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Chapter 0

Introduction

The solution set of a linear equation in 3 variables can always be written in terms of two free parameters. Is this also true for polynomial equations? We start with a simple example.

The general solution of the equation of the unit sphere $x^2+y^2+z^2-1=0$ can be written as

$$(x,y,z)=(\frac{2s}{s^2+t^2+1},\frac{2t}{s^2+t^2+1},\frac{s^2+t^2-1}{s^2+t^2+1}).$$

For any two (real) values of s,t, one gets a point on the sphere. Vice versa, any point on the sphere except the north pole $(0,0,1)$ can be obtained this way. We have found a *parametrization* of the sphere in terms of rational functions of two parameters.

Writing X for the solution set (a surface in $\mathbf{A}^3$), a parametrization is a *rational map* τ from the plane $\mathbf{A}^2$ to X. The image is almost all of X, only the north pole is missing.

Now, it is possible to assign a numerical invariant $p_g{\geq}0$, called the *geometrical genus*, to each surface, that has the following property. If there exists a rational map $\tau : Y{\to}X$, such that the image is almost all of X (at most a finite number of curves missing), then $p_g(X){\leq}p_g(Y)$. This imposes a strong restriction for the existence of parametrization. Since $p_g(\mathbf{A}^2)=0$, a surface with $p_g>0$ cannot have a parametrization. A concrete example is $X : x^4+y^4+z^4-1=0$ $(p_g=1)$.

A surface X allowing a parametrization is called *unirational.* If the rational map $\tau : \mathbf{A}^2{\to}X$ has a rational inverse, then we call the parametrization *proper* and the surface *rational.* The above example is a proper parametrization, with the stereographic projection

$$(s,t) = (\frac{x}{1-z}, \frac{y}{1-z})$$

as inverse. The sphere is therefore a rational surface.

Facing the existence of non-unirational surfaces, we ask: how useful is it to know a parametrization? Let us consider an analogous question: the solvability of an algebraic equation in terms of radicals. Here, it turns out that, even if some particular equation is solvable, the solution is sometimes too complicated to be useful. Moreover, there is no convincing argument that radicals are easier to calculate than solutions of more general algebraic equations. Consequently, algorithms for solving an equation in terms of radicals do not have much applications in practice.

Let us turn to a more related example: the parametrizability of an algebraic curve. Although we also have non-rational curves, parametrization algorithms have been implemented in computer algebra systems. We mention the implementation in the MAPLE package CASA ([29]).

It has been observed that a solution of an equation two variables in terms of rational functions in one parameter (which may not always exist!) is useful in many respects. We mention the main applications.

A parametrization by rational functions allows one to compute as many rational points on the curve as we want. When the rational functions have algebraic numbers in their coefficients, we still have infinitely many rational solutions in a fixed algebraic number field. Note that by Falting's theorem, this is never the case for curves of genus $\geq$2.

Computing points on an algebraic curve is used in [29] for plotting algebraic curves.

For the computation of a closed form of the integral of an algebraic function, it is even necessary to compute a parametrization of the curve whose equation is the implicit functional equation. If such a parametrization does not exist, then there is no such closed form built from algebraic functions, logarithms, exponentials ([18]).

With our question (parametrizability of an algebraic surface), the situation concerning possible applications is similar. Many CAD-algorithms assume a parametric representation of a surface ([16]). Parametric representations are extremely useful for tackling any problem in the intrinsic geometry on a surface, like investigating configurations on a surface, computing surface integrals, internal offset curves, path finding. A parametrization makes it easy to find many rational points on a curve (supposing the surface and the parametrization are given over the field of rational numbers). For instance, using the above parametrization of the sphere, one can quickly write down many rational points.

Nevertheless, most of the above problems can also be solved (though not so efficiently) with an implicit representation of the surface, i.e. just knowing the equation of X. If it were very complicated to compute a solution in terms of parameters, one would want to use methods that work also with implicit equations.

This thesis contains an algorithm that decides unirationality of a surface, and computes the parametrization if the answer is yes. The algorithm is fast except for two steps: the resolution of the singularities (a task that has to be done for many applications anyway) and the search for some curve with special properties. The structure of the algorithm is such that the second step is not needed in many cases; in these cases, the algorithm is fast (provided that the singularities are not too complicated).

The search for some curve with special properties is equivalent to solving a certain Diophantine equation over the polynomials. It is possible to solve them using general methods for the solution of systems of algebraic equations. This approach is rather expensive, though. We plan to look for better methods in the future.

0.1 Historical remarks

The parametrization of algebraic surfaces was a central research topic in the algebraic geometry of the 19th century. A. Clebsch computed parametrizations for various surfaces of degree ≤ 6 ([9], [10]). M. Nöether proved that every surface with a linear system of rational curves is rational ([31]).

Their methods are quite explicit.

The problem was theoretically solved by the classical Italian geometers. G. Castelnuovo gave a necessary and sufficient criterion for the (uni)rationality of a surface ([8]). F. Enriques gave an algorithm and analyzed the question which algebraic extensions of the ground field has to be made ([14]).

The Italians also invented the useful tool of adjunction. They applied this concept also to other kinds of surfaces. Their efforts culminated in the birational classification of surfaces.

Quite generally, the equivalence between rational maps from X to some projective variety and linear systems of divisors on X is an extremely fruitful principle. Applied to parametrizations, it establishes a correspondence between parametrizations of some surface and linear systems of plane curves.

Many older results on surface parametrization were first formulated as statements on linear systems of plane curves. The achievements of the classical Italian schools was appreciated as a major breakthrough ([7]). The ball was taken by Nagata ([30]), who also laid a more solid foundation to the birational theory of linear systems of plane curves.

A typical linear systems question is the following: Given are two linear systems of plane curves. Is there a birational transformation of the projective plane that transforms one to the other? A related question is to find "normal forms" of linear systems modulo birational auto-maps of the plane.

Rather soon, some numerical characters were discovered that are invariant under birational transformations: the genus, the self-intersection number, the dimension, the superabundance. These characters are related by (a variant of) the Riemann-Roch theorem. Their equality is a necessary condition for birational equivalence of two linear systems.

The following remarkable results were known to the Italians:

- Any linear system with dimension ≥ 1 of rational curves can be transformed to a system of curves of degree n with an $n-1$-fold base point, or to a system of conics.
- Any linear system with dimension ≥ 2 of elliptic curves can be transformed to a system of cubic curves, or to a system of quartic curves with two double base points.

Traditionally, these results have always been proven by constructing a sequence of quadratic transformations (this is a very simple birational auto-maps of the plane). This is very constructive, but also very tedious. Recently, some techniques have been found that allow another approach (Reider's lemma, see [33]). This caused a revival of classification results of linear systems ([25],[32],[37]).

Our contribution is a new proof without quadratic transformations of the results above. (This simply could not be avoided.) To my knowledge, it is the first such proof.

0.2 The state of the art

The methods of Castelnuovo and Enriques are more geometric than algebraic in nature. If one wants to parametrize a specific surface given by an equation, these methods are sometimes not very useful. The main reason for this is that the methods assume two things: the surface has to be smooth, and it has to have a sufficiently high degree in some projective space (of high dimension). The second assumption can be eliminated without changing much, but to generalize to singular surfaces, the concept of adjunction has to be changed.

The monography of F. Conforto [11] gives a complete account on rational surface of degree up to four and their parametrizations, based on the work of Clebsch and Nöether. It also works out the details of Enriques' algorithm. However, it is interesting to see that Conforto does not really use this method for the parametrization of concrete surfaces.

In more recent times, people have written algorithms that compute parametrizations for quadric and cubic surfaces ([1] and [35]). The general problem was also posed as an "open problem" in D. Eisenbud's famous paper [12]. There, he suggests considering partial solutions, namely the parametrization of certain well-known types of rational surfaces.

Having the history of the parametrization problem in mind, it appears as a pleasant surprise that there exists now an efficient parametrization algorithm that is relatively simple. This is mainly due to the fact that the concept of adjunction has a good generalization to the singular case, based on the notion of "Italian pairs".

Actually, the definition of Italian pairs (section 2.2) refers to smooth surfaces. The spirit is somehow similar to the treatment of singular plane curves: the algebraic manipulations are done on the plane singular curve, but the geometric reasoning is done on the normalization, which is a smooth curve. Usually, one does not care about an explicit embedding of the smooth curve. It is only important that it exists, and its geometric properties have some consequences for the singular plane curve.

We will treat algebraic surfaces in a similar way: algebraic manipulations are done on the singular surface in 3-space, but the geometric reasoning is done on some desingularization. A difference to the curve case is that desingularizations are not unique. The difference between two desingularizations of a singular surface can be expressed in terms of exceptional curves.

In the classical understanding of adjunction of divisors, exceptional curves are a mess because they defy many nice properties. This holds especially for divisors describing a projection to a singular surface in $\mathbf{P}^3$, since these divisors are not very ample.

In chapter 2, we remedy this situation not by correcting the adjoint divisor, but changing the surface where it lives! The idea is to force the nice properties mentioned above by blowing down bad exceptional curves.

An essential subtask is the resolution of singularities, which is needed to compute the adjoints. There are different existence proofs, and none of them is easy. An excellent overview on the various approaches is [27].

An algorithm for the resolution is posed as another open problem in [12]. The algorithm presented in chapter 4 is a essentially due to H. Hironaka ([22]), based on work of O. Zariski ([40]) and B. Levi.

One difficulty in the resolution algorithm is the technical difficulty of blowing up smooth curves. Geometrically, this is completely analogous to the blowing up of lines (which is quite easy). Algebraically, lines and other smooth curves may look very different.

There are several solutions to this technical problem (a very explicit one can be found in [24], pp 31-32). Here, we apply a trick transforming a smooth curve to a line, or at least to something that can be treated like a line (section 4.3).

Given a resolution, it is easy to compute the adjoints. The algorithm presented in chapter 4 is an example of applied sheaf theory in disguise: the main task is the merging of local results.

In examples, the representation of the adjoints in the result is often much easier than the resolution. Therefore, it is desirable to have an algorithm for the computation of adjoints without a resolution. I was looking for such an algorithm without much success, but I will continue this research in future.

The algebraic concept of adjoints (adjoint ideals, adjoint vector-spaces, adjoint sheafs) is very old (for instance, it is explicitly used in [8]). The only modern treatment of adjoints I know is [5] (thanks to Emil Volcheck for finding this reference). I have the feeling that this concept deserves much more attention in modern algebraic geometry.

0.3 Input and output specification

Algebraic surfaces can be represented in various ways. When writing an algorithm doing something on surfaces, it is necessary to fix a particular representation. Choosing a convenient representation may simplify the algorithm. On the other hand, the representation should not be too restrictive, because then one might loose generality.

Here we assume that the surface is given as a hypersurface in 3-space. This has the advantage that the surface can be described by a single equation. The loss of generality is not severe, because there are algorithms that produce a birationally equivalent hypersurface for algebraic varieties in higher dimensional space ([21]).

We also assume that the surface is *absolutely irreducible*, i.e. the equation does not factor over the algebraic closure of the ground field k. This assumption is necessary to ensure that we have a function field that remains a field when the ground field is algebraically extended.

The assumption that the surface be projective is here for the sake of convenience. Homogenizing an inhomogeneous equation is a rather easy process, and it does not change anything from a birational point of view.

The field of definition k should be algebraically closed. If not, then it might be necessary to pass to an algebraic extension in order to finish the parametrization task. As mentioned above, it is sometimes necessary to find a curve with special properties. The equation of this curve can have coefficients in an algebraic extension.

Since field extension is costly and makes the output more complicated, we try to avoid it if it is not necessary. It helps when a smooth point on the surface is known.

The characteristic zero assumption is certainly a loss of generality. It is made because it simplifies some proofs and algorithms. A generalization to positive characteristic is a goal of future research.

On the algorithmic side, we use certain operations. These include field operations, Gröbner bases calculation and certain operations on ideals (given by a basis) that can be done with Gröbner bases: intersection of ideals, elimination of variables, and the like. See [6] or [4] for a description of these algorithms.

For the resolution of singularities, we use decomposition of a variety into irreducible varieties (also described in [4]). For the search for a curve with special properties, we have to find a solution of an algebraic system of equations. By elimination of variables, this can be reduced to the solution of equations in one variable.

The output parametrization will be a 4-tuple of polynomials in two variables s, t, satisfying the homogeneous equation of the surface. Note that in the projective case we can always clear the denominators of a parametrization in terms of rational functions.

I have no implementation of the algorithm. The examples were computed by applying the algorithm interactively in Maple and Macaulay [3] sessions.

0.4 Guideline for reading

Language has become quite an issue in algebraic geometry (cf. [13]). Intrinsically, there is the algebraic language (speaking of polynomials) and the geometric language (speaking of curves). Searching for a suitable formalism, geometers have introduced several other linguistic concepts into the topic. The modern language of specialists is the language of schemes and sheaves.

In this thesis, I tried to stick to the two common (i.e. not for specialists) languages as close as possible. Other concepts are introduced only when it is necessary, and they will only be used where they are necessary. Here is the result of this try:

The algorithms are explained in algebraic language.

Many (new) proofs are given in geometric language. The geometric terminology of divisors has to be introduced. Some classical theorems have to be used. They had to be "translated" into geometric language.

Two relevant classical theorems are *not* used, because we want to give constructive proofs for them. These are Castelnuovo's rationality criterion and the resolution of singularities theorem.

Some proofs make essential use of sheaf theory. They are, however, concentrated in a single section (4.4). In section 4.1, we use several results from sheaf theory, but translated into algebraic language.

Here is a summary of the particular chapters.

1: Algorithms and Examples. We describe an algorithm for deciding (uni)rationality. If the surface is rational or unirational (this is actually the same for algebraically closed ground fields), then we describe an algorithm that reduces the parametrization task to the parametrization of some simpler sort of surface (base cases). Some of these are simple enough so that for them we can finish the parametrization.

The algorithms described in this chapter are simple and efficient, modulo a black box (*adjoint vector-spaces*). To compute them, a resolution of singularities must be known; but this topic is treated later (chapter 5). For singularities of certain simple types, it is transparent what these adjoint vector-spaces are.

Most statements are given without proofs.

2: Geometry on a surface. In the first section, we introduce the required geometric concepts, which may also be found in books on algebraic surfaces. Then, we define Italian pairs and adjunction of pairs. The main part of the proof of Castelnuovo's criterion is given here. Especially, we prove the correctness of the algorithms presented in chapter 1.

In section 5, we discuss ruled surfaces more closely. We also give an improvement of the sub-algorithm from chapter 1 dealing with the ruled case.

3: Del Pezzo surfaces. This case was left out in the chapters 1 and 2. It is somewhat special, as you will see.

Here, we finish the proof of Castelnuovo's criterion.

4: Resolution of the singularities. Here we give a method for the desingularization of a surface in space, by blowing up points and smooth curves in the singular locus.

