COMPUTER ALGEBRA APPLICATION TO THE DISTRIBUTION OF SAMPLE CORRELATION COEFFICIENT

Shigekazu NAKAGAWA[1),2)], Naoto NIKI[3)] and Hiroki HASHIGUCHI[3)]

[1)] Department of Computer Science and Mathematics,
Kurashiki University of Science and the Arts, Kurashiki 712, JAPAN
nakagawa@soft.kusa.ac.jp

[2)] The author is in RISC-Linz until the end of March 1997 as a guest.
Shigekazu.Nakagawa@risc.uni-linz.ac.at

[3)] Department of Management Science, Science University of Tokyo,
Shinjuku-ku, Tokyo 162, JAPAN
niki@ms.kagu.sut.ac.jp, hiro@ms.kagu.sut.ac.jp

Abstract

Let r be the sample correlation coefficient. In this paper, we clarify the role of computer algebra to obtain an asymptotic expansion for a probability integrals of r. A key technique is change of bases of the module of symmetric polynomials. We describe an algorithm for accomplishing it. We give the asymptotic cumulants of r in terms of population cumulants. An approximate distribution of r is also given.

1 Introduction

The statistic known as the sample correlation coefficient r, based on independent bivariate observations $\boldsymbol{X}_1 = (x_1, y_1)'$, $\boldsymbol{X}_2 = (x_2, y_2)'$, $\cdots$, $\boldsymbol{X}_n = (x_n, y_n)'$ from a population with the distribution function F is defined by

$$r = \frac{s_{xy}}{\sqrt{s_x^2 s_y^2}}, \tag{1}$$

where

$$s_x^2 = \frac{1}{n}\left\{\sum_{i=1}^{n} x_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)^2\right\},$$

$$
\begin{aligned}
s_y^2 &= \frac{1}{n}\left\{\sum_{i=1}^{n} y_i^2 - \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right)^2\right\}, \\
s_{xy} &= \frac{1}{n}\left\{\sum_{i=1}^{n} x_i y_i - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)\right\}.
\end{aligned}
$$

Clearly, the statistic r has symmetric property with respect to $\boldsymbol{X}_1 = (x_1, y_1)'$, $\boldsymbol{X}_2 = (x_2, y_2)'$, $\cdots$, $\boldsymbol{X}_n = (x_n, y_n)'$; that is, the value of r remains unchanged however observations may be interchanged or permuted.

Intensive works have been made, concerning the distributions of r on the derivation of both exact and approximate distributions. An excellent introduction to the distributions of r is contained in Johnson, Kotz and Balakrishnan [8]. An extensive bibliography can be found in it.

We are interested in the case that F is nonnormal population, where we usually use the word 'nonnormal population' in the sense that it has finite population cumulants of requisite order. In this situation, Cook [3] [4] obtained the first four moments of r up to and including terms of order n^{-2} with the help of k-statistics. Nakagawa and Niki [17] have extended his results, obtaining expressions for cumulants of the standardized variate of r up to and including terms of order n^{-3} and also given the Edgeworth expansion for probability integrals and the Cornish-Fisher inverse expansion for percentiles of order $1/n$. A key technique is change of bases of the module of symmetric polynomials and is called symmetric polynomial calculus.

In the present paper, a survey on the symmetric polynomial calculus is given based on Nakagawa and Niki [15] [16] [17], Nakagawa [10] and Niki, Nakagawa and Hashiguchi [21]. In section 2, 2-system symmetric polynomials are defined and the algorithm for change of bases is described. In section 3, approximate cumulants of the standardized variate of r and the Edgeworth expansion of order $1/n$ are given. In the case that F is elliptical population, section 4 is devoted. Finally, in section 5, some remarks about generalization of algorithms are made.

2 Symmetric polynomial calculus

2.1 symmetric polynomials

Let $x_1, y_1, \ldots, x_n, y_n$ be $N = 2n$ independent variables and consider the matrix

$$
\boldsymbol{X} = (\, \boldsymbol{X}_1 \quad \boldsymbol{X}_2 \quad \ldots \quad \boldsymbol{X}_n \,) = \begin{pmatrix} x_1 & x_2 & \ldots & x_n \\ y_1 & y_2 & \ldots & y_n \end{pmatrix}.
$$

The symmetric group S_n acts on the ring $\mathbf{Q}[\boldsymbol{X}] = \mathbf{Q}[x_1, y_1, \ldots, x_n, y_n]$ of polynomials in N independent variables with integer coefficients by permuting the column vectors $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots, \boldsymbol{X}_n$ of $\boldsymbol{X}$. We call a polynomial 2(-dimensionally)-symmetric if it is invariant under this action.

The family of 2-symmetric polynomials form a subring $\Lambda = \Lambda(2, n)$ of the ring of ordinary symmetric functions. The ring Λ is a graded ring

$$\Lambda = \bigoplus_{\boldsymbol{d} \succ \boldsymbol{0}} {}^{\boldsymbol{d}}\Lambda,$$

where each $\boldsymbol{d}$ is a column vector $\boldsymbol{d} = (d_1, d_2)' \in \mathbf{N}_0^2$ of non-negative integers; and the free module ${}^{\boldsymbol{d}}\Lambda$ consists of the homogeneous 2-symmetric polynomials of degree d_1 with respect to $x_1, \ldots, x_n$, of degree d_2 with respect to $y_1, \ldots, y_n$ including the zero polynomial. Any linear ordering $\succ$ of vectors is acceptable, if it satisfies the following conditions for any $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c} \in \mathbf{N}_0^2$:

$$\boldsymbol{a} \succ \boldsymbol{b} \Rightarrow \boldsymbol{a} + \boldsymbol{c} \succ \boldsymbol{b} + \boldsymbol{c} \quad \text{and} \quad \boldsymbol{a} \neq \boldsymbol{0} \Rightarrow \boldsymbol{a} \succ \boldsymbol{0}.$$

We shall also use the symbol $\succeq$ for denoting "$\succ$ or $=$".

Corresponding to the ordinary partitions, we define 2-partitions

$$\boldsymbol{K} = (\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots)$$

to be the sequences of column vectors $\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots \in \mathbf{N}_0^2$ in descending order, say $\boldsymbol{k}_1 \succeq \boldsymbol{k}_2 \succeq \ldots$, provided that any 2-partition contains only finitely many, say $\ell(\boldsymbol{K})$, non-zero vectors. For the sake of convenience, we shall not distinguish between two such sequences of vectors which differ only by a string of zero vectors at the tails; and, if need be, we shall also use the notation $\boldsymbol{K} = (\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots) = (\tilde{\boldsymbol{k}}_1^{\pi_1}, \ldots, \tilde{\boldsymbol{k}}_s^{\pi_s})$, where $\pi_1, \ldots \pi_s$ are the multiplicities of the distinct non-zero parts in $\boldsymbol{K}$, $\tilde{\boldsymbol{k}}_1 \succ \ldots \succ \tilde{\boldsymbol{k}}_s \succ \boldsymbol{0}$, respectively, and which satisfy $\pi_1 + \cdots + \pi_s = \ell(\boldsymbol{K})$. For $n \geq \ell = \ell(\boldsymbol{K})$, we can regard $\boldsymbol{K}$ as an $2 \times n$ matrix and use the same symbol $\boldsymbol{K}$ for denoting

$$\boldsymbol{K} = (\boldsymbol{k}_1 \ldots \boldsymbol{k}_\ell \overbrace{\boldsymbol{0}_2 \ldots \boldsymbol{0}_2}^{n-\ell}).$$

The weight vector $w(\boldsymbol{K})$ of the 2-partition $\boldsymbol{K}$ is defined as $w(\boldsymbol{K}) = \boldsymbol{K}\boldsymbol{1}_n$.

For any $2 \times n$ matrix $\boldsymbol{C} = \begin{pmatrix} c_{ij} \end{pmatrix}$ of non-negative integers, we write as $\boldsymbol{X}^{\boldsymbol{C}}$ the monomial

$$\boldsymbol{X}^{\boldsymbol{C}} = x_1^{c_{11}} y_1^{c_{21}} \cdots x_n^{c_{1n}} y_n^{c_{2n}}.$$

Let $\boldsymbol{K}$ be any 2-partition of length $\ell = \ell(\boldsymbol{K}) \leq n$. The polynomial $A_{\boldsymbol{K}}$ in $\boldsymbol{X}$

$$A_{\boldsymbol{K}} = \frac{1}{(n-\ell)!} \sum_{\omega \in S_n} (\omega \boldsymbol{X})^{\boldsymbol{K}}$$

is clearly 2-symmetric and is called the augmented (monomial) 2-symmetric polynomial with argument $\boldsymbol{K}$. If $\boldsymbol{K} = (\tilde{\boldsymbol{k}}_1^{\pi_1}, \ldots, \tilde{\boldsymbol{k}}_s^{\pi_s})$, then $A_{\boldsymbol{K}}$ has the integral factor $\pi_1! \ldots, \pi_s!$ and can be written as

$$A_{\boldsymbol{K}} = \pi_1! \ldots, \pi_s! \, M_{\boldsymbol{K}}.$$

The 2-symmetric polynomial $M_{\boldsymbol{K}}$ consists of distinct monic monomials each other in $\boldsymbol{x}$ and is called a monomial 2-symmetric polynomial.

