Characterization of Bernstein Coefficients

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Abstract

It is well known that a polynomial of degree n can be uniquely written as a linear combination of the Bernstein basis polynomials of degree n. The coefficients of the combination (Bernstein coefficients) depend on the interval over which the Bernstein basis polynomials are formed. In this paper, we study various properties of Bernstein coefficients as viewed as functions in the two end points of the interval.

1 Introduction

Bernstein bases [1] play a fundamental role in various areas such as approximation theory [11, 3, 16, 4], interval mathematics [2, 12, 13, 14, 10, 15, 8] geometric modeling [6, 7, 5]. Thus further understanding of the properties of Bernstein bases often lead to progress in those areas.

It is well known that the set of Bernstein basis polynomials of degree n forms a basis of the vector space of all the polynomials of degree at most n. Thus, every polynomial of degree at most n can be uniquely written as a linear combination of the Bernstein basis polynomials of degree n. We will call the coefficients in the linear combination *Bernstein coefficients*. For a given polynomial, its Bernstein coefficients depends on the interval over which the Bernstein basis polynomials are formed. In this paper, we study various properties of Bernstein coefficients as viewed as functions in the two end points of the interval. Let's make this goal more precise.

Definition 1 (Bernstein Basis Polynomial) *The i-th* Bernstein basis polynomial *of degree n over the interval $[\text{L}, \text{U}]$, written as $B_i^n(\text{L}, \text{U}; x)$, is defined by*

$$B_i^n(\text{L}, \text{U}; x) = \binom{n}{i} \frac{(x - \text{L})^i (\text{U} - x)^{n-i}}{(\text{U} - \text{L})^n}. \ \Box$$

Definition 2 (Bernstein Coefficients) *Let f be a polynomial of degree n. The i-th* Bernstein coefficients *of f over the range $[\text{L}, \text{U}]$, written as $b_i(\text{L}, \text{U})$, is uniquely defined by*

$$f(x) = \sum_{i=0}^{n} \ b_i(\text{L}, \text{U}) \ \ B_i^n(\text{L}, \text{U}; x). \ \Box$$

It turns out that b_i is a polynomial in L and U, of degree i in U and $n-i$ in L. The goal of this paper is study the Bernstein coefficients b_i's as viewed as polynomials in L and U. In particular, we will ask/answer the following questions:

Section 3: What does $b_i(\mathrm{L},\mathrm{U})$ look like in the power basis? In other words, find $C_{i,p,q}$ such that

$$b_i(\mathrm{L},\mathrm{U}) = \sum_{p,q} C_{i,p,q}\ U^p L^q.$$

Actually, it is easy to get an expression for $C_{i,p,q}$ and indeed at least two expressions are already known as a folklore (see for instance [9]). But they are not so elegant from my point of view (not symmetric), and thus the goal is to find a more elegant (symmetric) expression, which turns out to be a more difficult task. One such expression is given in Theorem 1.

Section 4: What does $b_i(\mathrm{L},\mathrm{U})$ look like in the Bernstein basis (over some interval $[\mathrm{L}',\mathrm{U}']$)? In other words, find $D_{i,\mu,\nu}$ such that

$$b_i(\mathrm{L},\mathrm{U}) = \sum_{\mu,\nu} D_{i,\mu,\nu}\ B^i_\mu(\mathrm{L}',\mathrm{U}';\mathrm{U})\ B^{n-i}_\nu(\mathrm{L}',\mathrm{U}';\mathrm{L}).$$

Again, we are looking for an elegant expression for it. One very simple expression is given in Theorem 2. We emphasize that the symmetric expression for $C_{i,p,q}$ played an essential role in finding the simple form for $D_{i,\mu,\nu}$.

Section 5: Are there any interesting/beautiful relationships among the Bernstein coefficients? Can one coefficient be expressed in terms of others elegantly? A nice relationship is given in Theorems 3 and 4. Further, we are curious about how the surfaces (graphs) defined Bernstein coefficients relate/intersect with each other. Several interesting geometric characterizations are given in Corollaries 5, 6, and 7 and Example 1.

2 Review of Basic Binomial Identities

In the subsequent sections, we will frequently use the following well known binomial identities.

