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Decomposing Many-valued Logics: An Experimental Case Study

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Abstract

The idea of this paper is to illustrate and motivate a parallelization/decomposition method based on logical fiberings which was suggested by experimenting with several examples. We start with some concrete small examples and then generalize the method to a whole class of many valued logics - e.g. the Lukasiewicz case, in general. In that way we hope to pave the way for future work aiming towards a representation methods of m-valued systems as logical fiberings.

1 Introduction

The notion of a logical fibering had been introduced in the framework of an industrial case study on applications of so-called "polycontextural logics". These polycontextural systems actually were under discussion in the Biological Computing Laboratory (Urbana, Illinois) in the sixtieth when an interdisciplinary group of scientists started an initiative to work on so-called second order cybernetics. This should extend considerably classical cybernetics especially in the direction of modeling complex communications systems and cooperating autonomous agents. The basic idea behind the polycontextural logic (PCL) was to provide each agent with a local individual logic which was assumed to be a classical 2-valued first order logic. All the subsystems are composed in a specific way by describing how they form as a whole a many-valued system (via a so-called "mediation scheme" which imposes constraints on the collection of all the classical truth values of local 2-valued systems which are labeled by the index of the corresponding system. Univariate and bivariate operations can be introduced which are built up by classical operations coming from the individual 2-valued subsystems. A typical non-classical bivariate connective, called transjunction, arises. A transjunction is an operation linking different subsystems logically. For the interested reader in the report [Pfa91b] we included some quotations and typical motivating remarks and comments from the PCL literature in the report [Pfa91b]. For a recently published monograph by one of the key researchers in PCL we recommend the work by Kaeher and Mahler [KM94].

The development of logical fiberings originates in work on the previously mentioned case study. It has been motivated by the classical theory of fiber bundles, a powerful modeling language

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from geometry and topology, where typical local-global interaction of different structures can be integrated in one concept. Thus, for example, in a vector bundle the fibers of such a fiber bundle are vector spaces of a fixed dimension. Roughly spoken, vertically one does algebra and horizontally (across the fibers) one is doing geometry/topology.

Concerning the (notion of) *logical fiberings*, as introduced in [Pfa91a], the idea was to replace the fibers of a fiber bundle by a logical space (e.g. a 2-valued logic) and try to preserve all the expressive power of the classical fiber bundle notion as good as possible. A typical problem point arising here is that we are going to mix discrete with continuous structures in one integrated concept, but this is part of the challenge. More generally spoken we want to extend this logical fiberings approach to a generic modeling principle which allows to mix various logics in the sense that we take various different logical spaces as fibers, putting them together (as a bundle of fibers) over a base space manifold that serves as an index system with its own structure. The development of such logical fiberings in terms of a flexible logical operational modeling tool is a difficult task.

As a first result we established the fact that a PCL is a special class of a logical fibering determined by a specified equivalence relation on the global set of truth values (which describes the corresponding "mediation scheme"). In this sense the logical fiberings provide a framework for a systematic construction of many-valued logics. This was pointed out by D. Gabbay and he further stated that they point to a general semantics for his extended theory of labeled deductive systems (LDS), cf. [Gab90b, Gab90a, Gab94]. He coined the notion of "fibered semantics". Moreover he suggested to select special well-known 3-, or 4-valued logical connectives and to express them in the framework of a special suitable logical fibering.

Some experimental studies in this direction were successful in the sense that they are suggesting to think at a representation of many-valued logics by an associated logical fibering such that corresponding bivariate operations of a specific many-valued system can be decomposed into classical operations (corresponding to classical systems as fibers, respectively) and eventually some transjunctions. The advantage of such a representation theory would be a "parallelization" of a many-valued system and the reduction of corresponding operations to "fiberwise" classical and some elementary non-classical operations (so-called transjunctions), respectively.

Further work in the direction of algebraic semantics where the possible world semantics in modal logics is a starting point is under development. It is also linked to the framework of the notion of logical fiberings.

Concerning general semantical modeling aspects the previous considerations are also connected with other work which deals with semantical models for relational structures (using the language of category theory) where sheaf semantics appear in a natural way (cf. [Pfa94] for a proposed program of work in this direction).

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2 Introductory and Motivating Examples

In the following considerations we use the same notation as in [Pfa91a]. Recalling notation, in that work we briefly discussed an example of a bivariate logical function like

$$X \wedge \vee \wedge Y = \begin{pmatrix} x_1 \wedge y_1 \\ x_2 \vee y_2 \\ x_3 \wedge y_3 \end{pmatrix}.$$

With that it shall be expressed that the whole bivariate logical function (operation) is formed by "putting together" local components defined in a 2−valued classical logical space (subsystem) – considered as a "fiber" of the whole system which as a whole forms the underlying *logical fibering.*

Here we are going to interpret the fiberings model exactly in the opposite direction: we intend to use this approach to *decompose* a given m−valued bivariate logical function into a number of 2−valued components based on a corresponding underlying logical fibering which has to be constructed, respectively.

Such a decomposition procedure might be of interest at least with respect to the following aspects which arise naturally.

- Decomposing m−valued operations into components as mentioned above results in a *parallelization* of logical operations where one can work with classical operations in the components in parallel ("fiberwise"), respectively.
- A general representation theory of many valued logics by means of logical fiberings is expected where it should be possible to represent a many valued space by a fibering of 2−valued spaces. More general, it should be possible also to "mix" logics and to model "carse" and "fine grain" decompositions, i.e. having also many valued logics as fibers (and not only 2−valued) – conceptually this is possible, in principle, with our approach.

It is the purpose of this contribution to to collect some practical experience and to acquire a feeling how to proceed. To this end we are going to discuss some concrete examples as presented below.

We start with the following motivating example of a 3-valued logic $\mathcal{L}$ taken from [RB88], p.169, as a first illustration of our method. The bivariate operations AND, OR, IMPLY of that example are given by the tables (cf. [RB88], loc.cit.)

