Testing Positiveness of Polynomials

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Abstract

Many problems in mathematics, logic, computer science, and engineering can be reduced to the problem of testing positiveness of polynomials. Although the problem is decidable (Tarski 1930), the general decision methods are not always practically applicable due to their high computational time requirements. Thus several *partial* methods were proposed in the field of term rewriting systems. In this paper, we exactly determine how much partial these methods are, and propose simpler and/or more efficient methods with the same power.

1 Introduction

Many problems in mathematics, logic, computer science, and engineering can be reduced to the problem of testing positiveness of polynomials. In 1930, Tarski [33, 34] showed that the problem is decidable. In fact, he gave a decision method for a more general problem than just testing positiveness. Since then, many improvements and new methods were proposed [7, 1, 26, 2, 3, 27, 13, 35, 30, 14, 15, 16, 9, 29, 17, 28, 6, 20, 19, 18, 25, 36, 8, 21, 22, 5, 12, 23].

However, these methods are computationally expensive due to their generality. Naturally one is interested in devising more efficient methods for the sub-problem: testing positiveness. But then, this sub-problem turns out to be still difficult. Thus, several authors (mainly from the field of term rewriting systems, for proving termination) proposed some *partial* methods, partial in the sense that these method might not be able to decide the positiveness of some polynomials [24, 4, 10, 10, 31, 32, 11].

In this paper, we determine how much partial these methods are. In other words, we find proper subsets of the set of all positive polynomials that are decided by those methods. Equivalently, we determine restrictions on the positiveness that are decided by them. Further, we propose simpler and/or more efficient methods for the same sets of polynomials.

The structure of the paper is as follows. In Section 2, we list three different restrictions on the property of positiveness (eventual, absolute, eventual absolute). In the subsequent sections, we show that each of the existing methods is a decision procedure for one of the above restricted positiveness properties. We also give several new methods.

2 Several Types of Positiveness

Let $x = (x_1, \ldots, x_n)$ and let P be a non-zero polynomial in $\mathbb{R}[x]$, and let $\mu \in \mathbb{R}$.

Definition 2.1 (Types of Positiveness) *We say that P is*

- positive *from μ iff $\forall x_1 \geq \mu \ \cdots \forall x_n \geq \mu \ P(x) > 0$.*
- eventually positive[1] *iff there exists a μ such that P is positive from μ.*
- absolutely positive *from μ iff P is positive from μ and every partial derivative P^* is non-negative from μ, that is, $\forall x_1 \geq \mu \ \cdots \forall x_n \geq \mu \ P^*(x) \geq 0$.*
- eventually absolutely positive *iff exists a μ such that P is absolutely positive from μ.* □

Example 2.1 Let $P = x^3 + 1$ be a polynomial. The polynomial P is positive from 0. In fact one sees easily that it is positive from all $\mu > -1$. Therefore it is also eventually positive. However, it is absolutely positive only from $\mu \geq 0$ because its second derivative $(6x)$ is non-negative only for non-negative values of x. As it is absolutely positive from some μ it is also eventually absolutely positive. □

Example 2.2 Let $P = (x - y)^2 + 1$. It is easy to see that the polynomial P is positive from every μ and therefore it is eventually positive. On the contrary, it is not absolutely positive from any μ, because the derivative $\frac{\delta^2 P}{\delta x \delta y} = -2$ is always negative. Consequently it is not eventually absolutely positive. □

[1] The notion of eventual positiveness was first introduced by Lankford [24].

Naturally, we are interested in the relations among the properties. Let $\mathbf{P}_\mu$, $\mathbf{EP}$, $\mathbf{AP}_\mu$, $\mathbf{EAP}$ respectively stand for positiveness from μ, eventual positiveness, absolute positiveness from μ and eventual absolute positiveness. The following proposition shows the relations among them.

Proposition 2.1 *The following diagram describes the implication relation among the different kinds of positiveness property. An arrow goes from A to B if A implies B.*

$$\begin{tikzcd}
& \mathbf{AP}_\mu \arrow[dl] \arrow[dr] & \\
\mathbf{EAP} \arrow[dr] & & \mathbf{P}_\mu \arrow[dl] \\
& \mathbf{EP} &
\end{tikzcd}$$

□

Proof.
The proofs of all indicated implications follow easily from the definition. It is equally easy to see that the indicated implications are the only implications valid for the given properties. We can use polynomials from Examples 2.1 and 2.2 as counter examples to other implications. □

3 Methods for Absolute Positiveness

We start with presenting a necessary and sufficient condition for absolute positiveness of a polynomial. The theorem will be used in several following subsections and will form a base for the new method we present later.