Proposition 1 (Basic Binomial Identities)

(A) $\binom{n-1}{i-1}\frac{n}{i} = \binom{n}{i}$. *(Absorption Rule)*

(B) $\binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n}{i}$. *(Addition Rule)*

(C) $\sum_{p+q=k}\binom{r}{p}\binom{s}{q} = \binom{r+s}{k}$. *(Vandermonde Convolution Rule)*

3 Bernstein Coefficients in Power Bases

In this section, we answer the question: What does $b_i(\mathrm{L},\mathrm{U})$ look like in the power basis? We give an elegant answer in the following theorem.

Theorem 1 (Bernstein Coefficients in Power Bases) *Let*

$$f(x) = \sum_{k=0}^{n} a_k x^k.$$

Then we have

$$b_i = \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} \, a_{p+q} \, \mathrm{U}^p \mathrm{L}^q. \;\Box$$

Proof. The proof given here is "constructive" in that it shows how to "discover" the theorem, rather than just verifying it. The main technique is that of generating functions. As often is the case, the main difficulty was in devising a suitable generating function.

The goal is to find an "elegant" expression for b_i such that

$$\sum_i b_i B_i^n = \sum_j a_j x^j.$$

Let

$$b_i = \sum_j T_{ij} a_j. \tag{1}$$

Then we have

$$\sum_i b_i B_i^n = \sum_i \sum_j T_{ij} a_j B_i^n = \sum_j a_j \sum_i T_{ij} B_i^n = \sum_j a_j x^j.$$

Thus, it is sufficient to find a nice expression for T_{ij} such that

$$\forall j \;\left[\; \sum_i T_{ij} B_i^n = x^j \;\right]. \tag{2}$$

Now we eliminate the universal quantification by introducing a new indeterminate y, that is, we take the approach of generating function. Formula (2) is equivalent to

$$\sum_j \left[\sum_i T_{ij} B_i^n\right] \binom{n}{j} y^{n-j} = \sum_j x^j \; \binom{n}{j} y^{n-j}.$$

The reason for introducing the binomial coefficient $\binom{n}{j}$ is that we can immediately simplify the right hand side as

$$\sum_j x^j \binom{n}{j} y^{n-j} = \sum_j \binom{n}{j} x^j y^{n-j} = (x+y)^n.$$

Thus, we have

$$\sum_j \left[\sum_i T_{ij} B_i^n\right] \binom{n}{j} y^{n-j} = (x+y)^n.$$

Recalling the definition of B_i^n, we have

$$\sum_j \sum_i T_{ij} \binom{n}{i} \frac{(x-\mathrm{L})^i (\mathrm{U}-x)^{n-i}}{(\mathrm{U}-\mathrm{L})^n} \binom{n}{j} y^{n-j} = (x+y)^n.$$

By moving the denominator $(\mathrm{U}-\mathrm{L})^n$ to the right hand side, we get

$$\sum_j \sum_i T_{ij} \binom{n}{i} (x-\mathrm{L})^i (\mathrm{U}-x)^{n-i} \binom{n}{j} y^{n-j} = (x+y)^n (\mathrm{U}-\mathrm{L})^n.$$

Rearranging the left hand side by collecting terms with same indices as much as possible, we get

$$\sum_i \binom{n}{i} (x - \mathrm{L})^i (\mathrm{U} - x)^{n-i} \sum_j T_{ij} \binom{n}{j} y^{n-j} = (x + y)^n (\mathrm{U} - \mathrm{L})^n. \tag{3}$$

Now we make a tricky[1] step of expanding the right hand side, "imitating" the form of the left hand side.

$$\begin{aligned}
(x+y)^n(\mathrm{U}-\mathrm{L})^n &= [(x+y)(\mathrm{U}-\mathrm{L})]^n \\
&= [x\mathrm{U} - x\mathrm{L} + y\mathrm{U} - y\mathrm{L}]^n \\
&= [x\mathrm{U} - x\mathrm{L} + y\mathrm{U} - y\mathrm{L} + xy - xy + \mathrm{UL} - \mathrm{UL}]^n \\
&= [(x-\mathrm{L})(y+\mathrm{U}) + (\mathrm{U}-x)(y+\mathrm{L})]^n \\
&= \sum_i \binom{n}{i} [(x-\mathrm{L})(y+\mathrm{U})]^i \, [(\mathrm{U}-x)(y+\mathrm{L})]^{n-i} \\
&= \sum_i \binom{n}{i} (x-\mathrm{L})^i (\mathrm{U}-x)^{n-i} \; (y+\mathrm{U})^i (y+\mathrm{L})^{n-i}.
\end{aligned}$$