AND	T	*	F
T	T	*	F
*	*	*	F
F	F	F	F

In the following considerations we shall use a technically shorter notation for displaying the truth table of such bivariate logical functions just using the 3×3−matrix consisting of the image values of such a function:

T	*	F
*	*	F
F	F	F

Analogously, the OR function is given by

T	T	T
T	*	*
T	*	F

and the IMPLY function is defined by

T	*	F
T	T	F
T	T	T

By technical reasons we rename the symbols for the truth values as follows, setting $a :=$ T, $b := *$, $c :=$ F. Symbolically, we then obtain the set of ("global") values $\Omega = \{a, b, c\}$.

Below we are briefly describing the principle how the given 3-valued logic $\mathcal{L}$ can be derived from the free parallel system $\mathcal{L}^3$, the logical fibering consisting of 3 classical 2-valued subsystems denoted L_1, L_2, L_3 (fibers) — cf.[Pfa91a] for the details.

The *global values* of $\mathcal{L}^3$ are given by the 6 *local values* $\{T_i, F_i\}$, $i = 1, 2, 3$, of the three *subsystems*:

$$\Omega^3 = \{T_1, F_1, T_2, F_2, T_3, F_3\}.$$

The procedure of deriving $\mathcal{L}$ from $\mathcal{L}^3$ can be briefly formulated as follows:

- Find a suitable equivalence relation $\equiv$ on Ω^3 such that the set of residue classes $\Omega := \Omega^3/\equiv$ yields the global value set $\{a, b, c\}$ of the given logic $\mathcal{L}$.
- Express each given logical connective AND, OR, IMPLY in terms of a family of local classical connectives (i.e. triples) defined in each of the three local subsystems L_1, L_2, L_3, respectively.
- These representations have to be compatible with the equivalence relation on Ω^3 ([Pfa91a]).

In some respect this procedure can be considered to be a a construction following the *generators and relations principle.*

Method of decomposition:

¿From a 3-valued bivariate logical operation represented by a 3×3-value matrix consisting of the image values of a corresponding logical function $\{a, b, c\} \times \{a, b, c\} \rightarrow \{a, b, c\}$ we derive three 2-valued classical operations given by the three 2×2-submatrices along the diagonal of the 3×3-schema — interpreted in $\mathcal{L}^3$.

That means each of the 2×2-submatrices belongs to the possible 4 index pairs formed by a selected pair of indices i, j, namely $\{(i, i), (i, j), (j, i), (j, j)\}$, where $i < j$ and i, j is running through $1, 2, 3$.

Thus, for a general m-valued bivariate operation we obtain the amount of $n := \binom{m}{2}$ suboperations defined in a corresponding 2−valued "local subsystem" (forming a fiber of the decomposition). For the logical values of each 2×2-submatrix we use the same total ordering which we define on the global values $\{a, b, c\}$, here we take $a < b < c$. For example, the AND operation leads to the first 2×2-submatrix ($\{1, 2\}$-submatrix)

a	b
b	b

which is interpreted in the first subsystem L_1 of $\mathcal{L}^3$, hence in terms of the local truth values $\{T_1, F_1\}$ of L_1. We obtain locally: $a = T_1, b = F_1$. Obviously this represents locally a classical conjunction – symbolically expressed by $x_1 \wedge y_1$.
Analogously we derive the $\{2,3\}$-submatrix and the $\{1,3\}$-submatrix and obtain the local conjunctions $x_2 \wedge y_2$ and $x_3 \wedge y_3$, respectively.
These are all considered as local logical operations in the free parallel logical system (fibering) $\mathcal{L}^3$. Summarizing, we obtain the bivariate operation:

$$X \wedge\wedge\wedge Y = \begin{pmatrix} x_1 \wedge y_1 \\ x_2 \wedge y_2 \\ x_3 \wedge y_3 \end{pmatrix}.$$

In order to find the representation of the originally given OR in $\mathcal{L}$ derived from that decomposition $X \wedge\wedge\wedge Y$ in $\mathcal{L}^3$ we have to define a suitable $\equiv$-relation on Ω^3, to identify the corresponding equivalence classes and to check whether this is compatible with all the three originally given bivariate operations. We demonstrate this with the example AND:
The second submatrix w.r.t. the indices $\{2,3\}$ is given by:

b	c
c	c

This induces the local T,F-scheme

T_2	F_2
F_2	F_2

w.r.t. the correspondence $T_2 \leftrightarrow b$, $F_2 \leftrightarrow c$. This again yields a classical conjunction $x_2 \wedge y_2$ in the subsystem L_2 of $\mathcal{L}^3$.
The (third) $\{1,3\}$-submatrix which can be derived from the given 3×3-matrix is

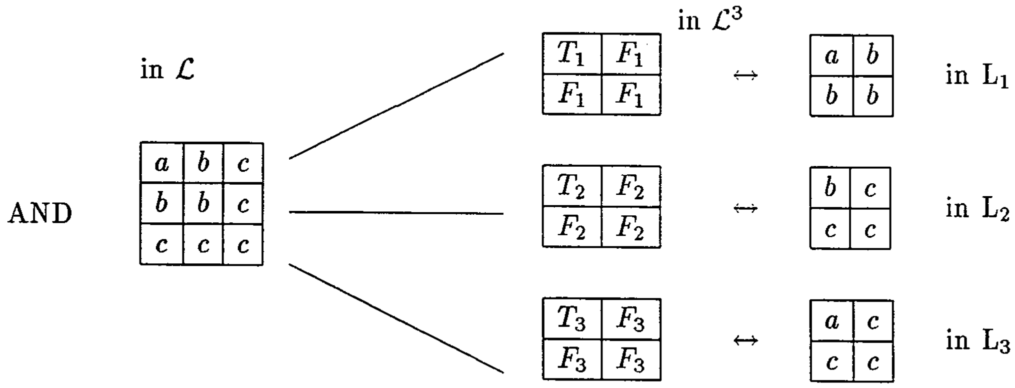
a	c
c	c

leading to

T_3	F_3
F_3	F_3

w.r.t. $T_3 \leftrightarrow b$, $F_3 \leftrightarrow c$ in subsystem L_3 of $\mathcal{L}^3$. This yields $x_3 \wedge y_3$. Altogether we obtain $X \wedge\wedge\wedge Y$ in $\mathcal{L}^3$.
Noticing that $\Omega = \{a, b, c\}$ is the set of global values which shall be obtained from Ω^3 as a set of equivalence classes $\Omega^3/\equiv$ we can read off the following equivalence relation on Ω^3 from the above identities:
$T_1 \equiv T_3$ leads to class a. Let $[T_1]$ denote the equivalence class of T_1 then $[T_1] = [T_3] = a$, analogously $F_1 \equiv T_2$, $F_2 \equiv F_3$ such that $[F_1] = [T_2] = b$, and $[F_2] = [F_3] = c$.