The product

$$P_{\boldsymbol{K}} = M_{(\boldsymbol{k}_1)} \cdots M_{(\boldsymbol{k}_n)} = A_{(\boldsymbol{k}_1)} \cdots A_{(\boldsymbol{k}_n)}$$

is also 2-symmetric, where each $M_{(\boldsymbol{r})}$ has a one-part 2-partition as its argument and is given by the summation

$$M_{(\boldsymbol{r})} = \sum_{i=1}^{n} \boldsymbol{x}_i^{\boldsymbol{r}} = \sum_{i=1}^{n} x_{1i}^{r_1} \cdots x_{mi}^{r_m}.$$

REMARK 1 *We rewrite (1) by means of* $\{P_{\boldsymbol{K}}\}$ *as follows:*

$$\begin{aligned} s_x^2 &= \frac{1}{n} P_{((20)')} - \frac{1}{n^2} P_{\left((10)'^2\right)}, \\ s_y^2 &= \frac{1}{n} P_{((02)')} - \frac{1}{n^2} P_{\left((01)'^2\right)}, \\ s_{xy} &= \frac{1}{n} P_{((11)')} - \frac{1}{n^2} P_{((10)'(01)')}. \end{aligned}$$

It is not so difficult to prove the following proposition;

PROPOSITION 1
(a) $\{A_{\boldsymbol{K}} \mid \ell(\boldsymbol{K}) \leq n,\ w(\boldsymbol{K}) = \boldsymbol{d}\}$ *forms a* $\mathbf{Q}$*-basis of* ${}^{\boldsymbol{d}}\Lambda$*, and*
(b) $\{P_{\boldsymbol{K}} \mid \ell(\boldsymbol{K}) \leq n,\ w(\boldsymbol{K}) = \boldsymbol{d}\}$ *forms a* $\mathbf{Q}$*-basis of* ${}^{\boldsymbol{d}}\Lambda$*.*

2.2 Algorithm for taking expectation

Let $\boldsymbol{X}_1 = (x_1, y_1)'$, $\boldsymbol{X}_2 = (x_2, y_2)'$, $\cdots$, $\boldsymbol{X}_n = (x_n, y_n)'$ be mutually independent bivariate observations. In arithmetic aspect, taking expectation of $P_{\boldsymbol{K}}$ is not so easy as $\ell(\boldsymbol{K})$ increases. The advantage of $A_{\boldsymbol{K}}$ can be seen as follows:

LEMMA 1 *Let* $\mu_{\boldsymbol{r}}$ *denote the population (product) moment* $\mu_{\boldsymbol{r}} = E(x_{1i}^{r_1} \cdots x_{mi}^{r_m})$ *for* $\boldsymbol{r} = (r_1, \ldots, r_m) \in \mathbf{N}_0^m$*, then*

$$E(A_{\boldsymbol{K}}) = n(n-1)\cdots(n-\ell)\, \mu_{\boldsymbol{k}_1} \cdots \mu_{\boldsymbol{k}_\ell},$$

where $\ell = \ell(\boldsymbol{K})$*.*

We will use this lemma in section 3.1.3.

Before discussing change of basis from (a) to (b) in Proposition 1, we need the following lemma:

LEMMA 2 *Disregarding the order of parts in m-partitions, it holds, for* $\ell = \ell(\boldsymbol{K}) < n$*, that*

$$P_{(\boldsymbol{r})} A_{(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_\ell)} = A_{(\boldsymbol{r}, \boldsymbol{k}_1, \ldots, \boldsymbol{k}_\ell)} + A_{(\boldsymbol{r}+\boldsymbol{k}_1, \boldsymbol{k}_2, \ldots, \boldsymbol{k}_\ell)} + \ldots + A_{(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{\ell-1}, \boldsymbol{r}+\boldsymbol{k}_\ell)}.$$

From Lemma 2, noting $P_{(\boldsymbol{r})} = A_{(\boldsymbol{r})}$ for any one-part m-partition $(\boldsymbol{r})$, we can compose an algorithm to get the expression of a P in terms of A's with rational number coefficients. The sketch of the algorithm is as follows:

ALGORITHM 1 (`PtoA`)

Input: $\boldsymbol{K}$: *m-partition designating a* P.
Output: *weighted sum of* A*'s.*

Step 1. **If** $\ell(\boldsymbol{K}) = 1$ **then** *return* $A_{\boldsymbol{K}}$ *as output.*

Step 2. $U \leftarrow P_{(\boldsymbol{k}_1)}$; $i \leftarrow 2$.

Step 3. $U \leftarrow P_{(\boldsymbol{k}_i)} \times U$ *by using Lemma 2.*

Step 4. $i \leftarrow i+1$; **if** $i > \ell(\boldsymbol{K})$ **then** *return* U *as output* **else** *go back to Step 3.*

It is worth to note that Algorithm `PtoA` for transformation from P to A's yields the moments (and also the cumulants) of r represented in the population cumulants; see section 3.1.4.

Algorithms `PtoA` are implemented on computers as Lisp programs with interface to REDUCE [7], a computer algebra system used worldwide.

3 Asymptotic distributions of R

We now turn to derive an asymptotic distribution of r. Assume that the bivariate population F has finite cumulants and product cumulants κ_{10}, κ_{01}, κ_{20}, κ_{11}, κ_{02}, $\cdots$ up to total order eight. Consider the standardized variate

$$R = \frac{\sqrt{n}\,(r - \rho)}{\sigma} \tag{2}$$

of r, where ρ is the population correlation coefficient, that is

$$\rho = \frac{\kappa_{11}}{\sqrt{\kappa_{20}}\sqrt{\kappa_{02}}}, \tag{3}$$

and we can choose σ^2 such that the limiting distribution of R is the standard normal (it will be shown in (9) exactly).

3.1 The delta method

For obtaining the Edgeworth expansion for probability integral $\Pr[R < x]$ of R up to order $O(n^{-1})$, we follow the so-called *delta method.* It is to calculate approximate moments by expanding R into power series of $1/n$, keeping a certain number of terms, raising to appropriate power and taking expectations term by term.

3.1.1 Expand R into power series of $1/n$

Introducing the following auxiliary random variables, as a first step,

$$w_x = \frac{\sqrt{n}(s_x^2 - \kappa_{20})}{\kappa_{20}},\ w_y = \frac{\sqrt{n}(s_y^2 - \kappa_{02})}{\kappa_{02}},\ w_{xy} = \frac{\sqrt{n}(s_{xy} - \kappa_{11})}{\kappa_{11}},$$

we expand R as power series in terms of $1/n$ and get truncated power series $\tilde{R}$ up to order $O(n^{-1})$:

$$\begin{aligned}
\tilde{R} \quad = \quad & \frac{\rho}{\sigma}\left\{-\frac{1}{2}w_y - \frac{1}{2}w_x + w_{xy}\right. \\
& + \frac{1}{\sqrt{n}}\left(\frac{3}{8}w_y^2 + \frac{1}{4}w_y w_x - \frac{1}{2}w_y w_{xy} + \frac{3}{8}w_x^2 - \frac{1}{2}w_x w_{xy}\right) \\
& + \frac{1}{n}\left(-\frac{5}{16}w_y^3 - \frac{3}{16}w_y^2 w_x + \frac{3}{8}w_y^2 w_{xy} + \frac{1}{4}w_x w_y w_{xy}\right. \\
& \left.\left. \quad - \frac{5}{16}w_x^3 - \frac{3}{16}w_x^2 w_y + \frac{3}{8}w_x^2 w_{xy}\right)\right\}. \qquad (4)
\end{aligned}$$

We remark that each auxiliary variable has the limiting normal distribution and that $w_x \sim O_p(1)$, $w_y \sim O_p(1)$, $w_{xy} \sim O_p(1)$.

Moreover, we compute $\tilde{R}^j$ $(j = 2, 3, 4)$ as truncated power series in terms of $1/n$ as same as (4).

3.1.2 Rewrite R by means of the base $\{P_{\boldsymbol{K}}\}$

Next step is rewriting $\tilde{R}^j$ $(j = 1 \sim 4)$ by means of the base $\{P_{\boldsymbol{K}}\}$. See Remark 1.

3.1.3 Express each $P_{\boldsymbol{K}}$ in terms of the base $\{A_{\boldsymbol{K}}\}$

As a third step, express each $P_{\boldsymbol{K}}$ in terms of the base $\{A_{\boldsymbol{K}}\}$ by applying Algorithm `PtoA`.

3.1.4 Taking expectations term by term

Now each $w_x^i w_y^j w_{xy}^k$ $(i + j + k \leq 6,\ i, j, k \geq 0)$ in right hand side of (4) is expressed in terms of $\{A_{\boldsymbol{K}}\}$. Taking expectation term by term, that is,

$$\mathrm{E}[w_x^i w_y^j w_{xy}^k]$$

and applying Lemma 1 we have the first approximate moment of R. The second, third and fourth moments are given in similar fashion.

