial x_i^{f_i}}(a_i) \\ &= \prod_{i=1}^{n} e_i!\, \delta_{e_i f_i} \\ &= \mathbf{e}!\ \delta_{\mathbf{ef}}. \ \blacksquare\end{aligned}$$

□

LEMMA 4.3. (PARTIAL DERIVATIVE OF MIXED PRODUCT)

$$\left((\mathbf{x}-\mathbf{a})^{\mathbf{e}} Q(x)\right)^{(\mathbf{f})}(\mathbf{a}) = \begin{cases} 0 & \text{if } \mathbf{e} \not\leq \mathbf{f} \\ \frac{\mathbf{f}!}{(\mathbf{f}-\mathbf{e})!} Q^{(\mathbf{f}-\mathbf{e})}(\mathbf{a}) & \text{if } \mathbf{e} \leq \mathbf{f} \end{cases} \quad \blacksquare$$

PROOF. The lemma is immediate from the following repeated rewritings (using the previous lemmas):

$$\begin{aligned}&\left((\mathbf{x}-\mathbf{a})^{\mathbf{e}} Q(x)\right)^{(\mathbf{f})}(\mathbf{a}) \\ =& \left[\mathbf{f}! \sum_{\mathbf{f}=\mathbf{f}'+\mathbf{f}''} \frac{\left((\mathbf{x}-\mathbf{a})^{\mathbf{e}}\right)^{(\mathbf{f}')}}{\mathbf{f}'!} \frac{Q^{(\mathbf{f}'')}}{\mathbf{f}''!}\right](\mathbf{a}) \\ =& \ \mathbf{f}! \sum_{\mathbf{f}=\mathbf{f}'+\mathbf{f}''} \frac{\left((\mathbf{x}-\mathbf{a})^{\mathbf{e}}\right)^{(\mathbf{f}')}(\mathbf{a})}{\mathbf{f}'!} \frac{Q^{(\mathbf{f}'')}(\mathbf{a})}{\mathbf{f}''!} \\ =& \ \mathbf{f}! \sum_{\mathbf{f}=\mathbf{f}'+\mathbf{f}''} \frac{\mathbf{e}!\, \delta_{\mathbf{ef}'}}{\mathbf{f}'!} \frac{Q^{(\mathbf{f}'')}(\mathbf{a})}{\mathbf{f}''!} \\ =& \begin{cases} 0 & \text{if } \mathbf{e} \not\leq \mathbf{f} \\ \frac{\mathbf{f}!}{(\mathbf{f}-\mathbf{e})!} Q^{(\mathbf{f}-\mathbf{e})}(\mathbf{a}) & \text{if } \mathbf{e} \leq \mathbf{f} \end{cases} \quad \blacksquare\end{aligned}$$

□

Proof of Theorem 3.1. We need to show that

$$w_{ij}^{(\mathbf{e}_{i'j'})}(\mathbf{a}_{i'}) = \delta_{ii'}\delta_{jj'}.$$

The proof is divided into three cases.

<u>CASE: $i' \neq i$.</u>

First note that w_{ij} can be written as

$$w_{ij} = (\mathbf{x}-\mathbf{a}_{i'})^{\mathbf{h}_{i'}} Q(x)$$

for some polynomial Q. Recall that $\mathbf{h}_{i'}$ is chosen so that $\mathbf{h}_{i'} \not\leq \mathbf{e}_{i'j'}$. Thus from Lemma 4.3, we have

$$w_{ij}^{(\mathbf{e}_{i'j'})}(\mathbf{a}_{i'}) = 0 = \delta_{ii'}\delta_{jj'}.$$

<u>CASE: $i' = i$ and $e_{ij} \not\leq e_{i'j'}$.</u>

First note that w_{ij} can be written as

$$w_{ij} = (\mathbf{x} - \mathbf{a}_i)^{\mathbf{e}_{ij}} Q(x)$$

for some polynomial Q. From the assumption of this case, we have $\mathbf{e}_{ij} \not\leq \mathbf{e}_{i'j'}$. Thus from Lemma 4.3, we have

$$w_{ij}^{(\mathbf{e}_{i'j'})}(\mathbf{a}_{i'}) = 0 = \delta_{ii'}\delta_{jj'}.$$