Recalling Formula (3), we have

$$\sum_i \binom{n}{i} (x - \mathrm{L})^i (\mathrm{U} - x)^{n-i} \; \sum_j T_{ij} \binom{n}{j} y^{n-j} = \sum_i \binom{n}{i} (x-\mathrm{L})^i (\mathrm{U}-x)^{n-i} \; (y+\mathrm{U})^i (y+\mathrm{L})^{n-i}.$$

By comparing the both sides, we see immediately

$$\sum_j T_{ij} \binom{n}{j} y^{n-j} = (y + \mathrm{U})^i (y + \mathrm{L})^{n-i}. \tag{4}$$

By expanding the right hand side, we get

$$\begin{aligned}
(y+\mathrm{U})^i (y+\mathrm{L})^{n-i} &= \sum_p \binom{i}{p} y^{i-p} \mathrm{U}^p \sum_q \binom{n-i}{q} y^{n-i-q} \mathrm{L}^q \\
&= \sum_{p,q} \binom{i}{p} \binom{n-i}{q} \mathrm{U}^p \mathrm{L}^q y^{n-(p+q)} \\
&= \sum_j \sum_{p+q=j} \binom{i}{p} \binom{n-i}{q} \mathrm{U}^p \mathrm{L}^q \; y^{n-j}.
\end{aligned}$$

Recalling Formula (4), we have

$$\sum_j T_{ij} \binom{n}{j} y^{n-j} = \sum_j \sum_{p+q=j} \binom{i}{p} \binom{n-i}{q} \mathrm{U}^p \mathrm{L}^q \; y^{n-j}.$$

By comparing the both sides, we see

$$T_{ij} = \sum_{p+q=j} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{j}} \mathrm{U}^p \mathrm{L}^q.$$

[1] I call this step tricky because I found this particular one after numerous trial-and-error search.

Finally, by Formula (1), we have

$$b_i = \sum_j T_{ij} a_j = \sum_j \sum_{p+q=j} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{j}} \mathrm{U}^p \mathrm{L}^q a_j = \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} \mathrm{U}^p \mathrm{L}^q a_{p+q}. \ \Box$$

4 Bernstein Coefficients in Bernstein Bases

In this section, we answer the question: What does $b_i(\mathrm{L}, \mathrm{U})$ look like in the Bernstein basis (over some interval $[\mathrm{L}', \mathrm{U}']$)? We give a very simple expression for it in the following theorem.

Theorem 2 (Bernstein Coefficients in Bernstein Bases)

$$b_i(\mathrm{L}, \mathrm{U}) \;=\; \sum_{\mu,\nu} b_{\mu+\nu}(\mathrm{L}', \mathrm{U}') \;\; B^i_\mu(\mathrm{L}', \mathrm{U}'; \mathrm{U}) \;\; B^{n-i}_\nu(\mathrm{L}', \mathrm{U}'; \mathrm{L}). \ \Box$$

Proof. From Theorem 1, we have

$$b_i(\mathrm{L}, \mathrm{U}) = \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} \mathrm{U}^p \mathrm{L}^q a_{p+q}.$$

Now, we need to represent $b_i(\mathrm{L}, \mathrm{U})$ (viewed as a polynomial in the variables L and U) in the Bernstein basis over the interval $[\mathrm{L}', \mathrm{U}']$. We will do so first for the variable L. Note

$$b_i(\mathrm{L}, \mathrm{U}) = \sum_{q=0}^{n-i} \bar{a}_q \ \mathrm{L}^q$$

where

$$\bar{a}_q = \sum_p \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} \mathrm{U}^p a_{p+q}. \tag{5}$$

Now let

$$b_i(\mathrm{L}, \mathrm{U}) = \sum_\nu \bar{b}_\nu B^{n-i}_\nu(\mathrm{L}', \mathrm{U}'; \mathrm{L}). \tag{6}$$