We summarize the previously introduced decomposition of AND in the following schematic drawing:



AND leads to

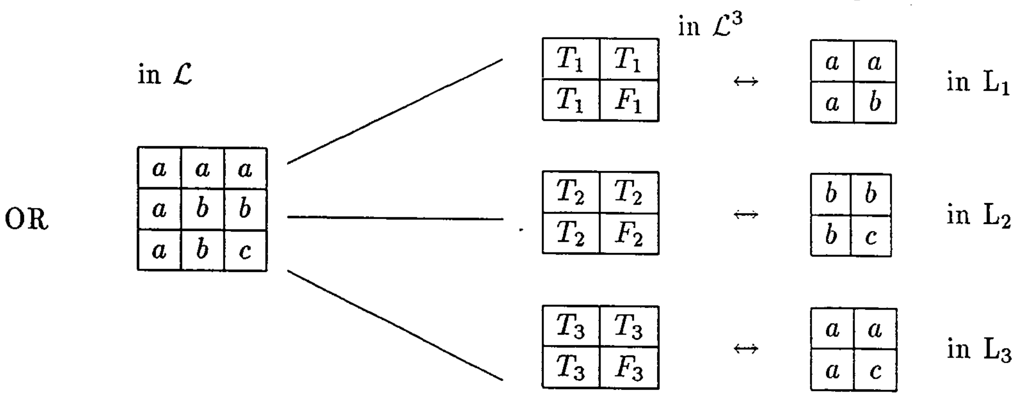
$$X \wedge\wedge\wedge Y = \begin{pmatrix} x_1 \wedge y_1 \\ x_2 \wedge y_2 \\ x_3 \wedge y_3 \end{pmatrix}$$

in $\mathcal{L}^3$ with $\equiv$ on Ω^3 given by

$$a = [\mathrm{T}_1] = [\mathrm{T}_3], b = [\mathrm{F}_1] = [\mathrm{T}_2], c = [\mathrm{F}_2] = [\mathrm{F}_3].$$

We can therefore represent AND by the operation $X \wedge\wedge\wedge Y$ interpreted in $\mathcal{L}$ w.r.t. $\Omega = (\Omega^3/\equiv) = \{a, b, c\}$ corresponding to the above identities on the truth values of Ω^3.

In this setting we have *parallelized* AND by the operation $X \wedge\wedge\wedge Y$ consisting of 3 conjunctions deduced from $\mathcal{L}^3$.
Analogously, we give the presentation of OR and IMPLY following the above method.



The corresponding identifications are:

$$a = [\mathrm{T}_1] = [\mathrm{T}_3], b = [\mathrm{F}_1] = [\mathrm{T}_2], c = [\mathrm{F}_2] = [\mathrm{F}_3].$$

These are the same relations as obtained from AND — in this sense both representations are compatible with the given equivalence relation $\equiv$.

OR is thus representable as

$$X \vee\vee\vee Y = \begin{pmatrix} x_1 \vee y_1 \\ x_2 \vee y_2 \\ x_3 \vee y_3 \end{pmatrix},$$

similarly as AND.

in $\mathcal{L}$ | in $\mathcal{L}^3$

$$\text{IMPLY} \quad \begin{array}{|c|c|c|} \hline a & b & c \\ \hline a & a & c \\ \hline a & a & a \\ \hline \end{array} \qquad \begin{array}{l} \begin{array}{|c|c|} \hline T_1 & F_1 \\ \hline T_1 & T_1 \\ \hline \end{array} \leftrightarrow \begin{array}{|c|c|} \hline a & b \\ \hline a & a \\ \hline \end{array} \ \text{in } \mathrm{L}_1 \\ \begin{array}{|c|c|} \hline T_2 & F_2 \\ \hline T_2 & T_2 \\ \hline \end{array} \leftrightarrow \begin{array}{|c|c|} \hline a & c \\ \hline a & a \\ \hline \end{array} \ \text{in } \mathrm{L}_2 \\ \begin{array}{|c|c|} \hline T_3 & F_3 \\ \hline T_3 & T_3 \\ \hline \end{array} \leftrightarrow \begin{array}{|c|c|} \hline a & c \\ \hline a & a \\ \hline \end{array} \ \text{in } \mathrm{L}_3 \end{array}$$

In $\mathcal{L}^3$ we obtain:

$$X \longrightarrow\longrightarrow\longrightarrow Y = \begin{pmatrix} x_1 \longrightarrow y_1 \\ x_2 \longrightarrow y_2 \\ x_3 \longrightarrow y_3 \end{pmatrix}.$$

The induced identifications on the value set Ω^3 are given by

$$a = [\mathrm{T}_1] = [\mathrm{T}_2] = [\mathrm{T}_3], b = [\mathrm{F}_1], c = [\mathrm{F}_2] = [\mathrm{F}_3].$$

This differs from the above $\equiv$-relation in the class represented by a, namely for imply we have to identify $\mathrm{T}_2 \equiv \mathrm{T}_1$ in contrast to $\mathrm{T}_2 \equiv \mathrm{F}_1$ in AND and OR.
This yields an incompatibility: with respect to the previous $\equiv$-relation IMPLY cannot be represented as $X \longrightarrow\longrightarrow\longrightarrow Y$ in the same way as we did this for OR, AND.

