c̃W10-Creative Sets and Noncappable r.e. T-degrees

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Abstract. A theorem by Shoenfield asserts that a set A is quasiproductive if and only if there exists a recursive function f such that $D_{f(x)} \subseteq A \Leftrightarrow D_{f(x)} \not\subseteq W_x$ for all x. We define a set A to be c̃W10-productive, iff A satisfies the analogue of the latter condition with $W_{f(x)}$ in place of $D_{f(x)}$. Also, A is c̃W10-creative, iff A is r.e. and $\overline{A}$ is c̃W10-productive. Clearly, every quasiproductive set is c̃W10-productive, and every c̃W10-productive set is semiproductive. We show that the Turing degrees of c̃W10-creative sets are precisely the noncappable r.e. Turing degrees; this provides a new and simple characterization of this class of r.e. Turing degrees. In addition we show how this property behaves under the join operator. Namely, if $A \oplus B$ is c̃W10-productive and B is immune, then A is c̃W10-productive. There exist r.e. sets A, B such that neither A nor B is c̃W10-creative but $A \oplus B$ is quasicreative. Finally, within any given noncappable r.e. Turing degrees **a**, **b** we construct r.e. sets A, B such that neither A nor B is c̃W10-creative but $A \oplus B$ is c̃W10-creative.

1. Introduction

In Shoenfield [8] the following result is shown:

Proposition 1.1: *The following two assertions are equivalent:*

(1.1) *A is quasiproductive, that is, there exists a recursive function f such that for all x,*

$$W_x \subseteq A \Leftrightarrow D_{f(x)} \subseteq A \ \ \& \ \ D_{f(x)} \not\subseteq W_x.$$

(1.2) *There exists a recursive function f such that for all x*

$$D_{f(x)} \subseteq A \Leftrightarrow D_{f(x)} \not\subseteq W_x.$$

This result parallels the well-known theorem which states that every productive set is completely productive (Rogers [5], theorem 11.VI). A set A which satisfies the analogue of (1.1), but with $D_{f(x)}$ replaced by $W_{f(x)}$ is called *semiproductive* (Dekker [3]). In general, let '[...]-productive' a any variant of the notion of productivity. Then a set A is *[...]-creative*, iff A is r.e. and $\overline{A}$ is [...]-productive.

In Rolletschek [6], [7] it is shown that the analogue of Proposition 1.1 does not hold with 'quasiproductive' replaced by 'semiproductive' and $D_{f(x)}$ replaced by $W_{f(x)}$. The following property is introduced which is a natural analogue of the notion of *completely productive set*, and which is stronger than the analogue of (1.1) but weaker than that of (1.2): A set A is *cW10-productive* with *cW10-productive function* f, iff f is recursive and for all x,

$$W_{f(x)} \cap (A \triangle W_x) \neq \emptyset \ \ \& \ \ W_{f(x)} \cap \overline{A \cup W_x} = \emptyset$$

(where $A \triangle W_x$ denotes the symmetric difference of A and W_x, see below). Then it is shown that not all semicreative sets are cW10-creative. It is also noted that not all cW10-creative sets are quasicreative; this can be seen easily in various ways.

In this paper we are primarily concerned with the creativity property which is given in the following definition, and which corresponds to the full analogue of (1.2):

Definition: A set A is *$\tilde{c}$W10-productive* with *$\tilde{c}$W10-productive function* f, iff f is recursive and for all x,

$$W_{f(x)} \subseteq A \Leftrightarrow W_{f(x)} \not\subseteq W_x.$$

A is $\tilde{c}$W10-creative, iff A is r.e. and $\overline{A}$ is $\tilde{c}$W10-productive.

Our terminology is based on a general definition scheme for productivity- and creativity properties used in Neubauer [4], Buchberger/Neubauer/Rolletschek/ Seeber [1] and Rolletschek [6]. Under this scheme productive, quasiproductive and semiproductive sets are also called *P00-productive*, *D10-productive* and *W10-productive* respectively. Let Γ stand for one of the strings P00, D10, $\tilde{c}$W10, cW10 or W10. Then we denote the property of being Γ-productive by (Γ), the property of being Γ-creative by (Γ^*). Using proposition 1.1, we immediately have the implications

$$(\mathrm{P00}) \Rightarrow (\mathrm{D10}) \Rightarrow (\tilde{\mathrm{c}}\mathrm{W10}) \Rightarrow (\mathrm{c\ W10}) \Rightarrow (\mathrm{W10}).$$

It is well-known that (D10*) does not imply (P00*), and, as noted above, (W10*) does not imply (cW10*).

One of the main goals of this paper is to show that the Turing degrees of $\tilde{c}$W10-creative sets are precisely the noncappable degrees (Corollary 2.3). This result is of interest for various reasons. To the author's knowledge it is the only result which associates a creativity property with a nontrivial class of r.e. Turing degrees. For most creativity properties P, r.e. sets satisfying P either have to be T-complete or occur within every nonrecursive r.e. Turing degree.

Corollary 2.3 also provides one of the simplest characterizations of noncappable degrees by a structural property of r.e. sets, though many other such characterizations exist (see Soare [9], chapter XIII). For instance, the noncappable degrees coincide with the degrees of promptly simple sets A, but the definition of the latter property involves a recursive enumeration of A.

Finally, an immediate consequence of Corollary 2.3 is that there exist $\tilde{c}$W10-creative sets which are not quasicreative and cW10-creative sets which are not $\tilde{c}$W10-creative. Indeed, quasicreative sets are q-complete and hence T-complete, see Rogers [5], Exercise 8–26, p. 123 and Exercise 11–8, p. 215. On the other hand, Yates [10] showed that semicreative sets occur within every nonrecursive r.e. T-degree, and the sets constructed in his paper are easily seen to be cW10-creative.

In section 3 we investigate the behavior of $\tilde{c}$W10-creative sets under the join operator; this was done for other productivity- and creativity properties in Rolletschek [7]. If A or B satisfies any of the productivity properties (Γ) considered, then $A \oplus B$ also satisfies (Γ). This rather trivial fact will be used tacitly later on; in Rolletschek [7] it was shown for properties other than ($\tilde{c}$W10), but the argument carries over readily to ($\tilde{c}$W10) as well. Theorem 3.2 asserts that if $A \oplus B$ is $\tilde{c}$W10-productive and B is immune, then A is $\tilde{c}$W10-productive. On the other hand, if $A \oplus B$ is $\tilde{c}$W10-creative or even quasicreative, then it cannot be concluded that A or B must be $\tilde{c}$W10-creative (Theorem 3.3). Finally, if $A \oplus B$ is only required to be $\tilde{c}$W10-creative, then the r.e. sets A, B forming a counterexample can be constructed within any given noncappable r.e. T-degrees (Theorem 3.4). This parallels a result from Rolletschek [7] which asserts that the semicreativity of $A \oplus B$ does not imply that of A or B, and that corresponding sets A and B can be constructed within any given nonrecursive r.e. degrees.

In our notation we follow mostly standard texts in recursive function theory, like Rogers [5] or Soare [9]. $\subseteq$ denotes the subset-relation, $\subsetneq$ or $\subset$ the *proper* subset relation; the latter symbol is used when the condition $A \neq B$ is obvious or unimportant and need not be emphasized particularly. Also, ω denotes the set of natural numbers including 0, $A \triangle B$ the symmetric difference of A and B.

As usual, $(\phi_0, \phi_1, \dots)$ is a Gödel numbering of all partial recursive functions, and $\phi_{e,s}(x) = y$ means that the computation of $\phi_e(x)$ yields the output y in at most s steps. W_e and $W_{e,s}$ are the domains of ϕ_e and $\phi_{e,s}$ respectively. We assume that the functions $\phi_{e,s}$ are defined such that the convention from Soare [9], p. 34 is satisfied. It implies

$$(\forall e, x, y, s)(\phi_{e,s}(x) = y \Leftrightarrow e, x, y < s), \quad \text{and}$$
$$(\forall e, s)(|W_{e,s+1} - W_{e,s}| \leq 1).$$

Furthermore, we make the convention that $W_{\phi_{e,s}(x),s}$ has a meaning even if $\phi_{e,s}(x)\uparrow$; in this case $W_{\phi_{e,s}(x),s} = \emptyset$.

If a construction in stages describes the enumeration of a set A, then A_s automatically denotes the set of all elements that have been enumerated by the end of stage s. If, on the other hand, an r.e. set A is given, then by default $\{ A_s \}_{s \in \omega}$ denotes an arbitrary recursive enumeration of A, but in several cases the (speed of) the enumeration is controlled by the construction, and A_s is again the set of all elements enumerated in A by the end of stage s.