We apply this method to compute the adjoint vector-spaces that were used in the previous chapters. It is essential to divide the problem into local problems, and then merge the local results.

In the last section, we give a sheaf-theoretic treatment of adjoints and prove all previously unproven theorems.

0.5 Acknowledgments

First of all I have to thank Bruno Buchberger for creating the fruitful scientific environment in the "Research Institute for Symbolic Computation" (RISC) of the Johannes Kepler University at Linz, Austria, which provided a good frame for this project.

The comments and hints of my advisor Jochen Pfalzgraf have influenced this thesis a lot. I also want to thank him for the warm and heartful atmosphere in our cooperation.

Franz Winkler was the first who suggested the project of surface parametrization. The parametrization algorithms developed by him, Michal Mnuk and Rafael Sendra ([29],[28]) for algebraic curves of genus zero have in many respects served me as a model for this algorithm.

I had many enlightening discussions with Emil Volcheck about singularities . His research and experiences and computation with singularities [38] have been of great value.

Yasushige Watase is partially responsible for the "sheaf section", by hinting to the sheaf theoretic nature of certain issues.

Hoon Hong made many critical remarks about language. If you are a non-expert and nevertheless you are able to read this, thank him.

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Finally, I want to thank Hélène Esnault and Eckart Vieweg, who were teacher's at a summer school ([15]). They taught a lot of useful things. In particular, they explained two theorems which simplified some proofs and algorithms a lot (the Grauert-Riemenschneider vanishing theorem and Reider's theorem).

0.6 Acknowledgments for the Second Edition

I finished this thesis in January, 1996. In the meantime, I received positive echoes from various people. Especially, I want to thank Emil Volcheck, who gave me a large list of errors (mainly concerning language). The main difference to the original version is the correction of these errors. Emil also found a modern treatment of adjoints [5] in the literature.

Parts of the thesis is submitted to the Journal of Symbolic Computation, but the refereeing process is still going on.

Some of the algorithms have been improved in the meantime. The problem of finding rational points on a conic over $k(t)$ has been solved theoretically in [20]. Currently research is done for improving the algorithm that computes the adjoints.

Chapter 1

Algorithms and examples

We describe an algorithm for the parametrization of algebraic surfaces. The task is divided into several subtasks (for instance, the first subtask is to decide whether the given surface is rational, i.e. whether there exists a parametrization). Several tasks are done here, some more difficult tasks must be delayed to later chapters.

The main part of the parametrization algorithm is presented in the first chapter, because I think that it should be possible to compute examples and to have a systematical overview at a very early stage.

Here is a summary of the particular sections.

1: Vector-spaces and rational maps. We define an appropriate data structure for the description of rational maps between projective varieties. Rational maps are interesting for us, because parametrizations and their inverses (if they exist) are rational maps.

2: Numerical invariants. We introduce numerical characters attached to a surface. Rationality can always be decided just by looking at some of these numbers (this is Castelnuovo's rationality criterion).

3: Adjoint maps. We introduce a set of distinguished rational maps from X to some projective space. In the lucky case, some of these maps lead to a parametrization very quickly.

4: More adjoint maps. Here we discuss the two other not so lucky cases. In one of them, the adjoint maps solve "half of the parametrization task". For the second half, we have to solve a Diophantine equation over the polynomials.

The third case must be delayed to chapter 3.

5: A parametrization algorithm. Finally, we present the parametrization algorithm. It just finds out in which of the three cases we are, and then forks to the appropriate sub-algorithm.

1.1 Vector-spaces and rational maps

In this section we introduce a convenient algebraic description for rational maps between projective varieties: such a map is given by a vector-space of forms in the graded function ring.

Let X be a surface in $\mathbf{P}^3$, given by a homogeneous equation $F(x, y, z, w)$ of degree d. The graded ring $S(X)$ of forms on X is the graded polynomial ring $k[x, y, z, w]$ modulo the homogeneous ideal generated by F. The grading is a decomposition

$$S = \bigoplus_{i \in \mathbf{N}} S_i, S_i S_j \subset S_{i+j}.$$

The homogeneous components S_i are finite dimensional vector-spaces over k. Let n be a natural number. Let $V \subset S_n$ be a subspace of positive dimension $r+1$. V has a basis $\{v_0, \ldots, v_r\}$ consisting of forms (homogeneous polynomials modulo F) of degree n.

V defines a rational map $i_V : X \to \mathbf{P}^r$ by

$$(x : y : z : w) \mapsto (v_0(x, y, z, w) : \ldots : v_r(x, y, z, w)).$$

Since the forms in the basis are linearly independent, the image $i_V(X)$ does not lie in a proper linear subspace. The map is defined on all points on X where at least one form in V does not vanish. The closure of the image in $\mathbf{P}^r$ is a projective variety. Its equations can be computed by eliminating x, y, z, w in the ideal generated by

$$F, x_0 - u_0(x, y, z, w), \ldots, x_r - v_r(x, y, z, w).$$

It helps if a good degree bound m for the equations is known. In this case, it suffices to compute the linear relations of the vector-space V^m (the multiplication has to be understood as multiplication of forms).

The domain of definition of i_V may actually also contain points where all forms in V do vanish. The forms may have a common factor that vanishes at p. Division by the common factor does not affect the ratio, and after division some form may have a nonzero value.

To see the common factor, it may be necessary to pass to the local ring at p. The common factor is not always a globally defined form. Cf. chapter 4 for a discussion on local and global equations.

Let $p \in X$ be such that i_V is defined at p. Let $V{\downarrow}p \subset V$ be the sub-vector-space of all forms that vanish at p. Its dimension is $\dim V - 1$. Its associated map is the composite of i_V with the projection from $i_V(p)$. We say that $V{\downarrow}p$ is obtained by *assigning p as a base point.*

Example 1: Let X be the unit sphere, homogeneously given by

$$F(x, y, z, w) = x^2 + y^2 + z^2 - w^2 = 0.$$

Let

$$V = S_1 = \langle x, y, z, w \rangle.$$

The associated map is the embedding $X \hookrightarrow \mathbf{P}^3$. Let $p := (0 : 0 : 1 : 1) \in X$. Then the vector-space with base point p is

$$V{\downarrow}p = \langle x, y, w - z \rangle.$$

Its associated map $\pi = i_{V\downarrow p} : X \to \mathbf{P}^2$

$$(x : y : z : w) \mapsto (s : t : 1) = (x : y : w - z)$$

is the projection from p. Since the generic line through p meets X in exactly one other point, the map is birational.

We can exploit this to parametrize X properly, by computing the inverse of π. To do this, we look for linear polynomials in the $k(s, t)$-ideal

$$(F, x - s, y - t, w - z - 1).$$

We compute the Gröbner base with total degree order, treating s, t as constants. The result is

$$\{y - t, x - s, 2z + (1 - s^2 - t^2), 2w - (1 + s^2 + t^2)\}.$$

All elements are linear. We can read off the parametrization

$$(x : y : z : w) = (2s : 2t : s^2 + t^2 - 1 : s^2 + t^2 + 1).$$

Any surface of degree 2 can be parametrized in this way: by inverting the projection from a smooth point on it.

Remark: The computation of a parametrization via its inverse is quite common in curve parametrization. Of course, only proper parametrizations can be computed that way.

1.2 Numerical invariants

We consider some particular vector-spaces and define several numerical invariants in terms of their dimensions. Using them, we describe an algorithm that decides whether a given surface has a parametrization.

Let m be a positive integer. We give an (oversimplifying!) definition of the *adjoint ideal* $A_m \subset S$ as follows. A_m is the ideal of all forms that have an $m(r-1)$-fold curve on each r-fold curve singularity of X, and an $m(r-2)$-fold point on each r-fold singular point of X. A precise definition and an algorithm for the computation is given in chapter 6. The given definition must be taken with a grain of salt, but it gives some reasonable intuition. For instance, it suggests the guess that $A_{m_1} \cdot A_{m_2} \subset A_{m_1+m_2}$. And this is indeed the case. Also, $A_0 = (1)$, and $A_m = (1)$ for all m when X is smooth.

For $n \in \mathbf{N}$, we define the *adjoint vector-space* $V_{n,m}$ as the vector-space of forms in A_m of degree $n + m(d-4)$. It is a finite dimensional vector-space.

A numerical character of a surface is called a *birational invariant* if it is preserved under birational maps. Since in particular proper parametrizations are birational maps from the plane, we see that rational surfaces share all birational invariants with the plane.

Many birational invariants can be defined in terms of the numbers $v_{n,m} := \dim V_{n,m}$, called *adjoint dimensions.*

We start with the *geometric genus* $p_g := v_{0,1}$. A generalization is the m-th *plurigenus* $P_m := v_{0,m}$ $(m > 0)$. The most significant birational invariant is the *Kodaira dimension* κ, defined as $\limsup_{m\to\infty} \frac{\log P_m}{\log m}$. It takes the values $-\infty, 0, 1, 2$. The statement $\kappa = -\infty$ is just another way of saying $P_m = 0$ for all m.

Example 2: Let X be smooth. Then all adjoint ideals are (1), because there are no singularities. Hence $V_{n,m} = S_{n+m(d-4)}$. The adjoint dimensions can be computed by subtracting the dimension of the vector-space consisting of multiples of F from the vector-space of all homogeneous polynomials of degree $n+m(d-4)$:

$$v_{n,m} = \binom{n + m(d-4) + 3}{3} - \binom{n + m(d-4) - d + 3}{3}.$$

For $d = 1$ (the plane), we have

$$v_{n,m} = \binom{n - 3m + 2}{2}$$

and $P_m = 0$ for all $m > 0$.

For $d=2$ (the smooth quadric), we have

$$v_{n,m} = \begin{cases} (n-2m+1)^2 & \text{for } n\geq 2m, \\ 0 & \text{otherwise,} \end{cases}$$

and $P_m = 0$ for all $m > 0$.
For $d = 3$, we have

$$v_{n,m} = \begin{cases} \frac{3(n-m)(n-m+1)+2}{2} & \text{for } n\geq m, \\ 0 & \text{otherwise,} \end{cases}$$

and again $P_m = 0$ for all $m > 0$.
For $d = 4$, we have

$$v_{n,m} = \begin{cases} 2(n^2+1) & \text{for } n > 0, \\ 1 & \text{for } n = 0, \end{cases}$$

and $P_m = 1$ for all $m > 0$ ($\kappa = 0$). These surfaces cannot be rational.
For $d\geq 5$, the binomial coefficients in the above formula can be replaced by a polynomial expression except for $m = 1$, $n = 0$. We have

$$p_g = \binom{d-1}{3}$$

and

$$P_m = ((d-4)^3 + 4(d-4)^2)\frac{m(m-1)}{2} + \frac{(d-4)^3 + 6(d-4)^2 + 11(d-4) + 2}{6}.$$

Asymptotically speaking, P_m grows proportional with m^2. Therefore, $\kappa = 2$. These surfaces cannot be rational, either.

In the above examples, $v_{n,1}$ is a polynomial function $P(n)$ for $n > 0$. This is true for all surfaces. $P(n)$ has degree 2, and the leading coefficient is $\frac{d}{2}$. The constant term is even more interesting, because it is another birational invariant. It is easy to evaluate it:

$$P(0) = d + 2P(1) - P(2) = d + 2v_{1,1} - v_{2,1}.$$

We define the *arithmetrical genus*

$$p_a = P(0) - 1.$$

The difference $q := p_g - p_a$ is called the *irregularity* of X (the second most significant birational invariant). It is always nonnegative. For smooth surfaces in $\mathbf{P}^3$, we can quickly verify that $p_a = p_g$ and $q = 0$.

Rationality and unirationality can be characterized in terms of the invariants introduced so far.

Theorem 1.1 (Castelnuovos rationality criterion) *Let X be a surface over an algebraically closed field of characteristic 0. The following are equivalent:*

a) *X is unirational.*

b) *X is rational.*

c) $p_a = P_2 = 0$.

d) $\kappa = -\infty$ *and* $q = 0$.

The implications (b)→(a) and (d)→(c) are trivial. Accepting the fact that the numerical characters are birational invariants, (b)→(d) is clear, since we have verified that (d) holds for the plane. The stronger assertion (a)→(d) follows from the fact that the invariants P_m and q do not increase under rational maps with dense image.

The hard part of the theorem is the implication (c)→(b). We will give a constructive proof for it in the chapters 2-4, along with a discussion of other (shorter) proofs.

Remark: Castelnuovo's criterion is a cornerstone in the birational classification of surfaces. The numbers κ, q, p_g, P_m are dimensions of birationally invariant objects (varieties or vector-spaces) attached to the surface. Since rational surfaces have the lowest possible value for all these invariants, this birational equivalence class is the one with the "poorest possible birationally invariant structure".

For the sake of completeness, we write down the obvious algorithm for decision of unirationality of a given surface.

Algorithm *deciderationality.*

Input:
F, an absolutely irreducible homogeneous polynomial in 4 variables, of degree d.

Output:
"yes" if the surface of F allows a parametrization, possibly in an algebraic extension of the ground field.
"no" otherwise.

compute A_1;
compute $v_{0,1}, v_{1,1}, v_{2,1}$;
if $v_{0,1} > 0$ then
 p_g **does not vanish.**

```
    return "no"; exit;
if d + 2v_{1,1} − v_{2,1} + 1 < 0 then
    p_a is negative.
    return "no"; exit;
compute A_2;
compute v_{0,2};
if v_{0,2} > 0 then
    P_2 does not vanish.
    return "no"; exit;
return "yes"; exit;
```

As a first application of Castelnuovo's criterion, we see that a smooth surface in $\mathbf{P}^3$ of degree d is rational iff $d \leq 3$.

1.3 Adjoint maps

Using the adjoint vector-spaces, we can do more than merely decide rationality. They can also be used to compute a parametrization efficiently. This section explains how.

Knowing that there exists a parametrization, it is possible to search for it by brute force. We could make an ansatz of a parametrization with rational functions with some degree N and indeterminate coefficients. The statement that we have a parametrization can be expressed as a finite set of algebraic equations (the functions have to satisfy the equations of X) and inequalities (the Jacobi determinant must not be identically zero, in order to ensure that the image is 2-dimensional). If this system of equations has no solution, we increase N. This algorithm is correct, but it is far too inefficient to be of any use. Even the parametrization of the sphere (example 1) becomes a huge calculation.

A much better method uses adjoint vector-spaces.

The embedding $i : X \hookrightarrow \mathbf{P^3}$ is obviously associated to the vector-space

$$V = S_1 = \langle x, y, z, w \rangle.$$

It is consistent to interpret V as the adjoint vector-space $V_{n,m} = V_{1,0}$. The idea is that increasing m simplifies i.