But it is possible to repair this, i.e. to make all 3 connectives compatible with the originally chosen $\equiv$-relation on Ω^3 if we apply the concept of transjunction (cf. [Pfa91a] for the definition). Recalling, a transjunction in subsystem L_i is a local bivariate operation, defined on the values of L_i, which distributes its image values over different subsystems. In accordance with our notation it is defined by a value matrix like, e.g.

$$\boxed{\begin{array}{cc} T_i & F_i \\ F_i & F_i \end{array}} \longrightarrow \boxed{\begin{array}{cc} T_\alpha & F_\beta \\ F_\gamma & F_\delta \end{array}}$$

In terms of valuations it can be described locally by the formula

$$\begin{aligned} \mathrm{w}_{\alpha\beta\gamma\delta}(,) = \quad & \chi_{(T_i,T_i)}(,) \cdot w_\alpha \phi_{\alpha i} \vartheta(,) \quad + \\ & \chi_{(T_i,F_i)}(,) \cdot w_\beta \phi_{\beta i} \vartheta(,) \quad + \\ & \chi_{(F_i,T_i)}(,) \cdot w_\gamma \phi_{\gamma i} \vartheta(,) \quad + \\ & \chi_{(F_i,F_i)}(,) \cdot w_\delta \phi_{\delta i} \vartheta(,) \quad . \end{aligned}$$

Application of this to the above problem yields:

The second submatrix $\begin{array}{|c|c|}\hline a & c \\ \hline a & a \\ \hline\end{array}$ corresponding to $\begin{array}{|c|c|}\hline T_2 & F_2 \\ \hline T_2 & T_2 \\ \hline\end{array}$ in subsystem L_2 leads to an incompatibility. But it would be compatible with the $\equiv$-relation obtained from the AND, OR representation if it were of the form $\begin{array}{|c|c|}\hline b & c \\ \hline b & b \\ \hline\end{array}$.

We can achieve a representation of IMPLY of the form $X \rightharpoonup\!\Rightarrow_t\!\rightharpoonup Y$ deducing it from the parallel system $\mathcal{L}^3$ where the second local operation $x_2 \Rightarrow_t y_2$ is a suitable transjunction defined by a bivariate operation defined as $\Omega_2 \times \Omega_2 \rightarrow \Omega_1 \cup \Omega_2$ given by $\begin{array}{|c|c|}\hline T_1 & F_2 \\ \hline T_1 & T_1 \\ \hline\end{array}$ and denoted by $\Rightarrow_t$. Note that this is a local bivariate operation defined in L_2 with values distributed over the two subsystems L_1 and L_2. As a T-F-pattern this is of the type of an implication table, we therefore also can describe this transjunction by: $\Rightarrow_t = \mathcal{D} \circ \rightharpoonup$, where $\rightharpoonup : \Omega_2 \times \Omega_2 \rightarrow \Omega_2$ is a classical implication and $\mathcal{D} : \Omega_2 \rightarrow \Omega_1 \cup \Omega_2$ distributes the values over 2 subsystems corresponding to $T_2 \mapsto T_1$, $F_2 \mapsto F_2$.
In this way we can express IMPLY by $X \rightharpoonup\!\Rightarrow_t\!\rightharpoonup Y$ and this is compatible with the original equivalence relation.

We verify this only for the local input $[T_2, F_2]$ to the second operation $\Rightarrow_t$ since this is the only critical situation:
Recall that IMPLY is evaluated by the local evaluations of $\rightharpoonup$ and $\Rightarrow_t$ and $\rightharpoonup$ corresponding to our representation, which can be performed in parallel. Note that we must take into account the given $\equiv$-relation.
Inputing the four possible pairs formed by $b = [T_2], c = [F_2]$ we obtain the correct second submatrix of IMPLY.

Remark: We point out again that the whole 3×3-value matrix that defines IMPLY is represented by the evaluation procedures of the three bivariate operations $\rightharpoonup, \Rightarrow_t, \rightharpoonup$; this can be done in parallel.

Remark: The compatibility condition with respect to the three suboperations (submatrices) can be expressed as follows (cf. [Pfa91a]):
The three 2×2-matrices have to be merged to a 3×3-matrix scheme along the diagonal of the 3×3-matrix such that the corresponding diagonal elements match (i.e. the 2×2-matrices are the suitable submatrices).
In this sense our decomposition method is the reverse process to this merge.

We conclude this section by illustrating the method described above with a more complicated example which actually is the implication operation of a 4-valued Lukasiewicz Logic. We include it here as an exercise and a preparation of the more general discussions following below.
The decomposition of the IMPLY operation leads to the following result containing transjunctions as shown in the figure:

in $\mathcal{L}$

IMPLY

a	b	c	d
a	a	b	c
a	a	a	b
a	a	a	a

in $\mathcal{L}^3$

$\begin{array}{|c|c|} \hline T_1 & F_1 \\ \hline T_1 & T_1 \\ \hline \end{array} \leftrightarrow \begin{array}{|c|c|} \hline a & b \\ \hline a & a \\ \hline \end{array}$ in $Ł_1$

$\begin{array}{|c|c|} \hline T_2 & F_2 \\ \hline T_2 & T_2 \\ \hline \end{array} \leftrightarrow \begin{array}{|c|c|} \hline a & c \\ \hline a & a \\ \hline \end{array}$ in $Ł_2$

$\begin{array}{|c|c|} \hline T_3 & F_3 \\ \hline T_3 & T_3 \\ \hline \end{array} \leftrightarrow \begin{array}{|c|c|} \hline a & d \\ \hline a & a \\ \hline \end{array}$ in $Ł_3$

$\begin{array}{|c|c|} \hline T_1 & F_1 \\ \hline T_1 & T_1 \\ \hline \end{array} \leftrightarrow \begin{array}{|c|c|} \hline a & b \\ \hline a & a \\ \hline \end{array}$ in $Ł_4$ - Transjunction!

$\begin{array}{|c|c|} \hline T_2 & F_2 \\ \hline T_2 & T_2 \\ \hline \end{array} \leftrightarrow \begin{array}{|c|c|} \hline a & c \\ \hline a & a \\ \hline \end{array}$ in $Ł_5$ - Transjunction!

$\begin{array}{|c|c|} \hline T_1 & F_1 \\ \hline T_1 & T_1 \\ \hline \end{array} \leftrightarrow \begin{array}{|c|c|} \hline a & b \\ \hline a & a \\ \hline \end{array}$ in $Ł_6$ - Transjunction!

After these "experimental considerations", the following section discusses a more general case.

3 The n-valued Łukasiewicz Logic $Ł_n$

The procedure described above for the 3-valued Lukasiewicz logics can be easily generalized to the n -valued case (for a presentation of Lukasiewicz n -valued logics see for example [Luk30], [Luk70] and [Urq86]).