2. The Turing degrees of c̃W10-creative sets

Although the following result will be superseded by the stronger Theorem 3.4, it seems appropriate to provide a proof at this point.

Theorem 2.1: *Every noncappable r.e. Turing degree contains a c̃W10-creative set.*

Proof: Let $\mathbf{b}$ be a noncappable degree, B an r.e. set of degree $\mathbf{b}$, and p a recursive function such that

(2.1) $(\forall s)[p(s) \geq s]$, and for all e with W_e infinite,

$$(\exists x, s)[x \in W_{e,\text{ at } s} \ \& \ B_s \restriction x \neq B_{p(s)} \restriction x]$$

(Soare [9], theorem XIII.1.6). We construct a c̃W10-creative set $A \leq_T B$ and a c̃W10-productive function f for $\overline{A}$; then $A \oplus B$ is a c̃W10-creative set of the same degree as B.

$U_e = W_{f(e)}$ is a subset of $\omega^{[e]}$, and the enumeration of U_e together with that of $A^{[e]}$ proceeds independently for all e, as described in the construction below. If by the end of stage s no element of U_e has been enumerated in A, then only the last element enumerated in U_e so far has not yet appeared in W_e; denote this last element by $u_{e,s}$. At the same time a set $X_e \subseteq W_e$ will be enumerated, and by the recursion theorem it may be assumed that an index x_e is given in advance such that

$$W_{x_e} = X_e \ \ \& \ \ (\forall s)(W_{x_e,s} \subseteq X_{e,s}).$$

According to our conventions from the end of section 1, $X_{e,s}$ denotes the set of all elements enumerated by the end of stage s of the following construction.

Construction:

Stage 0. Put $u_{e,0} = \langle 0, e \rangle$ and enumerate this element in U_e. Note that by our general conventions, $W_{e,0} = \emptyset$.

Stage $s+1$. Assume that $u_{e,s} \in W_{e,s}$; in this case $u_{e,s} \in W_{e,\text{ at } s}$. Enumerate $u_{e,s}$ in X_e and find t such that $u_{e,s} \in W_{x_e,\text{ at } t}$; by the definition of x_e we have $t \geq s+1$. If $B_s \restriction u_{e,s} \neq B_{p(t)} \restriction u_{e,s}$, then enumerate $u_{e,s}$ in A, otherwise put $u_{e,s+1} = \langle s+1, e \rangle$ and enumerate this element in U_e; again by our conventions we have $u_{e,s+1} \notin W_{e,s+1}$. •

The condition $A \leq_T B$ is clearly satisfied by construction. Also, U_e contains no element of $A - W_e$. Suppose that f is not c̃W10-creative for $\overline{A}$, and let e be such that U_e does not satisfy the required conditions. Then $U_e \subseteq W_e$, $U_e = X_e \subseteq \overline{A}$, and $X_e = W_{x_e}$ is infinite. Let $u \in X_e$ and t be such that $u \in W_{x_e,\text{ at } t}$ and $B_t \restriction u \neq B_{p(t)} \restriction u$. Also, let $s+1$ be the stage where $u = u_{e,s}$ is enumerated in U_e; we have $s+1 \leq t$. But then $B_s \restriction u \neq B_{p(t)} \restriction u$, and by construction u will be enumerated in A, a contradiction. Hence f is indeed c̃W10-productive for $\overline{A}$. □

Conversely, every c̃W10-creative set has noncappable degree. We show a stronger result which forms a link to section 4.

Theorem 2.2: *If $A \oplus B$ is c̃W10-creative but B is not, then A has noncappable Turing degree.*

Proof: Let f be a c̃W10-productive function for $\overline{A \oplus B}$. We have to show that there exist a recursive enumeration $\{ A_s \}_{s\in\omega}$ of A and a recursive function p satisfying (2.1). For this purpose we enumerate sets $X_{e,u} = W_{x_{e,u}}$. Again we may assume that the indices $x_{e,u}$ are given in advance. Let $\tilde{Y}_{e,u} = W_{f(x_{e,u})}$; we also enumerate sets $Y_{e,u} \subseteq \tilde{Y}_{e,u}$. The condition that f is c̃W10-productive for $\overline{A \oplus B}$ implies certain set-theoretical relations between $A \oplus B$, $X_{e,u}$ and $\tilde{Y}_{e,u}$, and for W_e infinite this will either force $\{ A_s \}_{s\in\omega}$ to satisfy (2.1), or else it will make B c̃W10-creative, contrary to assumption.

$X_{e,u,s}$ is not necessarily the same as $W_{x_{e,u},s}$, since $X_{e,u,s}$ and $Y_{e,u,s}$ denote the sets of elements enumerated in $X_{e,u}$ and $Y_{e,u}$ respectively by the end of stage s of the following construction. The enumerations $\{ A_s \}_{s\in\omega}$, $\{ B_s \}_{s\in\omega}$ are also controlled by the construction, that is, it will be specified how far given (arbitrary) enumerations should be carried out by the end of stage s, and the results are A_s and B_s.

After stage s, an index $u_{e,s}$ is defined for all e, and the enumeration of $X_{e,u}$ and $Y_{e,u}$ can be active only for $u = u_{e,s}$. Also, after stage s a *status* $S(e,u,s)$ is defined which describes the relationship between $X_{e,u,s}$ and $Y_{e,u,s}$ as follows.

We always have $X_{e,u,s} \subseteq Y_{e,u,s}$, that is, no element will be enumerated in $X_{e,u}$ unless it has previously been enumerated in $Y_{e,u}$. Also, an odd element $2z+1$ may be enumerated in $X_{e,u}$ only if z has appeared in the enumeration of W_u. If $X_{e,u,s} = Y_{e,u,s} = \emptyset$, then $S(e,u,s) = 0$. As a matter of fact, there is a number $\hat{u}_{e,s}$ such that $(\forall u)[S(e,u,s) = 0 \Leftrightarrow u \geq \hat{u}_{e,s}]$ and $u_{e,s} \leq \hat{u}_{e,s}$.

Assume $X_{e,u,s} = Y_{e,u,s} \neq \emptyset$. Then $A_s \oplus B_s$ contains at least one element z of $X_{e,u,s}$. If the first such z to appear in $A \oplus B$ is odd (that is, an element of $\emptyset \oplus B$), then $S(e,u,s) = \mathrm{B0}$; in this case $u_{e,s'} \neq u$ for all $s' \geq s$. If on the other hand the first element of $X_{e,u,s}$ to appear in $A \oplus B$ is even, that is, belongs to $A \oplus \emptyset$, then $S(e,u,s) = \mathrm{A0}$. In the latter case it will be seen that the required condition (2.1) is satisfied for the index e under consideration, we say that e is *settled*, and the enumeration of $X_{e,u'}$ and $Y_{e,u'}$ will be terminated for all u'.

Now assume $X_{e,u,s} \subsetneq Y_{e,u,s}$. Then $Y_{e,u,s} - X_{e,u,s}$ contains only a single element, which we denote by $y_{e,u,s}$. Also, $A_s \oplus B_s$ cannot contain any element of $Y_{e,u,s}$, otherwise the construction will be terminated immediately, and f would fail to be c̃W10-productive for $\overline{A}$. If $y_{e,u,s} = 2z+1$, then $S(e,u,s)$ equals B2 or B1, depending on whether z has appeared in W_u or not, while if $y_{e,s}$ is even, then $S(u,e,s) = \mathrm{A1}$. If $S(e,u,s)$ equals A1 or B2, then $u_{e,s}$ may equal u. If $S(e,u,s) = \mathrm{B1}$, then $u_{e,s} \neq u$, but $u_{e,s'}$ may equal u for some $s' > s$ (namely, if z appears in W_u).

Actually, $u_{e,s}$ always equals the smallest u such that $S(e,u,s)$ equals 0, A1 or B2. Define $F_s = \{\, e \mid (\exists u)(S(e,u,s) \neq 0 \,\} = \{\, e \mid \hat{u}_{e,s} > 0 \,\}$. F_s will be finite, in fact, all elements of F_s will be $< \lceil s/3 \rceil$.

Construction:

Stage 0. Put $u_{e,s} = 0$ and $S(e,u,0) = 0$ for all e and u.