From Theorem 1 and Formula (5), we have

$$\begin{aligned}
\bar{b}_\nu &= \sum_{\bar{p},\bar{q}} \frac{\binom{\nu}{\bar{p}}\binom{n-i-\nu}{\bar{q}}}{\binom{n-i}{\bar{p}+\bar{q}}} \mathrm{U}'^{\bar{p}} \mathrm{L}'^{\bar{q}} \;\; \bar{a}_{\bar{p}+\bar{q}} \\
&= \sum_{\bar{p},\bar{q}} \frac{\binom{\nu}{\bar{p}}\binom{n-i-\nu}{\bar{q}}}{\binom{n-i}{\bar{p}+\bar{q}}} \mathrm{U}'^{\bar{p}} \mathrm{L}'^{\bar{q}} \;\; \sum_p \frac{\binom{i}{p}\binom{n-i}{\bar{p}+\bar{q}}}{\binom{n}{p+\bar{p}+\bar{q}}} \mathrm{U}^p a_{p+\bar{p}+\bar{q}} \\
&= \sum_p \sum_{\bar{p},\bar{q}} \frac{\binom{\nu}{\bar{p}}\binom{n-i-\nu}{\bar{q}}\binom{i}{p}}{\binom{n}{p+\bar{p}+\bar{q}}} \mathrm{U}'^{\bar{p}} \mathrm{L}'^{\bar{q}} a_{p+\bar{p}+\bar{q}} \ \mathrm{U}^p \\
&= \sum_{p=0}^{i} \hat{a}_p \mathrm{U}^p
\end{aligned}$$

where

$$\hat{a}_p = \sum_{\bar{p},\bar{q}} \frac{\binom{\nu}{\bar{p}}\binom{n-i-\nu}{\bar{q}}\binom{i}{p}}{\binom{n}{p+\bar{p}+\bar{q}}} \mathrm{U}'^{\bar{p}}\mathrm{L}'^{\bar{q}} a_{p+\bar{p}+\bar{q}}. \tag{7}$$

Next, we will work on the variable U. For this let,

$$\bar{b}_\nu = \sum_\mu \hat{b}_\mu \; B^i_\mu(\mathrm{L}', \mathrm{U}'; \mathrm{U}). \tag{8}$$

From Theorem 1 and Formula (7), we have

$$\begin{aligned}
\hat{b}_\mu &= \sum_{\hat{p},\hat{q}} \frac{\binom{\mu}{\hat{p}}\binom{i-\mu}{\hat{q}}}{\binom{i}{\hat{p}+\hat{q}}} \mathrm{U}'^{\hat{p}}\mathrm{L}'^{\hat{q}} \;\; \hat{a}_{\hat{p}+\hat{q}} \\
&= \sum_{\hat{p},\hat{q}} \frac{\binom{\mu}{\hat{p}}\binom{i-\mu}{\hat{q}}}{\binom{i}{\hat{p}+\hat{q}}} \mathrm{U}'^{\hat{p}}\mathrm{L}'^{\hat{q}} \;\; \sum_{\bar{p},\bar{q}} \frac{\binom{\nu}{\bar{p}}\binom{n-i-\nu}{\bar{q}}\binom{i}{\hat{p}+\hat{q}}}{\binom{n}{\hat{p}+\hat{q}+\bar{p}+\bar{q}}} \mathrm{U}'^{\bar{p}}\mathrm{L}'^{\bar{q}} a_{\hat{p}+\hat{q}+\bar{p}+\bar{q}} \\
&= \sum_{\hat{p},\hat{q},\bar{p},\bar{q}} \frac{\binom{\mu}{\hat{p}}\binom{i-\mu}{\hat{q}}\binom{\nu}{\bar{p}}\binom{n-i-\nu}{\bar{q}}}{\binom{n}{\hat{p}+\hat{q}+\bar{p}+\bar{q}}} \mathrm{U}'^{\hat{p}+\bar{p}}\mathrm{L}'^{\hat{q}+\bar{q}} \;\; a_{\hat{p}+\hat{q}+\bar{p}+\bar{q}} \\
&= \sum_{p,q} \sum_{\hat{p}+\bar{p}=p,\hat{q}+\bar{q}=q} \frac{\binom{\mu}{\hat{p}}\binom{\nu}{\bar{p}}\binom{i-\mu}{\hat{q}}\binom{n-i-\nu}{\bar{q}}}{\binom{n}{p+q}} \mathrm{U}'^{p}\mathrm{L}'^{q} \;\; a_{p+q} \\
&= \sum_{p,q} \left[\sum_{\hat{p}+\bar{p}=p} \binom{\mu}{\hat{p}}\binom{\nu}{\bar{p}}\right] \left[\sum_{\hat{q}+\bar{q}=q} \binom{i-\mu}{\hat{q}}\binom{n-i-\nu}{\bar{q}}\right] \frac{\mathrm{U}'^{p}\mathrm{L}'^{q}}{\binom{n}{p+q}} a_{p+q} \\
&\quad \textit{By applying Proposition 1 (C),} \\
&= \sum_{p,q} \binom{\mu+\nu}{p}\binom{n-(\mu+\nu)}{q} \frac{\mathrm{U}'^{p}\mathrm{L}'^{q}}{\binom{n}{p+q}} a_{p+q} \\
&= \sum_{p,q} \frac{\binom{\mu+\nu}{p}\binom{n-(\mu+\nu)}{q}}{\binom{n}{p+q}} \mathrm{U}'^{p}\mathrm{L}'^{q} a_{p+q} \\
&\quad \textit{From Theorem 1,} \\
&= b_{\mu+\nu}(\mathrm{L}', \mathrm{U}').
\end{aligned}$$