Let $Ł_n = \{0, \frac{1}{n-1}, \ldots, \frac{n-2}{n-1}, 1\}$ be the set of truth values for the n-valued Lukasiewicz logic, with the usual order, $0 < \frac{1}{n-1} < \frac{2}{n-1} < \cdots < \frac{n-2}{n-1} < 1$. The truth tables of the logical operations can be established on basis of the following definitions:

$\neg a = 1 - a$

$a \vee b = \max(a, b)$

$a \wedge b = \min(a, b)$

$a \rightarrow b = \min(1, 1 + b - a)$

$a \leftrightarrow b = 1 - |a - b|$.

Note that the usual order on $Ł_n$ coincides with the (total) order induced by $\vee$ (or, equivalently, by $\wedge$). Consider the set $I = \{(i,j) \in Ł_n | \, i < j\}$. Let $L_{i,j}$, $(i,j) \in I$ be the $\binom{n}{2}$ local subsystems, with local truth values $\Omega_{i,j} = \{F_{i,j}, T_{i,j}\}$.

Let $\Omega = \coprod_{(i,j)\in I} \Omega_{i,j}$ be the set of global values. Consider the equivalence relation $\equiv$ on Ω that induces the identifications

$$[\mathrm{T}_{i,k}] = [\mathrm{F}_{k,j}] \text{ for every } i, j, k \text{ such that } i < k \text{ and } k < j$$

in the quotient set $\Omega/\equiv$ such that we have a bijective correspondence between $\Omega/\equiv$ and $Ł_n$ i.e. for all i, j with $i < k < j$, $[\mathrm{T}_{i,k}] = [F_{k,j}]$ corresponds to $k \in Ł_n$. This implies that the relation $\equiv$ can be chosen such that:

$\mathrm{T}_{i,k} \equiv \mathrm{T}_{j,k}$, for all i, j, k such that $i < k$ and $j < k$,

$F_{k,i} \equiv F_{k,j}$ for all i, j, k such that $k < i$ and $k < j$,
$T_{i,j} \equiv F_{j,k}$ for all i, j, k such that $i < k < j$.

For the case $n = 4$, the situation is graphically described in Figure 1.

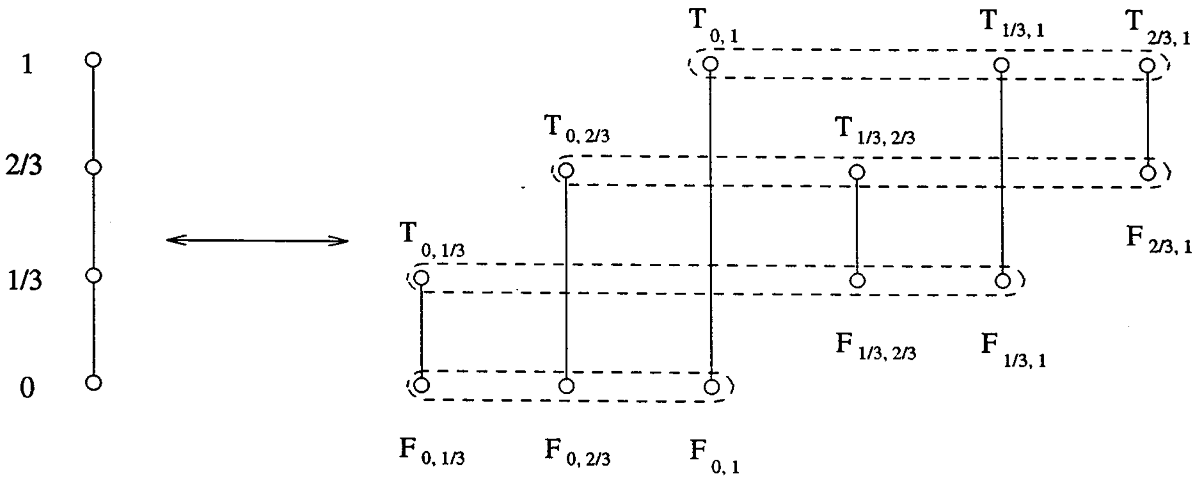


Figure 1:

This graphical illustration is related to the use of mediation schemes in PCL (cf. [KM94], p. 166-168 and [Pfa91b]).

We refer here also to Nicolas Zabel's work who started own investigations in that direction, cf. [Zab92].

As before, the operations induced by AND and OR in these subsystems are the classical conjunctions and disjunctions; the attempt of decomposing the IMPLY operation leads to the appearance of transjunctions.

First consider the AND operation, with the truth table given by $i \wedge j = \min(i, j)$.

The induced conjunction on the local subsystem $\mathcal{L}_{i,j}$, where $i < j$ is the classical conjunction:

$\wedge$	j	i
j	j	i
i	i	i

hence in $\Omega_{i,j}$ we have

$\wedge_{i,j}$	$T_{i,j}$	$F_{i,j}$
$T_{i,j}$	$T_{i,j}$	$F_{i,j}$
$F_{i,j}$	$F_{i,j}$	$F_{i,j}$

.

The operation OR has the truth table given by $i \vee j = \max(i, j)$. The induced disjunction on the local subsystem $\mathcal{L}_{i,j}$, where $i < j$ is the classical disjunction:

$\vee$	j	i
j	j	j
i	j	i

hence in $\Omega_{i,j}$ we have

$\vee_{i,j}$	$T_{i,j}$	$F_{i,j}$
$T_{i,j}$	$T_{i,j}$	$T_{i,j}$
$F_{i,j}$	$T_{i,j}$	$F_{i,j}$

.