Stage $s+1 = 3t+1$. Let e be minimal such that $S(e, u_{e,s}, s) = 0$. Carry out the enumeration of $\tilde{Y}_{e,u_{e,s}}$ until the first element $y_{e,u_{e,s},s+1}$ appears. ($\tilde{Y}_{e,u_{e,s}}$ cannot be empty, otherwise f would fail to be c̃W10-productive for $\overline{A \oplus B}$.) Enumerate $y_{e,u_{e,s},s+1}$ in $Y_{e,u_{e,s}}$. If $y_{e,u_{e,s},s+1} \in A_s \oplus B_s$, then terminate the whole construction. (Actually, case cannot happen, since it would imply that f is not c̃W10-productive for $\overline{A}$, but in order to obtain this contradiction, we have to specify explicitly that the construction is terminated in this case.) If $y_{e,u_{e,s},s+1}$ is even, put $S(e, u_{e,s}, s+1) = \mathrm{A1}$. If $y_{e,u_{e,s},s+1}$ is odd, then put $S(e, u_{e,s}, s+1) = \mathrm{B1}$ and $u_{e,s+1} = u_{e,s} + 1$. (Then $S(e, u_{e,s+1}, s+1)$ is again 0.)

Stage $s+1 = 3t+2$. Carry out the following action for all $e \in F_s$. Let $G_{e,s}$ be the set of all u such that $S(e,u,s) = B1$ and $y_{e,u,s} \in \emptyset \oplus W_{u,s}$. Put $S(e,u,s+1) = B2$ for every $u \in G_{e,s}$. If $G_{e,s} \neq \emptyset$ and $\min(G_{e,s}) < u_{e,s}$, then put $u_{e,s+1} = \min(G_{e,s})$.

Stage $s+1 = 3t+3$. Let $H_s \subseteq F_s$ be the set of all e such that (a) $S(e, u_{e,s}, s)$ equals A1 or B2, and (b) $W_{e,t+1} - W_{e,t}$ contains some $z > \max\{\, x \mid 2x \in Y_{e,u_{e,s},s} \,\}$. Enumerate $y_{e,u_{e,s},s}$ in $X_{e,u_{e,s},s}$ for all $e \in H_{e,s}$. Then continue the enumeration of the sets A, B, and $\tilde{Y}_{e,u_{e,s},s}$ ($e \in H_{e,s}$) simultaneously, until for each $e \in H_{e,s}$ one of the following two events has happened: (α) an element z_e of $Y_{e,u_{e,s},s} = X_{e,u_{e,s},s+1}$ has appeared in $A \oplus B$, or (β) a new element $y_{e,u_{e,s},s+1} \notin Y_{e,u_{e,s},s}$ has appeared in $\tilde{Y}_{e,u_{e,s}}$. Either (α) or (β) must happen for each $e \in H_{e,s}$, since otherwise $X_{e,u_{e,s}} = \tilde{Y}_{e,u_{e,s}} \subseteq \overline{A \oplus B}$, contradicting the assumption that F is čW10-productive for $\overline{A \oplus B}$. At this point we say that case (ω) holds iff either (i) there exists some $e \in H_{e,s}$ such that both (α) and (β) have happened, or (ii) there exist e and u such that $e \notin H_e \vee u \neq u_{e,s}$, $S(e,u,s)$ equals A1, B1 or B2, and some element of $Y_{e,u,s}$ has appeared in $A \oplus B$. If this is the case, then terminate the construction. Actually, case (ω) cannot hold, since it would imply that for some e and u, $\tilde{Y}_{e,u}$ either contains an element of $(A \oplus B) - X_{e,u}$, or an element of $(A \oplus B) \cap X_{e,u}$ and also of $\overline{A \oplus B} \cap \overline{X}_{e,u}$; however, similarly as in stage $3t+1$ we have to specify that the construction is terminated in case (ω). If case (ω) does not hold, then perform the following action for every $e \in H_{e,s}$. If event (α) has happened, then set $S(e, u_{e,s}, s+1)$ to A0 or B0, depending on whether z_e is even or odd, and $u_{e,s+1}$ to the smallest $u > u_{e,s}$ such that $S(e,u,s) = S(e,u,s+1)$ equals 0, A1 or B2. If (β) has happened, then enumerate $y_{e,u_{e,s},s+1}$ in $Y_{e,u_{e,s}}$, set $S(e, u_{e,s}, s+1)$ to A1 or B1, depending on whether $y_{e,u_{e,s},s+1}$ is even or odd, and in the second case set $u_{e,s+1}$ again to the smallest $u > u_{e,s}$ such that $S(e,u,s) = S(e,u,s+1)$ equals 0, A1 or B2.

This completes the definition of the sets $X_{e,u}$ and $Y_{e,u}$. •

We justify the claim above, that an odd element $2z+1$ may be enumerated in $X_{e,u}$ only if z has appeared in W_u. By construction every element of $X_{e,u}$ equals $y_{e,u,s}$ for some s, so assume $2z+1 = y_{e,u,s}$. If $2z+1$ is enumerated in $X_{e,u}$, then this happens at some stage $s'+1 > s$ with $S(e,u,s') = \text{B2}$, which implies the claim $z \in W_u$.

Suppose that for some e and u, $y_{e,u,s}$ takes infinitely many different values. Then every element of $\tilde{Y}_{e,u}$ is eventually enumerated in $Y_{e,u}$ and then in $X_{e,u}$, and $X_{e,u} = \tilde{Y}_{e,u} \subseteq \overline{A \oplus B}$, otherwise $S(e,u,s)$ would be A0 or B0 for large s, and $y_{e,u,s}$ would not take any new values. This would contradict the fact that f is čW10-productive for $\overline{A \oplus B}$. Hence $y_{e,u,s}$ takes only finitely many values, and by construction this applies also that $S(e,u,s)$ changes only a finite number of times. Accordingly, let $S(e,u)$ be such that $S(e,u) = S(e,u,s)$ for all sufficiently large s. If $S(e,u)$ equals A1, B1 or B2, then let $y_{e,u} = \lim_{s\to\infty} y_{e,u,s}$.

If $S(e,u) = \text{B0}$, then $A \oplus B$ contains an odd element $2z+1$ of $X_{e,u} = Y_{e,u}$, and $z \in W_u$. If $S(e,u) = \text{B1}$, then $y_{e,u} = 2z+1 \in Y_{e,u} - X_{e,u}$ with $z \notin W_u$, and $Y_{e,u} \cap A \oplus B = \emptyset$. Suppose that $S(e,u)$ equals B0 or B1 for all u. Then the recursive function g defined by $W_{g(u)} = \{\, z \mid 2z+1 \in Y_{e,u} \,\}$ would be čW10-productive for $\overline{B}$, contradicting the assumption that B is not čW10-creative. Hence let u_e be minimal such that $S(e,u_e)$ does not equal B0 or B1.

Now assume that W_e is infinite. Suppose that e never gets settled, so $S(e,u) \neq$ A0 for all u. Then it follows that $u_e = \lim_{s\to\infty} u_{e,s}$, since $u_{e,s}$ is the smallest u such that $S(e,u,s)$ equals 0, A1, or B2, and the action at stages $3t+1$ guarantees that $S(e,u_e,s) \neq 0$ for all sufficiently large s. However, in this case the action at stages $3t+3$ would force a change of $y_{e,u_e,s}$ infinitely often, a contradiction.

Hence e actually gets settled at some stage $s = 3t+3$, $W_{e,t+1} - W_{e,t}$ contains an element $z > \max\{\, x \mid 2x \in Y_{e,u_{e,s},s} \,\}$, and $(A_{3t+1} \oplus B_{3t+1}) - (A_{3t} \oplus B_{3t})$ contains an even element of $Y_{e,u_{e,s},s}$, that is, $A_{3t+1} - A_{3t}$ contains an element $< z$. Hence (2.1) is indeed satisfied, as claimed. □

Corollary 2.3: *The T-degrees of $\tilde{c}$W10-creative sets are precisely the noncappable r.e. T-degrees.*

3. c̃W10-creative sets and joins

We first recall the following results from Rolletschek [7], which describe the behavior of other productivity properties than (c̃W10) with respect to the join operator:

Proposition 3.1: *(i) If $A \oplus B$ is productive, then A or B is productive.*

(ii) If $A \oplus B$ is cW10-productive, then A or B is cW10-productive.

(iii) If $A \oplus B$ is quasiproductive or semiproductive and B is immune, then A is quasiproductive or semiproductive respectively.

(iv) There exist r.e. sets A, B such that neither A nor B is quasicreative but $A \oplus B$ is quasicreative.

(v) There exist r.e. sets A, B such that neither A nor B is semicreative but $A \oplus B$ is semicreative. Moreover, such A and B can be constructed within any given nonrecursive r.e. T-degrees.