Recalling Formulas (6) and (8), we finally have

$$\begin{aligned}
b_i(\mathrm{L}, \mathrm{U}) &= \sum_\nu \;\; \bar{b}_\nu \;\; B^{n-i}_\nu(\mathrm{L}', \mathrm{U}'; \mathrm{L}) \\
&= \sum_{\mu,\nu} \;\; \hat{b}_\mu \;\; B^i_\mu(\mathrm{L}', \mathrm{U}'; \mathrm{U}) \;\; B^{n-i}_\nu(\mathrm{L}', \mathrm{U}'; \mathrm{L}) \\
&= \sum_{\mu,\nu} \;\; b_{\mu+\nu}(\mathrm{L}', \mathrm{U}') \;\; B^i_\mu(\mathrm{L}', \mathrm{U}'; \mathrm{U}) \;\; B^{n-i}_\nu(\mathrm{L}', \mathrm{U}'; \mathrm{L}). \;\square
\end{aligned}$$

Corollary 1 (When Lower Bound is Zero)

$$b_i(0, \mathrm{U}) \;\; = \;\; \sum_\mu \;\; b_\mu(0,1) \;\; B^i_\mu(0, 1; \mathrm{U}).\square$$

Proof. From Theorem 2, we have

$$\begin{aligned} b_i(0,\text{U}) &= \sum_{\mu,\nu} b_{\mu+\nu}(0,1)\ \ B^i_\mu(0,1;\text{U})\ \ B^{n-i}_\nu(0,1;0) \\ &= \sum_{\mu,\nu} b_{\mu+\nu}(0,1)\ \ B^i_\mu(0,1;\text{U})\ \ \delta_{\nu,0} \\ &= \sum_{\mu} b_{\mu}(0,1)\ \ B^i_\mu(0,1;\text{U}).\ \square \end{aligned}$$

Corollary 2 (When Upper Bound is One)

$$b_i(\text{L},1) \ = \ \sum_{\nu} \ b_{i+\nu}(0,1)\ \ B^{n-i}_\nu(0,1;\text{L}).\square$$

Proof. From Theorem 2, we have

$$\begin{aligned} b_i(\text{L},1) &= \sum_{\mu,\nu} b_{\mu+\nu}(0,1)\ \ B^i_\mu(0,1;1)\ \ B^{n-i}_\nu(0,1;\text{L}) \\ &= \sum_{\mu,\nu} b_{\mu+\nu}(0,1)\ \ \delta_{\mu,i}\ \ B^{n-i}_\nu(0,1;\text{L}) \\ &= \sum_{\nu} b_{i+\nu}(0,1)\ \ B^{n-i}_\nu(0,1;\text{L}).\ \square \end{aligned}$$

5 Relationship among Bernstein Coefficients

In this section, we ask whether there are any interesting/beautiful relationships among the Bernstein coefficients. In particular, we ask whether one coefficient be expressed in terms of others elegantly. A nice relationship is given in Theorems 3 and 4. Further, we provide an interesting geometric interpretation in Corollaries 5, 6, and 7 and Example 1.