From now on, let σ_μ stand for "shifting" by μ. Precisely, $\sigma_\mu(P)$ is the polynomial obtained from P by replacing each x_i with $x_i + \mu$. Every non-zero polynomial P can be rewritten as $\sum_e c_e x^e$ where the exponent vectors (e) are distinct and $c_e \neq 0$ (distributed representation). We will call those c_e's *coefficients* of P.

Theorem 3.1 *Let P be a non-zero polynomial. Then P is absolutely positive from μ iff every coefficient of the polynomial $\sigma_\mu(P)$ is positive and the constant term is non-zero.* □

Proof.
First note that since P is a non-zero polynomial, $\sigma_\mu(P)$ is also a non-zero polynomial.

$(\Longleftarrow)$

Assume that every coefficient of $\sigma_\mu(P)$ is positive and the constant term is non-zero. Then obviously $\sigma_\mu(P)$ is positive from 0. Note that the graph of P is obtained by shifting that of $\sigma_\mu(P)$ by μ to the positive direction. Thus the polynomial P is positive from μ.

Let ∂ be an arbitrary but fixed partial differentiation operator as $\frac{\partial^{e_1+\cdots+e_n}}{\partial x_1^{e_1}\cdots\partial x_n^{e_n}}$. Since the polynomial $\sigma_\mu(P)$ has all coefficients positive, the polynomial $\partial(\sigma_\mu(P))$ also has all coefficients positive. Therefore $\partial(\sigma_\mu(P))$ is non-negative from 0. Observing that differentiation and shifting commutes, that is, $\partial(\sigma_\mu(P)) = \sigma_\mu(\partial(P))$, we see that $\sigma_\mu(\partial(P))$ is non-negative from 0. Note again that the graph of $\partial(P)$ is obtained by shifting that of $\sigma_\mu(\partial(P))$ by μ to the positive direction. Thus $\partial(P)$ is non-negative from μ. Therefore P is absolutely positive from μ.

$(\Longrightarrow)$

We can use the same arguments as in the first part of the proof. The assumption of the polynomial P being absolutely positive from μ is then equivalent to the fact of polynomial $\sigma_\mu(P)$ being absolutely positive from 0. Therefore we want to prove that if the polynomial $\sigma_\mu(P)$ is absolutely positive from 0 then it has all coefficients positive and non-zero constant term. The proof will be done by contradiction.

Assume the constant term of $\sigma_\mu(P)$ is zero. Then the value of $\sigma_\mu(P)$ at 0 is 0, contradicting the fact that $\sigma_\mu(P)$ is absolutely positive from 0. Assume there exists a term with negative coefficient in $\sigma_\mu(P)$. Then there exists a derivative[2] of $\sigma_\mu(P)$ with a negative constant term (it is the derivative eliminating all variables of the term with the negative coefficient). The derivative is negative at 0, contradicting the absolute positiveness of $\sigma_\mu(P)$. □

3.1 Partial Derivative Method

Giesl [11] gave a partial method for testing the positiveness, by extending the idea of Lankford [24]. The method is based on the following implications (called differentiation rules):

Rule-1: $\mathbf{A1} \wedge \mathbf{B1}$ implies $\mathbf{C1}$, where

$$\begin{array}{ll}
\mathbf{A1}: & \forall x_1 \geq \mu \ \cdots \forall x_{j-1} \geq \mu \ \forall x_{j+1} \geq \mu \ \cdots \forall x_n \geq \mu \\
& P(x_1, \ldots, x_{j-1}, \mu, x_{j+1}, \ldots, x_n) > 0 \\
\mathbf{B1}: & \forall x_1 \geq \mu \ \cdots \forall x_n \geq \mu \ \dfrac{\partial P(x_1, \ldots, x_j, \ldots, x_n)}{\partial x_j} \geq 0 \\
\mathbf{C1}: & \forall x_1 \geq \mu \ \cdots \forall x_n \geq \mu \ P(x_1, \ldots, x_n) > 0
\end{array}$$