Approximate cumulants of R will be described in the next subsection and an asymptotic expansion for probability integrals $\Pr[R < x]$ will be given in section 3.3

3.2 Approximate cumulants of R

The cumulants λ_j $(j = 1, \cdots, 4)$ of the distribution of R obtained from the relations between moments and cumulants are as follows:

$$\lambda_1 = \frac{1}{\sqrt{n}}\left(\frac{\lambda_{11}}{\sigma}\right) + O\left(\frac{1}{n\sqrt{n}}\right), \tag{5}$$

$$\lambda_2 = 1 + \frac{1}{n}\left(\frac{\lambda_{22}}{\sigma^2}\right) + O\left(\frac{1}{n^2}\right), \tag{6}$$

$$\lambda_3 = \frac{1}{\sqrt{n}}\left(\frac{\lambda_{31}}{\sigma^3}\right) + O\left(\frac{1}{n\sqrt{n}}\right), \tag{7}$$

$$\lambda_4 = \frac{1}{n}\left(\frac{\lambda_{42}}{\sigma^4}\right) + O\left(\frac{1}{n^2}\right), \tag{8}$$

where, $\eta = 1/\sqrt{\kappa_{20}\kappa_{02}}$ and

$$\begin{aligned}
\sigma^2 &= \eta^2\Big(\tfrac{1}{4}\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^2 - \kappa_{31}\kappa_{20}^{-1}\kappa_{11} + \tfrac{1}{2}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^2\kappa_{02}^{-1} \\
&\quad + \kappa_{22} + \kappa_{20}\kappa_{02} + \kappa_{20}^{-1}\kappa_{11}^4\kappa_{02}^{-1} - \kappa_{13}\kappa_{11}\kappa_{02}^{-1} + \tfrac{1}{4}\kappa_{11}^2\kappa_{04}\kappa_{02}^{-2} - 2\kappa_{11}^2\Big), \\
\lambda_{11} &= \eta\Big(\tfrac{3}{8}\kappa_{40}\kappa_{20}^{-2}\kappa_{11} - \tfrac{1}{2}\kappa_{31}\kappa_{20}^{-1} + \tfrac{1}{4}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}\kappa_{02}^{-1} \\
&\quad + \tfrac{1}{2}\kappa_{20}^{-1}\kappa_{11}^3\kappa_{02}^{-1} - \tfrac{1}{2}\kappa_{13}\kappa_{02}^{-1} + \tfrac{3}{8}\kappa_{11}\kappa_{04}\kappa_{02}^{-2} - \tfrac{1}{2}\kappa_{11}\Big), \\
\lambda_{22} &= \eta^2\Big(-\tfrac{3}{8}\kappa_{60}\kappa_{20}^{-3}\kappa_{11}^2 + \tfrac{5}{4}\kappa_{51}\kappa_{20}^{-2}\kappa_{11} - \kappa_{42}\kappa_{20}^{-1} + \cdots\Big), \\
\lambda_{31} &= \eta^3\Big(-\tfrac{1}{8}\kappa_{60}\kappa_{20}^{-3}\kappa_{11}^3 + \tfrac{3}{4}\kappa_{51}\kappa_{20}^{-2}\kappa_{11}^2 - \tfrac{3}{2}\kappa_{42}\kappa_{20}^{-1}\kappa_{11} + \cdots\Big), \\
\lambda_{42} &= \eta^4\Big(\tfrac{1}{16}\kappa_{80}\kappa_{20}^{-4}\kappa_{11}^4 - \tfrac{1}{2}\kappa_{71}\kappa_{20}^{-3}\kappa_{11}^3 + \cdots\Big).
\end{aligned} \tag{9}$$

The coefficients λ_{11}, λ_{22}, λ_{31}, λ_{42}, in detail, are shown in Appendix A. We remark that λ_{11}, λ_{22} and λ_{31} can be derived from the results in Cook [4] but, as far as the author knows, their explicit forms appear first in Nakagawa and Niki [17]. The coefficient λ_{42} is extremely lengthy, containing some 345 terms.

3.3 Asymptotic expansions

The Edgeworth expansion of order $1/n$ is given by

$$\begin{aligned}
\Pr[R < x] \sim\ & \Phi(x) - \phi(x)\Bigg[\frac{1}{\sqrt{n}}\left\{\frac{1}{6}\frac{\lambda_{31}}{\sigma^3}H_2(x) + \frac{\lambda_{11}}{\sigma}\right\} \\
& + \frac{1}{n}\left\{\frac{1}{72}\frac{\lambda_{31}^2}{\sigma^6}H_5(x) + \left(\frac{1}{6}\frac{\lambda_{11}\lambda_{31}}{\sigma^4} + \frac{1}{24}\frac{\lambda_{42}}{\sigma^4}\right)H_3(x) \right. \\
& \left. + \left(\frac{1}{2}\frac{\lambda_{11}^2}{\sigma^2} + \frac{1}{2}\frac{\lambda_{22}}{\sigma^2}\right)H_1(x)\right\}\Bigg] + O\left(\frac{1}{n\sqrt{n}}\right),
\end{aligned} \tag{10}$$

where $\Phi(x)$, $\phi(x)$ and $H_j(x)$ denote the standard normal distribution function, its density function and j-th Hermite polynomial, respectively. The validity of this expansion is proved from the results of Bhattacharya and Ghosh [2].

4 Bivariate elliptical case

In this section, we consider the case that F is a bivariate elliptical population with the correlation coefficient ρ and the general kurtosis parameters $\kappa_{(i)}$ defined by Berkane and Bentler [1].

When an elliptical distribution is normal, the $\kappa_{(i)}$ vanishes for each i. In this situation Fisher [5] introduced the normalizing and variance stabilizing transformation

$$z(r) = \tanh^{-1} r = \frac{1}{2} \log \frac{1+r}{1-r}$$

which is known as Fisher's z-transformation. Higher order asymptotic expansions of $z(r)$ are given in Niki and Konishi [18].

For the elliptical population, Muirhead [9] suggested that

$$\sqrt{n}(z(r) - z(\rho))$$

is approximated by a normal variate with mean 0 and variance $1 + \kappa_{(2)}$. The Edgeworth expansion of order $1/\sqrt{n}$ of the distribution for $z(r)$ can be found in Hayakawa [6]. This section, mainly due to Nakagawa and Niki [14], gives us the Edgeworth expansion of order $1/n$.

4.1 Cumulants

Consider the standardized variate R of r

$$R = \frac{\sqrt{n}(r-\rho)}{\sqrt{\kappa_{(2)}+1}(1-\rho^2)}.$$

The limiting distribution of R is a normal with mean 0 and variance 1. Let λ_j be the j-th cumulant of R. Then,

$$\begin{aligned}
\lambda_1 &= \frac{1}{\sqrt{n}}\left(-\sqrt{\tilde{\kappa}_{(2)}}\rho/2\right) + O\left(\frac{1}{n\sqrt{n}}\right), \\
\lambda_2 &= 1 + \frac{1}{n}\left(11/2\rho^2\tilde{\kappa}_{(2)}^2 + 7\tilde{\kappa}_{(2)}^2 - \tilde{\kappa}_{(2)} - 6\tilde{\kappa}_{(3)} + 1\right)/\tilde{\kappa}_{(2)} + O\left(\frac{1}{n^2}\right), \\
\lambda_3 &= \frac{1}{\sqrt{n}}\left(-6\sqrt{\tilde{\kappa}_{(2)}}\rho\right) + O\left(\frac{1}{n\sqrt{n}}\right), \\
\lambda_4 &= \frac{3}{n}\left(24\rho^2\tilde{\kappa}_{(2)}^3 + 8\tilde{\kappa}_{(2)}^3 - \tilde{\kappa}_{(2)}^2 - 12\tilde{\kappa}_{(2)}\tilde{\kappa}_{(3)} + 3\tilde{\kappa}_{(4)}\right)/\tilde{\kappa}_{(2)}^2 + O\left(\frac{1}{n^2}\right),
\end{aligned}$$

where $\tilde{\kappa}_{(i)} = \kappa_{(i)} + 1$.

Let Z be the transformed variate of r

$$Z = \frac{\sqrt{n}(z(r) - z(\rho))}{\sqrt{\tilde{\kappa}_{(2)}}}.$$

The cumulants of Z, ν_j $(j = 1, 2, \cdots)$ take the same form as λ's. As well as for the normal population, Fisher's z-transformation is a normalizing transformation in the sense of Niki and Konishi [20] which makes the coefficient of $1/\sqrt{n}$ in ν_3 be

$$\frac{3\sqrt{\tilde{\kappa}_{(2)}}(1-\rho^2)z''(\rho) - 6\sqrt{\tilde{\kappa}_{(2)}}\rho z'(\rho)}{z'(\rho)} = 0.$$

The same transformation makes the lowest order coefficients in the asymptotic expansions of ν_1 be

$$\frac{\sqrt{\tilde{\kappa}_{(2)}}(1-\rho^2)z''(\rho) - \sqrt{\tilde{\kappa}_{(2)}}\rho z'(\rho)}{2z'(\rho)} = \frac{\sqrt{\tilde{\kappa}_{(2)}}\rho}{2}.$$

The cumulants ν_j of Z are given in the following;

$$\begin{aligned}
\nu_1 &= \frac{1}{\sqrt{n}}\left(\sqrt{\tilde{\kappa}_{(2)}}\rho/2\right) + O\left(\frac{1}{n\sqrt{n}}\right), \\
\nu_2 &= 1 - \frac{1}{2n}\left(\rho^2\tilde{\kappa}_{(2)}^2 - 18\tilde{\kappa}_{(2)}^2 + 2\tilde{\kappa}_{(2)} + 12\tilde{\kappa}_{(3)} - 2\right)/\tilde{\kappa}_{(2)} + O\left(\frac{1}{n^2}\right), \\
\nu_3 &= O\left(\frac{1}{n\sqrt{n}}\right), \\
\nu_4 &= \frac{1}{n}\left(32\tilde{\kappa}_{(2)}^3 - 3\tilde{\kappa}_{(2)}^2 - 36\tilde{\kappa}_{(2)}\tilde{\kappa}_{(3)} + 9\tilde{\kappa}_{(4)}\right)/\tilde{\kappa}_{(2)}^2 + O\left(\frac{1}{n^2}\right).
\end{aligned}$$

4.2 Asymptotic expansions

We obtain an asymptotic expansion for the distribution of Z as follows:

$$\begin{aligned}
&\Pr[Z < x] \sim \Phi(x) - \phi(x)\left[\frac{1}{\sqrt{n}}\left(\sqrt{\tilde{\kappa}_{(2)}}\rho/2\right)\right. \\
&+ \frac{1}{n}\left\{\left(H_3(x)\ \left(32\tilde{\kappa}_{(2)}^3 - 3\tilde{\kappa}_{(2)}^2 - 36\tilde{\kappa}_{(2)}\tilde{\kappa}_{(3)} + 9\tilde{\kappa}_{(4)}\right)/\left(24\tilde{\kappa}_{(2)}^2\right)\right.\right. \\
&\left.\left. + H_1(x)\ \left(-\rho^2\tilde{\kappa}_{(2)}^2 + 36\tilde{\kappa}_{(2)}^2 - 4\tilde{\kappa}_{(2)} - 24\tilde{\kappa}_{(3)} + 4\right)/\left(8\tilde{\kappa}_{(2)}\right)\right\}\right] + O\left(\frac{1}{n\sqrt{n}}\right).
\end{aligned}$$

This formula is extended version of Hayakawa [6].