Theorem 3 (Relationship among Bernstein Coefficients)

(A) $b_n(\text{L},\text{U}) = f(\text{U})$.

$b_{i-1} = b_i - \frac{(\text{U}-\text{L})}{i}\frac{\partial}{\partial \text{U}} b_i$.

(B) $b_0(\text{L},\text{U}) = f(\text{L})$.

$b_{i+1} = b_i - \frac{(\text{L}-\text{U})}{n-i}\frac{\partial}{\partial \text{L}} b_i$. $\square$

Proof.
(A): First note

$$\begin{aligned} f(\text{U}) &= \sum_{i=0}^{n} b_i(\text{L},\text{U}) B^n_i(\text{L},\text{U};\text{U}) \\ &= \sum_{i=0}^{n} b_i(\text{L},\text{U}) \binom{n}{i} \frac{(\text{U}-\text{L})^i(\text{U}-\text{U})^{n-i}}{(\text{U}-\text{L})^n} \\ &= b_n(\text{L},\text{U}). \end{aligned}$$

Next follow the repeated rewriting of the right hand side into the left hand side, using Theorem 1 and Proposition 1.

$$
\begin{aligned}
\textbf{R.H.S} &= b_i - \frac{(\mathrm{U}-\mathrm{L})}{i}\frac{\partial}{\partial \mathrm{U}} b_i \\
&\quad \textit{By Theorem 1,} \\
&= \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q - \frac{(\mathrm{U}-\mathrm{L})}{i}\frac{\partial}{\partial \mathrm{U}} \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q \\
&\quad \textit{By carrying out the differentiation,} \\
&= \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q - \frac{(\mathrm{U}-\mathrm{L})}{i} \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ p\mathrm{U}^{p-1}\mathrm{L}^q \\
&\quad \textit{Moving to the constant factors } p \textit{ and } i \textit{ to the suitable place,} \\
&= \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q - (\mathrm{U}-\mathrm{L}) \sum_{p,q} \frac{\frac{p}{i}\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^{p-1}\mathrm{L}^q \\
&\quad \textit{Applying Proposition 1 (A),} \\
&= \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q - (\mathrm{U}-\mathrm{L}) \sum_{p,q} \frac{\binom{i-1}{p-1}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^{p-1}\mathrm{L}^q \\
&\quad \textit{Multiplying out } (\mathrm{U}-\mathrm{L}), \\
&= \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q - \sum_{p,q} \frac{\binom{i-1}{p-1}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q + \sum_{p,q} \frac{\binom{i-1}{p-1}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^{p-1}\mathrm{L}^{q+1} \\
&\quad \textit{Adjusting the indices in the third summation: } p-1 \to p \textit{ and } q+1 \to q, \\
&= \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q - \sum_{p,q} \frac{\binom{i-1}{p-1}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q + \sum_{p,q} \frac{\binom{i-1}{p}\binom{n-i}{q-1}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q \\
&\quad \textit{Collecting the coefficients together,} \\
&= \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q} - \binom{i-1}{p-1}\binom{n-i}{q} + \binom{i-1}{p}\binom{n-i}{q-1}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q \\
&\quad \textit{Applying Proposition 1 (B) on the first two binomial terms,} \\
&= \sum_{p,q} \frac{\binom{i-1}{p}\binom{n-i}{q} + \binom{i-1}{p}\binom{n-i}{q-1}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q \\
&\quad \textit{Applying Proposition 1 (B) again on the resulting two binomial terms,} \\
&= \sum_{p,q} \frac{\binom{i-1}{p}\binom{n-i+1}{q}}{\binom{n}{p+q}} a_{p+q}\ \mathrm{U}^p\mathrm{L}^q \\
&\quad \textit{Recalling Theorem 1,} \\
&= b_{i-1} \\
&= \textbf{L.H.S.}
\end{aligned}
$$