Rule-2: $\mathbf{A2} \wedge \mathbf{B2}$ implies $\mathbf{C2}$, where

$$\begin{array}{ll}
\mathbf{A2}: & \forall x_1 \geq \mu \ \cdots \forall x_{j-1} \geq \mu \ \forall x_{j+1} \geq \mu \ \cdots \forall x_n \geq \mu \\
& P(x_1, \ldots, x_{j-1}, \mu, x_{j+1}, \ldots, x_n) \geq 0 \\
\mathbf{B2}: & \forall x_1 \geq \mu \ \cdots \forall x_n \geq \mu \ \dfrac{\partial P(x_1, \ldots, x_j, \ldots, x_n)}{\partial x_j} \geq 0 \\
\mathbf{C2}: & \forall x_1 \geq \mu \ \cdots \forall x_n \geq \mu \ P(x_1, \ldots, x_n) \geq 0
\end{array}$$

[2] It might be the 0-th order derivative, that is, the polynomial $\sigma_\mu(P)$ itself.

The method, starting from the sentence

$$\forall x_1 \geq \mu \ \cdots \forall x_n \geq \mu \ P(x_1, \ldots, x_n) > 0,$$

repeatedly applies the implications (backward) until it results in a conjunction of inequalities among numbers. If all those numeric inequalities are satisfied, then the method outputs 'yes' (P is positive from μ). If not, it outputs 'don't know'.

Since it is a partial method, a question naturally arises: How much partial is the method? In other words, we want to know the proper subset of the set of all positive polynomials that are decided by the method. As the reader might have guessed (from the fact we put the method into the section about absolute positiveness), the method is a decision procedure for *absolute* positiveness (Theorem 3.2).

The rest of the subsection is devoted to the proving the theorem. For this we start with a lemma which will simplify the proof of the theorem.

Lemma 3.1 *The partial derivative method outputs 'yes' on an input P and μ iff it outputs 'yes' on the input $\sigma_\mu(P)$ and 0.* $\square$

Proof.
Suppose that by applying Rule-1 on an inequality $Q > 0$ we generate two new inequalities $Q_1 \geq 0$ and $Q_2 > 0$. It is easy to see, from the commutativity of shifting and differentiation/evaluation, that by applying the same rule on the inequality $\sigma_\mu(Q) > 0$, we will generate two new inequalities $\sigma_\mu(Q_1) \geq 0$ and $\sigma_\mu(Q_2) > 0$. The same holds for applying Rule-2 on $Q \geq 0$.

The computation of the partial derivative method on input P and μ ends in a set A of numerical inequalities. Following the observation above, we easily see that applying the same computation path on $\sigma_\mu(P)$ and 0 results in the set A' of numerical inequalities, which are obtained by shifting the inequalities in A. But then shifting a numerical inequality results in the same inequality. Thus we conclude that $A = A'$, and so their truths are the same. $\square$

Now we can prove the exact power of the partial derivative method.

Theorem 3.2 (Power of Partial Derivative Method) *The partial derivative method outputs 'yes' for an input P and μ iff the polynomial P is absolutely positive from μ.*

Proof.
$(\Longrightarrow)$

Assume the partial derivative method outputs 'yes' on P and μ, it means P is positive from μ. From Lemma 3.1 we know it also outputs 'yes' on $\sigma_\mu(P)$ and 0, and $\sigma_\mu(P)$ is positive from 0.

Now we will show, by contradiction, that $\sigma_\mu(P)$ has all coefficients positive and non-zero constant term. Assume that the polynomial $\sigma_\mu(P)$ has either a negative monomial or constant term equal to zero.

We start with the case when the constant term is zero. Then we know $\sigma_\mu(P)$ is not positive at 0 and it contradicts the positiveness of $\sigma_\mu(P)$ from 0.