Consider now the implication, with the following truth table:

$\rightarrow$	1	$\frac{n-2}{n-1}$	$\frac{n-3}{n-1}$	$\cdots$	$\frac{2}{n-1}$	$\frac{1}{n-1}$	0
1	1	$\frac{n-2}{n-1}$	$\frac{n-3}{n-1}$	$\cdots$	$\frac{2}{n-1}$	$\frac{1}{n-1}$	0
$\frac{n-2}{n-1}$	1	1	$\frac{n-2}{n-1}$	$\cdots$	$\frac{3}{n-1}$	$\frac{2}{n-1}$	$\frac{1}{n-1}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$
$\frac{2}{n-1}$	1	1	1	$\cdots$	1	$\frac{n-2}{n-1}$	$\frac{n-3}{n-1}$
$\frac{1}{n-1}$	1	1	1	$\cdots$	1	1	$\frac{n-2}{n-1}$
0	1	1	1	$\cdots$	1	1	1

We note that the operation induced on the local subsystem $\mathcal{L}_{i,j}$, where $i < j$ is:

$\rightarrow$	j	i
j	1	$1-(j-i)$
i	1	1

hence in $\Omega_{i,j}$ we have

$\rightarrow_{i,j}$	$\mathrm{T}_{i,j}$	$\mathrm{F}_{i,j}$
$\mathrm{T}_{i,j}$	$\mathrm{T}_{1-(j-i),1}$	$\mathrm{F}_{1-(j-i),1}$
$\mathrm{F}_{i,j}$	$\mathrm{T}_{1-(j-i),1}$	$\mathrm{T}_{1-(j-i),1}$

this means that the induced operation is a transjunction

$$\rightarrow_{i,j}: \Omega_{i,j} \times \Omega_{i,j} \longrightarrow \Omega_{1-(j-i),1}.$$

Note that $\Omega_{1-(j-i),1} = \Omega_{j\rightarrow i,1}$.

Remark: The construction described above can also be applied for structures of the form $(P, \leq)$, where $\leq$ is a partial order on P. The P can be possibly endowed with additional operations. Let $I = \{(i,j) | i,j \in P, i < j\}$ be the index set and consider the local subsystems $P_{i,j}, (i,j) \in I$, with sets of local truth values $\Omega_{i,j} = \{T_{i,j}, F_{i,j}\}$ for $(i,j) \in I$. The equivalence relation on the disjoint union of the local truth values, $\Omega = \coprod_{(i,j)\in I} \Omega_{i,j}$ can be defined similarly, namely by imposing $[\mathrm{T}_{i,k}] = [\mathrm{F}_{k,j}]$ for every $i,j,k \in P$ such that $i < k < j$ in the quotient set $\Omega/\equiv$. The equivalence class $[\mathrm{T}_{i,k}] = [\mathrm{F}_{k,j}]$ will be put into correspondence with the element $k \in P$. The induced (total) order relation in the 2-element subsystems, $F_{i,j} < T_{i,j}$, will be compatible with the partial ordering in the given partially ordered set $(P, \leq)$. Decomposing the additional operations on P may lead to some transjunctions.

3.1 Kleene's System

We will show how the method previously described can be applied to Kleene's 3-valued logic, introduced by Kleene in [Kle38], see also [Kle52] and [Urq86]. The decomposition of this logical system has also been studied by Nicolas- Zabel in [Zab92].

The set of truth values in Kleene's 3-valued logic is $\mathcal{K} = \{0, \frac{1}{2}, 1\}$, where 0 stands for "false", 1 for "true" and $\frac{1}{2}$ for "undefined". The motivation for Kleene's logic arises from the theory of recursive functions. If we consider a machine designed to answer "true" or "false" to certain questions, then for certain inputs the machine may not provide an answer (by going into an infinite loop, or by exhausting its computing capacity). In that case, we think of the machine's response as “undefined”. The truth tables for the connectives reflect this motivation. We will show how the method presented before can be applied in order to obtain a decomposition for the Kleene's system.

- The given set of truth values is $\mathcal{K} = \{0, \frac{1}{2}, 1\}$ with the usual ordering $0 < \frac{1}{2} < 1$.
- The index set for the subsystem (fibers) will be in this case $I = \{(0, \frac{1}{2}), (0, 1), (\frac{1}{2}, 1)\}$.
- We obtain 3 local subsystems, with sets of local truth values $\Omega_{i,j} = \{T_{i,j}, F_{i,j}\}$ for $(i, j) \in I$.
- Let $\Omega = \coprod_{i,j} \Omega_{i,j}$ be the global set of truth values.
- Let the equivalence relation $\equiv$ on Ω be defined by

  $$T_{i,j} \equiv T_{k,j}, F_{i,j} \equiv F_{i,k}, F_{i,j} \equiv T_{k,i}$$

  (then there is a bijective correspondence between $\Omega/\equiv$ and $\mathcal{K}$.)
- The task remains to express each given logical connective in terms of a family of local classical connectives defined in each of the three local subsystems.

The decomposition of the logical operations is summarized in the following schematic drawings. The AND operation decomposes into local classical conjunctions:

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0	0

$\wedge$	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	0
0	0	0

$\leftrightarrow$ in $L_{0,\frac{1}{2}}$

$\wedge_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$
$T_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$
$F_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$

$\wedge$	1	0
1	1	0
0	0	0

$\leftrightarrow$ in $L_{0,1}$

$\wedge_{0,1}$	$T_{0,1}$	$F_{0,1}$
$T_{0,1}$	$T_{0,1}$	$F_{0,1}$
$F_{0,1}$	$F_{0,1}$	$F_{0,1}$

$\wedge$	1	$\frac{1}{2}$
1	1	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$\leftrightarrow$ in $L_{\frac{1}{2},1}$

$\wedge_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$T_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$
$F_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$

The OR operation decomposes into local classical disjunctions:

$\vee$	1	$\frac{1}{2}$	0
1	1	1	1
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\vee$	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{2}$	0

$\leftrightarrow$ in $\mathrm{L}_{0,\frac{1}{2}}$

$\vee_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$
$T_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$
$F_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$

$\vee$	1	0
1	1	0
0	0	0

$\leftrightarrow$ in $\mathrm{L}_{0,1}$

$\vee_{0,1}$	$T_{0,1}$	$F_{0,1}$
$T_{0,1}$	$T_{0,1}$	$T_{0,1}$
$F_{0,1}$	$T_{0,1}$	$F_{0,1}$

$\vee$	1	$\frac{1}{2}$
1	1	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$\leftrightarrow$ in $\mathrm{L}_{\frac{1}{2},1}$

$\vee_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$T_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$
$F_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$