Example 3: Let X be of degree 5, given by

$$F = x(y^2 - xz)(z^2 - yw) + w(xw - yz)^2.$$

X has a double curve C, the twisted cubic curve $(x : y : z : w) = (1 : t : t^2 : t^3)$. The first two adjoint ideals are:

$$A_1 = (y^2 - xz, z^2 - yw, xw - yz), A_2 = A_1^2.$$

(A_1 is the ideal of C.) Since

$$v_{0,1} = v_{0,2} = 0, v_{1,1} = 3, v_{2,1} = 12,$$

we know that $p_a = P_2 = 0$, and a parametrization exists.
The rational map $i_{1,1} : X \to \mathbf{P}^2$ associated to $V_{1,1}$

$$(x : y : z : w :) \mapsto (s : t : 1) = (y^2 - xz : z^2 - yw : xw - yz)$$

happens to be birational. A proper parametrization can be found by inverting $i_{1,1}$.
To do this, we compute a Gröbner base of the ideal

$$(F, y^2 - xz - s, z^2 - yw - t, xw - yz - 1),$$

treating s, t as constants (total degree order). The base contains the three homogeneous linear polynomials

$$w + stx, (ts^2 - s)y + (s^3 + 1)w, (ts^2 - s)z + (-t - s^2)w.$$

Solving for x, y, z, w yields

$$(x : y : z : w) = (st - 1 : t(s^3 + 1) : -t(t + s^2) : -st(st - 1)).$$

In general, the *adjoint map* $i_{1,1}$ might still be too complicated. In this case, we keep increasing m.

The maximal sequence of nonzero vector-spaces

$$V_{1,0}, V_{1,1}, V_{1,2}, \ldots$$

is called the *adjoint series* of X. It is necessarily finite (lemma 2.3). Let μ be such that $V_{n,\mu}$ is the last element in the sequence (i.e. $v_{1,\mu} > 0$ and $v_{1,\mu+1} = 0$). We will call μ the *adjoint depth.* $i_{1,\mu}$ is the simplest possible adjoint map that can be obtained in the described way.

Recall that a rational normal curve is a curve $C \subset \mathbf{P^n}$ of degree n. It is necessarily rational and smooth. Its ideal is generated by forms of degree 2. With a suitable projective coordinate transformation in $\mathbf{P}^n$, one can achieve that

$$C = \{(1 : t : \ldots : t^n) \mid t \in k \cup \infty\},$$

supposing that at least a point on C is known or n is odd.

Theorem 1.2 *Let X be a rational surface. Let μ the adjoint depth. Suppose that $Y := i_{1,\mu}(X) \subset \mathrm{P}^{v_{\mu,1}-1}$ is two-dimensional. Then $i_{1,\mu}$ is birational onto Y. The generic hyperplane section of Y is a rational curve. Moreover, one of the following is true:*

a) *$v_{1,\mu} = 3$ and Y is the plane.*

b) *$v_{2,2\mu+1} = 3$ and $i_{2,2\mu+1}$ is birational onto the plane.*

c) *$v_{1,\mu} = 4$ and $Y \subset \mathrm{P}^3$ has degree 2.*

d) *Y has a* line ruling*, i.e. an infinite set of lines, parametrized by P^1. The image of $i_{2,2\mu+1}$ is a rational normal curve. The fibers of $i_{2,2\mu+1}$ on X correspond birationally (via $i_{1,\mu}$) to the lines of the ruling.*

Example 4: Let X be of degree 4, given by

$$F = (y^2 - xz)(z^2 - yw) + (xw - yz)^2.$$

X has goes doubly through the twisted cubic curve. The first two adjoint ideals are again:

$$A_1 = (y^2 - xz, xw - yz, z^2 - yw), A_2 = A_1^2.$$

X passes the test for rationality. Since $v_{1,1} = 0$, we have $\mu = 0$. The "adjoint map" $i_{1,0}$ is the embedding in $\mathbf{P}^3$, so it has two-dimensional image X and the theorem applies.
Case (a) is out. Since $v_{2,1} = 3$, we cannot decide whether we are in case (b) or in case (c) by looking at the adjoint dimensions. We compute the equations of the image of

$$i_{2,1} : X \to \mathbf{P}^3,$$

$$(x : y : z : w) \mapsto (w_0 : w_1 : w_2) = (y^2 - xz : xw - yz : z^2 - yw)$$

by reducing $\{w_0^2, w_0w_1, w_0w_2, w_1^2, w_1w_2, w_2^2\}$ modulo F and obtain the equation

$$w_0w_2 + w_1^2 = 0.$$

The image is thus a conic $C \subset \mathbf{P}^3$, and case (c) holds.
To see the line ruling, we compute a proper parametrization of C. In our case, $t = \frac{w_0}{w_1}$ is a parameter, and

$$C = \{(t^2, t, -1) \mid t \in k \cup \infty\}.$$

The map $\pi : X \to \mathbf{P}^1$, obtained by composing the inverse parametrization with $i_{2,1}$, is

$$(x : y : z : w) \mapsto (t : 1) = (w_0 : w_1) = (y^2 - xz : xw - yz).$$

A Gröbner base with total degree order of the $k(t)$-ideal

$$(y^2 - xz - t, xw - yz - 1, F)$$

contains the linear homogeneous polynomials

$$tz + t^2 w + y, -t^3 w + x$$

representing a line on X for each t.
For a parametrization, set $z = s$, $w = 1$ and solve for x, y:

$$(x : y : z : w) = (t^3 : -t^2 - st : s : 1).$$

As suggested by example 4, the theorem can be used to compute a parametrization of X (supposing that the theorem applies). When $\mu \geq 1$, then it is not necessary to compute the equations of Y.

In case (a), there is no equation.

In case (b), we skip Y anyhow.

In case (c), we compose $i_{1,\mu}$ with the projection from a smooth point of Y. For any quadric, projection from a smooth point is birational onto the plane. Hence, the composition $X \to \mathbf{P}^2$ is birational. The parametrization can be computed by inverting.

It is possible to compute the composed map without computing the equation of Y, by assigning a smooth point of X as a base point to the adjoint map $i_{1,\mu}$.

Case (d) must be examined more closely. Let n be the degree of the rational normal curve $C = i_{2,2\mu+1}(X)$. Then $v_{2,2\mu+1} = n + 1$. We may choose a basis of $W := V_{2,2\mu+1}(X)$ such that

$$C = \{(1 : \ldots : t^n) \mid t \in k \cup \infty\}.$$

If an algorithm for curve parametrization is available, one may start computing an arbitrary proper parametrization

$$C = \{(p_0(t) : \ldots : p_n(t)) \mid t \in k \cup \infty\}$$

By clearing denominators and dividing by the greatest common divisor, we may assume that $p_0, \ldots, p_n$ are polynomials without a common divisor. The maximum degree is equal to $\deg C = n$. Moreover, the p_i are linearly independent. The rest is simple linear algebra.

The quotient $\frac{w_1}{w_0}$ of the first two basis elements of the basis is one parameter for the surface X.

The second parameter is a quotient of two forms $v_0, v_1 \in V := V_{1,\mu}$. Not any two forms work. A sufficient condition is that the two vector-spaces

$$\langle v_0 w_0, v_1 w_0 \rangle, V \cdot \langle w_1, \ldots, w_n \rangle$$

be complementary vector-spaces in $V \cdot W$. Such forms can be found within a basis $\{v_0, \ldots, v_N\}$. Just take v_i when $v_i w_0$ is not contained in the vector-space

$$V \cdot \langle w_1, \ldots, w_n \rangle.$$

The following algorithm parametrizes a rational surface if $i_{1,\mu}(X)$ is two-dimensional.

Algorithm *adj2dim.*

Input:

An equation $F(x, y, z, w)$, representing a rational surface X.

The integer m. It must be the adjoint depth of X.

A smooth point $p = (x_0 : y_0 : z_0 : w_0)$ on X might or might not be required.

Output:

4 polynomials $\overline{x}, \overline{y}, \overline{z}, \overline{w} \in k[s, t]$, representing a proper parametrization of X.

The algorithm works only if $i_{1,\mu}(X)$ is two-dimensional.

$V := V_{1,m}$;
if $v_{1,m} = 3$ then
  Case (a).
  return($inverse(i_V)$);
  exit;
$W := V_{2,2m+1}$;
if $v_{2,2m+1} = 1$ then
  Case (c).
  require(p);
  return($inverse(i_{V\downarrow p})$);
  exit;
Compute the equations of $i_W(X)$.
if $i_{2,2m+1} \geq 3$ then
  RATNOR := linear relations in W^2;
if $i_{2,2m+1} = 3$ and RATNOR is empty then
  Case (b).
  return($inverse(i_W)$);
  exit;
Case (d).
$n := v_{2,2m+1} - 1$;
Computation of the first parameter by parametrizing RATNOR.
CPARAM := *parametrize*(RATNOR);

W:=*basetrafo*(CPARAM,W);
TFUNCT := w_1/w_0;
Computation of the second parameter.
INDEP := empty;
SUB := $V\cdot\langle w_1,\ldots,w_n\rangle$;
$i := 0$;
while *length*(INDEP)<2 do
    if $v_i w_0 \notin$ SUB then
        append v_i to INDEP;
    $i := i+1$;
SFUNCT := INDEP[2]/INDEP[1];
Parametrization via inverse.
BIRAT := $(x:y:z:w)\mapsto$(SFUNCT,TFUNCT);
return(*inverse*(BIRAT));
exit;

Example 5: Let X be of degree 6, given by

$$F(x,y,z,w) = yz^3w^2 + (xy+z^2)^3.$$

X passes triply through the line $x=z=0$ and doubly through the plane conic $xy+z^2=w=0$. Rationality is easily checked. Since

$$V_{1,1} = \langle x^2w, z^2w, xzw, x^2y+xz^2, xyz+z^3\rangle, V_{1,2}=0,$$

we see that $\mu=1$. By

$$W = V_{2,3} = \langle z^6w^2, xz^5w^2\rangle,$$

we are in case (c), deg$Y=3$ and $n=1$. The rational normal curve is a line, and we take

$$t = \frac{xz^6w^2}{z^6w^2} = \frac{x}{z}$$

as the first parameter.
We test sequentially the basis elements of $V=V_{1,1}$: $x^2w\cdot x$ is not in $V\cdot z$, hence we buy it. The next two are in $V\cdot z$. The element x^2y+xz^2 is not. The second parameter is

$$s = \frac{x^2w}{x^2y+xz^2} = \frac{xw}{xy+z^2}.$$

To invert the rational map

$$(x,y,z,w)\mapsto(\frac{xw}{xy+z^2},\frac{x}{z}) = (\frac{xzw}{z(xy+z^2)},\frac{x(xy+z^2)}{z(xy+z^2)}),$$

we compute the Gröbner basis of the ideal

$$(F, z(xy+z^2)-1, xzw-s, x(xy+z^2)-t),$$

treating s, t as constants (total degree order). The base contains the three linear homogeneous polynomials

$$s^3y + t^3w, (-s^2t^2 - t^5)w + s^3x, s^3tz + (-s^2t^2 - t^5)w.$$

Solving for x, y, z, w yields

$$(x : y : z : w) = (s^2t^2 + t^5 : -t^3 : s^2t + t^4 : s^3).$$

We note that a parametrization for the two-dimensional case can be done within the ground field or a field extension in a degree 2 extension of the ground field. Degree 2 suffices to parametrize any rational curve or to find a smooth point on a quadric.

The highest degree N of the polynomials $\overline{x}, \overline{y}, \overline{z}, \overline{w}$ occurring in the parametrization is called the *degree of the parametrization.*

The examples show that there is no obvious relation between the degree d of X and the degree N of the parametrization. This is a contrast to the curve case, where the degree of any proper parametrization coincides with the degree of the curve. The degree of an improper parametrization is always a multiple of the degree of the curve.

In the surface case, however, we have already seen several parametrizations where N is smaller than d. One can prove that $N^2 \geq d$ always holds, and there exist rational surfaces of degree N^2 where this bound is sharp.

The smallest possible degree of a proper parametrization is closely related to the adjoint depth.

Proposition 1.1 *Let X be a rational surface of degree d. Let μ be its adjoint depth. Suppose that there exists a parametrization of degree N. Then $N \geq 3\mu$.*

Vice versa, the parametrizations computed by adj2dim have degree smaller or equal $1 + 3\mu$ in case (a), $2 + 3\mu$ in case (b) and $2 + 4\mu$ in case (c).

Using more sophisticated linear algebra, one can also decrease the degree of the parametrization in case (d). This may be a good idea for larger examples, because we save then a lot of time in the last step (computation of the inverse). The improvement concerns the part "computation of the second parameter". Instead of choosing denominator and numerator in the basis, we choose them more carefully (especially the denominator). Using the refined version, we can compute a parametrization of degree $\leq n + 2 + (n + 4)\mu$.

1.4 More adjoint maps

The algorithm in the last section is simple and fast, but not complete. Here we discuss what must be done when its assumption, namely that the image of the last adjoint map is 2-dimensional, does not hold.

If the image is 1-dimensional, then we the last adjoint map gives on of the two parameters.

We have already seen adjoint maps with images that are curves. The algorithm *adj2dim* will fail if the image of $v_{1,\mu}(X)$ is a curve: it assumes that $v_{1,\mu}(X)$ is a surface.

But recall the treatment of case (d) in the algorithm. We used the adjoint maps onto a curve to compute one parameter. The fibers turned out to be rational (namely lines in Y), and we could finish the parametrization task. Geometrically speaking, we simply parametrized the general fiber.

The same idea can also be employed in the case when $C := v_{1,\mu}(X)$ is a curve.

Example 6: Let X be the surface of degree 4, given by

$$F(x, y, z, w) = x^2zw + y^2zw + x^2y^2.$$

X has a double line $x = y = 0$. The first two adjoint ideals are

$$A_1 = (x, y), A_2 = A_1^2.$$

The surface is rational. We have

$$V_{1,1} = \langle x, y\rangle, V_{1,2} = 0.$$

Hence $\mu = 1$. The adjoint map $i_{1,1} : X \rightarrow \mathbf{P}^1$,

$$(x : y : z : w) \mapsto (t : 1) = (w_0 : w_1) = (x : y)$$

is onto, but of course not birational. To compute its fibers, we compute the Gröbner base of

$$(x - t, y - 1, F)$$

as usual and get

$$\{(x - t, (t^2 + 1)zw + t^2, y - 1)\}.$$

We do not see any homogeneous polynomial, because the degree of some inhomogeneous polynomials are too low. Combining the inhomogeneous elements, we get

$$x - ty, (t^2 + 1)zw + y^2.$$

We see that the generic fiber is a plane conic, indeed a rational curve.

Conics are usually parametrized by projection from a point on it. The difficulty is that the coordinates of the point must be rational functions in

t. The field $k(t)$ is not algebraically closed, so the Nullstellensatz does not guarantee that there exists such a point.

In example 6, however, it is easy: A point is (for instance) $(y : z : w) = (0 : 1 : 0)$. We set $s = \frac{w}{y}$ and obtain the parametrization

$$(x : y : z : w) = (st(t^2 + 1) : s(t^2 + 1) : -1 : s^2(t^2 + 1))$$

for X.