Next we assume that a monomial in $\sigma_\mu(P)$ has a negative coefficient. We would like to show that the partial derivative method does not prove the positiveness of $\sigma_\mu(P)$ from 0. For this, let us first observe the following. Let Q be a polynomial in an inequality and let M be a monomial in Q with negative coefficient. We claim that after applying a rule (Rule-1 or Rule-2) one of the resulting polynomials still contains a monomial with negative coefficient. To see this, let x_i be the variable on which the rule is applied. Let Q_1 be the polynomial resulting from substituting 0 into x_i in Q and let Q_2 be the polynomial resulting from differentiating Q in x_i. Assume that x_i occurs in M. Then the monomial $\frac{\partial M}{\partial x_i}$ occurs in Q_2 since it cannot be canceled by other monomials. Further, the monomial obviously has a negative coefficient. Assume that x_i does not occur in M. Then M occurs in Q_1 because there cannot be any cancelation among different monomials after substituting 0 into x_i.

Repeated applications of the rules results in a set of numerical inequalities. From the observation above we know that at least one of the inequalities must be of the form $a \geq 0$ or $a > 0$ where $a < 0$. Thus the partial derivation method does not output 'yes' for $\sigma_\mu(P)$ and 0. Contradiction.

Thus, we showed that $\sigma_\mu(P)$ has all coefficients positive and the constant term is non-zero. From Theorem 3.1 we conclude that P is absolutely positive from μ.

$(\Longleftarrow)$

Assume P is absolutely positive from μ. From Theorem 3.1 we know that $\sigma_\mu(P)$ has all coefficients positive and non-zero constant term. We will show that the method proves the positiveness of $\sigma_\mu(P)$ from 0.

In the final set of numerical inequalities there is only one strict inequality (resulting from the application of the first part of Rule-1). Note that the left hand side of the strict numerical inequality is the constant term of $\sigma_\mu(P)$. Because the constant term is positive, the strict inequality is true. All other numerical inequalities are non-strict. Note that the applications of the rules cannot create a negative term. Thus all the numbers in the non-strict numerical inequalities are non-negative. Hence, all non-strict inequalities are true,

We showed that the initial inequality is transformed into a valid set of numerical inequalities. Thus the partial derivative method outputs 'yes' for the input $\sigma_\mu(P)$ and 0. From Lemma 3.1 we conclude that the method outputs 'yes' also on the input P and μ. $\square$

3.2 Merging Method

The method was first described by Ben Cherifa and Lescanne [4]. Steinbach [31, 32] increased the power of the method and removed some restriction imposed by the original version.

The main idea of this method is deciding the positiveness of polynomial P by finding a sequence of polynomials $P_1, \ldots, P_n$ such that

$$\forall x \geq \mu \quad P = P_0 \geq P_1 \geq \cdots \geq P_n > 0.$$

The positiveness of polynomial P_n is checked with the "all coefficients are positive" rule. Of course this rule decides the positiveness of a polynomial only from positive μ.

The transformation of P_i into P_{i+1} is done in two main steps. Firstly, a pair of monomials, one having negative and one positive coefficient are *chosen*. Secondly, the two monomials are *merged* into a new monomial. The procedures *CHOOSE* and *CHANGE* are the essential part of the method and for a detailed discussion on them we refer to Section 5.3 of [32].

We are again interested in the relation between the method and the positiveness properties we have defined. Giesl in [11] proved the following facts regarding the comparison of his method with the merging methods of Steinbach [32] and Ben Cherifa and Lescanne [4].

Theorem 3.3 (Power of Merging Method (Giesl 95))

- *If the merging method can prove a polynomial P to be positive from μ, then the partial derivative method can do so as well.*
- *If partial derivative method can prove P to be positive from μ, then there exists a $\mu' \geq \mu$ such that the merging method can prove P to be positive from μ'.* □

Giesl also gives an example for the second observation, where the merging method cannot prove positiveness of P from μ though the partial derivative method does. Therefore the partial derivative method is stronger than the merging method.

We see that the merging method besides being a partial method for deciding positiveness of polynomial also only partially decides absolute positiveness of polynomial. Naturally one can be again interested in an exact characterization of the power of the method. We do not know a simple characterization for the polynomials positiveness of which is decided by the merging method and leave it as an open problem.

3.3 Shifting Method (New)

We present a new and simple method for checking absolute positiveness of polynomial. The method is based on Theorem 3.1 which gives the necessary and sufficient condition for absolute positiveness. From the theorem the method is obvious.

1. Shift P by μ, that is, compute the polynomial P' such that $P'(x_1, \ldots, x_n) = P(x_1 + \mu, \ldots, x_n + \mu)$.
2. If all the coefficients of P' are positive and the constant term is non-zero then output 'yes', else output 'no'.