The attempt to decompose the IMPLY operation gives rise to transjunctions:

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	1	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1	1

$\rightarrow$	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	1

$\leftrightarrow$ in $\mathrm{L}_{0,\frac{1}{2}}$

$\rightarrow_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$
$T_{0,\frac{1}{2}}$	$F_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$F_{0,\frac{1}{2}}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$

$\rightarrow$	1	0
1	1	0
0	1	1

$\leftrightarrow$ in $\mathrm{L}_{0,1}$

$\rightarrow_{0,1}$	$T_{0,1}$	$F_{0,1}$
$T_{0,1}$	$T_{0,1}$	$F_{0,1}$
$F_{0,1}$	$T_{0,1}$	$T_{0,1}$

$\rightarrow$	1	$\frac{1}{2}$
1	1	$\frac{1}{2}$
$\frac{1}{2}$	1	$\frac{1}{2}$

$\leftrightarrow$ in $\mathrm{L}_{\frac{1}{2},1}$

$\rightarrow_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$T_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$F_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$

3.2 Bochvar's System

Bochvar uses many-valued logic as a means of avoiding the logical paradoxes. His system contains the truth values $\mathcal{B} = \{0, \frac{1}{2}, 1\}$, where 0 stands for "false", 1 for "true" and $\frac{1}{2}$ for "meaningless". Bochvar's tables for his connectives are as follows: when only the values 0 and 1 are involved, they are exactly the same as their classical counterparts; any formula containing "meaningless" is meaningless. The truth tables reflect these motivations. For further details we refer to [Urq86]. The application of our decomposition method yields:

- The given set of truth values is $\mathcal{B} = \{0, \frac{1}{2}, 1\}$, with the usual ordering $0 < \frac{1}{2} < 1$.
- The index set is $I = \{(0, \frac{1}{2}), (0, 1), (\frac{1}{2}, 1)\}$.
- The 3 resulting local subsystems are $\Omega_{i,j} = \{T_{i,j}, F_{i,j}\}$ for $(i,j) \in I$ and the set of global truth values is $\Omega = \coprod_{i,j} \Omega_{i,j}$.

We will give a short discussion about the choice of the equivalence relation on Ω; and we will show why in this case for one of the connectives AND / OR, not all the induced local operations will be of the same type (i.e. all local conjunctions, resp. disjunctions). It follows a schematic representation of the decomposition of such systems.
The AND connective decomposes in the following way (note that in this case the induced equivalence relation is defined as before by imposing $T_{i,j} \equiv T_{k,j}$, $F_{i,j} \equiv F_{i,k}$, $F_{i,j} \equiv T_{k,i}$).

$\wedge$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	0	$\frac{1}{2}$	0

$\wedge$	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{2}$	0

$\leftrightarrow$ in $\mathrm{L}_{0,\frac{1}{2}}$

$\wedge_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$
$T_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$
$F_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$

$\wedge$	1	0
1	1	0
0	0	0

$\leftrightarrow$ in $\mathrm{L}_{0,1}$

$\wedge_{0,1}$	$T_{0,1}$	$F_{0,1}$
$T_{0,1}$	$T_{0,1}$	$F_{0,1}$
$F_{0,1}$	$F_{0,1}$	$F_{0,1}$

$\wedge$	1	$\frac{1}{2}$
1	1	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$\leftrightarrow$ in $\mathrm{L}_{\frac{1}{2},1}$

$\wedge_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$T_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$F_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$

Let $\leq_\vee$ be the order induced on $\mathcal{B}$ by the OR operation. It is obvious that $\leq_\vee$ does not coincide with the given order on $\mathcal{B}$, $\leq$. Therefore the local operations induced by $\vee$ on the subsystems $L_{i,j}, i < j$, are not all classical disjunctions. Namely, in the order $\leq_\vee$ induced by $\vee$ we have $0 <_\vee 1 <_\vee \frac{1}{2}$. The local operations corresponding to the pairs $(0, \frac{1}{2})$ and $(0, 1)$ (on which the initial ordering coincides with the ordering induced by $\vee$) are classical disjunctions, but the local

operation induced in the subsystem $L_{\frac{1}{2},1}$ is a conjunction, because we have $1 < \frac{1}{2}$, but on the other hand $\frac{1}{2} <_{\vee} 1$.
The decomposition of the OR connective can be summarized in the following schematic drawing:

$\vee$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	1
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	0

$\vee$	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{2}$	0

$\leftrightarrow$ in $L_{0,\frac{1}{2}}$

$\vee_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$
$T_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$
$F_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$

$\vee$	1	0
1	1	1
0	1	0

$\leftrightarrow$ in $L_{0,1}$

$\vee_{0,1}$	$T_{0,1}$	$F_{0,1}$
$T_{0,1}$	$T_{0,1}$	$T_{0,1}$
$F_{0,1}$	$T_{0,1}$	$F_{0,1}$

$\vee$	1	$\frac{1}{2}$
1	1	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$\leftrightarrow$ in $L_{\frac{1}{2},1}$

$\vee_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$T_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$F_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$

The decomposition of the IMPLY operation can be schematically represented as follows:

$\rightarrow$	1	$\frac{1}{2}$	0
1	1	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	1	$\frac{1}{2}$	1

$\rightarrow$	$\frac{1}{2}$	0
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$
0	$\frac{1}{2}$	1

$\leftrightarrow$ in $L_{0,\frac{1}{2}}$

$\rightarrow_{0,\frac{1}{2}}$	$T_{0,\frac{1}{2}}$	$F_{0,\frac{1}{2}}$
$T_{0,\frac{1}{2}}$	$F_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$F_{0,\frac{1}{2}}$	$F_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$

$\rightarrow$	1	0
1	1	0
0	1	1

$\leftrightarrow$ in $L_{0,1}$

$\rightarrow_{0,1}$	$T_{0,1}$	$F_{0,1}$
$T_{0,1}$	$T_{0,1}$	$F_{0,1}$
$F_{0,1}$	$T_{0,1}$	$T_{0,1}$

$\rightarrow$	1	$\frac{1}{2}$
1	1	$\frac{1}{2}$
$\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$