The Edgeworth expansions for the distribution of R is more complicated:

$$\Pr[R < x] \ \sim \Phi(x) - \phi(x)\left[-\frac{1}{\sqrt{n}}\left\{\sqrt{\tilde{\kappa}_{(2)}}\rho\left(2H_2(x) - 1\right)/2\right\}\right.$$

$$+\frac{1}{n}\left\{H_5(x)\,\rho^2\tilde{\kappa}_{(2)}/2+H_3(x)\,\left(28\rho^2\tilde{\kappa}_{(2)}^3+8\tilde{\kappa}_{(2)}^3-\tilde{\kappa}_{(2)}^2\right.\right.$$
$$\left.-12\tilde{\kappa}_{(2)}\tilde{\kappa}_{(3)}+3\tilde{\kappa}_{(4)}\right)/\left(8\tilde{\kappa}_{(2)}^2\right)+H_1(x)\,\left(23\rho^2\tilde{\kappa}_{(2)}^2\right.$$
$$\left.\left.\left.+28\tilde{\kappa}_{(2)}^2-4\tilde{\kappa}_{(2)}-24\tilde{\kappa}_{(3)}+4\right)/\left(8\tilde{\kappa}_{(2)}\right)\right\}\right]+O\left(\frac{1}{n\sqrt{n}}\right).$$

5 Concluding remarks

The emphasis of this paper has been on explicit formulation of procedures on deriving the Edgeworth expansion of r. An extension is found in Niki, Nakagawa and Hashiguchi [21]. Approximate distributions of other statistics with symmetric property can be derived. For related work, see Niki and Konishi [19] (coefficient of variation), Nakagawa and Niki [13] (sample regression coefficient), Nakagawa, Fukuhara, Niki and Ichimura [11] ($\sqrt{b_1}$).

An inverse algorithm of Algorithm `PtoA` is easily obtained. It leads us a method of obtaining the distribution of k-statistics: see Nakagawa, Hashiguchi and Niki [12].

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A Coefficients of approximate cumulants of R

The coefficients λ_{11}, λ_{22}, λ_{31}, λ_{42} have the following form in terms of population cumulants κ_{ij} $(0 \leq i+j \leq 8)$:

$$
\begin{aligned}
\lambda_{11} \;=\;& \eta \Big(\tfrac{3}{8}\kappa_{40}\kappa_{20}^{-2}\kappa_{11} - \tfrac{1}{2}\kappa_{31}\kappa_{20}^{-1} + \tfrac{1}{4}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}\kappa_{02}^{-1} \\
&+ \tfrac{1}{2}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{02}^{-1} - \tfrac{1}{2}\kappa_{13}\kappa_{02}^{-1} + \tfrac{3}{8}\kappa_{11}\kappa_{04}\kappa_{02}^{-2} - \tfrac{1}{2}\kappa_{11} \Big) \\
\lambda_{22} \;=\;& \eta^{2} \Big(-\tfrac{3}{8}\kappa_{60}\kappa_{20}^{-3}\kappa_{11}^{2} + \tfrac{5}{4}\kappa_{51}\kappa_{20}^{-2}\kappa_{11} - \kappa_{42}\kappa_{20}^{-1} - \tfrac{5}{8}\kappa_{42}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ \tfrac{39}{32}\kappa_{40}^{2}\kappa_{20}^{-4}\kappa_{11}^{2} - \tfrac{15}{4}\kappa_{40}\kappa_{31}\kappa_{20}^{-3}\kappa_{11} + \kappa_{40}\kappa_{22}\kappa_{20}^{-2} \\
&+ \tfrac{15}{8}\kappa_{40}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{2}\kappa_{02}^{-1} - \tfrac{5}{4}\kappa_{40}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}\kappa_{02}^{-1} \\
&+ \tfrac{7}{16}\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-2} - \tfrac{11}{4}\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^{2} + \tfrac{15}{4}\kappa_{40}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&+ \tfrac{3}{2}\kappa_{33}\kappa_{20}^{-1}\kappa_{11}\kappa_{02}^{-1} + \tfrac{7}{4}\kappa_{31}^{2}\kappa_{20}^{-2} - 3\kappa_{31}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}\kappa_{02}^{-1} \\
&+ \tfrac{3}{2}\kappa_{31}\kappa_{20}^{-1}\kappa_{13}\kappa_{02}^{-1} - \tfrac{5}{4}\kappa_{31}\kappa_{20}^{-1}\kappa_{11}\kappa_{04}\kappa_{02}^{-2} \\
&+ 7\kappa_{31}\kappa_{20}^{-1}\kappa_{11} - 11\kappa_{31}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-1} - \tfrac{3}{2}\kappa_{30}^{2}\kappa_{20}^{-3}\kappa_{11}^{2} \\
&+ 5\kappa_{30}\kappa_{21}\kappa_{20}^{-2}\kappa_{11} - 2\kappa_{30}\kappa_{20}^{-1}\kappa_{12} - \tfrac{5}{8}\kappa_{24}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{02}^{-2} - \kappa_{24}\kappa_{02}^{-1} \\
&+ \kappa_{22}^{2}\kappa_{20}^{-1}\kappa_{02}^{-1} + \tfrac{11}{8}\kappa_{22}^{2}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{02}^{-2} - 3\kappa_{22}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}\kappa_{02}^{-2} \\
&+ \tfrac{15}{8}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-3} + \tfrac{11}{2}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ \tfrac{11}{2}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{02}^{-2} + \kappa_{22}\kappa_{04}\kappa_{02}^{-2} - 5\kappa_{22} - 2\kappa_{21}^{2}\kappa_{20}^{-1} \\
&- \tfrac{5}{2}\kappa_{21}^{2}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{02}^{-1} + 6\kappa_{21}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}\kappa_{02}^{-1} - 2\kappa_{21}\kappa_{03}\kappa_{02}^{-1} \\
&+ \kappa_{20}\kappa_{02} - 11\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-2} - \tfrac{5}{2}\kappa_{20}^{-1}\kappa_{12}^{2}\kappa_{11}^{2}\kappa_{02}^{-2} \\
&+ \tfrac{15}{4}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-3} - 10\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{02}^{-1} + \tfrac{11}{2}\kappa_{20}^{-2}\kappa_{11}^{6}\kappa_{02}^{-2} \\
&+ \tfrac{5}{4}\kappa_{15}\kappa_{11}\kappa_{02}^{-2} + \tfrac{7}{4}\kappa_{13}^{2}\kappa_{02}^{-2} - \tfrac{15}{4}\kappa_{13}\kappa_{11}\kappa_{04}\kappa_{02}^{-3} \\
&+ 7\kappa_{13}\kappa_{11}\kappa_{02}^{-1} - 2\kappa_{12}^{2}\kappa_{02}^{-1} + 5\kappa_{12}\kappa_{11}\kappa_{03}\kappa_{02}^{-2} - \tfrac{3}{8}\kappa_{11}^{2}\kappa_{06}\kappa_{02}^{-3} \\
&+ \tfrac{39}{32}\kappa_{11}^{2}\kappa_{04}^{2}\kappa_{02}^{-4} - \tfrac{11}{4}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-2} - \tfrac{3}{2}\kappa_{11}^{2}\kappa_{03}^{2}\kappa_{02}^{-3} + \tfrac{7}{2}\kappa_{11}^{2} \Big) \\
\lambda_{31} \;=\;& \eta^{3} \Big(-\tfrac{1}{8}\kappa_{60}\kappa_{20}^{-3}\kappa_{11}^{3} + \tfrac{3}{4}\kappa_{51}\kappa_{20}^{-2}\kappa_{11}^{2} - \tfrac{3}{2}\kappa_{42}\kappa_{20}^{-1}\kappa_{11}
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{3}{8}\kappa_{42}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-1} + \tfrac{9}{16}\kappa_{40}^{2}\kappa_{20}^{-4}\kappa_{11}^{3} - 3\kappa_{40}\kappa_{31}\kappa_{20}^{-3}\kappa_{11}^{2} \\
&+ \tfrac{3}{2}\kappa_{40}\kappa_{22}\kappa_{20}^{-2}\kappa_{11} + \tfrac{3}{2}\kappa_{40}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{3}\kappa_{02}^{-1} - \tfrac{3}{2}\kappa_{40}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ \tfrac{3}{8}\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-2} - 3\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^{3} + 3\kappa_{40}\kappa_{20}^{-3}\kappa_{11}^{5}\kappa_{02}^{-1} + \tfrac{3}{2}\kappa_{33}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ \kappa_{33} + \tfrac{15}{4}\kappa_{31}^{2}\kappa_{20}^{-2}\kappa_{11} - 3\kappa_{31}\kappa_{22}\kappa_{20}^{-1} - \tfrac{9}{2}\kappa_{31}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ \tfrac{9}{2}\kappa_{31}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}\kappa_{02}^{-1} - \tfrac{3}{2}\kappa_{31}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-2} \\
&+ 12\kappa_{31}\kappa_{20}^{-1}\kappa_{11}^{2} - 12\kappa_{31}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{02}^{-1} - \tfrac{1}{2}\kappa_{30}^{2}\kappa_{20}^{-3}\kappa_{11}^{3} \\
&+ 3\kappa_{30}\kappa_{21}\kappa_{20}^{-2}\kappa_{11}^{2} - 3\kappa_{30}\kappa_{20}^{-1}\kappa_{12}\kappa_{11} + \kappa_{30}\kappa_{03} - \tfrac{3}{8}\kappa_{24}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{02}^{-2} - \tfrac{3}{2}\kappa_{24}\kappa_{11}\kappa_{02}^{-1} \\
&+ 3\kappa_{22}^{2}\kappa_{20}^{-1}\kappa_{11}\kappa_{02}^{-1} + \tfrac{3}{2}\kappa_{22}^{2}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-2} - \tfrac{9}{2}\kappa_{22}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-2} \\
&+ \tfrac{3}{2}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-3} + 6\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{02}^{-1} + 6\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{5}\kappa_{02}^{-2} - 3\kappa_{22}\kappa_{13}\kappa_{02}^{-1} \\
&+ \tfrac{3}{2}\kappa_{22}\kappa_{11}\kappa_{04}\kappa_{02}^{-2} - 12\kappa_{22}\kappa_{11} - 3\kappa_{21}^{2}\kappa_{20}^{-1}\kappa_{11} - \tfrac{3}{2}\kappa_{21}^{2}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-1} \\
&+ 6\kappa_{21}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{2}\kappa_{02}^{-1} + 3\kappa_{21}\kappa_{12} - 3\kappa_{21}\kappa_{11}\kappa_{03}\kappa_{02}^{-1} - 6\kappa_{20}\kappa_{11}\kappa_{02} \\
&- 12\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{4}\kappa_{02}^{-2} - \tfrac{3}{2}\kappa_{20}^{-1}\kappa_{12}^{2}\kappa_{11}^{3}\kappa_{02}^{-2} \\
&+ 3\kappa_{20}^{-1}\kappa_{11}^{5}\kappa_{04}\kappa_{02}^{-3} - 18\kappa_{20}^{-1}\kappa_{11}^{5}\kappa_{02}^{-1} + 6\kappa_{20}^{-2}\kappa_{11}^{7}\kappa_{02}^{-2} \\
&+ \tfrac{3}{4}\kappa_{15}\kappa_{11}^{2}\kappa_{02}^{-2} + \tfrac{15}{4}\kappa_{13}^{2}\kappa_{11}\kappa_{02}^{-2} - 3\kappa_{13}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-3} \\
&+ 12\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-1} - 3\kappa_{12}^{2}\kappa_{11}\kappa_{02}^{-1} + 3\kappa_{12}\kappa_{11}^{2}\kappa_{03}\kappa_{02}^{-2} - \tfrac{1}{8}\kappa_{11}^{3}\kappa_{06}\kappa_{02}^{-3} \\
&\left. + \tfrac{9}{16}\kappa_{11}^{3}\kappa_{04}^{2}\kappa_{02}^{-4} - 3\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-2} - \tfrac{1}{2}\kappa_{11}^{3}\kappa_{03}^{2}\kappa_{02}^{-3} + 18\kappa_{11}^{3}\right)
\end{aligned}
$$