Example 6 is quite typical. The general situation is described in the following theorem.

Theorem 1.3 *Let X be a rational surface. Let μ be its adjoint depth. Suppose that $C := i_{1,\mu}(X)$ is one-dimensional. Then C is a rational normal curve of degree $n := v_{1,\mu} - 1$. Also, $v_{2,2\mu} = 2n + 1$, $V_{2,2\mu} = V_{1,\mu}^2$, and $v_{2,2\mu+1} = 0$.*

Let $Y := i_{1,\mu-1}(X)$ and let $Y_2 := i_{2,2\mu-1}(X)$.

a) *$i_{2,2\mu-1}$ is birational onto Y_2. The generic fiber of*

$$i_{2,2\mu-1} \circ i_{1,\mu}^{-1} : Y_2 \to C$$

is a conic.

b) *If $i1, \mu - 1$ is birational onto Y, then the generic fiber of*

$$i_{1,\mu-1} \circ i_{1,\mu}^{-1} : Y \to C$$

is a conic.

The calculation of the equation of the conic is already half of the parametrization. A parametrization of the rational normal curve C gives the first parameter t. The equation of degree 2, which is the aim of our interest, has coefficients depending on the first parameter t. As in the easy case where we had lines as fibers, we can do this without computing the equation of Y or Y_2 explicitly.

We can again choose a basis $w_0, \ldots, w_n$ of $W := V_{1,\mu}$ such that the quotient $\frac{w_1}{w_0}$ is a parameter for C (see also the last section). We take this quotient as the first parameter for X.

Now, set $V := V_{1,\mu-1}$. We try to compute three elements v_0, v_1, v_2 such that the two vector-spaces

$$\langle v_0 w_0, v_1 w_0, v_1 w_0 \rangle, V \cdot \langle w_1, \ldots, w_n \rangle$$

be complementary vector-spaces in $V \cdot W$. Such forms can be found within a basis $\{v_0, \ldots, v_N\}$, iff they exist at all. To find them, add v_i when $v_i w_0$ is linearly independent of the vector-space

$$V \cdot \langle w_1, \ldots, w_n \rangle.$$

If there are no three such elements, then the map $i_{1,\mu,1}$ cannot be birational (this will be obvious from theorem 2.11b). In this case, set $V := V_{2,2mu-1}$ and start over again.

The so computed three elements are suitable for coordinates of a conic. By the way they were obtained, they do not satisfy a linear relation with coefficients in $k(t)$. Hence, they have to satisfy an irreducible quadratic equation over $k(t)$.

Let us write Y_t for the conic over $k(t)$, which is birationally equivalent to X when considered as a surface over k. We will give another interpretation of this notation, namely considering the subscript t as an index, and Y_t as a fiber, in the next chapter.

Here is the first half of the algorithm *adj1dim*.

Algorithm *adj1dim*.
Part 1.

Input:
An equation $F(x, y, z, w)$, representing a rational surface X.
The integer m. It must be the adjoint depth of X.

Output (intermediate result):
A rational function τ in x, y, z, w, homogeneous of degree 0, representing the first parameter.
Three homogeneous forms U, V, W in x, y, z, w, of the same degree.
An irreducible quadratic equation $G_t(X, Y, Z)$, representing a conic Y_t over $k(t)$ and at the same time a surface in $\mathbf{P}^1 \times \mathbf{P}^2$.
$((U : V : W); T)$ is a birational map from X to Y_t.
The algorithm works only if $i_{1,\mu}(X)$ is one-dimensional.

$W := V_{1,m}$;
$n := v_{1,m} - 1$;
Compute the equations of C.
RATNOR := linear relations in W^2;
Computation of the first parameter by parametrizing RATNOR.
CPARAM := *parametrize*(RATNOR);
W:=*basetrafo*(CPARAM,W);
T := w_0/w_1; return(T);

Computation of 3 suitable coordinate elements.
VSPACE:=$V_{1,m-1}$;
$*$ **starting point when first try fails.**
COOR := empty;
SUB := VSPACE$\cdot\langle w_1,\ldots,w_n\rangle$;
$i := 0$;
while *length*(COOR)$<$ 3 and $i\leq$dim VSPACE do
    if $v_i w_0\in$ SUB
        append v_i to COOR;
    $i := i+1$;
if *length*(COOR)$<$ 3 do
    VSPACE:=$V_{2,2m-1}$;
    goto $*$
(U,V,W) := COOR;
$I := (F, w_0 - t, w_1 - 1, X - U, Y - V, Z - W)$;
compute a Gröbner base of I, treating t as constant
    with term-order that eliminates x, y, z, w;
G_t:= equation in X, Y, Z); return(G_t);
exit;

Example 7: Let X be the surface of degree 4 given by the equation

$$F(x,y,z,w) = w^2x^2 + 2wxz^2 + 2wxyz + 2wy^3 + 2z^3y + x^4 + y^4 + z^4.$$

X has a non-ordinary double point $(0:0:0:1)$. The first three adjoint ideals are

$$A_1 = (x,y,z), A_2 = (x,y,z^2),$$
$$A_3 = (x^2, xy, y^2, xz, yz, xw + z^2, z^3).$$

The surface is rational. We have $\mu = 2$. Since $v_{1,2} = 2$, the image of the adjoint map $i_{1,1} = i_W : X \to \mathbf{P}^1$

$$(x:y:z:w) \mapsto (t:1) = (x:y)$$

is the projective line $C = \mathbf{P}^1$.
We try to find suitable coordinate elements in $V := V_{1,1} = \langle x,y,z\rangle$. $x\cdot x$ is linearly independent of

$$y\cdot\{x,y,z\} = \{xy, y^2, yz\},$$

hence we take it. $y\cdot x$ is dependent. $z\cdot x$ is again independent. Hence we could only find two suitable elements, which implies that i_V is not birational. (Since this is the projection from the double point, it is generically two to one.)
We have to look for the coordinate elements in

$$V := V_{2,3} = \langle x^2, xy, y^2, xz, yz, xw + z^2\rangle.$$

Again, we check if x times a basis element gives something new modulo

$$\langle x^2y, xy^2, y^3, xyz, y^2z, xyw + yz^2\rangle.$$

Sequential testing gives the coordinate elements

$$v_0 := x^2, v_1 := xz, v_2 := xw + z^2.$$

We eliminate x, y, z, w in the $k(t)$-ideal

$$(F, x - t, y - 1, x^2 - v_0, xz - v_1, xw + z^2 - v_2).$$

The Gröbner base contains the two elements

$$v_0 - t^2, 2v_1^2 - t^7 - t^3 - 2t^2v_2 - t^3v_2^2 - 2t^2v_1v_2,$$

which we can easily combine to get the homogeneous quadratic equation

$$G_t(v_0, v_1, v_2) = 2tv_1^2 - (t^4 + 1)v_0^2 - 2tv_0v_2 - t^4v_2^2 - 2t^3v_1v_2.$$

The second half of the parametrization is the parametrization of Y_t. The way to parametrize conics is to project them from a point (and invert this birational map onto $\mathbf{P}^1$). So, it is crucial to find a point on Y_t. This is a nontrivial polynomial solution of G_t.

The field $k(t)$ is never algebraically closed, so we have to worry about the existence of solutions. The following theorem is surprising.

Theorem 1.4 *Let k be algebraically closed. Then any conic over $k(t)$ has a $k(t)$-rational point.*

Proof: We can write the equation as a homogeneous polynomial of degree 2 in x, y, z with polynomial coefficients:

$$F(x, y, z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz = 0,$$

$$a_i \in k[t]$$

Let n be an integer such that $\deg_t a_i \leq n$ for all i. We make the ansatz

$$x = x_0 + \ldots + x_{n-1}t^{n-1},$$

$$y = y_0 + \ldots + y_{n-1}t^{n-1},$$

$$z = z_0 + \ldots + z_{n-1}t^{n-1},$$

and plug it into the equation. By comparing coefficients, we get at most $3n - 1$ homogeneous equations (of degree 2) in $3n$ indeterminates. Hence there must be a solution, because k is algebraically closed.

We have algorithms to solve equation systems like the above. However, the system has so many unknowns that the time needed to solve is rather long. For this reason, I was unable to come up with a solution of the quadratic equation G_t in example 7.

One should think of faster solution methods for this special case. It seems to be promising to apply more theoretical results on quadratic forms ([17]).

For the moment let us assume that we have an algorithm *diophant* that takes a Diophantine quadratic homogeneous equation $G_t(X_0, X_1, X_2)$ over $k(t)$ and computes a polynomial solution $(X_0 : X_1 : X_2) = (a(t) : b(t) : c(t))$.

The projection

$$(X_0 : X_1 : X_2) \mapsto (a(t)X_2 - c(t)X_0 : b(t)X_2 - c(t)X_1)$$

is birational. In our context, Y_t is also considered as a surface. The birational equivalence is consequently considered as a birational equivalence to the plane, coordinatized by s, t:

$$(X_0 : X_1 : X_2; t) \mapsto (t, s) = (t, \frac{a(t)v_2 - c(t)v_0}{b(t)v_2 - c(t)v_1}).$$

In other words, s is the second parameter. All we have to do is to invert the composed map from X to the plane.

Algorithm *adj1dim.*

Part 2.

Input (intermediate result):

An equation $F(x, y, z, w)$, representing a rational surface X.

A rational function T in x, y, z, w, homogeneous of degree 0, representing the first parameter.

Three homogeneous rational functions U, V, W in x, y, z, w, of the same degree.

An irreducible quadratic equation $G_t(X, Y, Z)$, representing a conic Y_t over $k(t)$ and at the same time a surface in $\mathbf{P}^1 \times \mathbf{P}^2$.

It is assumed that $(\xi_0 : \xi_1 : \xi_2); \tau$ is a birational map from X to Y_t.

Output:

4 polynomials $\overline{x}, \overline{y}, \overline{z}, \overline{w} \in k[s, t]$, representing a proper parametrization of X.

(A,B,C) := *diophant*(G_t);

Compose the two birational maps.
BIRAT := $(x:y:z:w)\mapsto(T,\frac{A(T)W-C(T)X}{B(T)W-C(T)Y})$;

return(*inverse*(BIRAT));
exit;

Remark: The theorem of the existence of a $k(t)$-rational point on a conic can be generalized to arbitrary function fields of algebraic curves over an algebraically closed field. This fact is fundamental for the theory of ruled surfaces. In geometric terms, the theorem says that any map $\pi : X\to C$ from a surface to curve with rational fibers has a cross section.

We have discussed methods where the image of $i_{1,\mu}$ is one- and two-dimensional. One case is still missing. If the image is a point, then the map $i_{1,\mu}$ gives nothing at all. This happens iff $v_{1,\mu}=1$ (note that the image is not contained in a proper linear subspace).

Example 8: Let X be a smooth cubic surface. The adjoint ideals are all trivial. We have

$$V_{1,1}=\langle 1\rangle, V_{1,2}=0,$$

and $\mu=0$. The map $i_{1,1}$ collapses X to $\mathbf{P}^0$.

One might think that this case is the most difficult one. But the good news is that we can construct a birationally equivalent surface of bounded degree. This implies that we will not be faced with arbitrarily large (with regard to number of variables and degree) equation systems, unlike in the algorithm *adj1dim*.

Theorem 1.5 *Let X be a rational surface. Let μ be the adjoint depth. Suppose that $v_{1,\mu}=1$. Then one of the following is true:*

a) *$v_{1,\mu-1}\geq 4$, and $i_{1,\mu-1}$ is birational onto a surface of degree $d\leq 9$.*

b) *$v_{1,\mu-1}=3$, and $i_{1,\mu-1}$ is generically two to one onto the projective plane. $v_{2,2\mu-2}=7$, and $i_{2,2\mu-2}$ is birational onto the image.*

c) *$v_{1,\mu-1}=2$, $v_{2,2\mu-2}=4$, and $i_{2,2\mu-2}$ is generically two to one onto a singular quadric surface. $v_{3,3\mu-3}=7$, and $i_{3,3\mu-3}$ is birational onto the image.*

The surfaces occuring in the theorem are called Del Pezzo surfaces. They have many interesting geometric properties, which will be discussed in detail in chapter 4. At this place, we just want to show an example.

Example 9: Let X be the surface of degree 5 given by the equation

$$F(x,y,z,w) = x^2w^3 + y^3w^2 + z^5.$$

There is an isolated double point $(0:0:0:1)$ and a double line $L: z = w = 0$ with a triple point $(1:0:0:0)$ on it. The double point, although not ordinary, does not impose conditions on the adjoint ideals. The line does. The first adjoint ideals are

$$A_1 = (yw, z^2, zw, w^2), A_2 = (y^2w^2, yz^2w, z^4, zw^2, w^3),$$

$$A_3 = (y^2z^2w^2, yz^4w, yzw^3, y^3w^3, z^6, z^3w^2, z^2w^3, w^4).$$

Hence we get the adjoint vector-spaces

$$V_{1,1} = \langle yw, z^2, zw, w^2\rangle,$$

$$V_{1,2} = \langle c_0, c_1\rangle = \langle z^2w, w^3\rangle,$$

$$V_{1,3} = \langle w^4\rangle.$$

In chapter 2, we will see that adjoint dimension 1 is sufficient for the termination of the adjoint series, hence $\mu = 3$. Since $v_{1,3} = 1$ and $v_{1,2} = 2$, we are in case (c) of the above theorem.

We may compute $V_{2,4}$ without computing the adjoint ideal A_4, by using the facts

$$V_{1,3} \cdot V_{1,1} \subset V_{2,4} \text{ and } \dim V_{2,4} = 4:$$

by multiplying the basis of $V_{1,1}$ with w^4, we get linearly independent elements, which must span the four-dimensional vector-space.

$$V_{2,4} = \langle s_0, c_0^2, c_0c_1, c_2^2\rangle = \langle yw^5, z^2w^4, zw^5, w^6\rangle.$$

The image of the associated map is indeed a singular quadric, because the relation

$$c_0^2 \cdot c_2^2 - (c_0c_1)^2 = 0$$

holds.

Similarly, the 6 linearly independent forms

$$c_0^3, c_0^2c_1, c_0c_1^2, c_1^3, s_0c_0, s_0c_1$$

in $V_{1,2} \cdot V_{2,4}$ span a 6-dimensional subspace of $V_{3,6}$. A seventh element can be found in

$$V_{1,0} \cdot V_{1,3}^2 = \langle xw^8, yw^8, zw^8, w^9\rangle,$$

namely $u_0 := xw^8$ (the three other elements are already known).

The algebraic relations of c_0, c_1, s_0, n_0 can be found by elimination. There is exactly one, namely

$$u_0^2 + s_0^3 + c_0^5c^1 = 0.$$

We will see that it is possible to give a degree bound (with weighted degrees), so that the relation can be found by linear algebra.