(B): This can be proved in the exactly same way as for **(A)**. □

Corollary 3 (Diagonal) *For $0 \le i \le n$, we have*

$$b_i(z, z) = f(z). \ \square$$

Proof. From Theorem 3 (A), we have

$$i(b_i(z,z) - b_{i-1}(z,z)) = (z-z)\frac{\partial b_i}{\partial \mathrm{U}}(z,z) = 0.$$

Thus for $i \geq 1$, we have $b_i(z,z) = b_{i-1}(z,z)$. So, by induction, we have $b_n(z,z) = b_{n-1}(z,z) = \cdots = b_0(z,z)$. Recalling Theorem 3 (B), we have $b_0(z,z) = f(z)$. Thus $b_i(z,z) = f(z)$ for every i. □

Corollary 4 (Relationship between Partial Derivatives) *For $1 \leq i \leq n$, we have*

$$\frac{1}{i}\frac{\partial b_i}{\partial \mathrm{U}} = \frac{1}{n-i+1}\frac{\partial b_{i-1}}{\partial \mathrm{L}}. \;\square$$

Proof. From Theorem 3 (A) and (B), we see

$$\begin{aligned}
\frac{1}{i}\frac{\partial b_i}{\partial \mathrm{U}} &= \frac{1}{\mathrm{U}-\mathrm{L}}(b_i - b_{i-1}) \\
&= \frac{1}{\mathrm{L}-\mathrm{U}}(b_{i-1} - b_i) \\
&= \frac{1}{n-(i-1)}\frac{n-(i-1)}{\mathrm{L}-\mathrm{U}}(b_{i-1} - b_i) \\
&= \frac{1}{n-i+1}\frac{\partial b_{i-1}}{\partial \mathrm{L}}. \;\square
\end{aligned}$$

Corollary 5 (Intersection of Surfaces) *Let*

$$\begin{aligned}
D &= \{(\mathrm{L},\mathrm{U}) \mid \mathrm{L} = \mathrm{U}\} \\
E &= \{(\mathrm{L},\mathrm{U}) \mid \frac{\partial b_i}{\partial \mathrm{U}} = 0\} \\
F &= \{(\mathrm{L},\mathrm{U}) \mid \frac{\partial b_{i-1}}{\partial \mathrm{L}} = 0\}.
\end{aligned}$$

Then, $E = F$ and the graphs of b_i and b_{i-1} intersect exactly over $D \cup E$. □

Proof. From Corollary 4, wee see immediately that $E = F$. From Theorem 3, we also see immediately that the graphs of b_i and b_{i-1} intersect exactly over $D \cup E$. □

Corollary 6 (Cross Section for Fixed Lower Bound) *Let L be a fixed constant. Then the curves defined by $b_{i-1}(\mathrm{L},x)$ and $b_i(\mathrm{L},x)$ intersect exactly over L and the roots of $\frac{db_i(\mathrm{L},x)}{dx}$ (extremum points).* □

Proof. Immediate from looking at the cross section of the graph of b_i for a fixed L. □.

Corollary 7 (Cross Section for Fixed Upper Bound) *Let U be a fixed constant. Then the curves defined by $b_{i+1}(x,\mathrm{U})$ and $b_i(x,\mathrm{U})$ intersect exactly over U and the roots of $\frac{db_i(x,\mathrm{U})}{dx}$ (extremum points).* □

Proof. Immediate from looking at the cross section of the graph of b_i for a fixed U. □.

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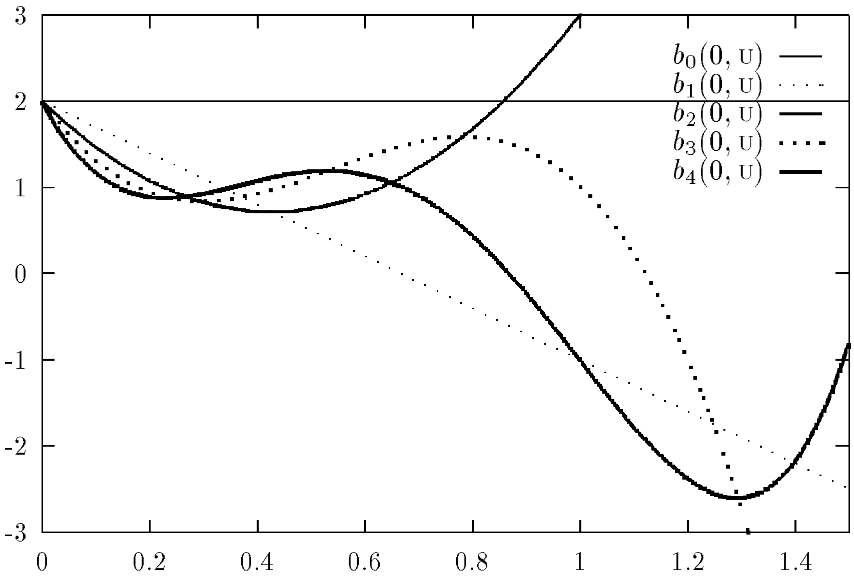