$$
\begin{aligned}
\lambda_{42} \quad = \quad \eta^{4} \Big(&\tfrac{1}{16}\kappa_{80}\kappa_{20}^{-4}\kappa_{11}^{4} - \tfrac{1}{2}\kappa_{71}\kappa_{20}^{-3}\kappa_{11}^{3} \\
&+ \tfrac{3}{2}\kappa_{62}\kappa_{20}^{-2}\kappa_{11}^{2} + \tfrac{1}{4}\kappa_{62}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-1} - \tfrac{9}{8}\kappa_{60}\kappa_{40}\kappa_{20}^{-5}\kappa_{11}^{4} \\
&+ 3\kappa_{60}\kappa_{31}\kappa_{20}^{-4}\kappa_{11}^{3} - \tfrac{3}{2}\kappa_{60}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{2} - \tfrac{3}{2}\kappa_{60}\kappa_{22}\kappa_{20}^{-4}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&+ \tfrac{3}{2}\kappa_{60}\kappa_{20}^{-3}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-1} - \tfrac{3}{8}\kappa_{60}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-2} \\
&+ 3\kappa_{60}\kappa_{20}^{-3}\kappa_{11}^{4} - 3\kappa_{60}\kappa_{20}^{-4}\kappa_{11}^{6}\kappa_{02}^{-1} - 2\kappa_{53}\kappa_{20}^{-1}\kappa_{11} - \tfrac{3}{2}\kappa_{53}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-1} \\
&+ \tfrac{21}{4}\kappa_{51}\kappa_{40}\kappa_{20}^{-4}\kappa_{11}^{3} - \tfrac{27}{2}\kappa_{51}\kappa_{31}\kappa_{20}^{-3}\kappa_{11}^{2} + 6\kappa_{51}\kappa_{22}\kappa_{20}^{-2}\kappa_{11} \\
&+ \tfrac{15}{2}\kappa_{51}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{3}\kappa_{02}^{-1} - \tfrac{15}{2}\kappa_{51}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ \tfrac{9}{4}\kappa_{51}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-2} - 18\kappa_{51}\kappa_{20}^{-2}\kappa_{11}^{3} + 18\kappa_{51}\kappa_{20}^{-3}\kappa_{11}^{5}\kappa_{02}^{-1} \\
&+ 2\kappa_{50}\kappa_{30}\kappa_{20}^{-4}\kappa_{11}^{4} - 6\kappa_{50}\kappa_{21}\kappa_{20}^{-3}\kappa_{11}^{3} - 2\kappa_{50}\kappa_{20}^{-1}\kappa_{11}\kappa_{03} \\
&+ 6\kappa_{50}\kappa_{20}^{-2}\kappa_{12}\kappa_{11}^{2} + 3\kappa_{44}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{02}^{-1} + \tfrac{3}{8}\kappa_{44}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{02}^{-2} \\
&+ \kappa_{44} - \tfrac{15}{2}\kappa_{42}\kappa_{40}\kappa_{20}^{-3}\kappa_{11}^{2} - \tfrac{21}{8}\kappa_{42}\kappa_{40}\kappa_{20}^{-4}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&+ 18\kappa_{42}\kappa_{31}\kappa_{20}^{-2}\kappa_{11} + \tfrac{15}{2}\kappa_{42}\kappa_{31}\kappa_{20}^{-3}\kappa_{11}^{3}\kappa_{02}^{-1} - 6\kappa_{42}\kappa_{22}\kappa_{20}^{-1} \\
&- \tfrac{33}{2}\kappa_{42}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{02}^{-1} - \tfrac{9}{2}\kappa_{42}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-2}
\end{aligned}
$$