Example 3.1 Check if the following polynomial[3] P is absolutely positive from 2:

$$P(x, y, z) = xyz + xy - xz.$$

We apply the shifting method. For this, we first shift P by 2,

$$\begin{aligned} & P'(x, y, z) \\ = \; & P(x+2, y+2, z+2) \\ = \; & (x+2)(y+2)(z+2) + (x+2)(y+2) - (x+2)(z+2) \\ = \; & xyz + 3xy + xz + 4x + 2yz + 6y + 2z + 8. \end{aligned}$$

Next, we find that all the coefficients are positive and the constant term is non-zero (8), thus we conclude that P is absolutely positive from 2. □.

3.4 Comparing the Complexity of the Methods

Since all the above methods are of equal testing power, we are interested in complexity differences. In the following theorem, we recall the analysis of the merging method by Steinbach [31] and the analysis of the partial derivate method by Giesl [11]. Then we give the analysis of the shifting method. Below we assume that every arithmetic operation takes a constant time.

Theorem 3.4 (Computing Times) *The computing times for worst cases are bounded from above as follows:*

- *Partial Derivative Method* $O(d^n \ell)$
- *Merging Method* $O(\ell^\ell)$
- *Shifting Method* $O(d^n \ell)$

where

n *the number of variables,*
ℓ *the number of monomials,*
d *the maximum of the degrees in each variable.* □

Proof.

Partial Derivative Method: In [11], Giesl gave the following derivation: Let $T(n, d, \ell)$ be the computing time for an input characterized by n, d and ℓ. First we compute $\mu, \mu^2, \mu^3, \ldots, \mu^d$ and store the results for the later use. This

[3] This polynomial arises from a certain polynomial interpretation of the term rewrite system for associativity and endomorphism [11].

takes $O(d)$. Let $S(n, d, \ell)$ be the computing time for the remaing part (successive elimination of variables). The elimination of one variable involves the following: d partial differentiations, where each differentiation takes $O(\ell)$, d evaluations, where each evaluation takes $O(\ell)$, using the stored values of the powers of μ. Further this process produces $(d+1)$ new polynomials in $n-1$ variables to work on. Thus, we get the following recurrence formula:

$$S(n, d, \ell) = O(d\ell) + (d+1)S(n-1, d, \ell).$$

By solving the recurrence formula, we see that $S(n, d, \ell) = O(d^n \ell)$. Thus, $T(n, d, \ell) = O(d) + O(d^n \ell) = O(d^n \ell)$.

Merging Method: Steinbach [31] proved the following bound

$$O(s!\, t!\, s^{t-s})$$

where s is the number of the positive monomials and t is the number of the negative monomials. He then, using the formula of Stirling($s!$ aproximates to $(\frac{s}{e})^s \sqrt{2\pi s}$), simplified it to

$$O((st)^t).$$

Since $\ell = s + t$, it can be rewritten in terms of ℓ as

$$O(\ell^\ell).$$

Shifting Method: First, we need to shift each monomial.[4] For this, we need to shift the power of each variable, that takes $O(d^2)$ for each variable. Then we need to multiply out the shifted powers, that takes $O(d^n)$. Thus, shifting each monomial takes $O(d^2 n + d^n) = O(d^n)$, and so, shifting all the monomials takes $O(d^n \ell)$. Next, we need to sum all the shifted monomials, which takes $O(d^n \ell)$. Thus, all together, the shifting method takes $O(d^n \ell + d^n \ell) = O(d^n \ell)$. □

These are asymptotic results (with the big O's), and thus one should be careful not to draw any too definitive conclusion about their performances on small/moderate size inputs, which often arise in application domains.

4 Methods for Eventual Absolute Positiveness

Now we shift our focus to the (known and new) methods for testing eventual absolute positiveness.

4.1 Partial Derivative Method (eventual positiveness version)

We have already mentioned the Lankford's method [24] which was suggested as a partial decision method for eventual positiveness. Let P be a non-constant polynomial. Then the method is based on a simple fact that if every non-vanishing first order partial derivative of P is eventually positive then P is eventually positive. We can apply the same reasoning on the partial derivatives repeatedly until all polynomials are numbers. If all of them are positive then the input polynomial is eventually positive. Otherwise, we don't know.