Figure 1: Cross Section at $\text{L} = 0$

Example 1 Let

$$f(x) = \sum_{i=0}^{4} c_i B_i^4(0, 1; x).$$

Then from Corollary 1, we have

$$\begin{aligned}
b_4(0, \text{U}) &= \sum_{i=0}^{4} c_i B_i^4(0, 1, \text{U}) \\
b_3(0, \text{U}) &= \sum_{i=0}^{3} c_i B_i^3(0, 1; \text{U}) \\
b_2(0, \text{U}) &= \sum_{i=0}^{2} c_i B_i^2(0, 1; \text{U}) \\
b_1(0, \text{U}) &= \sum_{i=0}^{1} c_i B_i^1(0, 1; \text{U}) \\
b_0(0, \text{U}) &= \sum_{i=0}^{0} c_i B_i^0(0, 1; \text{U}).
\end{aligned}$$

When $[c_4, c_3, c_2, c_1, c_0] = [-1, 1, 3, -1, 2]$, we get the graphs in Figure 1. Note that $b_{i-1}(0, \text{U})$ passes through all the extremal points of $b_i(0, \text{U})$.

This sheds a new light on the quantities:

$$\max_i c_i \quad \text{and} \quad \min_i c_i.$$

It is well known that these respectively give an upper bound and a lower bound for the range of $f(x)$ over the interval $[0, 1]$. But it is also well known that these bounds are not (always) tight.

Thus, a question naturally arises: *what do they bound tightly?* From Corollary 1 and 6, we see that these are the tight bounds for the ranges of the polynomials $f_n(x), f_{n-1}(x), \ldots, f_0(x)$ over the interval $[0,1]$ where $f_n(x) = f(x)$ and $f_{i-1}(x)$ is the polynomial which intersect $f_i(x)$ over 0 and all the roots of $f_i'(x)$. $\Box$

Theorem 4 (When Lower Bound is Zero) *Let*

$$c_i(\mathrm{U}) = \frac{\mathrm{U}^i}{i!}\, b_i(0, 1/\mathrm{U}).$$

Then we have

$$\frac{d^k}{d\mathrm{U}^k}\, c_i = c_{i-k}. \ \Box$$

Proof. From Theorem 1, we have

$$c_i(\mathrm{U}) = \frac{\mathrm{U}^i}{i!}\, b_i(0, 1/\mathrm{U}) = \frac{\mathrm{U}^i}{i!} \sum_{p,q} \frac{\binom{i}{p}\binom{n-i}{q}}{\binom{n}{p+q}} a_{p+q} \mathrm{U}^{-p} 0^q = \sum_p \frac{1}{p!\,(i-p)!\,\binom{n}{p}} a_p \mathrm{U}^{i-p}.$$

We only need to follow the calculation below.

$$\begin{aligned}
\frac{d^k}{d\mathrm{U}^k} c_i &= \frac{d^k}{d\mathrm{U}^k} \sum_p \frac{1}{p!\,(i-p)!\,\binom{n}{p}} a_p \mathrm{U}^{i-p}. \\
&= \sum_p \frac{(i-p)!}{p!\,(i-p)!\,(i-p-k)!\,\binom{n}{p}} a_p \mathrm{U}^{i-p-k}. \\
&= \sum_p \frac{1}{p!\,(i-k-p)!\,\binom{n}{p}} a_p \mathrm{U}^{i-k-p}. \\
&= c_{i-k}. \ \Box
\end{aligned}$$

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