$$
\begin{aligned}
&+ 12\kappa_{42}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}\kappa_{02}^{-1} - \tfrac{9}{2}\kappa_{42}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-2} \\
&+ 36\kappa_{42}\kappa_{20}^{-1}\kappa_{11}^{2} + 6\kappa_{42}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-2} - \tfrac{15}{8}\kappa_{42}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-3} \\
&- 27\kappa_{42}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{02}^{-1} - 9\kappa_{42}\kappa_{20}^{-3}\kappa_{11}^{6}\kappa_{02}^{-2} - 10\kappa_{41}\kappa_{30}\kappa_{20}^{-3}\kappa_{11}^{3} \\
&+ 24\kappa_{41}\kappa_{21}\kappa_{20}^{-2}\kappa_{11}^{2} + 6\kappa_{41}\kappa_{21}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-1} - 18\kappa_{41}\kappa_{20}^{-1}\kappa_{12}\kappa_{11} \\
&+ 6\kappa_{41}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{03}\kappa_{02}^{-1} - 12\kappa_{41}\kappa_{20}^{-2}\kappa_{12}\kappa_{11}^{3}\kappa_{02}^{-1} + 4\kappa_{41}\kappa_{03} \\
&+ \tfrac{21}{8}\kappa_{40}^{3}\kappa_{20}^{-6}\kappa_{11}^{4} - 18\kappa_{40}^{2}\kappa_{31}\kappa_{20}^{-5}\kappa_{11}^{3} + \tfrac{15}{2}\kappa_{40}^{2}\kappa_{22}\kappa_{20}^{-4}\kappa_{11}^{2} \\
&+ 9\kappa_{40}^{2}\kappa_{22}\kappa_{20}^{-5}\kappa_{11}^{4}\kappa_{02}^{-1} - \tfrac{15}{2}\kappa_{40}^{2}\kappa_{20}^{-4}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-1} \\
&+ \tfrac{15}{8}\kappa_{40}^{2}\kappa_{20}^{-4}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-2} - \tfrac{29}{2}\kappa_{40}^{2}\kappa_{20}^{-4}\kappa_{11}^{4} \\
&+ 18\kappa_{40}^{2}\kappa_{20}^{-5}\kappa_{11}^{6}\kappa_{02}^{-1} + 3\kappa_{40}\kappa_{33}\kappa_{20}^{-2}\kappa_{11} + \tfrac{15}{2}\kappa_{40}\kappa_{33}\kappa_{20}^{-3}\kappa_{11}^{3}\kappa_{02}^{-1} \\
&+ \tfrac{153}{4}\kappa_{40}\kappa_{31}^{2}\kappa_{20}^{-4}\kappa_{11}^{2} - 27\kappa_{40}\kappa_{31}\kappa_{22}\kappa_{20}^{-3}\kappa_{11} - \tfrac{87}{2}\kappa_{40}\kappa_{31}\kappa_{22}\kappa_{20}^{-4}\kappa_{11}^{3}\kappa_{02}^{-1} \\
&+ \tfrac{69}{2}\kappa_{40}\kappa_{31}\kappa_{20}^{-3}\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-1} - \tfrac{21}{2}\kappa_{40}\kappa_{31}\kappa_{20}^{-3}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-2} \\
&+ 80\kappa_{40}\kappa_{31}\kappa_{20}^{-3}\kappa_{11}^{3} - 108\kappa_{40}\kappa_{31}\kappa_{20}^{-4}\kappa_{11}^{5}\kappa_{02}^{-1} - \tfrac{9}{2}\kappa_{40}\kappa_{30}^{2}\kappa_{20}^{-5}\kappa_{11}^{4} \\
&+ 21\kappa_{40}\kappa_{30}\kappa_{21}\kappa_{20}^{-4}\kappa_{11}^{3} + 3\kappa_{40}\kappa_{30}\kappa_{20}^{-2}\kappa_{11}\kappa_{03} - 15\kappa_{40}\kappa_{30}\kappa_{20}^{-3}\kappa_{12}\kappa_{11}^{2} \\
&- \tfrac{9}{2}\kappa_{40}\kappa_{24}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{02}^{-1} - \tfrac{15}{8}\kappa_{40}\kappa_{24}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-2} + 3\kappa_{40}\kappa_{22}^{2}\kappa_{20}^{-2} \\
&+ 21\kappa_{40}\kappa_{22}^{2}\kappa_{20}^{-3}\kappa_{11}^{2}\kappa_{02}^{-1} + \tfrac{27}{2}\kappa_{40}\kappa_{22}^{2}\kappa_{20}^{-4}\kappa_{11}^{4}\kappa_{02}^{-2} - 15\kappa_{40}\kappa_{22}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}\kappa_{02}^{-1} \\
&+ 6\kappa_{40}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-2} - 42\kappa_{40}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{2} - \tfrac{57}{2}\kappa_{40}\kappa_{22}\kappa_{20}^{-3}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-2} \\
&+ 9\kappa_{40}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-3} + 30\kappa_{40}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&+ 54\kappa_{40}\kappa_{22}\kappa_{20}^{-4}\kappa_{11}^{6}\kappa_{02}^{-2} - 15\kappa_{40}\kappa_{21}^{2}\kappa_{20}^{-3}\kappa_{11}^{2} - \tfrac{21}{2}\kappa_{40}\kappa_{21}^{2}\kappa_{20}^{-4}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&+ 9\kappa_{40}\kappa_{21}\kappa_{20}^{-2}\kappa_{12}\kappa_{11} - 9\kappa_{40}\kappa_{21}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{03}\kappa_{02}^{-1} \\
&+ 30\kappa_{40}\kappa_{21}\kappa_{20}^{-3}\kappa_{12}\kappa_{11}^{3}\kappa_{02}^{-1} - 4\kappa_{40}\kappa_{20}^{-1}\kappa_{13}\kappa_{11} - 6\kappa_{40}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{02} \\
&+ \tfrac{9}{4}\kappa_{40}\kappa_{20}^{-2}\kappa_{15}\kappa_{11}^{3}\kappa_{02}^{-2} + \tfrac{57}{4}\kappa_{40}\kappa_{20}^{-2}\kappa_{13}^{2}\kappa_{11}^{2}\kappa_{02}^{-2} - \tfrac{21}{2}\kappa_{40}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-3} \\
&+ 48\kappa_{40}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-1} - 9\kappa_{40}\kappa_{20}^{-2}\kappa_{12}^{2}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ 9\kappa_{40}\kappa_{20}^{-2}\kappa_{12}\kappa_{11}^{3}\kappa_{03}\kappa_{02}^{-2} - \tfrac{3}{8}\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{06}\kappa_{02}^{-3} \\
&+ \tfrac{15}{8}\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{04}^{2}\kappa_{02}^{-4} - 12\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-2} - \tfrac{3}{2}\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{03}^{2}\kappa_{02}^{-3} \\
&+ 66\kappa_{40}\kappa_{20}^{-2}\kappa_{11}^{4} - 72\kappa_{40}\kappa_{20}^{-3}\kappa_{13}\kappa_{11}^{5}\kappa_{02}^{-2} - \tfrac{15}{2}\kappa_{40}\kappa_{20}^{-3}\kappa_{12}^{2}\kappa_{11}^{4}\kappa_{02}^{-2} \\
&+ 18\kappa_{40}\kappa_{20}^{-3}\kappa_{11}^{6}\kappa_{04}\kappa_{02}^{-3} - 114\kappa_{40}\kappa_{20}^{-3}\kappa_{11}^{6}\kappa_{02}^{-1} + 54\kappa_{40}\kappa_{20}^{-4}\kappa_{11}^{8}\kappa_{02}^{-2} + \kappa_{40}\kappa_{04} \\
&- \tfrac{3}{2}\kappa_{35}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{02}^{-2} - 2\kappa_{35}\kappa_{11}\kappa_{02}^{-1} - 6\kappa_{33}\kappa_{31}\kappa_{20}^{-1} \\
&- 21\kappa_{33}\kappa_{31}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{02}^{-1} + 18\kappa_{33}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}\kappa_{02}^{-1} \\
&+ 15\kappa_{33}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-2} - 21\kappa_{33}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-2} \\
&+ \tfrac{15}{2}\kappa_{33}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-3} - 12\kappa_{33}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{02}^{-1} + 36\kappa_{33}\kappa_{20}^{-2}\kappa_{11}^{5}\kappa_{02}^{-2} - 6\kappa_{33}\kappa_{13}\kappa_{02}^{-1} \\
&+ 3\kappa_{33}\kappa_{11}\kappa_{04}\kappa_{02}^{-2} - 24\kappa_{33}\kappa_{11} + 18\kappa_{32}\kappa_{30}\kappa_{20}^{-2}\kappa_{11}^{2}
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\kappa_{32}\kappa_{30}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-1} - 30\kappa_{32}\kappa_{21}\kappa_{20}^{-1}\kappa_{11} - 30\kappa_{32}\kappa_{21}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-1} \\
&+ 42\kappa_{32}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{2}\kappa_{02}^{-1} - 6\kappa_{32}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{03}\kappa_{02}^{-2} + 6\kappa_{32}\kappa_{20}^{-2}\kappa_{12}\kappa_{11}^{4}\kappa_{02}^{-2} \\
&+ 12\kappa_{32}\kappa_{12} - 14\kappa_{32}\kappa_{11}\kappa_{03}\kappa_{02}^{-1} - 24\kappa_{31}^{3}\kappa_{20}^{-3}\kappa_{11} + 18\kappa_{31}^{2}\kappa_{22}\kappa_{20}^{-2} \\
&+ \tfrac{99}{2}\kappa_{31}^{2}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{2}\kappa_{02}^{-1} - 36\kappa_{31}^{2}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}\kappa_{02}^{-1} \\
&+ \tfrac{57}{4}\kappa_{31}^{2}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-2} - 108\kappa_{31}^{2}\kappa_{20}^{-2}\kappa_{11}^{2} + 164\kappa_{31}^{2}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&+ 12\kappa_{31}\kappa_{30}^{2}\kappa_{20}^{-4}\kappa_{11}^{3} - 54\kappa_{31}\kappa_{30}\kappa_{21}\kappa_{20}^{-3}\kappa_{11}^{2} - 6\kappa_{31}\kappa_{30}\kappa_{20}^{-1}\kappa_{03} \\
&+ 36\kappa_{31}\kappa_{30}\kappa_{20}^{-2}\kappa_{12}\kappa_{11} + 12\kappa_{31}\kappa_{24}\kappa_{20}^{-1}\kappa_{11}\kappa_{02}^{-1} \\
&+ 6\kappa_{31}\kappa_{24}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-2} - 42\kappa_{31}\kappa_{22}^{2}\kappa_{20}^{-2}\kappa_{11}\kappa_{02}^{-1} - 36\kappa_{31}\kappa_{22}^{2}\kappa_{20}^{-3}\kappa_{11}^{3}\kappa_{02}^{-2} \\
&+ 24\kappa_{31}\kappa_{22}\kappa_{20}^{-1}\kappa_{13}\kappa_{02}^{-1} - 15\kappa_{31}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}\kappa_{04}\kappa_{02}^{-2} \\
&+ 108\kappa_{31}\kappa_{22}\kappa_{20}^{-1}\kappa_{11} + 75\kappa_{31}\kappa_{22}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-2} \\
&- \tfrac{57}{2}\kappa_{31}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-3} - 96\kappa_{31}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-1} - 180\kappa_{31}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{5}\kappa_{02}^{-2} \\
&+ 36\kappa_{31}\kappa_{21}^{2}\kappa_{20}^{-2}\kappa_{11} + 30\kappa_{31}\kappa_{21}^{2}\kappa_{20}^{-3}\kappa_{11}^{3}\kappa_{02}^{-1} - 18\kappa_{31}\kappa_{21}\kappa_{20}^{-1}\kappa_{12} \\
&+ 24\kappa_{31}\kappa_{21}\kappa_{20}^{-1}\kappa_{11}\kappa_{03}\kappa_{02}^{-1} - 84\kappa_{31}\kappa_{21}\kappa_{20}^{-2}\kappa_{12}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&- \tfrac{15}{2}\kappa_{31}\kappa_{20}^{-1}\kappa_{15}\kappa_{11}^{2}\kappa_{02}^{-2} - 36\kappa_{31}\kappa_{20}^{-1}\kappa_{13}^{2}\kappa_{11}\kappa_{02}^{-2} \\
&+ \tfrac{69}{2}\kappa_{31}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-3} - 144\kappa_{31}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ 24\kappa_{31}\kappa_{20}^{-1}\kappa_{12}^{2}\kappa_{11}\kappa_{02}^{-1} - 30\kappa_{31}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{2}\kappa_{03}\kappa_{02}^{-2} \\
&+ \tfrac{3}{2}\kappa_{31}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{06}\kappa_{02}^{-3} - \tfrac{15}{2}\kappa_{31}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{04}^{2}\kappa_{02}^{-4} + 48\kappa_{31}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-2} \\
&+ 6\kappa_{31}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{03}^{2}\kappa_{02}^{-3} - 264\kappa_{31}\kappa_{20}^{-1}\kappa_{11}^{3} + 252\kappa_{31}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{4}\kappa_{02}^{-2} \\
&+ 24\kappa_{31}\kappa_{20}^{-2}\kappa_{12}^{2}\kappa_{11}^{3}\kappa_{02}^{-2} - 72\kappa_{31}\kappa_{20}^{-2}\kappa_{11}^{5}\kappa_{04}\kappa_{02}^{-3} \\
&+ 456\kappa_{31}\kappa_{20}^{-2}\kappa_{11}^{5}\kappa_{02}^{-1} - 216\kappa_{31}\kappa_{20}^{-3}\kappa_{11}^{7}\kappa_{02}^{-2} + 4\kappa_{31}\kappa_{13} - 4\kappa_{31}\kappa_{11}\kappa_{04}\kappa_{02}^{-1} \\
&+ 24\kappa_{31}\kappa_{11}\kappa_{02} - 6\kappa_{30}^{2}\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{2} - 6\kappa_{30}^{2}\kappa_{22}\kappa_{20}^{-4}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&+ 6\kappa_{30}^{2}\kappa_{20}^{-3}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-1} - \tfrac{3}{2}\kappa_{30}^{2}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-2} \\
&+ 12\kappa_{30}^{2}\kappa_{20}^{-3}\kappa_{11}^{4} - 12\kappa_{30}^{2}\kappa_{20}^{-4}\kappa_{11}^{6}\kappa_{02}^{-1} - 14\kappa_{30}\kappa_{23}\kappa_{20}^{-1}\kappa_{11} - 6\kappa_{30}\kappa_{23}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{02}^{-1} \\
&+ 24\kappa_{30}\kappa_{22}\kappa_{21}\kappa_{20}^{-2}\kappa_{11} + 30\kappa_{30}\kappa_{22}\kappa_{21}\kappa_{20}^{-3}\kappa_{11}^{3}\kappa_{02}^{-1} - 12\kappa_{30}\kappa_{22}\kappa_{20}^{-1}\kappa_{12} \\
&+ 6\kappa_{30}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}\kappa_{03}\kappa_{02}^{-1} - 24\kappa_{30}\kappa_{22}\kappa_{20}^{-2}\kappa_{12}\kappa_{11}^{2}\kappa_{02}^{-1} - 30\kappa_{30}\kappa_{21}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ 9\kappa_{30}\kappa_{21}\kappa_{20}^{-2}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-2} - 72\kappa_{30}\kappa_{21}\kappa_{20}^{-2}\kappa_{11}^{3} \\
&+ 72\kappa_{30}\kappa_{21}\kappa_{20}^{-3}\kappa_{11}^{5}\kappa_{02}^{-1} + 6\kappa_{30}\kappa_{20}^{-1}\kappa_{14}\kappa_{11}^{2}\kappa_{02}^{-1} \\
&+ 24\kappa_{30}\kappa_{20}^{-1}\kappa_{13}\kappa_{12}\kappa_{11}\kappa_{02}^{-1} - 9\kappa_{30}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-2} \\
&+ 72\kappa_{30}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{2} + 24\kappa_{30}\kappa_{20}^{-1}\kappa_{11}^{3}\kappa_{03}\kappa_{02}^{-1} - 72\kappa_{30}\kappa_{20}^{-2}\kappa_{12}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&+ 4\kappa_{30}\kappa_{14} - 6\kappa_{30}\kappa_{13}\kappa_{03}\kappa_{02}^{-1} - 2\kappa_{30}\kappa_{11}\kappa_{05}\kappa_{02}^{-1} \\
&+ 3\kappa_{30}\kappa_{11}\kappa_{04}\kappa_{03}\kappa_{02}^{-2} - 24\kappa_{30}\kappa_{11}\kappa_{03} + \tfrac{1}{4}\kappa_{26}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{02}^{-3} \\
&+ \tfrac{3}{2}\kappa_{26}\kappa_{11}^{2}\kappa_{02}^{-2} - \tfrac{33}{2}\kappa_{24}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{02}^{-2} - \tfrac{9}{2}\kappa_{24}\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{02}^{-3} - 6\kappa_{24}\kappa_{22}\kappa_{02}^{-1} \\
&+ \tfrac{15}{2}\kappa_{24}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-3} - \tfrac{21}{8}\kappa_{24}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-4} - 27\kappa_{24}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{02}^{-2}
\end{aligned}
$$