In this chapter we show similar results concerning c̃W10-productivity and -creativity. The analogue of Proposition 3.1 (iii) holds for c̃W10-productivity, though in this case the proof is a trifle trickier. It will be used in the proof of Theorem 3.4.

Theorem 3.2: *If $A \oplus B$ is c̃W10-productive and B is immune, then A is c̃W10-productive.*

Proof: Let f be a c̃W10-productive function for $A \oplus B$; we construct a c̃W10-productive function g for A. Given x, we define indices $x_0, x_1, \ldots, x_m$, where m will be determined later. W_{x_i} is of the form $U_i \oplus F_i$, where F_i is a finite set to be defined prior to x_i. The enumeration of U_i, to be described below, depends on x_i, but by the recursion theorem it may be assumed that x_i is known in advance. We will have $U_i \subseteq W_x$. Along with U_i we enumerate a set V_i, where $V_i \oplus \emptyset \subseteq W_{f(x_i)}$. For each s, $V_{i,s}$ equals either $U_{i,s}$ or $U_{i,s} \cup \{u_{i,s}\}$, where $u_{i,s} \notin U_{i,s}$.

Put $F_0 = \emptyset$. After F_i has been specified, the enumeration of U_i and V_i proceeds in stages. Put $U_{i,0} = V_{i,0} = \emptyset$. The following action is taken at stage $s+1$:

(a) Assume $U_{i,s} = V_{i,s}$. If $W_{f(x_i),s}$ contains an odd element $2z+1$ with $z \notin F_i$, then pick the first such element in the enumeration of $W_{f(x_i)}$, denote it by $2z_i + 1$, terminate the enumeration of U_i and V_i, put $F_{i+1} = F_i \cup \{z_i\}$ and start the enumeration of U_{i+1} and V_{i+1}. If no such element $2z+1$ exists but $W_{f(x_i),s}$ contains an even element $2y$ with $y \notin U_{i,s}$, then pick the first such element $2y$ to appear in the enumeration of $W_{f(x_i)}$, put $u_{i,s+1} = y$ and enumerate y in $V_{i,s+1}$.

(b) Now assume $U_{i,s} \subsetneq V_{i,s}$. If $u_{i,s} \in W_{x,s}$, then enumerate $u_{i,s}$ in $U_{i,s+1}$. If in addition $W_{f(x_i),s}$ contains an odd element $2z+1$ with $z \notin F_i$, then pick the first such element in the enumeration of $W_{f(x_i)}$, denote it by $2z_i+1$, terminate the enumeration of U_i and V_i, put $F_{i+1} = F_i \cup \{z_i\}$ and start the enumeration of U_{i+1} and V_{i+1}.

Define g by $W_{g(x)} = \bigcup_i V_i$, where the union is taken over all i for which x_i is defined; we have to show that g is indeed čW10-productive for A. First note that if z_i is defined, then $2z_i+1 \in W_{f(x_i)} - W_{x_i}$. This implies $2z+1 \in A \oplus B$, since f is čW10-productive for $A \oplus B$, hence $z_i \in B$. Also, the elements z_i are distinct, since $z_i \in F_{i+1} - F_i$. Since B is immune, it follows that there exist only finitely many z_i, hence there exists a largest index $i = m$ for which x_i is defined.

Assume $i < m$. Then $U_i = V_i \subseteq W_x$ by construction. Also, $2z_i+1 \in W_{f(x_i)} - W_{x_i}$ implies $V_i \oplus \emptyset \subseteq W_{f(x_i)} \subseteq A \oplus B$, that is, $V_i \subseteq W_x \cap A$, and V_i contains no element which witnesses $W_x \neq A$.

The enumeration of U_m and V_m is never terminated. The set $W_{f(x_m)}$ must satisfy the analogue of (1.2) for $A \oplus B$ in place of A; hence one of the following conditions holds: (α) $W_{f(x_m)}$ contains an even element $2y \in (A \oplus B) - W_{x_m}$; ($\beta$) $W_{f(x_m)}$ contains an odd element $2z+1 \in (A \oplus B) - W_{x_m}$; or ($\gamma$) $W_{f(x_m)}$ contains an element v of $W_{x_m} - (A \oplus B)$.

In cases (α) and (β), the analogue of (1.2) also implies $V_m \oplus \emptyset \subseteq W_{f(x_m)} \subseteq A \oplus B$, hence $V_m \subseteq A$. Assume (α), and let $2y$ be the first element of $(A \oplus B) - W_{x_m}$ to appear in the enumeration of $W_{f(x_m)}$. Then $y \in A - U_m$, by construction we have $y \notin W_x$, and y will eventually be enumerated in V_m. Hence $W_g(x) \subseteq A$, and $W_{g(x)}$ contains an element of $A - W_x$,

Next assume (β). Then V_m contains an element u_m which never appears in W_x, otherwise the enumeration of U_m and V_m would be terminated; u_m equals $u_{m,s}$ for all sufficiently large s. Hence (α) holds as well, and the argument for (α) applies.

Finally, assume (γ). Then (α) and (β) do not hold, and every element of V_m will eventually be enumerated in U_m. The element v from condition (γ) can only be even, since the odd elements $2z+1 \in W_{f(x_m)}$ satisfy $z \in F_m \subseteq B$. Thus $v = 2y$ with $y \in U_m = V_m \subseteq W_x$, so $W_{g(x)} \subseteq W_x$, and $y \notin A$.

In each case $W_{g(x)}$ satisfies the required conditions. □

Next we strengthen Proposition 3.1 (iv):

Theorem 3.3: *There exist r.e. sets A, B such that neither A nor B is čW10-creative, but $A \oplus B$ is quasicreative.*

Proof: The requirement that $A \oplus B$ is quasicreative will be witnessed by the function f, defined by $D_{f(e)} = \{2e, 2e+1\}$. In fact, f will even satisfy (1.2) (with $\overline{A \oplus B}$ in place of A), which is just as easy to accomplish. So the rules of the game are as follows: If both $2e$ and $2e+1$ appear in W_e, then at least one of these two elements has to be enumerated in $A \oplus B$, otherwise neither one of these two elements must be enumerated in $A \oplus B$.

At the same time we have to meet the requirements

$$P_e\colon \phi_e \text{ is not a čW10-productive function for } \overline{A},$$

$$Q_e\colon \phi_e \text{ is not a čW10-productive function for } \overline{B}.$$

The ordering be decreasing priority is P_0, Q_0, P_1, … .

For this purpose we enumerate r.e. sets $U_{e,i}$, $V_{e,j}$ as potential witnesses. As usual we assume that indices $u_{e,i}$ and $v_{e,j}$ with $U_{e,i} = W_{u_{e,i}}$ and $V_{e,j} = W_{v_{e,j}}$ are given in advance, and we put $X_{e,i} = W_{\phi_e(u_{e,i})}$, $Y_{e,j} = W_{\phi_e(v_{e,j})}$. $X_{e,i,s}$ denotes the set of elements of $X_{e,i}$ enumerated by the end of stage s of the following construction (which is still in accordance with our general conventions), but this time $\{ X_{e,i,s} \}_{s\in\omega}$ is not necessarily a recursive enumeration of $X_{e,i}$, since the enumeration of $X_{e,i}$ may be terminated, and $\bigcup_{s\in\omega} X_{e,i,s}$ may be a proper subset of $X_{e,i}$. The same remark applies for $\{ Y_{e,j,s} \}_{s\in\omega}$.

By the end of stage s, the enumeration of $U_{e,i}$ and $X_{e,i}$ is active only for one index i, which we denote by $i_{e,s}$, and $u_{e,i_{e,s}}$ is currently regarded as the witness for P_e. For any e, i and s we have $U_{e,i,s} \subseteq X_{e,i,s}$, that is, an element is enumerated in $U_{e,i}$ only if it has previously been enumerated in $X_{e,i}$. If $U_{e,i,s} \neq X_{e,i,s}$, then $|X_{e,i,s} - U_{e,i,s}| = 1$. The only element of $X_{e,i,s} - U_{e,i,s}$ (if any) plays a rôle only for $i = i_{e,s}$, in this case it is denoted by $x_{e,s}$. The index $u_{e,i_{e,s}}$ 'appears' to be a witness for P_e in the following sense. Recall that ϕ_e would be čW10-productive for $\overline{A}$ iff for all x, $W_{\phi_e(x)} \subseteq \overline{A} \Leftrightarrow W_{\phi_e(x)} \nsubseteq W_x$. However, this condition will fail with $U_{e,i_{e,s},s}$ and $X_{e,i_{e,s},s}$ in place of W_x and $W_{\phi_e(x)}$. This leaves several possibilities, and a *status* $S(e,s)$ is defined accordingly.