X is birational equivalent to a projective surface given by the weighted projective coordinates

$$\deg c_0 = \deg c_1 = 1, \deg s_0 = 2, \deg u_0 = 3,$$

satisfying the homogenous equation of degree 6 above. This is typical for case (c) in the above theorem. The equation example 9 looks very similar to the input equation F, we essentially just got rid of the double line by a different homogenization of the affine equation $x^2 + y^3 + z^5 = 0$.
Note that we also achieved that the equation is linear in the variable c_1. This allows one to do the parametrization very quickly:

$$(c_0 : c_1; s_0; u_0) = (1 : -s^2 - t^3; t; s).$$

A parametrization of the affine surface is now easily derived:

$$(x, y, z) = (\frac{u_0}{c_1^3}, \frac{s_0}{c_1^2}, \frac{c_0}{c_1}) = (\frac{s}{(-s^2 - t^3)^3}, \frac{t}{(-s^2 - t^3)^2}, \frac{1}{-s^2 - t^3}).$$

Moreover, we may homogenize and obtain a parametrization of the original projective surface:

$$(x : y : z : w) = (s : t^2(-s^2 - t^3) : 1 : (-s^2 - t^3)^3).$$

For many Del Pezzo surfaces, it is easier to compute an improper parametrizations, especially when a point on it is known.

1.5 A parametrization algorithm

The algorithm presented in this section computes the adjoint depth and calls to the appropriate parametrization method.

Until now, we have a parametrization algorithm *adj2dim* and a sub-algorithm *adj1dim*, both working in exclusive cases. The algorithm for the Del Pezzo case will be referred to as *adj0dim*.

Let μ be the adjoint depth of an input surface X. To figure out the appropriate parametrization method among those three, it is not always necessary to compute the image of $i_{1,\mu}$ (in order to find its dimension). The adjoint dimensions already give a lot of useful information. For instance, a Del Pezzo case (fork to *adj0dim*) can be easily recognized by the necessary and sufficient criterion $v_{1,\mu} = 1$.

If $v_{1,\mu} = 2$, then we necessarily have a one-dimensional image $\mathbf{P}^1$, and we go to *adj1dim*.

If $v_{1,\mu} = 3$, then we have either a two-dimensional image and case (a) holds in theorem 2, or a one-dimensional image, namely a conic in $\mathbf{P}^2$

(theorem 3). Here it is wise to look for linear relations in $V^2_{1,\mu}$, since we need the equation (if the image is one-dimensional) anyway. (if the image is one-dimensional) anyway. If $v_{1,\mu}\geq 4$, then we have to compute $v_{2,2\mu+1}$. The image of $v_{1,\mu}$ is one-dimensional iff $v_{2,2\mu+1}=0$. Note that in the other case we did compute something useful, because *adj2dim* uses the adjoint vector-space $V_{2,2\mu+1}$.

Here is the algorithm.

Algorithm *parametrize.*

Input:

$F(x,y,z,w)$, homogeneous of degree d.

We assume that F passed the test for rationality.

Output:

4 polynomials $\overline{x},\overline{y},\overline{z},\overline{w}\in k[s,t]$, representing a parametrization of X.

The algorithm is not yet complete, some sub-algorithms are missing.

Compute the adjoint depth.

$m := 0$;
while $v_{1,m+1} > 0$ do $m := m+1$;

if $v_{1,m}=1$ then
    return(*adj0dim*(X,m));
    exit;
if $v_{1,m}=2$ then
    return(*adj1dim*(X,m));
    exit;
if $v_{1,m}=3$ then
    RATNOR := compute relations in $V^2_{1,m}$;
    if RATNOR is empty then
        return(*adj2dim*(X,m));
        exit;
    else
        return(*adj1dim*(X,m));
        exit;
if $v_{2,2m+1}=0$ then
    return(*adj2dim*(X,m));
    exit;
else
    return(*adj1dim*(X,m));
    exit;

We could give a good estimation for the degree of the parametrization computed by *adj2dim*. The situation is even better for *adj0dim*: Here we may always compute parametrizations of degree $\leq 4\mu$. It would probably also be possible to control the degree in the algorithm *adj1dim* somehow.

Chapter 2

Geometry on a surface

This chapter is the mathematical corpus of this thesis. It contains a constructive proof of Castelnuovo's criterion, with two gaps which are closed in the two following chapters. More precisely, we prove the correctness of the algorithm presented in the previous chapter.

In contrast to the previous chapter, we will formulate and prove most statements in a geometrical language. One does not find many polynomials in this chapter. The reason is that this language is much more "thought-efficient" than the algebraic language. On the other hand, it does not make so clear how to manipulate the equations.

Here is a summary of the particular sections.

1: Preliminaries. We develop a minimal apparatus for doing geometry on a smooth surface. Most of the material presented here is also covered by any textbook on algebraic surfaces ([39], [24], [19], [18]). Here are the exceptions:

The Grauert-Riemenschneider vanishing theorem ([15]) and Reider's lemma ([33]) are too recent to be mentioned in older textbooks.

The notions rigid, large, and very large introduced in subsection 2.1.1 differ from the usual terminology (which is more complicated).

2: Italian pairs and their adjoint pairs. The concept of adjunction is the most important ingredience in the proof of Castelnuovo's criterion. We introduce the notion of Italian pairs, to have a property that is passed along to the adjoint pair.

We also show the finiteness of adjoint series: we cannot pass to the adjoint pair infinitely often.

3: Singular surfaces. Here we explain the connection between adjoint series of Italian pairs and the adjoint maps from chapter 1. We also prove a bound for the length of the adjoint series, in terms of the smallest possible degree of a proper parametrization.

4: Classification results. We give a classification of all Italian pairs that may arise at the end of an adjoint series with positive self-intersection. We prove some results about the last but one pair in the adjoint series. We apply these results to show the correctness of the algorithms in the previous chapter.

A side result is the classification of Italian pairs with genus zero.

5: Rulings. Here we analyze a particular class of rational surfaces more closely, namely the ruled surfaces. We apply the analysis to improve one of the sub-algorithms in chapter 1.

6: Historical and philosophical remarks. This is a discussion of other proofs of Castelnuovo's criterion. My point is that some other proofs are shorter, but this proof gives the best algorithm.

2.1 Preliminaries

We consider smooth projective algebraic surfaces over an algebraically closed field. What this means should be clear, since we already saw enough in the previous chapter.

2.1.1 Divisors

The objects of internal geometry on a smooth algebraic surface are points and algebraic curves. For two different algebraic curves, we may count the number of intersection points. The result is called the *intersection number*. A proper counting takes into account multiple intersection points. For instance, if the two curves have the same tangent at an intersection point, the intersection should count at least doubly.

If the ground field k is not algebraically closed, then we also have to count all the points which have have coordinates in an algebraic extension of the ground field k. In other words, the geometric entities live in the algebraic closure of the ground field k.

We want to introduce an equivalence relation on the set of curves. It can be regarded as a sort of homotopy equivalence. The set of curves also has to be enlarged to contain also reducible curves with multiple components. The idea is best explained by a simple example.

Let $X = \mathbf{A}^2$ be the affine plane. An algebraic curve is given by an absolutely irreducible polynomial equation $f(x, y) = 0$. Now, let

$$f(x, y) = f_1(x, y)^{r_1} \cdots f_n(x, y)^{r_n}$$

be an arbitrary nonzero polynomial. The corresponding "generalized curve"

$$C = r_1 C_1 + \ldots + r_n C_n$$

is called a *divisor*. The irreducible components C_i are called *prime divisors*. We form the additive group Div$X$ by allowing also negative coefficients r_i. The divisors are partially ordered by saying that $D_1 \leq D_2$ iff any coefficient in D_1 is less than or equal to the corresponding coefficient in D_2. A divisor has nonnegative coefficients iff it is greater than or equal to the zero divisor 0. Such a divisor is called *effective*. The effective divisors are those that are given by polynomial equations.

To explain the word "divisor", consider the hyperboloid in 3-space, given by the equation

$$x^2 + y^2 - z^2 - 1 = 0.$$

Let D_1 be the divisor given by the equation $y - 1 = 0$. It splits into two prime divisors, the lines $L_1 : x = z, y = 1$ and $L_2 : x = -z, y = 1$. Let D_2 be the divisor given by the equation $x - z = 0$. It splits into two prime divisors, the lines L_1 and $L_3 : x = z, y - 1$. The prime divisor L_1 does not have a globally defined equation. It appears as the greatest common divisor of D_1 and D_2.

If a divisor happens to have a global equation f, then we denote it by (f). We use this notation also for homogeneous equations defining divisors on projective varieties.

To define equivalence, let us return to the affine plane. Let D_1 and D_2 be two different effective divisors given by polynomials f_1 and f_2. Since the polynomials must be linearly independent over k (otherwise we would have $D_1 = D_2$), we may form a linear homotopy

$$D_{(\lambda_1:\lambda_2)} : \lambda_1 f_1 + \lambda_2 f_2 = 0.$$

For any ratio $(\lambda_1 : \lambda_2) \in \mathbf{P}^1$, we get a divisor (figure 1). For $(1 : 0)$, we get D_1, and for $(0 : 1)$, we get D_2. In this situation, we call D_1 and D_2 (linearly) equivalent. In this example, we can do this for any two divisors. Consequently, all effective divisors on the affine plane are equivalent.

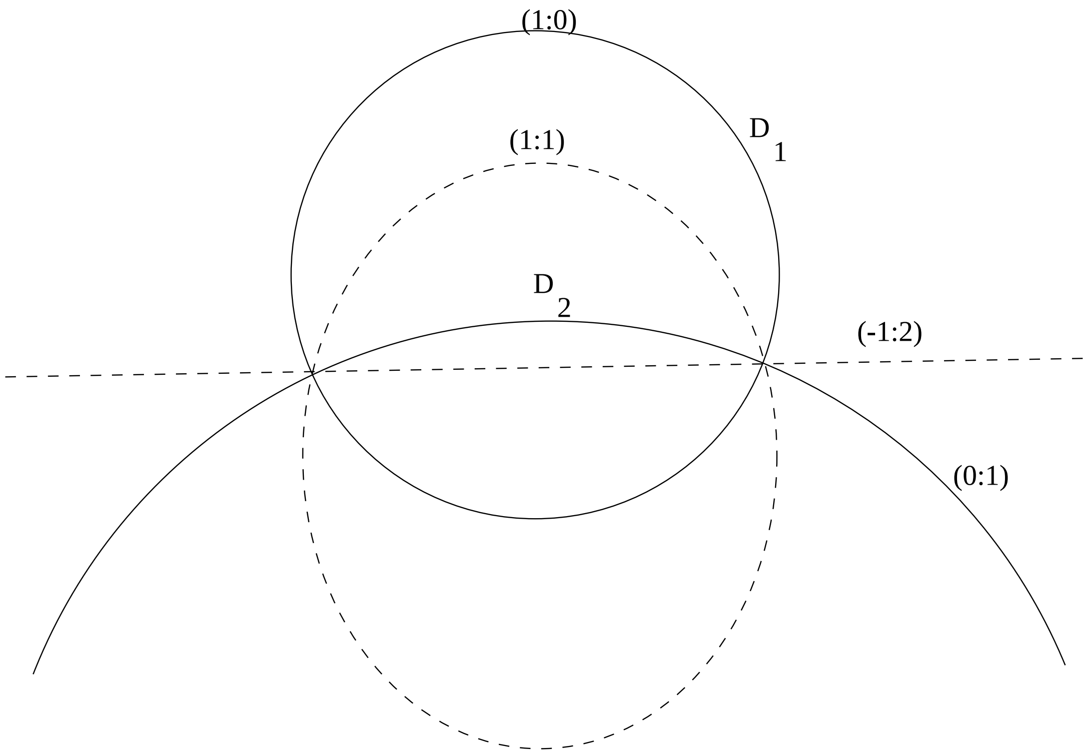

Figure 2.1: Linear equivalence of divisors

We say that two (not necessarily effective) divisors D_1 and D_2 are equivalent iff there exists a divisor D_3, such that D_1+D_3 and D_2+D_3 are effective and linear equivalent by a construction like the above.

A more interesting example is the projective plane. Here, a linear homotopy can be made iff the defining equations, namely homogeneous polynomials in 3 variables x, y, z, have the same degree. Hence, the class of a divisor is uniquely determined by the degree.

It can be proved easily that the addition and subtraction of divisors is well-defined on classes. The classes form again an Abelian group $\mathrm{Cl}X$. For instance, $\mathrm{Cl}\mathbf{P}^2 \cong \mathbf{Z}$ with the usual addition.

If X is projective, we have two remarkable facts.

First, the intersection number of D_1 and D_2 depends only on the classes of D_1 and D_2. This is called Bezout's theorem.

The fact can be used to define self-intersection, by intersecting D with an equivalent divisor. Intersection number gives then a bilinear symmetric product on divisors and on classes.

Second, the effective divisors in an equivalence class carry the structure of a projective space.

Let D be a divisor. The set of divisors

$$|D| := \{D' | D' \geq 0 \text{ and } D' \sim D\}$$

is called the *linear system* of D. Assume that $|D|$ is not empty. In a suitable affine subset of X, the equations of divisors in $|D|$ may be written as

$$\lambda_0 f_0 + \ldots + \lambda_n f_n, (\lambda_0 : \ldots : \lambda_n) \in \mathbf{P}^n$$

for some suitable equations $f_0, \ldots, f_n$. The divisors in $|D|$ are in bijective correspondence with a projective space. The integer $n := \dim|D|$ is called the dimension of $|D|$.

Any linear subspace Γ of the projective space $|D|$ is called an incomplete linear system. As above, the local equations of divisors in Γ form a vector-space, and Γ is the projectivization of this vector-space.

If X is the projective plane, then we can write down global equations of the divisors. $|D|$ is not empty iff the degree d of D is nonnegative. In this case, the homogeneous forms of degree d form a finite-dimensional vector-space. Two forms give the same divisor iff they are linearly dependent over k. Hence, the set $|D|$ is the projectivization of the finite dimensional vector-space of forms of degree d.

Let $|D|$ be the linear system of a divisor D. The greatest common divisor F of all divisors in $|D|$ is called the *fixed part* of $|D|$. The *moving part*

$|D| - F = |D - F|$ is a linear system of the same dimension without fixed components.

An effective divisor D is called *rigid* iff $\dim|D| = 0$. Then D is the fixed part of $|D|$, and the moving part is zero. A prime divisor with negative self-intersection is necessarily rigid.

A point $p \in X$ is called a base point of a linear system $|D|$ iff it is contained in any divisor in $|D|$. The base locus consists of the fixed components and a finite set of isolated base points.

The *associated map* $i_D : X \to \mathbf{P}^n$ is defined locally by sending a point in the affine subset above to the ratio $(f_0 : \ldots : f_n)$. The ratio is defined outside the base locus. Subtracting fixed components does not change the ratio, hence the associated map is defined outside the base locus of the moving part, which is a finite set of base points.