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\begin{aligned}
&- 9\kappa_{24}\kappa_{20}^{-2}\kappa_{11}^{6}\kappa_{02}^{-3} + 18\kappa_{24}\kappa_{13}\kappa_{11}\kappa_{02}^{-2} - \frac{15}{2}\kappa_{24}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-3} \\
&+ 36\kappa_{24}\kappa_{11}^{2}\kappa_{02}^{-1} + 42\kappa_{23}\kappa_{21}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{02}^{-1} + 6\kappa_{23}\kappa_{21}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{02}^{-2} \\
&+ 12\kappa_{23}\kappa_{21} - 30\kappa_{23}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{3}\kappa_{02}^{-2} + 2\kappa_{23}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{03}\kappa_{02}^{-3} - 30\kappa_{23}\kappa_{12}\kappa_{11}\kappa_{02}^{-1} \\
&+ 18\kappa_{23}\kappa_{11}^{2}\kappa_{03}\kappa_{02}^{-2} + 6\kappa_{22}^{3}\kappa_{20}^{-1}\kappa_{02}^{-1} + 21\kappa_{22}^{3}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{02}^{-2} \\
&+ 9\kappa_{22}^{3}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-3} - 42\kappa_{22}^{2}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}\kappa_{02}^{-2} \\
&+ 21\kappa_{22}^{2}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-3} - 12\kappa_{22}^{2}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{02}^{-1} - 36\kappa_{22}^{2}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-3} \\
&+ \frac{27}{2}\kappa_{22}^{2}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-4} + 105\kappa_{22}^{2}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{02}^{-2} \\
&+ 54\kappa_{22}^{2}\kappa_{20}^{-3}\kappa_{11}^{6}\kappa_{02}^{-3} + 3\kappa_{22}^{2}\kappa_{04}\kappa_{02}^{-2} - 21\kappa_{22}^{2} - 12\kappa_{22}\kappa_{21}^{2}\kappa_{20}^{-1} \\
&- 42\kappa_{22}\kappa_{21}^{2}\kappa_{20}^{-2}\kappa_{11}^{2}\kappa_{02}^{-1} - 18\kappa_{22}\kappa_{21}^{2}\kappa_{20}^{-3}\kappa_{11}^{4}\kappa_{02}^{-2} \\
&+ 66\kappa_{22}\kappa_{21}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}\kappa_{02}^{-1} - 24\kappa_{22}\kappa_{21}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{03}\kappa_{02}^{-2} \\
&+ 60\kappa_{22}\kappa_{21}\kappa_{20}^{-2}\kappa_{12}\kappa_{11}^{3}\kappa_{02}^{-2} - 12\kappa_{22}\kappa_{21}\kappa_{03}\kappa_{02}^{-1} - 24\kappa_{22}\kappa_{20}\kappa_{02} \\
&+ \frac{15}{2}\kappa_{22}\kappa_{20}^{-1}\kappa_{15}\kappa_{11}^{3}\kappa_{02}^{-3} + \frac{99}{2}\kappa_{22}\kappa_{20}^{-1}\kappa_{13}^{2}\kappa_{11}^{2}\kappa_{02}^{-3} \\
&- \frac{87}{2}\kappa_{22}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-4} - 96\kappa_{22}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-2} - 42\kappa_{22}\kappa_{20}^{-1}\kappa_{12}^{2}\kappa_{11}^{2}\kappa_{02}^{-2} \\
&+ 30\kappa_{22}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{3}\kappa_{03}\kappa_{02}^{-3} - \frac{3}{2}\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{06}\kappa_{02}^{-4} + 9\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{04}^{2}\kappa_{02}^{-5} \\
&+ 30\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-3} - 6\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{03}^{2}\kappa_{02}^{-4} - 324\kappa_{22}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&- 180\kappa_{22}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{5}\kappa_{02}^{-3} - 18\kappa_{22}\kappa_{20}^{-2}\kappa_{12}^{2}\kappa_{11}^{4}\kappa_{02}^{-3} \\
&+ 54\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{6}\kappa_{04}\kappa_{02}^{-4} - 12\kappa_{22}\kappa_{20}^{-2}\kappa_{11}^{6}\kappa_{02}^{-2} + 108\kappa_{22}\kappa_{20}^{-3}\kappa_{11}^{8}\kappa_{02}^{-3} \\
&+ 6\kappa_{22}\kappa_{15}\kappa_{11}\kappa_{02}^{-2} + 18\kappa_{22}\kappa_{13}^{2}\kappa_{02}^{-2} - 27\kappa_{22}\kappa_{13}\kappa_{11}\kappa_{04}\kappa_{02}^{-3} \\
&+ 108\kappa_{22}\kappa_{13}\kappa_{11}\kappa_{02}^{-1} - 12\kappa_{22}\kappa_{12}^{2}\kappa_{02}^{-1} + 24\kappa_{22}\kappa_{12}\kappa_{11}\kappa_{03}\kappa_{02}^{-2} - \frac{3}{2}\kappa_{22}\kappa_{11}^{2}\kappa_{06}\kappa_{02}^{-3} \\
&+ \frac{15}{2}\kappa_{22}\kappa_{11}^{2}\kappa_{04}^{2}\kappa_{02}^{-4} - 42\kappa_{22}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-2} - 6\kappa_{22}\kappa_{11}^{2}\kappa_{03}^{2}\kappa_{02}^{-3} + 252\kappa_{22}\kappa_{11}^{2} \\
&+ 24\kappa_{21}^{2}\kappa_{20}^{-1}\kappa_{13}\kappa_{11}\kappa_{02}^{-1} - 9\kappa_{21}^{2}\kappa_{20}^{-1}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-2} + 72\kappa_{21}^{2}\kappa_{20}^{-1}\kappa_{11}^{2} \\
&+ 24\kappa_{21}^{2}\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-2} - \frac{15}{2}\kappa_{21}^{2}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-3} - 36\kappa_{21}^{2}\kappa_{20}^{-2}\kappa_{11}^{4}\kappa_{02}^{-1} \\
&- 36\kappa_{21}^{2}\kappa_{20}^{-3}\kappa_{11}^{6}\kappa_{02}^{-2} - 12\kappa_{21}\kappa_{20}^{-1}\kappa_{14}\kappa_{11}^{3}\kappa_{02}^{-2} - 84\kappa_{21}\kappa_{20}^{-1}\kappa_{13}\kappa_{12}\kappa_{11}^{2}\kappa_{02}^{-2} \\
&+ 30\kappa_{21}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-3} - 72\kappa_{21}\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{3}\kappa_{02}^{-1} - 72\kappa_{21}\kappa_{20}^{-1}\kappa_{11}^{4}\kappa_{03}\kappa_{02}^{-2} \\
&+ 144\kappa_{21}\kappa_{20}^{-2}\kappa_{12}\kappa_{11}^{5}\kappa_{02}^{-2} - 18\kappa_{21}\kappa_{14}\kappa_{11}\kappa_{02}^{-1} - 18\kappa_{21}\kappa_{13}\kappa_{12}\kappa_{02}^{-1} \\
&+ 36\kappa_{21}\kappa_{13}\kappa_{11}\kappa_{03}\kappa_{02}^{-2} + 9\kappa_{21}\kappa_{12}\kappa_{11}\kappa_{04}\kappa_{02}^{-2} - 72\kappa_{21}\kappa_{12}\kappa_{11} \\
&+ 6\kappa_{21}\kappa_{11}^{2}\kappa_{05}\kappa_{02}^{-2} - 15\kappa_{21}\kappa_{11}^{2}\kappa_{04}\kappa_{03}\kappa_{02}^{-3} + 72\kappa_{21}\kappa_{11}^{2}\kappa_{03}\kappa_{02}^{-1} - 6\kappa_{20}^{2}\kappa_{02}^{2} \\
&+ 24\kappa_{20}\kappa_{13}\kappa_{11} - 6\kappa_{20}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-1} + 96\kappa_{20}\kappa_{11}^{2}\kappa_{02} + 18\kappa_{20}^{-1}\kappa_{15}\kappa_{11}^{5}\kappa_{02}^{-3} \\
&+ 6\kappa_{20}^{-1}\kappa_{14}\kappa_{12}\kappa_{11}^{4}\kappa_{02}^{-3} + 164\kappa_{20}^{-1}\kappa_{13}^{2}\kappa_{11}^{4}\kappa_{02}^{-3} \\
&+ 30\kappa_{20}^{-1}\kappa_{13}\kappa_{12}^{2}\kappa_{11}^{3}\kappa_{02}^{-3} - 108\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{5}\kappa_{04}\kappa_{02}^{-4} \\
&+ 456\kappa_{20}^{-1}\kappa_{13}\kappa_{11}^{5}\kappa_{02}^{-2} - \frac{21}{2}\kappa_{20}^{-1}\kappa_{12}^{2}\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-4} - 36\kappa_{20}^{-1}\kappa_{12}^{2}\kappa_{11}^{4}\kappa_{02}^{-2} \\
&+ 72\kappa_{20}^{-1}\kappa_{12}\kappa_{11}^{5}\kappa_{03}\kappa_{02}^{-3} - 3\kappa_{20}^{-1}\kappa_{11}^{6}\kappa_{06}\kappa_{02}^{-4} \\
&+ 18\kappa_{20}^{-1}\kappa_{11}^{6}\kappa_{04}^{2}\kappa_{02}^{-5} - 114\kappa_{20}^{-1}\kappa_{11}^{6}\kappa_{04}\kappa_{02}^{-3} - 12\kappa_{20}^{-1}\kappa_{11}^{6}\kappa_{03}^{2}\kappa_{02}^{-4}
\end{aligned}
$$