Since it is a partial method, a question again naturally arises: How much partial is the method? We will answer this question by following the similar approach that we already used in studying the power of the partial methods for testing positiveness (Section 3).

Lemma 4.1 *The partial derivative method (Lankford) outputs 'yes' on an input P iff it outputs 'yes' on the input $\sigma_\mu(P)$.* □

Proof.
Suppose that by applying differentiation on an inequality $Q > 0$ we generate another inequality $Q^* > 0$. It is easy to see, from the commutativity of shifting and differentiation, that by differentiation in the same variable on the inequality $\sigma_\mu(Q) > 0$, we will generate a new inequality $\sigma_\mu(Q^*) > 0$.

The computation of the partial derivative method (Lankford) on input P ends in a set A of numerical inequalities. Following the observation above, we easily see that applying the same computation path on $\sigma_\mu(P)$ results in the set A' of numerical inequalities, which are obtained by shifting the inequalities in A. But then shifting a numerical inequality results in the same inequality. Thus we conclude that $A = A'$, and so their truths are the same. □

Now we can prove the exact power of the partial derivative method (Lankford).

Theorem 4.1 (Power of Partial Derivative Method) *The partial derivative method (Lankford) outputs 'yes' on input P iff the polynomial P is eventually absolutely positive.*

Proof.
$(\Longrightarrow)$

Assume that the partial derivative method outputs 'yes' on an input P. We need to show that P is eventually absolutely positive. From the assumption, we know that P is eventually positive. Therefore there exists a μ_0 such that P is positive from μ_0. Further from the details of the execution of the partial derivative method, we observe that every partial derivative of P is either a zero polynomial or eventually positive. Thus for every non-zero partial derivative P^* of P, there exists a μ^* such that P^* is positive (thus non-negative) from μ^*. Now let μ be the maximum of μ_0 and all μ^*'s. Then, P is absolutely positive from μ. Thus, P is eventually absolutely positive.

$(\Longleftarrow)$

[4] Actually, there are practically better ways, such as Honer evaluation or synthetic division, etc. But they do not seem to give a better bound, thus we will stick to the brute-force way since it easier to analyze.

Assume that the polynomial P is eventually absolutely positive. We need to show that the partial derivative method outputs 'yes' on this input. Since P is eventually absolutely positive, there exists μ such that P is absolutely positive from μ. Then, $\sigma_\mu(P)$ is absolutely positive from 0. By Theorem 3.1, we see that all coefficients of $\sigma_\mu(P)$ are positive and the constant term is non-zero. Note that the partial derivative method does not introduce negative terms during the execution on the input $\sigma_\mu(P)$. Thus, the method will output 'yes' on the input $\sigma_\mu(P)$. By Lemma 4.1, we conclude that the method will also output 'yes' on the input P. $\square$

4.2 Dominating Monomials Method (New)

Now we propose a new and more efficient method for testing eventual absolute positiveness of polynomial. First note that the eventual (absolute) positiveness of a univariate polynomial can be trivially decided by checking whether its leading coefficient is positive. We generalize the observation to multivariate case. For this, the notion of leading monomial must be generalized. The new method is essentially based on one such generalization (which we call *dominating* monomial).

Definition 4.1 (Dominating monomial) *We say that a monomial s* dominates *a monomial m. iff m divides s and m is different from s. We say that s is a* dominating monomial *of P iff there is no monomial in P dominating s.* $\square$

Example 4.1 Let us consider polynomials $P = x^2 - 2xy + y^2 + 1$ and $Q = x^2y - xy + y^2$. There are three dominating monomials in P $(x^2, -2xy, y^2)$ and there are two in Q (x^2y, y^2). For univariate polynomials there is only one dominating monomial — the leading monomial. $\square$

Now we are ready to describe the new method:

1. Collect all the dominating monomials of P.
2. If their coefficients are all positive, then output 'yes', else output 'no'.

The proof of the correctness of the method will be given later (Theorem 4.2). Let us first get familiar with the method by working on several small examples.

Example 4.2 Check if the following polynomial is eventually absolutely positive:

$$xyz + xy - xz.$$

One sees immediately that the only dominating monomial is xyz. Since its coefficient is positive, the polynomial is eventually absolutely positive.