For one thing, it may be that $U_{e,i_{e,s},s} \subsetneq X_{e,i_{e,s},s}$, and $A_s \cap X_{e,i_{e,s},s} \neq \emptyset$, which implies $\neg(X_{e,i_{e,s},s} \subseteq \overline{A}_s \Leftrightarrow X_{e,i_{e,s},s} \nsubseteq U_{e,i_{e,s},s})$. In this case we say that P_e is *settled* and put $S(e,s) = \mathrm{T}$. Then $S(e,s)$ will not change, and P_e will never again require attention, since $x_{e,s}$ will not be enumerated in $U_{e,i_{e,s}}$ under any circumstances.

The second possibility is $U_{e,i_{e,s},s} = X_{e,i_{e,s},s}$ and $U_{e,i_{e,s},s} \cap A_s = \emptyset$. In this case put

$$M_{e,s} = \{ x \in U_{e,i_{e,s},s} \mid x \in B_s \ \ \& \ \ (\forall e' < e)(S(e',s) \neq \mathrm{T} \Rightarrow x \notin U_{e',i_{e',s},s}) \};$$

the motivation for this definition will be explaned shortly. Note that $x \in B_s$ implies that both elements of $D_{f(x)}$ have already appeared in W_x. Also, if P_{e_1} and P_{e_2} are not settled and $e_1 \neq e_2$, then $M_{e_1,s} \cap M_{e_2,s} = \emptyset$. Next define

$$S(e,s) = \begin{cases} \mathrm{R} & \text{if } M_{e,s} \neq \emptyset, \\ \mathrm{N} & \text{if } M_{e,s} = \emptyset. \end{cases}$$

In contrast to the proof of Theorem 2.2, there is no middle argument i for $S(e,s)$, since the relationship between A, $U_{e,i,s}$ and $X_{e,i,s}$ plays no rôle for $i \neq i_{e,s}$.

Define a predicate $P(e,x,s)$ to be equivalent to $S(e,s) \in \{\mathrm{N},\mathrm{R}\} \ \ \& \ \ x \in U_{e,i_{e,s},s}$. If $P(e,x,s)$ holds, and if for some reason x has to be enumerated in A at stage $s+1$, then $u_{e,i_{e,s}}$ can no longer serve as a witness for P_e, and we say that P_e is *injured*. In this case we put $i_{e,s+1} = i_{e,s} + 1$ and start the enumeration of $U_{e,i_{e,s+1}}$ etc.

Assume that $S(e,s)$ equals N or R, and that an element $z \notin U_{e,i_{e,s},s}$ appears in the enumeration of $X_{e,i_{e,s}}$ at stage $s+1$. Then one such z is enumerated in $X_{e,i_{e,s},s}$, and P_e requires attention. If (i) $z \in A_s$, then no further action is needed except to record that P_e is now settled. If (ii) $z \notin A_s$ but $z \in B_s$, and $P(e',z,s)$ does not hold for any $e' < e$, then both elements of $D_{f(z)}$ have appeared in W_z earlier, so we may enumerate z in A and record again that P_e is settled. Similarly, if

(iii) $S(e,s) = \mathrm{R}$ and $x \in M_{e,s}$, then x may be enumerated in A, and once again P_e gets settled. So the point is that $M_{e,s}$ consists of those elements of $U_{e,i_{e,s},s}$ which may be enumerated in A without violating the requirement that f satisfies (1.2) for $\overline{A \oplus B}$ in place of A, and without injuring a condition of higher priority than P_e. However, if none of the conditions (i)–(iii) holds, then P_e cannot be settled, and we have to enumerate z in $U_{e,i_{e,s}}$.

All remarks from the last five paragraphs carry over to $V_{e,j}$, $Y_{e,j}$ in the obvious way. We denote the only element of $Y_{e,j_{e,s},s} - V_{e,j_{e,s},s}$ (if any) by $y_{e,s}$, the status corresponding to Q_e after stage s by $S'(e,s)$; the latter equals again T, R or N. The following are the analogue of the set $M_{e,s}$ and of the predicate $P(e,x,s)$:

$$N_{e,s} = \{\, x \in V_{e,i_{e,s},s} \mid x \in A_s \ \ \& \ \ (\forall e' < e)(S'(e',s) \neq \mathrm{T} \Rightarrow x \notin V_{e',i_{e',s},s}) \,\},$$
$$Q(e,x,s) \equiv S'(e,s) \in \{\mathrm{N},\mathrm{R}\} \ \ \& \ \ x \in V_{e,j_{e,s},s}.$$

The discussion so far makes the following construction and the correctness proof for the most part straightforward. It only remains to indicate why each condition P_e, Q_e is injured only a finite number of times. An injury can happen in two types of situations. The first was described two paragraphs ago: $S(e,s)$ equals N or R, and at stage $s+1$ an element $z \notin U_{e,i_{e,s},s}$ appears in the enumeration $X_{e,i_{e,s}}$, either z or an element of $M_{e,s}$ is enumerated in A_{s+1}, and $P(e',z,s)$ holds for some $e' > e$. In this case we say that $P_{e'}$ is injured for the sake of P_e. Since P_e gets settled in this case, it is clear that $P_{e'}$ can be injured only a finite number of times in this fashion.

The second type of injury occurs at stage $s+1$ if for the first time $2x$ and $2x+1$ both appear in W_x. Then we enumerate x in A or in B, so that the conditions injured have lowest possible priority. Specifically, assume that e and e' are minimal such that $P(e,x,s)$ and $Q(e',x,s)$ hold. Assume further that $e \leq e'$, so that x is actually enumerated in B; we say that $Q_{e'}$ is injured for the sake of P_e. The key point is that $S(e,s+1)$ will be R, not N. As a consequence, P_e will be settled as soon as another element appears in $X_{e,i_{e,s}}$, and $U_{e,i_{e,s}}$ will be finite. But this means that none of the conditions $Q_{e'}$ can be injured an infinite number of times for the sake of P_e. This consideration provides a straightforward proof by induction for Lemma 2 below.

Construction:

Stage 0. Put $i_{e,0} = j_{e,0} = 0$, $S(e,0) = S'(e,0) = \mathrm{N}$ and $M_{e,s} = N_{e,s} = \emptyset$ for all e.

Stage $s+1 = 3t+1$. Let $e \leq t$ be minimal such that P_e is not settled, and $W_{\phi_e(u_{e,i_{e,s}}),s}$ contains an element $z \notin U_{e,i_{e,s},s}$. (If no such e exists, go to stage $3t+2$.) Enumerate the first such z in $X_{e,i_{e,s}}$. For the further action consider four cases.

Case (i): $z \in A_s$. Put $S(e,s+1) = \mathrm{T}$ and record that P_e is settled.

Case (ii): $z \in B_s - A_s$, and $P(e',z,s)$ does not hold for any $e' < e$. Enumerate z in A_{s+1}, and again put $S(e,s+1) = \mathrm{T}$ and record that P_e is settled. For every $e' > e$ for which $P(e',z,s)$ holds put $i_{e',s+1} = i_{e',s} + 1$, $S(e',s+1) = \mathrm{N}$ and $M_{e',s+1} = \emptyset$.

Case (iii): Neither case (i) nor (ii) holds, and $S(e,s) = \mathrm{R}$. Enumerate the smallest $x \in M_{e,s}$ in A, and once again put $S(e,s+1) = \mathrm{T}$ and record that P_e is settled. For every $e' > e$ for which $P(e',x,s)$ holds put $i_{e',s+1} = i_{e',s} + 1$, $S(e',s+1) = \mathrm{N}$ and $M_{e',s+1} = \emptyset$.

Case (iv): None of (i)–(iii) holds. Enumerate z in $U_{e,i_{e,s},s+1}$.

Stage $s+1 = 3t+2$. This is analogous to stage $3t+1$, but with Q_e, $V_{e,j_{e,s},s}$ etc. in place of P_e, $U_{e,i_{e,s},s}$ etc. Let $e \le t$ be minimal such that Q_e is not settled, and $W_{\phi_e(v_{e,j_{e,s}}),s}$ contains an element $z \notin V_{e,j_{e,s},s}$. (If no such e exists, go to stage $3t+3$.) Enumerate the first such z in $Y_{e,j_{e,s}}$. For the further action consider the following cases.

Case (i): $z \in B_s$. Put $S'(e,s+1) = \mathrm{T}$ and record that Q_e is settled.