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\begin{aligned}
&+ 456\kappa_{20}^{-1}\kappa_{11}^{6}\kappa_{02}^{-1} - 216\kappa_{20}^{-2}\kappa_{13}\kappa_{11}^{7}\kappa_{02}^{-3} - 36\kappa_{20}^{-2}\kappa_{12}^{2}\kappa_{11}^{6}\kappa_{02}^{-3} \\
&+ 54\kappa_{20}^{-2}\kappa_{11}^{8}\kappa_{04}\kappa_{02}^{-4} - 294\kappa_{20}^{-2}\kappa_{11}^{8}\kappa_{02}^{-2} + 72\kappa_{20}^{-3}\kappa_{11}^{10}\kappa_{02}^{-3} \\
&- \frac{1}{2}\kappa_{17}\kappa_{11}^{3}\kappa_{02}^{-3} - \frac{27}{2}\kappa_{15}\kappa_{13}\kappa_{11}^{2}\kappa_{02}^{-3} + \frac{21}{4}\kappa_{15}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-4} - 18\kappa_{15}\kappa_{11}^{3}\kappa_{02}^{-2} \\
&+ 24\kappa_{14}\kappa_{12}\kappa_{11}^{2}\kappa_{02}^{-2} - 10\kappa_{14}\kappa_{11}^{3}\kappa_{03}\kappa_{02}^{-3} - 24\kappa_{13}^{3}\kappa_{11}\kappa_{02}^{-3} \\
&+ \frac{153}{4}\kappa_{13}^{2}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-4} - 108\kappa_{13}^{2}\kappa_{11}^{2}\kappa_{02}^{-2} + 36\kappa_{13}\kappa_{12}^{2}\kappa_{11}\kappa_{02}^{-2} - 54\kappa_{13}\kappa_{12}\kappa_{11}^{2}\kappa_{03}\kappa_{02}^{-3} \\
&+ 3\kappa_{13}\kappa_{11}^{3}\kappa_{06}\kappa_{02}^{-4} - 18\kappa_{13}\kappa_{11}^{3}\kappa_{04}^{2}\kappa_{02}^{-5} + 80\kappa_{13}\kappa_{11}^{3}\kappa_{04}\kappa_{02}^{-3} \\
&+ 12\kappa_{13}\kappa_{11}^{3}\kappa_{03}^{2}\kappa_{02}^{-4} - 264\kappa_{13}\kappa_{11}^{3}\kappa_{02}^{-1} - 15\kappa_{12}^{2}\kappa_{11}^{2}\kappa_{04}\kappa_{02}^{-3} \\
&+ 72\kappa_{12}^{2}\kappa_{11}^{2}\kappa_{02}^{-1} - 6\kappa_{12}\kappa_{11}^{3}\kappa_{05}\kappa_{02}^{-3} + 21\kappa_{12}\kappa_{11}^{3}\kappa_{04}\kappa_{03}\kappa_{02}^{-4} - 72\kappa_{12}\kappa_{11}^{3}\kappa_{03}\kappa_{02}^{-2} \\
&+ \frac{1}{16}\kappa_{11}^{4}\kappa_{08}\kappa_{02}^{-4} - \frac{9}{8}\kappa_{11}^{4}\kappa_{06}\kappa_{04}\kappa_{02}^{-5} + 3\kappa_{11}^{4}\kappa_{06}\kappa_{02}^{-3} + 2\kappa_{11}^{4}\kappa_{05}\kappa_{03}\kappa_{02}^{-4} \\
&+ \frac{21}{8}\kappa_{11}^{4}\kappa_{04}^{3}\kappa_{02}^{-6} - \frac{29}{2}\kappa_{11}^{4}\kappa_{04}^{2}\kappa_{02}^{-4} - \frac{9}{2}\kappa_{11}^{4}\kappa_{04}\kappa_{03}^{2}\kappa_{02}^{-5} \\
&+ 66\kappa_{11}^{4}\kappa_{04}\kappa_{02}^{-2} + 12\kappa_{11}^{4}\kappa_{03}^{2}\kappa_{02}^{-3} - 324\kappa_{11}^{4}\big) \,.
\end{aligned}
$$