$\leftrightarrow$ in $L_{\frac{1}{2},1}$

$\rightarrow_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$T_{\frac{1}{2},1}$	$T_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$
$F_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$	$F_{\frac{1}{2},1}$

We finish these experimental studies by giving a summarizing "metalevel language" description of the proposed decomposition procedure (algorithm). We are aware that in an exact description of a decomposition algorithm we should come to a "standard way" of constructing such representations. In what we are doing so far with the examples given above there are in principle various ways to get suitable equivalence relations depending on , among others, with which logical operation we start the decomposition. Heading towards a suitable "normal form" of a decomposition we would like to exploit our experimental experiences with the examples. With this in mind we tend to propose to start a decomposition procedure always with the AND and OR connectives and then "tune" the IMPLY and others accordingly since there more frequently problems arise (like the need to resort to transjunctions and so). Another important constraint will be to have as many classical local operations in the components of a decomposition as possible. This is of course a desirable goal as we pointed out in our previous discussions. Still far from having found a satisfying solution of a kind of "normal form" we formulate (in meta language) the following

Decomposition Procedure (preliminary version):
Let $\mathcal{L}$ be a logical system and L its set of truth values. Let $<$ be a given total order on the elements of L. Assume that the truth tables of selected logical connectives of L are given.
The decomposition procedure for L can be stated in meta-language as follows:

Input: L, the set of truth values of the logical system $\mathcal{L}$,
$<$, a total order on L,
a set of (bivariate) logical connectives of $\mathcal{L}$,
the truth tables for these logical connectives (e.g. $\vee$, $\wedge$, $\Rightarrow$, ...).

Task: Find a decomposition of the connectives into a number of 2-valued components, based on a corresponding underlying logical fibering.

Begin

Step 1 Let $I = \{(i,j) \mid i,j \in L, i < j\}$ be the set of all possible pairs of distinct elements of L.
Let $\Omega_{i,j} = \{T_{i,j}, F_{i,j}\}$ be the local truth values of the local subsystems corresponding to the elements of I.
Let $\Omega = \coprod_{i,j} \Omega_{i,j}$ be the set of global truth values.

Step 2 Express each given connective on L in terms of a family of local classical connectives defined on each of these logical subsystems.

Step 3 Find a suitable equivalence relation $\equiv$ on Ω (compatible with the operations), such that the quotient set, $\Omega/\equiv$ yields the set L of truth values of the given logical system $\mathcal{L}$.

End

With this vague description we would like to briefly summarize the decomposition procedure applied in the previous concrete cases. Using this principle of "factoring out" a suitable equivalence relation on the global set Ω, leading to the quotient $\Omega/\equiv$, we are thinking at the "generators and relations" principle which is often applied in mathematics (e.g. combinatorial group theory). In our case Ω plays the role of a free generator system.
Looking further for decomposition of more general formulas we can restrict our work to taking into account only a minimal set of "generating" logical operations (like NOT and OR in 2-valued system) and perform fiberwise operations (i.e. in parallel) being aware that handling of transjunctions needs a dedicated calculus (work in progress).

With a remark how it might be possible to apply the decomposition method to results presented in [CRAB91] we are finishing this section. In their work these authors develop a method to translate logical formulas of an m-valued logic into corresponding polynomials. This approach generalizes the *Stone isomorphism* in classical 2-valued logic. It is working with multivariate polynomials over finite fields (Galois fields). A crucial point in that article is that the authors can translate a *problem of derivability* of a formula from a given set of formulas into a corresponding *problem of ideal membership*. This naturally leads to the application of *Buchberger's Gröbner basis algorithm* and thus to the deployment of computer algebra systems. This is the point where we think that our decomposition method might be usefully exploited in the following direction. Assuming that the decomposition of a given many valued space into a fibering can also be transformed canonically into the polynomial algebra case leads to the following aspects: a given decision problem then can be fully parallelized, i.e. manipulations can be done fiberwise in parallel, and even more, in each fiber (component) we have in many cases classical logical formulas to handle that means that the corresponding polynomials have degree not greater than two (!). If we have to deal with certain transjunctions (as discussed in examples above) we only have to consider a well known restricted class of operations and, again, the corresponding polynomials have bounded degree (maximally degree four). Having bounded small polynomial degrees might be a big advantage in Gröbner basis applications, because high polynomial degrees can cause heavy problems to computer algebra systems.

4 Concluding Remarks

We conclude this experimental case study with some summarizing comments and prospects of work. As indicated in several places in the previous work the basic intention is to use the notion of logical fiberings (which form logical systems consisting of indexed subsystems) to decompose a given many valued logical space into subsystems (fibers) such that all the many valued logical operations can be decomposed, accordingly. In most concrete cases this results in getting classical 2-valued operations "fiberwise" (in the corresponding components or fibers), but it can happen that it is necessary to resort to so-called transjunctions. These transjunctions form in a certain way a "minimal" nonclassical bivariate extension of a classical 2-valued operation, because each transjunction has the same "T-F-pattern" for its truth table but the image values of such a logical function can be distributed over (maximally four) different truth values from other subsystems. This means that in all decomposition tasks we have to deal with classical connectives and possibly such transjunctions in the fibers (subsystems or components) of a decomposition, respectively. This reduces the amount of work to these restricted classes of operations. What we gain with decomposing many valued systems can be described as follows: the representation of a logical space as a fibering leads to a natural parallelization and the fiberwise consideration of the connectives. Moreover, in the fibers it is possible to work classically (2-valued) modulo the possible appearance of transjunctions, but these form a well known restricted class of operations (maximally 4-valued). Aiming at a general representation theory for many valued logics (with a general decomposition algorithm) would provide us with the previously mentioned advantages and furthermore would allow to combine decompositions in a flexible way in the sense that we can decompose a big many valued space into many valued components of less size up to two valued subsystems. Thinking even further, as already mentioned, the fiberings concept allows to "mix" logics of different type also. Concerning intended future implementation work we are thinking at the following two aspects of computer applications to the work discussed in this article. On one hand we are thinking at using the system Elf (on basis of standard ML) and furthermore at building a new module (for logical fiberings) within the powerful ATINF system (cf. [CH]) of Ricardo Caferra's group at LIFIA-IMAG, Grenoble. These concluding remarks form the basis of prospects for future work in that direction.

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