Case (ii): $z \in A_s - B_s$, and $Q(e',z,s)$ does not hold for any $e' < e$. Enumerate z in B_{s+1}, and again put $S'(e,s+1) = \mathrm{T}$ and record that Q_e is settled. For every $e' > e$ for which $Q(e',z,s)$ holds put $j_{e',s+1} = j_{e',s} + 1$, $S'(e',s+1) = \mathrm{N}$ and $N_{e',s+1} = \emptyset$.

Case (iii): Neither case (i) nor (ii) holds, and $S'(e,s) = \mathrm{R}$. Enumerate the smallest $x \in N_{e,s}$ in B, and once again put $S'(e,s+1) = \mathrm{T}$ and record that Q_e is settled. For every $e' > e$ for which $Q(e',x,s)$ holds put $j_{e',s+1} = j_{e',s} + 1$, $S'(e',s+1) = \mathrm{N}$ and $N_{e',s+1} = \emptyset$.

Case (iv): None of (i)–(iii) holds. Enumerate z in $V_{e,j_{e,s},s+1}$.

Stage $s+1 = 3t+3$. Let x be minimal such that $x \notin A_s \cup B_s$ and $D_{f(x)} \subseteq W_{x,s}$. (If no such x exists, then go to stage $3t+4$.) Let e be minimal such that $P(e,x,s)$ holds, or $e = \infty$ if $P(e,x,s)$ holds for no e. Similarly, let e' be minimal such that $Q(e',x,s)$ holds, or $e' = \infty$ if $Q(e',x,s)$ holds for no e'.

Assume first that $e \le e'$. Enumerate x in B_{s+1} and if $e < \infty$ put $M_{e,s+1} = M_{e,s} \cup \{x\}$, $S(e,s+1) = R$. For all e'' satisfying $Q(e'',x,s)$ put $j_{e'',s+1} = j_{e'',s} + 1$ and $S'(e'',s+1) = \mathrm{N}$; $Q_{e''}$ is injured for the sake of P_e.

Now assume $e > e'$. Enumerate x in A_{s+1} and if $e' < \infty$ put $N_{e',s+1} = N_{e',s} \cup \{x\}$, $S'(e',s+1) = R$. For all e'' satisfying $P(e'',x,s)$ put $i_{e'',s+1} = i_{e'',s} + 1$ and $S(e'',s+1) = \mathrm{N}$; $P_{e''}$ is injured for the sake of $Q_{e'}$. •

Lemma 1: The function f satisfies (1.2) with $\overline{A \oplus B}$ in place of A.

Proof: If $D_{f(x)} \not\subseteq W_x$, then by construction x will be enumerated neither in A nor in B, so $D_{f(x)} \cap (A \oplus B) = \emptyset$. On the other hand, if $D_{f(x)} \subseteq W_x$, then x will eventually be enumerated in A or in B at some stage $s+1$, where $s+1 = 3t+3$ is such that $D_{f(x)} \subseteq W_{x,s}$ and no $x' < x$ is enumerated in A or B at stage $s+1$. The claim follows.

Now assume that $i_e = \lim_{s\to\infty} i_{e,s}$ exists, which is equivalent to the assertion that P_e is injured only a finite number of times. Let s_0 be such that $i_{e,s} = i_e$ for all $s \ge s_0$. After stage s_0 the status $S(e,s)$ can change only from N to R or T or from R to T, hence there exists a stage s_1 and a status $S \in \{\mathrm{N},\mathrm{R},\mathrm{T}\}$ such that $S(e,s) = S$ for all $s \ge s_1$. If $S = \mathrm{T}$, then $U_{e,i_e} \subsetneq \bigcup_{s\in\omega} X_{e,i_e,s} \subseteq X_{e,i_e}$. If $S \in \{\mathrm{N},\mathrm{R}\}$, then every element of $X_{e,i_e} = W_{\phi_e(u_{e,i_e})}$ will eventually be enumerated in U_{e,i_e} and in $X_{e,i_e,s}$ at some stage s, and we have $U_{e,i_e} = X_{e,i_e} \subset \overline{A}$. In either case u_{e,i_e} witnesses that ϕ_e is not c̃W10-productive for $\overline{A}$. An analogous argument shows that if $j_e = \lim_{s\to\infty} j_{e,s}$ exists, then Q_e is satisfied. Thus it only remains to show:

Lemma 2: Each of the conditions P_e, Q_e is injured only a finite number of times.

Proof: We show that $Q_{e'}$ is injured only a finite number of times; the proof for P_e is analogous. Using induction, we may assume that the assertion holds for every condition of higher priority than $Q_{e'}$. By construction $Q_{e'}$ cannot be injured for the sake of any condition of lower priority, and $Q_{e'}$ can be injured at most once for the sake of Q_e, where $e < e'$. Hence it suffices to show for any $e \le e'$ that $Q_{e'}$ is injured only a finite number of times for the sake of P_e. Define s_1, i_e and $S(e)$ as above, so that $i_{e,s} = i_e$ and $S(e,s) = S(e)$ for all $s \ge s_1$. If $Q_{e'}$ is injured for the sake of P_e at stage $s \ge s_1$, then by construction $S(e,s) = \mathrm{R}$, so $S(e) = \mathrm{R}$, no further element is enumerated in U_{e,i_e}, and U_{e,i_e} is finite. Furthermore, $Q_{e'}$ is injured for the sake of P_e after stage s_1 only when an element $x \in U_{e,i_e}$ is enumerated in B, which happens only a finite number of times. The claim follows. $\Box$

The last result provides an analogue for part (v) of Proposition 3.1. If $A \oplus B$ is c̃W10-creative but A and B are not, then A and B must have noncappable Turing degrees by Theorem 2.2. Therefore the degree requirements in the following theorem cannot be strenghtened. The results of this section describe completely which productivity- or creativitity properties can be deduced for A or B if $A \oplus B$ satisfies one of the productivity- or creativity properties considered in this paper. The only question that remains open is the following. Find a necessary and sufficient condition for r.e. T-degrees **a**, **b** such that there exist r.e. sets A, B of degrees **a**, **b** with neither A nor B c̃W10-creative but $A \oplus B$ quasicreative? As a modification, we may only require that A and B are not quasicreative.

Theorem 3.4: *For any given noncappable r.e. T-degrees* **a**, **b**, *there exist r.e. sets A, B of degrees* **a** *and* **b** *such that neither A nor B is c̃W10-creative, but $A \oplus B$ is c̃W10-creative.*

Proof: Let C and D be r.e. sets of degrees **a** and **b** respectively, p be a recursive function which satisfies (2.1) for $\{ C_s \}_{s\in\omega}$ in place of $\{ B_s \}_{s\in\omega}$, and q a recursive function which satisfies the analogous condition for $\{ D_s \}_{s\in\omega}$. Similarly as in Theorem 2.1, but in contrast to some of the preceding proofs, $\{ C_s \}_{s\in\omega}$ and $\{ D_s \}_{s\in\omega}$ are arbitrary given recursive enumerations of C and D, not controlled by the following construction.

In view of Theorem 3.2 it suffices to construct r.e. sets $\tilde{A}$, $\tilde{B}$ such that $\tilde{A}$ and $\tilde{B}$ are not c̃W10-creative, $\tilde{A} \oplus \tilde{A}$ is c̃W10-creative, and $\tilde{A} \le_T C$, $\tilde{B} \le_T D$ (rather than $\tilde{A} \equiv_T C$ and $B \equiv_T D$). Then the requirements will be satisfied by $A = \tilde{A} \oplus \tilde{C}$ and $B = \tilde{B} \oplus \tilde{D}$, where $\tilde{C}$ and $\tilde{D}$ are simple sets of degrees **a** and **b** respectively. The existence of such $\tilde{C}$ and $\tilde{D}$ was shown by Yates [10]and Dekker [2]. In the sequel we write again A and B for $\tilde{A}$ and $\tilde{B}$ for notational simplicity.

We define a c̃W10-productive function f for $\overline{A \oplus B}$ by enumerating for every e the set $W_{f(e)} = G_e = E_e \oplus F_e$. Every element of E_e or F_e will be of the form $\langle x, e\rangle$, so that $G_e \cap G_{e'} = \emptyset$ for $e \ne e'$. For the formulation of the proof it is helpful to introduce for every e a condition R_e which asserts that G_e satisfies the required relations:

$$W_{f(e)} \cap (A \oplus B) = \emptyset \Leftrightarrow W_{f(e)} \not\subseteq W_e.$$

A status $\hat{S}(e,s)$ is associated with R_e after stage s. R_e always 'appears' to be satisfied, which implies that one of two conditions holds, and $\hat{S}(e,s)$ indicates which. The first possibility is that all elements of $G_{e,s}$ have already appeared in W_e, and $G_{e,s} \cap (A_s \oplus B_s) \neq \emptyset$. In this case $\hat{S}(e,s) = T$, we say that R_e is settled, and R_e requires no further attention. Otherwise $G_{e,s} \cap (A_s \oplus B_s) = \emptyset$, and there actually exists only one element $g_{e,s}$ of $G_{e,s}$ which has not yet appeared in W_e, since there is no need to enumerate in G_e more than one element that has not appeared in W_e. In this second case $\hat{S}(e,s)$ equals N. In contrast to the conditions P_e and Q_e, which are defined as in the proof of Theorem 3.3, there is no third value R for the status associated with R_e, and R_e is never injured.