As defined, the associated map depends on the chosen basis $(f_0, \ldots, f_n)$ of the vector-space of local equations. A different base results in a composition with a projective coordinate transformation in $\mathbf{P}^n$.

There is a natural duality between $|D|$ as a projective space and the codomain $\mathbf{P}^n$ of i_D. The inverse image of the hyperplanes in $\mathbf{P}^n$ correspond to divisors in $|D|$. i This duality can be used to give a coordinate invariant description of the associated map, in the following way. Let $|D|^*$ be the dual space of $|D|$. It is an n-dimensional projective space, whose points are the $n-1$-dimensional linear subsystems of $|D|$. Then we describe

$$i_D : X - \text{base locus} \to |D|^*$$

as sending p to the subsystem of all divisors passing through p. It is easy to show that this is indeed a linear subsystem of dimension $n-1$. In terms of local coordinates, one can show easily that the described map can be represented as in the definition above.

The concepts of base points, fixed and moving part, and associated map make sense also for incomplete linear systems. Any rational map from X to some projective space is associated to a (possibly incomplete) linear system. The linear system is not unique, but its moving part is unique.

The associated map gives some means to measure divisors. Here are some definitions that define "largeness properties" in terms of the associated rational map.

- D is called *very ample* iff i_D is an isomorphism onto the image.
- D is called *ample* iff nD is very ample for an $n \geq 1$.
- D is called *very large* iff i_D is a regular map, or equivalently iff $|D|$ does not have base points.

- D is called *large* iff nD is very large for an $n\geq 1$.

Obviously, ampleness implies largeness, very ampleness implies very largeness and ampleness, and very largeness implies largeness. All properties depend only on the class of D.

Remark: The usual terminology is "semi-ample" for what we call large, and "globally generated" or "spanned" for what we call very large. However, these words do not tell much in the purely geometric context, so we simplified terminology a little bit. Note, however, that the zero divisor is very large (a degenerate case).

2.1.2 Riemann-Roch and other theorems

We quote some classical theorems which we will use frequently in the following: the Riemann-Roch theorem for surfaces, the genus formula and the Hodge index theorem. Furthermore, we quote an instance of the Grauert-Riemenschneider vanishing theorem and Reider's lemma.

The most important numerical invariant for curves is the genus. It is a nonnegative integer. Rationality of a curve is characterized by the vanishing of the genus. Curves with genus 1 are called elliptic curves.

Let $C\subset X$ be a curve on X. The genus of C can be computed in terms of the intersection product and the singularities of C. The information of the self-intersection is, however, not enough. We need a second factor.

There exists a distinguished class $K\in \mathrm{Cl}X$, called the *canonical class*. Any divisor in this class is also called canonical, and often denoted with the same symbol K. The geometric genus and the plurigeni can geometrically be defined in terms of K, by

$$p_g := \dim|K|+1, P_m := \dim|mK|+1.$$

For surfaces of Kodaira dimension $\kappa=-\infty$, all plurigeni vanish, i.e. all sets $|mK|$ are empty.

Theorem 2.1 *If C is smooth, then $g(C)=\frac{C\cdot(C+K)}{2}-1$. In the general case,*

$$g(C)=\frac{C\cdot(C+K)}{2}-1-\sum\frac{m_p(m_p-1)}{2},$$

where the sum ranges over all singularities of C, including infinitely near.

For a divisor D, we define

$$g(D):=\frac{D\cdot(D+K)}{2}+1,$$

depending only of the class. Even if D is not effective, its genus is always an integer.

In case of prime divisors, the symbol g is ambiguous. We have to specify whether we mean the genus of the curve C or of the divisor C.

Corollary 2.1 *For any prime divisor C,*

$$C^2 + C \cdot K \geq -2,$$

and equality holds for smooth rational curves.

Example 1: Let $X = \mathbf{P}^2$. Let C be a curve of degree n. The canonical class has degree -3. The genus formula says

$$g(C) = \frac{(n-1)(n-2)}{2} - \sum \frac{m_p(m_p - 1)}{2},$$

the sum ranging over all singularities of C, including infinitely near.

The fundamental theorem in the intrinsic geometry on an algebraic surface is the Riemann-Roch theorem. It is useful to find effective divisors in various classes.

Theorem 2.2 *Let $D \in \mathrm{Div} X$ be a divisor. The following inequality holds:*

$$\dim|D| + \dim|K - D| \geq \frac{D \cdot (D - K)}{2} - 1 + p_a,$$

where p_a is the arithmetic genus of X.

Here is an application to surfaces with $p_g = 0$.

Corollary 2.2 *Let X be a surface with $p_g = 0$. Let D be an effective divisor. Then*

$$\dim|D + K| \geq g(D) - 1 + p_a.$$

Proof: Apply Riemann-Roch to the divisor $D + K$. The second summand on the left hand side $\dim|-D|$ is -1, because a nontrivial sum of effective divisors can never be equivalent to zero.

The difference between the left hand side and the right hand side in Riemann-Roch is called the *superabundance* $s(D)$ of D. It is always nonnegative and depends only on the class.

There are several vanishing theorems for the superabundance. We will use the Grauert-Riemenschneider vanishing theorem.

Theorem 2.3 *If D is large and $D^2 > 0$, then $s(D + K) = 0$.*

This allows to sharpen our previous corollary.

Corollary 2.3 *Let X be a surface with $p_g = 0$. Let D be a large effective divisor with $D^2 > 0$. Then*

$$\dim|D + K| = g(D) - 1 + p_a.$$

The Hodge index theorem says something about the intersection product itself.

Theorem 2.4 *Let D_1, D_2 be divisors such that $D_1^2 > 0$ and $D_1 \cdot D_2 = 0$. Then $D_2^2 \leq 0$. In case of equality, $D_2 \cdot D_3 = 0$ for all divisors D_3.*

The name of the theorem may be explained as follows. If we allow real coefficients to form "real divisors", then the intersection product gives a scalar product on a real vector-space. Let B be an orthogonal bases. The cardinality of vectors in B with positive square is independent of the orthogonal system. Hodge index says that this cardinality is 1.

The divisors orthogonal to the whole group Div$X$ form a subgroup. It can easily be seen that the subgroup does not contain nonzero effective divisors. The equivalence modulo this subgroup is called *numerical equivalence*, and the quotient group is called Num$X$. Since intersection product depends only on classes, Num$X$ is also a quotient of ClX. It is obviously torsion free. A deeper fact is that Num$X$ is a finitely generated Abelian group.

In addition to ample and large, we define a numerical property of divisors, depending only on the numerical class. D is called *numerically effective* or *nef* iff $D \cdot D' \geq 0$ for all effective divisors D'. Large implies nef: $D \cdot D' < 0$ can only hold if D' has components that are fixed in $|nD|$.

The following lemma, due to Reider ([33]), may sound a little bit technical, but it is very useful.

Lemma 2.1 *Let R be a nef divisor.*

a) If $R^2 \geq 5$, then $R + K$ is very large, unless there exists an effective divisor E such that

$$\textit{either } R \cdot E = 0, E^2 = -1,$$
$$\textit{or } R \cdot E = 1, E^2 = 0.$$

b) If $R^2 \geq 10$, then the rational map i_{R+K} is birational, unless there exists an effective divisor E such that

$$R \cdot E = 1 \textit{ or } 2, E^2 = 0.$$

Remark: After Reider's paper appeared, a lot of people came up with quick and elegant proofs for various classical theorems ([25],[32],[37]). Currently, there is a boom in revisiting the classical adjunction technique invented by the Italian school.

2.1.3 Rational maps and blowing up

This section gives a geometric analysis of rational maps in general and of blowing ups in particular.

Let $f : Y \to X$ be a regular birational map. f induces a map $f^* : \mathrm{Div}X \to \mathrm{Div}Y$, called the *pullback*, as follows. Let g be a local equation for $D \in \mathrm{Div}X$. Locally, we may represent f by polynomials in the coordinates of Y, one for each coordinate in X. Substitute these polynomials for the variables in g. The result is the local equation for f^*D.

The function f^* has a lot of nice properties.

- Set-theoretically, f^* maps D to the inverse image $f^{-1}(D)$.
- f^* is injective.
- f^* preserves the group structure, the partial order, linear equivalence and the intersection product. It induces therefore group monomorphisms $\mathrm{Cl}X \to \mathrm{Cl}Y$ and $\mathrm{Num}X \to \mathrm{Num}Y$, which are also denoted by f^*.
- f^* preserves $\dim|\_|$, superabundance, genus, and the properties large, very large, rigid and nef (i.e. D enjoys the property iff f^*D enjoys the property).
- For any D, the associated rational maps are rational maps are related by the commutative diagram

$$\begin{tikzcd} & Y \arrow[dl, "f"'] \arrow[dr, "i_{f^*D}"] & \\ X \arrow[rr, "i_D"] & & \mathbf{P}^N \end{tikzcd}$$

f^* does not preserve the canonical class and the properties ample and very ample. These properties will therefore not be very relevant to us. They are mainly here for contrast.

Neither does f^* preserve the property prime. A prime divisor C will in general be mapped to a composite divisor. One component, called the *proper transform*, is birationally equivalent to C via f. The other components are inverse images of points on X.

If f is not birational, then we still get an injective group homomorphism $f^* : \mathrm{Div}X \to \mathrm{Div}Y$. It preserves linear equivalence and the partial order. The intersection product is not preserved, but there is a natural number d, such that

$$f^*D_1 \cdot f^*D_2 = dD_1 \cdot D_2$$

for all $D_1, D_2 \in \mathrm{Div}X$. The number d is the cardinality of the generic fiber.

The properties nef, large, and very large are still preserved in the weak sense: If D is nef (large) (very large), then f^*D is nef (large) (very large).

Let p be a point on X. There is a birational regular map $\pi : \tilde{X} \to X$, where $\tilde{X}$ is called the *blowup* at p. π restricts to an isomorphism $\tilde{X} - \pi^{-1}p \to X - p$. The fiber $E := \pi^{-1}p$ is a projective line. Its points correspond to tangent directions at p (figure 2).

Example 2: Let $X = \mathbf{P}^2$. Let $p = (0:0:1)$. The blowup $\tilde{X}$ is the subvariety of $\mathbf{P}^1 \times \mathbf{P}^2$ given by the bi-homogeneous equation $x_0 y_1 - x_1 y_0 = 0$. Here, the coordinates of $\mathbf{P}^1 \times \mathbf{P}^2$ are $(x_0 : x_1, y_0 : y_1 : y_2)$.
The blowing down map $\pi : \tilde{X} \to X$ is the second projection. The exceptional line is $\mathbf{P}^1 \times (0:0:1)$.

The algebraic realization of the blowup is another story (cf. section 4.1). Here, we are concerned with geometric properties of blowing ups.

- $\mathrm{Div}\tilde{X}$ is a direct orthogonal sum of $\mathrm{Div}X$ and $\mathbf{Z}E$. The proper transform of a prime divisor C is $\pi^*C - mE$, where m is the multiplicity of C at p.
- $K_{\tilde{X}} = \pi^* K_X + E$.
- $E^2 = -1$.

Note that $E \cdot (E + K_{\tilde{X}}) = -2$, in agreement with the genus formula (E is smooth rational). Smooth rational curves with self-intersection -1 are called *exceptional curves*.

Blowing down maps are the "prime elements" in the algebra of rational maps with composition. In the following, we factor out a blowing down from a regular map of the form i_D, where D is very large.

Theorem 2.5 *Let X be a smooth surface. Let E be an exceptional curve. Let D be a very large divisor with $D \cdot E = 0$. Then there exists a smooth surface Y, and a point $p \in Y$, such that X is the blowup of Y at p. Moreover, the map*

$$i_D \circ \pi^{-1} : X \to \mathrm{P}^N$$

is regular. It is associated to the unique divisor $D' \in \mathrm{Div}X$ with $\pi^* D' = D$.

Setting $D = 0$, we obtain an interesting characterization for exceptional curves.

Corollary 2.4 *E is exceptional iff E can be blown down.*

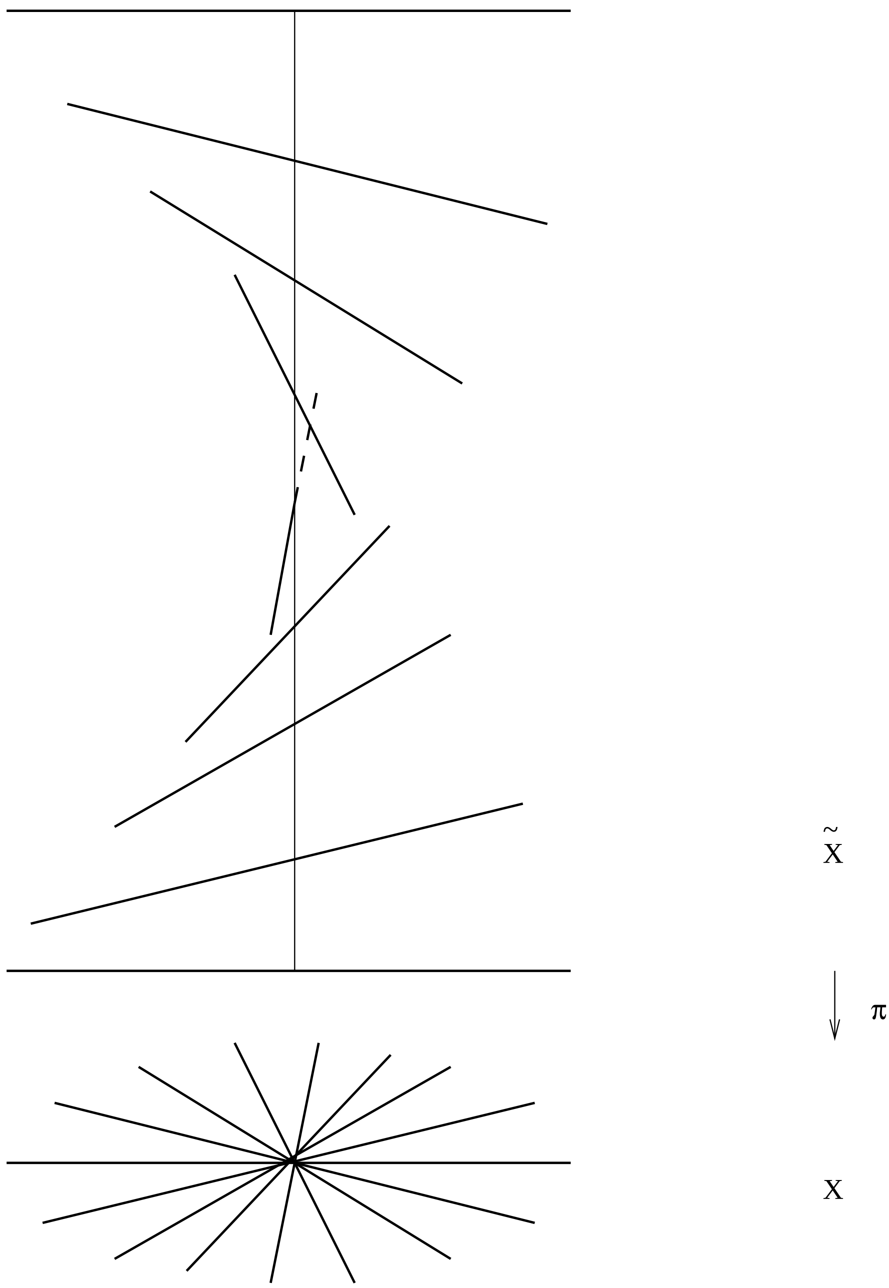

Figure 2.2: The blowup at a point

(The if part follows from the above analysis of blowing ups.)