<u>CASE: $i' = i$ and $e_{ij} \leq e_{i'j'}$.</u>

First note that w_{ij} can be written as

$$w_{ij} = \frac{(\mathbf{x} - \mathbf{a}_i)^{\mathbf{e}_{ij}}}{\mathbf{e}_{ij}!} Q(x)$$

where

$$Q(x) = \sum_{\mathbf{e} \in \mathbf{E}_{ij}} (\frac{1}{\Omega_i})^{(\mathbf{e})}(\mathbf{a}_i) \frac{(\mathbf{x} - \mathbf{a}_i)^{\mathbf{e}}}{\mathbf{e}!} \Omega_i.$$

Applying Lemma 4.3 on w_{ij}, we obtain

$$w_{ij}^{(\mathbf{e}_{ij'})}(\mathbf{a}_i) = \frac{\mathbf{e}_{ij'}!}{(\mathbf{e}_{ij'} - \mathbf{e}_{ij})!\ \mathbf{e}_{ij}!} Q^{(\mathbf{e}_{ij'} - \mathbf{e}_{ij})}(\mathbf{a}_i).$$

Applying Lemma 4.3 again on each summand of Q, we see that $Q^{(\mathbf{e}_{ij'} - \mathbf{e}_{ij})}(\mathbf{a}_i)$ is

$$\sum_{\substack{\mathbf{e} \in \mathbf{E}_{ij} \\ \mathbf{e} \leq \mathbf{e}_{ij'} - \mathbf{e}_{ij}}} \frac{(\mathbf{e}_{ij'} - \mathbf{e}_{ij})!}{(\mathbf{e}_{ij'} - \mathbf{e}_{ij} - \mathbf{e})!\ \mathbf{e}!} (\frac{1}{\Omega_i})^{(\mathbf{e})}(\mathbf{a}_i) \Omega_i^{(\mathbf{e}_{ij'} - \mathbf{e}_{ij} - \mathbf{e})}(\mathbf{a}_i).$$

Recall that $\mathbf{E}_{ij}$ is defined so that it contains all the $\mathbf{e} \leq \mathbf{e}_{ij'} - \mathbf{e}_{ij}$. Thus we can safely drop the condition $\mathbf{e} \in \mathbf{E}_{ij}$ from the above summation range. Further, by setting $\mathbf{e} + \mathbf{f} = \mathbf{e}_{ij'} - \mathbf{e}_{ij}$, we see that $Q^{(\mathbf{e}_{ij'} - \mathbf{e}_{ij})}(\mathbf{a}_i)$ is

$$\sum_{\mathbf{e} + \mathbf{f} = \mathbf{e}_{ij'} - \mathbf{e}_{ij}} \frac{(\mathbf{e}_{ij'} - \mathbf{e}_{ij})!}{\mathbf{e}!\ \mathbf{f}!} (\frac{1}{\Omega_i})^{(\mathbf{e})}(\mathbf{a}_i) \Omega_i^{(\mathbf{f})}(\mathbf{a}_i)$$

By factoring out the evaluation on $\mathbf{a}_i$, it can be written as

$$\left[\sum_{\mathbf{e} + \mathbf{f} = \mathbf{e}_{ij'} - \mathbf{e}_{ij}} \frac{(\mathbf{e}_{ij'} - \mathbf{e}_{ij})!}{\mathbf{e}!\ \mathbf{f}!} (\frac{1}{\Omega_i})^{(\mathbf{e})} \Omega_i^{(\mathbf{f})} \right] (\mathbf{a}_i).$$

By applying Lemma 4.1, it can be again written as

$$\left[\frac{1}{\Omega_i}\Omega_i\right]^{(\mathbf{e}_{ij'}-\mathbf{e}_{ij})}(\mathbf{a}_i)$$

which is obviously

$$1^{(\mathbf{e}_{ij'}-\mathbf{e}_{ij})}(\mathbf{a}_i),$$

and thus we see that

$$Q^{(\mathbf{e}_{ij'}-\mathbf{e}_{ij})}(\mathbf{a}_i) = \delta_{jj'}.$$

Using this, we finally get

$$w_{ij}^{(\mathbf{e}_{ij'})}(\mathbf{a}_i) = \frac{\mathbf{e}_{ij'}!}{(\mathbf{e}_{ij'} - \mathbf{e}_{ij})!\,\mathbf{e}_{ij}!}\delta_{jj'} = \delta_{jj'} = \delta_{ii'}\delta_{jj'}. \blacksquare$$

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