In many respects the currents proof resembles that of Theorem 3.3, and we refer to the latter by PT3. In particular, the following objects have the same significance as in PT3: the requirements P_e and Q_e, the predicates $P(e,x,s)$ and $Q(e,x,s)$, the sets $U_{e,i}$, $V_{e,i}$, $X_{e,i}$, $Y_{e,i}$, $M_{e,s}$ and $N_{e,s}$, the indices $u_{e,i}$ and $v_{e,i}$, the numbers $i_{e,s}$, $j_{e,s}$, $x_{e,s}$ and $y_{e,s}$, and, finally, $S(e,s)$ and $S'(e,s)$. The formal definition changes only for $M_{e,s}$ and $N_{e,s}$, since G_e now plays the rôle of $D_{f(e)} = \{2e, 2e+1\}$. Thus

$$M_{e,s} = \{\, x \in U_{e,i_{e,s},s} \mid (\forall e')(x \in E_{e',s} \Rightarrow \hat{S}(e',s) = \mathrm{T}) \;\;\&\;\; (\forall e' < e)(\neg P(e',x,s)) \,\},$$
$$N_{e,s} = \{\, x \in V_{e,j_{e,s},s} \mid (\forall e')(x \in F_{e',s} \Rightarrow \hat{S}'(e',s) = \mathrm{T}) \;\;\&\;\; (\forall e' < e)(\neg Q(e',x,s)) \,\},$$

The definition of $S(e,s)$ and $S'(e,s)$ is of course based on this new definition of $M_{e,s}$ and $N_{e,s}$.

We now consider the kind of situation where R_e requires attention. Thus assume that $\hat{S}(e,s) = \mathrm{N}$ and that $g_{e,s}$ appears in W_e at stage $s{+}1$. Then one of the following two actions has to be performed: (α) enumerate an element of $G_{e,s}$ in $A \oplus B$, or (β) enumerate a new element $g_{e,s+1}$ in $G_{e,s+1}$. Action (α) is analogous to the action at stage $t+3$ in PT3; it is preferable from the point of view of condition R_e, since it settles R_e. In fact, R_e would ultimately fail if (β) were performed an infinite number of times, and action (β) might be regarded as an injury to R_e (although we will not use this phraseology in this context). However, (α) can be carried out only if it is permitted by C or by D (depending on whether we wish to enumerate an element of $E_{e,s}$ in A or an element of $F_{e,s}$ in B). If the situation described arises an infinite number of times (for the same e), then eventually C or D will permit (α), as in Theorem 2.1. So it is option (β) which ultimately allows to meet the conditions $A \leq_T C$, $B \leq_T D$, in contrast to PT3.

We have to make sure that actions of type (α) do not cause infinitely many injuries for any condition $P_{e'}$ or $Q_{e'}$, which is accomplished analogously as in PT3. Namely, if $G_{e,s}$ contains no more than one element, then action (β) is chosen. Otherwise $E_{e,s}$ and $F_{e,s}$ are both nonempty, and we decide whether it is preferable to enumerate the largest $x \in E_{e,s}$ in A or the largest $y \in F_{e,s}$ in B, using the same criterion as in PT3. Suppose that it is preferable to enumerate y in B. Then is indeed enumerated in B if D permits it (promptly), otherwise action (β) is performed.

This is not the only way to proceed, but probably the one which allows the simplest formulation of the proof. We might consider enumerating a smaller element of $F_{e,s}$ than y in B, so as to injure conditions of lowest possible priorities. This is not necessary, however, and it would not simplify the argument. Yet another option would be to include the conditions R_e in the priority ranking, and to allow a condition $P_{e'}$ of lower priority to be injured for the sake of R_e, even if this does not guarantee $S'(e'', s+1) = \mathrm{R}$, where $Q_{e''}$ has higher priority than $P_{e'}$ (and similarly with $P_{\ldots}$ and $Q_{\ldots}$ interchanged).

If P_e (Q_e) requires attention at stage $s+1$ and $S(e,s)$ ($S'(e,s)$) equals R, then permissions by C (D) is again necessary in order to settle P_e or Q_e respectively. We have to make sure that every condition $\mathcal{C}$ will ultimately get settled if there are infinitely many stages where we wish to enumerate an element x in A or B in order to settle $\mathcal{C}$, pending permission by C or D. For this purpose we also enumerate sets $\mathcal{P}_e$, $\mathcal{Q}_e$, $\mathcal{R}_e$ and $\mathcal{R}'_e$ such that

$$\begin{aligned}
x \in \mathcal{P}_{e,s+1} - \mathcal{P}_{e,s} &\Leftrightarrow x \text{ wants to enter } A \text{ at stage } s+1 \text{ in order} \\ & \text{to settle } P_e, \\
x \in \mathcal{Q}_{e,s+1} - \mathcal{Q}_{e,s} &\Leftrightarrow x \text{ wants to enter } B \text{ at stage } s+1 \text{ in order} \\ & \text{to settle } Q_e, \\
x \in \mathcal{R}_{e,s+1} - \mathcal{R}_{e,s} &\Leftrightarrow x \text{ wants to enter } A \text{ at stage } s+1 \text{ in order} \\ & \text{to settle } R_e, \\
x \in \mathcal{R}'_{e,s+1} - \mathcal{R}'_{e,s} &\Leftrightarrow x \text{ wants to enter } B \text{ at stage } s+1 \text{ in order} \\ & \text{to settle } R_e.
\end{aligned}$$

The formal definition of these sets will be determined by the construction. Analogously as in the proof of Theorem 2.1 let p_e, q_e, r_e and r'_e be indices such that

$$\begin{aligned}
W_{p_e} &= \mathcal{P}_e \ \ \& \ \ W_{p_e,s} \subseteq \mathcal{P}_{e,s}, \\
W_{q_e} &= \mathcal{Q}_e \ \ \& \ \ W_{q_e,s} \subseteq \mathcal{Q}_{e,s}, \\
W_{r_e} &= \mathcal{R}_e \ \ \& \ \ W_{r_e,s} \subseteq \mathcal{R}_{e,s}, \\
W_{r'_e} &= \mathcal{R}'_e \ \ \& \ \ W_{r'_e,s} \subseteq \mathcal{R}'_{e,s}.
\end{aligned}$$

This involves another application of the recursion theorem.

Construction:

Stage 0. For all e put $i_{e,0} = j_{e,0} = 0$, $S(e,0) = S'(e,0) = \mathrm{N}$ and $M_{e,s} = N_{e,s} = \emptyset$, and enumerate $\langle 0, e\rangle$ in E_e.

Stage $s+1 = 3t+1$. Let $e \leq t$ be minimal such that P_e is not settled, and $W_{\phi_e(u_{e,i_{e,s}}),s}$ contains an element $z \notin U_{e,i_{e,s},s}$. (If no such e exists, go to stage $3t+2$.) Fix the first such z and enumerate it in $X_{e,i_{e,s}}$. If $z \in A_s$, then put $S(e,s+1) = \mathrm{T}$ and record that P_e is settled. Assume otherwise. Add z to M_e if $P(e', z, s)$ holds for no $e' < e$ and z is contained in none of the sets $E_{e'',s}$ with $\hat{S}(e'', s) = \mathrm{N}$. If $M_{e,s+1} - \mathcal{P}_{e,s} \neq \emptyset$, then let x be the largest element of $M_{e,s+1} - \mathcal{P}_{e,s}$, enumerate x in $\mathcal{P}_{e,s+1}$, and find $s' > s$ such that $x \in W_{p_e,s,\text{ at } s'}$. If $C_s \restriction x \neq C_{p(s')} \restriction x$, then enumerate x in A, and again put $S(e,s+1) = \mathrm{T}$ and record that P_e is settled. If $C_s \restriction x = C_{p(s')} \restriction x$, or if $M_{e,s+1} - \mathcal{P}_{e,s} = \emptyset$, then enumerate z in $U_{e,i_{e,s},s+1}$. In either case perform the following action for all $e' \neq e$ such that $P_{e'}$ is not settled; actually, this involves only indices $e' > e$. If x was enumerated in A and $P(e',x,s)$ holds, then put $i_{e',s+1} = i_{e,s}+1$; we say that $P_{e'}$ was injured for the sake of P_e. Also, update $M_{e',s+1}$ and $S(e',s+1)$ according to the definition of these objects; if $P_{e'}$ was not injured, then this involves removing z from $M_{e',s+1}$ if necessary and possibly adding elements if P_e got settled.