A composition of blowing ups and blowing downs is birational, of course. The converse also holds.

Theorem 2.6 *Any birational map between smooth surfaces can be decomposed into blowing ups and blowing downs.*

We can use this fact to construct birational invariants. We just have to take care that blowing up does not change a numerical character. If $\pi : \tilde{X} \to X$ is a blowing down map, then, by our analysis

$$\mathrm{rk}(\mathrm{Num}\tilde{X}) = \mathrm{rk}(\mathrm{Num}X) + 1,$$

$$K_{\tilde{X}}^2 = K_X^2 + E^2 = K_X^2 - 1,$$

hence the corollary:

Corollary 2.5 $\mathrm{rk}(\mathrm{Num}X) + K^2$ *is a birational invariant.*

As an example, we compute the value of this invariant for rational surfaces. It suffices to compute the value for $\mathbf{P}^2$. The group of numerical classes is isomorphic to $\mathbf{Z}$, so its rank is 1. The canonical class has degree -3, so $K^2 = 9$. Our birational invariant takes the value 10.

For rational surfaces, numerical equivalence coincides with linear equivalence, because this holds for $\mathbf{P}^2$ and is stable under blowing up and blowing down.

Remark: In contrary to the other birational invariants p_g, q, P_m, κ, the invariant in the corollary is artificial. There is no functorial interpretation of the invariance. The artificial invariant cannot be expressed in terms of the other invariants, and in general, its computation is an extremely hard problem.

2.1.4 Plane divisors with base points

In this section, we study divisors on the blowup of the plane at some points. Such divisors arise naturally from a given parametrization. The main advantage of this type of divisors is that they provide many interesting examples with an easy geometric description.

Let $X_0 := \mathbf{P}^2$. Let L_0 be a line in X_0. For $i = 1, \ldots r$, let $a_i \in X_{i-1}$ be a point. Let X_i be the blowing up of X_{i-1} at a_i, and let E_i be the exceptional curve. Let $\pi_i : X_i \to X_{i-1}$ be the blowing down map. We fix the notation

$$L := \pi_r^* \circ \ldots \circ \pi_1^* L_0,$$

$$A_i := \pi_r^* \circ \ldots \circ \pi_{i+1}^* E_i$$

for divisors on $\mathbf{Y} := X_r$. The classes $L, A_1, \ldots, A_r$ generate $\mathrm{Cl}\mathbf{Y}$, because L_0 generates $\mathrm{Cl}X_0$. Let

$$D := mL - m_1 A_1 - \ldots - m_r A_r$$

be a divisor. The linear system $|D|$ may be imagined as the system of all plane divisors of degree m passing with multiplicity m_i through the base point a_i. The points a_i may be imagined as points in the plane or as points infinitely near to some other base point a_j (namely on the exceptional curve E_j). If $m < 0$, then $|D|$ is empty.

If $\tau : \mathbf{P}^2 \to X$ is a parametrization, then we may construct a very large plane divisor with base points in a canonical way. The parametrization is associated to an incomplete linear system Γ of curves in $\mathbf{P}^2$ without fixed part. The base locus consists of finitely many points, which we blow up. The pullback of Γ has a fixed part, which is a sum of exceptional divisors. After removing them, we have again only finitely many base points, infinitely near to the first base points.

By the theorem on the resolution of rational maps ([19]), this cannot be done infinitely often. Eventually, we arrive at some linear system Γ' without base points. Its completion can be written as $|mL - m_1 A_1 - \ldots - m_r A_r|$.

Obviously, it is not necessary that the linear system Γ comes from a parametrization. It suffices to say that Γ has no fixed component.

The divisors $L, A_1, \ldots, A_r$ are mutually orthogonal, and $L^2 = 1$, $A_i^2 = -1$ (cf. the remark following the Hodge index theorem). The canonical divisor of Y is

$$K = -3L + A_1 + \ldots + A_r.$$

We may evaluate the self-intersection and the genus of D:

$$D^2 = m^2 - m_1^2 - \ldots - m_r^2,$$

$$g(D) = \frac{(m-1)(m-2)}{2} - \frac{m_1(m_1-1)}{2} - \ldots - \frac{m_r(m_r-1)}{2}.$$

Suppose that $m \geq 0$. The superabundance allows an interesting interpretation. The divisor mL has dimension $\frac{m(m+3)}{2}$ (this can be calculated on X_0). The condition that a curve has an m_i fold point imposes $\frac{m_i(m_i+1)}{2}$ linear conditions on the coefficients of the equation, by translating the point to the origin and setting all coefficients of monomials of degree $< m_i$ to zero. If these conditions are linearly independent, than

$$\dim|D| = \frac{m(m+3)}{2} - \frac{m_1(m_1+1)}{2} - \ldots - \frac{m_r(m_r+1)}{2} =$$

$$= \frac{D\cdot(D-K)}{2},$$

which is the right hand side of Riemann-Roch minus 1. The left hand side of Riemann-Roch is

$$\dim|D| + \dim|K-D| = \dim|D| - 1.$$

Hence equality holds in Riemann-Roch, in other words $s(D) = 0$. In general, the superabundance can be interpreted as the number of linear relations between the above linear conditions on the coefficients. Note that this interpretation gives $s(D)\geq 0$, hence we have proved Riemann-Roch for this case.

The divisors A_i are rigid. If $a_{i_1}, \ldots, a_{i_s}$ are the points infinitely near to a_i, then the proper transform of E_i is $E'_i := A_i - A_{i_1} - \ldots - A_{i_s}$, which is also rigid. There may be other rigid curves, too.

Example 3: Let $r = 2$, and suppose that a_1 and a_2 are plane points. Then $D_0 := L - A_1 - A_2$ is rigid, because there is only one line through a_1 and a_2.

If D is nef, then we must have

$$D\cdot A_i = m_i \geq 0,$$

$$D\cdot E'_i = m_i - m_{i_1} - \ldots - m_{i_s} \geq 0.$$

(The first inequality is of course the reason why we put $-m_i$ instead of m_i in the first place.)

The above conditions for nef-ness are necessary but not sufficient. In example 3, the divisor $D_1 = 3L - 2A_1 - 2A_2$ is not nef, because $D_1\cdot D_0 = 3-2-2 = -1$. Consequently, D_1 is not very large (not even large). Indeed, $|D_2|$ has the line through a_1 and a_2 as a fixed component.

Example 4: Here is a divisor which is large but not very large. Let $a_1, \ldots, a_8$ be generic plane points. Then the divisor

$$D := 3L - A_1 - \ldots - A_8$$

has superabundance zero (this holds always for generic base points). Therefore, $\dim|D| = 2$, and the linear system is generated by two different cubic curves through $a_1, \ldots, a_8$. Let F_1 and F_2 be their equations (cubic forms). The two cubics intersect in 9 points, 8 of them being $a_1, \ldots, a_8$. The ninth point is contained in all cubics of the form $\lambda_1 F_1 + \lambda_2 F_2 = 0$, i.e. in all divisors of $|D|$. We have therefore a base point in $\mathbf{Y}$, i.e. D is not very large.

On the other hand, $2D$ is very large (theorem 3.1).

Example 5: Here is a divisor which is nef but not large. Let $a_1, \ldots, a_9$ be generic plane points. Let

$$D := 3L - A_1 - \ldots - A_9.$$

For $n > 0$, we have $s(nD) = 0$ because the base points are generic. Therefore, $\dim|nD| = 0$ by an easy calculation. All divisors nD are rigid, and none of them is very large because the fixed part of $|nD|$ is not zero. Hence, D is not large.

The only curve in $|D|$ is the cubic D_0 determined by the nine base points. Since the base points are generic, D_0 is irreducible. Any prime divisor C with $C{\cdot}D_0 < 0$ must be a component of D_0, hence must be equal to D_0. But $D_0^2 = 0$, hence D is nef.

2.2 Italian pairs and their adjoint pairs

We describe a process of eliminating exceptional curves (minimalization). Then we introduce two important notions: Italian pairs and adjoint pairs.

We will see Italian pairs arising in two ways. First, one constructs - in a natural way - an Italian pair from a given parametrization. The second way is more important. It is explained in section 3.

Let X be a smooth surface. Let D be a divisor on X. X is called *D-minimal* iff there exists no exceptional curve E with $D{\cdot}E = 0$. In case $D = 0$, we call X simply *minimal*.

Let n be a positive integer such that nD is very large. Then the map i_{nD} is regular. If X is not D-minimal, then there exists an exceptional curve E with $D{\cdot}E = 0$. By theorem 5, E can be blown down by $\pi : X \to X_1$, and there exists a $D_1 \in \mathrm{Div}X_1$ with $\pi^* D$.

If X_1 is not D_1-minimal, we repeat this process. Since the rank of Num drops by one at each step, we arrive eventually at a situation (X_m, D_m), with D_m being large and X_n being D_m-minimal. The pair (X_m, D_m) is called a *minimalization* of the pair (X, D). In case $D = 0$, we call X_m simply a *minimalization* of X.

The main purpose of minimalization is to make classification more easy. In birational classification, we may assume minimality because a minimalization can be constructed as above. The absence of exceptional curves is an enormous simplification.

Minimalizations are not unique in general. We might get different minimalizations by different choices of exceptional curves to blow down. If $D^2 > 0$, then minimalization is unique up to isomorphism: Two different exceptional curves E_1, E_2 with $D{\cdot}E_1 = D{\cdot}E_2 = 0$ are disjoint (apply the Hodge index theorem to the divisors D and $E_1 + E_2$). Consequently, the blowing downs of E_1 and E_2 do not interfere, and the order does not matter.

Minimalization is stable under pullbacks: Let $f: Y \to X$ be a birational regular map. Let D be a divisor on X. Let (X_m, D_m) be a minimalization of (X, D). Then (X_m, D_m) is also a minimalization of (Y, f^*D).

The idea of adjunction for divisors is simply to add K. What we need is some property that is passed to the adjoint divisor. A first step into this direction is made in the following lemma.

Lemma 2.2 *Let X be a surface with $q = 0$. Let $D \not\sim 0$ be a nef divisor such that $|D|$ is not empty and X is D-minimal. The following are equivalent:*
a) $g(D) > 0$.
b) $|D + K|$ is not empty.
c) $D + K$ is nef.
If $D^2 > 0$, and the three are true, then $D + K$ is also large. More precisely, $2(D+K)$ is very large. If $D^2 > 0$ and $g(D) = 1$, then $D+K \sim 0$, hence $D + K$ is already very large.

Proof: (a)$\to$(b): If $p_g > 0$, then we may add an effective divisor in $|D|$ and an effective divisor in $|K|$ to get an effective divisor in $|D + K|$.

If $p_g = 0$, then also $p_a = 0$ because $q = 0$. The claim follows from corollary 2.

(c)$\to$(a): Since $|D|$ is not empty, $D \cdot (D + K) \geq 0$.

(b)$\to$(c): Assume indirectly, there exists a prime divisor C with $(D + K) \cdot C < 0$. Then C must be a fixed component of $|D + K|$. Especially, C must be rigid. Below, we prove that $C \cdot K \geq -1$, and equality holds iff C is exceptional. Since D is nef, we must have $C \cdot K = -1$ and $C \cdot D = 0$, contradicting the assumption that X is D-minimal.

Now, suppose that $D^2 > 0$ and $g(D) > 0$. If $g(D) = 1$, then $D \cdot (D + K) = 0$, and $(D + K)^2 \leq 0$ by Hodge index. But $D + K$ is nef and has effective divisors, hence $(D + K)^2 = 0$. Again by Hodge index, $D + K$ is numerically zero. Again, $|D + K|$ has effective divisors, hence $D + K \sim 0$.

If $g(D) \geq 2$, then $D \cdot (D + K) \geq 2$. Since D and $D + K$ are both nef, the sum $R := 2D + K$ is also. Moreover,

$$R^2 = D^2 + 2D \cdot (D + K) + (D + K)^2 \geq 1 + 4 = 5,$$

hence we can apply Reider's lemma. A divisor E with

$$\text{either } R \cdot E = 0, E^2 = -1,$$

$$\text{or } R \cdot E = 1, E^2 = 0,$$

cannot exist, because E^2, $E \cdot K$, and hence $E \cdot R$ have the same parity (by the genus formula). Therefore, $R + K = 2(D + K)$ is very large.

Here is the proposition that fills the gap concerned with rigid curves (and some more).

Proposition 2.1 *Let X be a surface with $q = 0$. Let C be a rigid curve. Then $C{\cdot}K \geq C^2$, $C{\cdot}K \geq -1$, and equality holds in the second iff C is exceptional.*

Proof: The second inequality follows from the first together with corollary 1 ($C^2 + C{\cdot}K \geq -2$). In case of equality, we have $C^2 = C{\cdot}K = -1$, which characterizes exceptional curves.

It remains to show the first inequality. By Riemann-Roch,

$$\frac{C{\cdot}(C-K)}{2} \leq \dim|C| + \dim|K-C| + 1 - p_a \leq$$

$$\leq 0 + \dim|K| + 1 - p_a = p_g - p_a = q = 0,$$

and we are done.

In view of the lemma, "good pairs" (X, D) are those for which D is nef, $|D|$ is not empty, and X is D-minimal. The first two properties pass to the divisor $D + K$. The third property can be forced by minimalization, without loosing the first two properties.

Since we are mainly concerned with surfaces with $p_a = p_g = P_2 = 0$, we put this assumption also into the next definition.

Definition 2.1 *A pair (X, D) of a smooth surface and a divisor on it is called* Italian *iff the following conditions are fulfilled.*

- *D is nef.*
- *$|D|$ is not empty.*
- *X is D-minimal.*
- *$p_a = p_g = P_2 = 0$.*

If (X, D) is a pair, then an adjoint pair *is a minimalization $(X, D)' = (X', D')$ of the pair $(X, D + K)$.*

Note that the adjoint pair is unique iff $(D + K)^2 > 0$, because in this case the minimalization is unique.