Let us try another:

$$x^4 + 2xy^3 + 3x^2y - 9y^3 - 7xy^2 - 5x^3 - 7y^3 - 6.$$

One sees that the dominating monomials are x^4, $2xy^3$ and $3x^2y$. Since their coefficients are positive, the polynomial is eventually absolutely positive.

Let us try still another (obtained by changing signs in the previous example),

$$x^4 - 2xy^3 + 3x^2y + 9y^3 + 7xy^2 + 5x^3 + 7y^3 + 6.$$

Again, the dominating monomials are x^4, $-2xy^3$ and $3x^2y$. Since one coefficient is negative, the polynomial is *not* eventually absolutely positive. $\square$

Remark: Note that one can use the dominating monomial method as a quick *pre*-pruning method while testing absolute positiveness: Suppose that we would like to check whether a polynomial P is absolutely positive from μ. We first apply the (very cheap) dominating monomial method. If it turns out that P is not eventually absolutely positive, then we do not need to carry out the more expensive absolute positiveness check, since we already know that the answer is 'no'. $\square$

The remaining of this section is devoted to proving the correctness of the dominating monomial method (Theorem 4.2). We begin with a lemma which "justifies" the notion of dominating monomial.

Lemma 4.2 *The set of dominating monomials of P and that of $\sigma_\mu(P)$ are the same.* $\square$

Proof.
Let m be a dominating monomial of P. We need to show that m is a dominating monomial of $\sigma_\mu(P)$. We prove it by contradiction. Assume that m is not a dominating monomial of $\sigma_\mu(P)$. There are two cases:

- m is not a monomial of $\sigma_\mu(P)$
- m is a monomial of $\sigma_\mu(P)$, but not dominating.

Case 1: m is not a monomial of $\sigma_\mu(P)$.

Note that m is a monomial of $\sigma_\mu(m)$. Since m is not a monomial in the $\sigma_\mu(P)$, there must be a monomial q (different from m) in P such that there is a monomial q' of $\sigma_\mu(q)$, that has the same degree vector as m. Every monomial of $\sigma_\mu(q)$ divides q, in particular, q' divides q. Since m and q' have the same degree vector, m divides q. Recall that m is different from q, thus q dominates m. Therefore m is not a dominating monomial of P. Contradiction.

Case 2: m is a monomial of $\sigma_\mu(P)$, but not dominating.

Thus there is a monomial q' in $\sigma_\mu(P)$ that dominates m. Hence there is a monomial q of P such that there is a monomial q^* of $\sigma_\mu(q)$, that has the same degree vector as q'. Every monomial of $\sigma_\mu(q)$ divides q, in particular, q^* divides q. Since q' and q^* have the same degree vector, q' divides q. Recall that q' dominates m. Thus m divides q' and they are different. Since q' divides q, we see that m divides q and they are different. Thus q dominates m, and m is not a dominating monomial of P. Contradiction.

Thus m must be a dominating monomial of $\sigma_\mu(P)$.

Let m be a dominating monomial of $\sigma_\mu(P)$. We need to show that m is a dominating monomial of P. This can be done in the exactly same way as before, since $P = \sigma_{-\mu}(\sigma_\mu(P))$. □

The following theorem relates the notions of eventual absolute positiveness and dominating monomials, proving the correctness of the dominating monomial method.

Theorem 4.2 *A polynomial P is eventually absolutely positive iff the coefficients of all dominating monomials of P are positive.* □

Proof.

$(\Longrightarrow)$

Assume that P is eventually absolutely positive. We show that the coefficients of all dominating monomials are positive.

From eventual absolute positiveness of P we know that there exists a μ such that P is absolutely positive from μ. From Theorem 3.1 we know that $\sigma_\mu(P)$ must have all coefficients positive and non-zero constant term. Therefore also all coefficients of the dominating monomials of $\sigma_\mu(P)$ are positive. By Lemma 4.2, the set of all dominating monomials of P is the same as that of $\sigma_\mu(P)$. Thus all dominating monomials of P have positive coefficients.

$(\Longleftarrow)$

Assume that the coefficients of all dominating monomials of P are positive. We need to prove that P is eventually absolutely positive.

We begin by taking care of a trivial case where all other (non-dominating) monomials of the polynomial P also have positive coefficients. It is easily seen, that such a polynomial is absolutely positive from any $\mu > 0$ and therefore eventually absolutely positive.