Stage $s+1 = 3t+2$. Let $e \leq t$ be minimal such that Q_e is not settled, and $W_{\phi_e(v_{e,j_{e,s}}),s}$ contains an element $z \notin V_{e,j_{e,s},s}$. (If no such e exists, go to stage $3t+3$.) Fix the first such z and enumerate it in $Y_{e,j_{e,s}}$. If $z \in B_s$, then put $S'(e,s+1) = \mathrm{T}$ and record that Q_e is settled. Assume otherwise. Add z to N_e if $Q(e',z,s)$ holds for no $e' < e$ and z is contained in none of the sets $F_{e'',s}$ with $\hat{S}(e'',s) = \mathrm{N}$. If $N_{e,s+1} - \mathcal{Q}_{e,s} \neq \emptyset$, then let x be the largest element of $N_{e,s+1} - \mathcal{Q}_{e,s}$, enumerate x in $\mathcal{Q}_{e,s+1}$, and find $s' > s$ such that $x \in W_{q_e,s,\text{ at } s'}$. If $D_s \restriction x \neq D_{q(s')} \restriction x$, then enumerate x in B, and again put $S'(e,s+1) = \mathrm{T}$ and record that Q_e is settled. If $D_s \restriction x = D_{q(s')} \restriction x$, or if $N_{e,s+1} - \mathcal{Q}_{e,s} = \emptyset$, then enumerate z in $V_{e,i_{e,s},s+1}$. In either case perform the following action for all $e' \neq e$ (actually $e' > e$) such that $P_{e'}$ is not settled. If x was enumerated in B and $Q(e',x,s)$ holds, then put $j_{e',s+1} = j_{e,s}+1$; we say that $Q_{e'}$ was injured for the sake of Q_e. Also, update $N_{e',s+1}$ and $S'(e',s+1)$ according to the definition of these objects.

Stage $s+1 = 3t+3$. Let e be minimal such that $\hat{S}(e,s) = \mathrm{N}$ and $g_{e,s} \in W_{e,s}$. (If no such e exists, then go to stage $3t+4$.) Let e_1 be minimal such that $P(e_1,x,s)$ holds for some $x \in E_{e,s}$, or $e_1 = \infty$ if $P(e_1,x,s)$ holds for no e_1 and $x \in E_{e,s}$. Similarly, let e_2 be minimal such that $Q(e_2,x,s)$ holds for some $x \in F_{e,s}$, or $e_2 = \infty$ if $Q(e_2,x,s)$ holds for no e_2 and $x \in F_{e,s}$.

Assume first that $e_1 \leq e_2$ and $|G_{e,s}| \geq 2$. Let x be the largest element of $F_{e,s} - \mathcal{R}'_{e,s}$, if it exists, enumerate it in $\mathcal{R}'_{e,s+1}$, and find $s' > s$ such that $x \in W_{r'_e,s,\text{ at } s'}$. If $D_s \restriction x \neq D_{q(s')} \restriction x$, then enumerate x in B, and record that R_e is settled. For every e' for which $Q(e',x,s)$ holds put $j_{e',s+1} = j_{e',s}+1$ and $S'(e',s+1) = \mathrm{N}$; we say that $Q_{e'}$ is injured for the sake of P_{e_1}. On the other hand, if $D_s \restriction x = D_{q(s')} \restriction x$, or $F_{e,s} - \mathcal{R}'_{e,s} = \emptyset$, or if $|G_{e,s}| \leq 1$, then enumerate a new element $g_{e,s+1}$ in G_e, where

$$g_{e,s+1} = \begin{cases} 2\langle s,e\rangle + 1 & \text{if } g_{e,s} \text{ is even,} \\ 2\langle s,e\rangle & \text{if } g_{e,s} \text{ is odd.} \end{cases}$$

Note that according to the conventions from section 1, $g_{e,s+1} \notin W_{u,s}$ for all u, which implies that $g_{e,s+1}$ has not been enumerated in any of the sets $U_{e',i_{e',s}}$, $X_{e',i_{e',s}}$ etc.

Now assume $e_1 \leq e_2$ and $|G_{e,s}| \geq 2$. Let x be the largest element of $E_{e,s} - \mathcal{R}_{e,s}$, if it exists, enumerate it in $\mathcal{R}_{e,s+1}$, and find $s' > s$ such that $x \in W_{r_e,s,\text{ at } s'}$. If $C_s \restriction x \neq C_{q(s')} \restriction x$, then enumerate x in A, and record that R_e is settled. For every e' for which $P(e',x,s)$ holds put $i_{e',s+1} = i_{e',s}+1$ and $S(e',s+1) = \mathrm{N}$; we say that $P_{e'}$ is injured for the sake of Q_{e_2}. On the other hand, if $C_s \restriction x = C_{q(s')} \restriction x$, or $F_{e,s} - \mathcal{R}'_{e,s} = \emptyset$, then enumerate a new element $g_{e,s+1}$ in G_e, where $g_{e,s+1}$ is defined as in the previous case. In each case update $M_{e',s+1}$, $N_{e',s+1}$, $S(e',s+1)$ and $S'(e',s+1)$ for all e'. •

Lemma 1: $A \leq_T C$ and $B \leq_T D$.

Proof: Immediatey by construction.

Lemma 2: Each of the sets G_e is finite, R_e is satisfied, and f is čW10-productive for $\overline{A \oplus B}$.

Proof: Suppose that G_e is infinite. Then E_e and F_e are both infinite, since even and odd elements are included alternatively in G_e. Also, $\mathcal{R}_e$ and $\mathcal{R}'_e$ are infinite. Since the enumerations $\{C_s\}_{s\in\omega}$ and $\{D_s\}_{s\in\omega}$ satisfy the analogue of (2.1), C or D will eventually permit to enumerate an element of G_e in $A \oplus B$, as in the proof of theorem 2.1. Then R_e will be settled, and G_e will after all be finite. This proves the first assertion. Then the remaining two assertions follow, since $G_{e,s}$ 'appears' to satisfy the required set relations after each stage s.

Lemma 3: If P_e is never settled, then $\bigcup_{s\in\omega} M_{e,s}$ is finite. Similarly, if Q_e is never settled, then $\bigcup_{s\in\omega} N_{e,s}$ is finite.

Proof: We prove the first assertion; the second is analogous. Suppose that e is minimal such that the assertion fails. If x is removed from $M_{e,s}$ at stage $s+1$, that is, $x \in M_{e,s} - M_{e,s+1}$, then $x \in M_{e',s+1}$ for some $e' < e$ by construction; this happens for only finitely many x. Hence there exist infinitely many x which belong to $M_{e,s}$ for all sufficiently large s, and also infinitely many stages s where another element is added to $U_{e,i_{e,s}}$, hence $\mathcal{P}_e$ is also infinite. But then, analogously as in the proof of Lemma 2, some $x \in \mathcal{P}_e$ will eventually be enumerated in A and P_e will be settled, a contradiction.

Lemma 4: Each of the conditions P_e, Q_e is injured only a finite number of times.

Proof: If P_e is injured for the sake of $P_{e'}$, then $e' < e$ and $P_{e'}$ gets settled, so this can happen only a finite number of times. Similarly, Q_e can be injured only a finite number of times for the sake of some $Q_{e'}$. If $Q_{e'}$ is injured at stage $s+1$ for the sake of P_{e_1}, then s is of the form $3t+3$, $e_1 \leq e'$, and by construction new elements are added to $M_{e_1,s}$. Namely, an element x is enumerated in B which belongs to $F_{e,s}$ for some e, and at least one element of $U_{e_1,i_{e_1,s},s}$ belongs to $E_{e,s}$ and is added to $M_{e_1,s}$, where e_1 is defined as in stage $3t+3$. It follows by Lemma 3 that $Q_{e'}$ can be injured only a finite number of times for the sake of P_{e_1}, and hence only a finite number of times altogether. Similarly, $P_{e'}$ can be injured only a finite number of times.

Now it follows analogously as in PT3 that the conditions P_e and Q_e are met and that A and B satisfy all required conditions. □

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