Therefore assume that there exists at least one monomial with negative coefficient (negative monomial) in P. We partition the polynomial P into a sum of several sub-polynomials such that one of the sub-polynomials consists of all positive non-dominating monomials of P and that each of the remaining sub-polynomials consists of exactly one dominating monomial of P (which we will call the leading monomial) and some (possible none) monomials dominated by the leading monomial (thus, these monomials are all negative.)

Obviously, if each sub-polynomial is eventually absolutely positive, so is P. Thus, from now on, we will show that each sub-polynomial is eventually absolutely positive. Let Q be one of sub-polynomials.

If Q consists of only positive monomials, it is absolutely positive from any $\mu > 0$ and therefore eventually absolutely positive. Thus, from now on assume that Q has at least one negative monomial.

First we will prove that Q is eventually positive. Let $Q = ax^m - a_1x^{e_1} - \cdots - a_\ell x^{e_\ell}$ where $x = (x_1, \ldots, x_n)$, $m = (m_1, \ldots, m_n)$, $e_i = (e_{i,1}, \ldots, e_{i,n})$ and $a, a_1, \ldots, a_\ell > 0$. Let $b = \max_i a_i/a$ and $\mu = \ell b + 1$. We claim that Q is positive from μ[5]. To see this, first recall that x^m dominates x^{e_i} for every i. Hence, we see immediately that for every i and for every $x \geq \mu$,

$$a_i x^{e_i} \leq abx^{e_i} \leq ab\frac{x^m}{\mu} \leq \frac{ab}{\ell b + 1}x^m < \frac{a}{\ell}x^m.$$

Thus, we see that for every $x \geq \mu$, $Q > ax^m - \ell\frac{a}{\ell}x^m = 0$. So Q is eventually positive.

It remains to show that for every partial derivative Q^* of Q there exists a μ^* such that Q^* is non-negative from μ^*. It is obvious that every vanishing derivative is non-negative from any μ. Thus from now on let Q^* be a non-vanishing partial derivative of the polynomial Q. Note that Q^* again consists of a positive leading monomial (a derivative of the leading monomial of Q) and some (possibly none) negative monomials dominated by the leading monomial. Thus, by applying the idea used for showing the eventual positiveness of Q, we see that there exists a μ^* such that Q^* is positive from μ^* and therefore it is also non-negative from μ^*. As there are only finitely many derivatives, we can take μ_m the maximum of the bound for positiveness of the polynomial Q and the bounds for non-negativeness of its derivatives. Then the polynomial Q is absolutely positive from μ_m. Thus Q is eventually absolutely positive. We showed that every sub-polynomial of P is eventually absolutely positive. Therefore P is also eventually absolutely positive. □

4.3 Comparing the Complexity of the Methods

We know that the partial derivative method (Lankford) and the dominating monomial method have the same power. The following theorem compares their asymptotic complexity bounds.

Theorem 4.3 (Computing Times) *The computing times for worst cases are bounded from above as follows:*

- *Partial Derivative Method (Lankford)* $O(d^n\ell)$
- *Dominating Monomial Method* $O(n\ell\log\ell)$

where

n *the number of variables,*
ℓ *the number of monomials,*
d *the maximum of the degrees in each variable.* □

Proof.

[5] Actually, one can find a much smaller value for μ than the one given here. But here we are only interested in proving the *existence* of a bound, and thus any bound suffices.

The bound for the partial derivative method is immediate from the fact that the method performs up to d^n differentiations of polynomials, where each differentiation takes $O(\ell)$.

The bound for the dominating monomial method is immediate from the number of comparisons needed for sorting the monomials according to the partial order defined by the domination property, where each comparison takes $O(n)$. $\square$

5 Conclusions

Several results were presented in the paper. We determined how much partial the several known methods are for testing positiveness of polynomials. After that we proposed simpler and/or more efficient methods with equal power.

There are several directions for the future work. Regarding the shifting method, one can use more sophisticated methods for the shifting. For example, the number of monomials necessary to be shifted can be significantly reduced by exploiting the structure of the polynomial (not all monomials have to be shifted, the shifting can be stopped after no negative monomials can be generated, summing new monomials need not be done for only positive monomials etc). Also, it will be interesting/important to find out experimentally how the new methods (shifting, dominating monomial) behave on small/moderate size inputs.

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