Note also the $D^2 \geq 0$ for an Italian pair (X, D).

By lemma 2, the adjoint pair of an Italian pair (X, D) is again Italian, or $|D'|$ is empty. In the second case, we say "the adjoint pair of (X, D) does not exist". This terminology dates back to the last century, when "real" divisors had to be effective.

We consider the case of plane divisors with base points. Let

$$D := mL - m_1A_1 - \ldots - m_rA_r$$

be such that $(\mathbf{Y}, D)$ is Italian. Normally, we have

$$D := (m-3)L - (m_1-1)A_1 - \ldots - (m_r-1)A_r$$

with subsequent blowing down of the A_i's where $m_i = 1$ (cancelation of simple base points). But sometimes there are other exceptional curves to blow down.

Example 6: Let $r = 2$. Let a_1, a_2 be two different plane points. Let $D := 5L - 2A_1 - 2A_2$. Then $D + K = 2L - A_1 - A_2$. Let L_1 be the line through a_1, a_2 ($L_1 \sim L - A_1 - A_2$). Then L_1 is exceptional. Also, $(D+K)\cdot L_1 = 0$, hence $\mathbf{Y}$ is not $D+K$-minimal. The adjoint pair is the blowing down of L_1, together with the unique divisor with pullback $D+K$. It happens to be a very ample divisor, producing an embedding as a smooth conic in $\mathbf{P}^3$.

Remark: I do not know an example of an Italian pair (X, D), such that $D^2 > 0$, D large, and $|D'|$ has fixed components. When reading the papers of the classical Italians, one gets the impression that they believed that such an example does not occur. It seems unlikely, indeed.

Remark: Rephrasing our terminology in the terms of the classical Italian school of geometry (and assuming Castelnuovos rationality criterion), an Italian pair is something like an equivalence class of linear systems of plane curves with base points, modulo birational transformations of the plane. The classical Italians have computed classification tables for systems of genus 0,1,2,3 ([11]).

Starting with an Italian pair, we may repeat the adjunction process until we arrive at a divisor with genus zero or smaller, or at a divisor in the zero class. If (X, D) is any pair, than the sequence

$$(X^0, D^0) := \text{ a minimalization of } (X, D),$$

$$(X^{i+1}, D^{i+1}) := (X^i, D^i)' \text{ if } |D^i + K_{X^i}| \text{ is not empty}$$

is called adjoint series of (X, D).

By lemma 1, the adjoint series of Italian pairs consists of Italian pairs, and if it terminates, then the last pair (X^μ, D^μ) satisfies either $g(D^\mu) \leq 0$ or $D^\mu \sim 0$.

Lemma 2.3 *The adjoint series of an Italian pair (X, D) is finite.*

Proof: Suppose indirectly that the adjoint series is infinite. Since we cannot blow down infinitely often, the rank of Num dropping at each blowing down, we may wlog assume that the surface never changes:

$$(X^i, D^i) = (X, D + iK).$$

All divisors are nef and have non-empty linear systems. Thus,

$$(D + iK)\cdot(D + jK) = D^2 + (i + j)D\cdot K + ijK^2 \geq 0$$

for all $i, j\geq 0$. This is only possible if $K^2\geq 0$.

Riemann-Roch, applied to $2K$, gives

$$\dim|2K| + \dim|-K| \geq K^2 - 1.$$

But $\dim|2K| = P_2 - 1 = -1$, hence $|-K|$ is not empty. Let A be a divisor in $|-K|$. A_0 cannot be zero, because $p_g = 0$.

There exists an effective divisor B such that $A\cdot B > 0$: Think of an embedding of X, and let B be the divisor of a hyperplane section. The number $A\cdot B$ is nothing else than the degree of A as a "generalized curve" (i.e. composite with multiple components) in the projective space.

For large i, $(D + iK)\cdot B < 0$, contradicting nef-ness of $D + iK$.

The adjoint sequence enjoys the following functorial property.

Proposition 2.2 *Let (X, D) be a pair. The length of the adjoint series of (X, D) is equal to the largest μ such that $|D|$, $|D + K|$, ..., $|D + \mu K|$ are non-empty. For each $i\leq m$, let $\pi_i : X\rightarrow X^i$ be the composition of the minimalization maps. Then the diagram of rational maps*

$$\begin{tikzcd} & X \arrow[dl, "\pi_i"'] \arrow[dr, "i_{D+iK}"] & \\ X' \arrow[rr, "i_{D^i}"] & & \mathrm{P}^N \end{tikzcd}$$

commutes.

Proof: By the effect of pullbacks on associated maps (cf. subsection 1.3), the composed map is $i_{\pi_i^* D^i}$. We have to show that the moving part of $|\pi_i^* D^i| =: |L|$ is equal to the moving part of $|D + iK| =: |R|$. This entails also the assertion on the lenghth of the adjoint series, because it shows that either both or no moving part is empty.

Every blowing down $\pi : \tilde{X} \to X$ in a minimalization causes a difference between L and R, because $K_{\tilde{X}}$ differs from $\pi^* K_X$ by the exceptional divisor E (cf. subsection 1.3). Dynamically speaking, the blowing down step subtracts $(i-j)E$ from L, while it leaves R intact. Here, $j \leq i$ is the index of the adjoint pair which is currently constructed. The point is that E is an $(i-j)$-fold fixed component of L, hence the moving parts are not changed.

To make this argument exact, we approximate R along the blowing downs which compose to π_i. Here is the procedure.

$Y := X$;
$\pi_i :=$ identity on X;
$H := D$;
$R := L\ (= D + iK\ )$;
for j from 0 to i do
    while X is not H-minimal do
        Now, $R = \pi_i^*(H + (i-j)K_Y)$.
        $E :=$ an exceptional curve with $E \cdot H = 0$;
        The critical step:
        $R := R - (i-j)\pi_i^* E$;
        blow E down by $\pi : Y \to Y'$;
        $\pi_i := \pi \circ \pi_i$;
        $Y := Y'$;
        $H := (\pi^*)^{-1}(H)$;
        Again, $R = \pi_i^*(H + (i-j)K_X)$.
    $H := D + K_Y$;

At the critical step, $(H + (i-j)K_Y) \cdot E = -(i-j)$, so E is a fixed component of $|H + (i-j)K_Y|$. We subtract E, but again $(H + (i-j)K_Y) \cdot E = 1 - (i-j)$ is negative, so E is fixed in $|H + (i-j)K_Y - E|$. We can do this $i-j$ times, thus showing that $(i-j)E$ is in the fixed part of $|H + (i-j)K_Y|$.

The property of being in the fixed part is preserved by pullbacks, because it can be expressed in terms of the group structure, the partial order, and $\dim|\_|$. Consequently, $(i-j)\pi_i^* E$ is in the fixed part of R. Hence, the critical step removes only fixed divisors.

For plane divisors with base points, we can give easily an explicit bound for the adjoint length. Let

$$D := mL - m_1 A_1 - \ldots - m_r A_r$$

be an arbitrary plane divisor with base points. The divisor $D + iK$ has degree $m - 3i$. The linear system $|D + iK|$ can only be non-empty if $m - 3i \geq 0$. The adjoint length is therefore bound by $\frac{m}{3}$, also in the not so good case where we have to blow down exceptional curves. This shows the first part of proposition 1.1.

We apply the property mainly to Italian pairs. However, it seems also promising to apply it to the case $q(X) = 0$ and D is nef. The adjoint series does not always terminate for this case. But it may be used to construct a (unique by [43]) minimal model for X. An exceptional curve E is blown down exactly in the $D{\cdot}E$-th adjunction step.

2.3 Singular surfaces

Here we explain the second, more important construction of Italian pairs, which is obtained by resolving the singularities of a singular surface in 3-space that fulfills Castelnuovo's criterion.

The algorithms in chapter 1 deal with singular surfaces in $\mathbf{P}^3$. On the other hand, we have only developed geometric concepts for the smooth case. This is the method of the madness: If Y is a singular projective surface, then there exists a *desingularization* $\pi : X{\to}Y$, with X smooth projective and π regular. We do the geometric reasoning on X. Instead of computing X explicitly, we use the functorial property of the adjoint series to prove something about the adjoint vector-spaces of Y.

Let $Y{\subset}\mathbf{P}^3$ be a singular surface, given by a homogeneous equation $F(x, y, z, w)$ of degree d. Let $\pi : X{\to}Y$ be a desingularization of X. To find a divisor D on X, somehow associated with π, we compose π with the embedding

$$X \to Y \hookrightarrow \mathbf{P}^3,$$

and construct a D by pulling back a divisor on $\mathbf{P}^3$, skipping Y completely.

The composed map $X{\to}\mathbf{P}^3$ is associated to an incomplete linear system Γ of dimension 3 without base points. We choose D to be an arbitrary divisor in Γ (note that we are mainly interested in the class of D, so the choice of D does not matter). Obviously, D is very large. Furthermore, we know that $\dim|D|{\geq}3$, the map associated to $|D|$ is birational, and D^2 is the degree d of the image surface Y. To prove the third assertion, we remark that the common points of two divisors in Γ are mapped to the intersection points of Y with a line. Generically, this mapping is bijective, and the line is generic, hence it intersects Y in d points.

Example 7: Let Y be the surface of degree 3, given by the equation

$$F(x, y, z, w) = x^2y - z^2w.$$

Y has a double line $x = z = 0$. Here is a desingularization of Y. Let $X \in \mathbf{P}^4$ be the surface given by the equations

$$x^2 y - z^2 w, u^2 - wy, ux - yz, uz - xw.$$

Let π be the projection

$$(x : y : z : w : u) \mapsto (x : y : z : w).$$

The projection is birational because $u = \frac{yz}{x}$. It is also regular, because the projection center $p = (0 : 0 : 0 : 0 : 1)$ is not in X.
The map π is associated to the linear system of all hyperplane sections of X with hyperplanes through p. Its completion is the linear system of all hyperplane sections, which has dimension 4.

To compute the genus of D, we can use Bertini's theorem: The generic divisor of a very large linear system is smooth. The genus of D coincides therefore with the genus, in the sense of genus for curves, of the generic divisor in D. But the generic divisor is birationally mapped to a generic plane section in Y, and curve genus is a birational invariant. Hence $g(D)$ is the genus of the generic plane section of Y.

In example 7, the generic plane section is a cubic with one double point, the intersection of the plane with the double line. Cubics with a double point are rational, hence $g(D) = 0$.

On X, there exists a divisor Δ, depending on π. We must delay the exact definition to chapter 6. Here, we can only tell that π maps its components into the singular locus of Y, and the multiplicity of the components are related to the multiplicity of the image singularity. This divisor is called the *adjoint divisor*.

Here is the geometric interpretation of the adjoint ideals A_m which occurred in chapter 1. It is the ideal of all forms with pullback divisor at least $m\Delta$.

The proof of the following fundamental theorem must also be delayed to chapter 6.

Theorem 2.7 *Let $Y \subset \mathrm{P}^3$ be a singular surface of degree $d > 1$. Let $\pi : X \to Y$ be a desingularization of Y. Let D be a divisor in the linear system whose associated map is π. The adjoint divisor and the canonical divisor are related by the equation*

$$(d-4)D - \Delta \sim K.$$

Let $n \geq 0$, $m > 0$. Then the linear system

$$\{(\pi^* G) - m\Delta \mid G \in V_{n,m}\}$$

is equal to the linear system $|nD + mK|$.

It is easy to show that the first statement implies that the linear system defined by $V_{n,m}$ as above is a subsystem of $|nD+mK|$. The nontrivial part of the second statement is the completeness of the linear system.

Here is the geometric interpretation of the adjoint dimensions.

Corollary 2.6

$$v_{n,m} = \dim|nD + mK| - 1,$$

especially

$$v_{0,m} = \dim|mK| - 1 = P_m.$$

Corollary 2.7 *The function from positive integers to integers*

$$n \mapsto v_{n,1}$$

is a polynomial of degree 2 with leading coefficient $\frac{d}{2}$ *and constant coefficient* $p_a + 1$.

Proof: By the corollary to Grauert-Riemenschneider vanishing,

$$v_{n,1} = \dim|nD + K| - 1 = g(nD) + p_a = \frac{D^2}{2}n^2 + \frac{D{\cdot}K}{2}n + 1 + p_a$$

for $n > 0$.

Note that the two corollaries, together with Castelnuovo's rationality criterion, yield a correctness proof for the algorithm *deciderationality*.

Example 7 continued: Recall the equations of $X{\in}\mathbf{P}^4$ of example 7:

$$x^2y - z^2w, u^2 - wy, ux - yz, uz - xw.$$

$\pi : X{\to}Y$ is the projection on the four coordinates x, y, z, w. The adjoint divisor is the inverse image of the double line $x = z = 0$, which is the conic

$$x = z = u^2 - wy = 0.$$

The adjoint ideal A_1 is (x, z).
The adjoint vector-space $V_{2,1}$ is $\langle x, z\rangle$.
Let $v_0 = x - tz$ be an generic element in $V_{2,1}$. The pullback is the sum of the adjoint divisor with the line

$$t^2y - w = ty - u = x - tz = 0.$$

For varying t, these lines make up $|2D + K|$, a system of dimension 1, genus 0 and self-intersection 0.

Here is the geometric interpretation of the adjoint maps.

Corollary 2.8 *Let X, π, Y, D be as in theorem 7 above. Let $n \geq 0$, $m > 0$. Then $v_{n,m} = 0$ iff the $m-th$ adjoint pair of (X, nD) does not exist.*

Suppose that $v_{n,m} > 0$. Let $i_{n,m} : Y \rightarrow \mathbf{P}^{v_{n,m}-1}$ be the adjoint map. Let (X_m, D_m) be the m-th adjoint pair of the pair (X, nD). Then the diagram of rational maps

$$
\begin{tikzcd}
& Y \arrow[dr, "i_{n,m}"] & \\
X \arrow[ur, "\pi"] \arrow[rr, "i_{nD+mK}"] \arrow[dr] & & \mathbf{P}^{v_{n,m}-1} \\
& X^m \arrow[ur, "i_{D^m}"'] &
\end{tikzcd}
$$

commutes.

Proof: The lower triangle commutes by proposition 2.

Let Γ be the linear system defined by the adjoint vector-space $V_{n,m}$. Obviously, $i_{n,m} \circ \pi = i_\Gamma$. By theorem 7, $i_{nD+mK} = i_\Gamma$ because the linear system differ by a fixed divisor $m\Delta$.

If X satisfies Castelnuovo's criterion, then all pairs in occuring in the above diagram are either Italian, or "do not exist" (the linear system is empty).

As a first application, we prove the bound for the adjoint length of a parametrized surface, mentioned before (chapter 1).

Theorem 2.8 *Let $\overline{x}, \overline{y}, \overline{z}, \overline{w}$ by homogeneous forms of degree N in 3 variables s, t, u, defining a proper parametrization*

$$\sigma : \mathbf{P}^2 \rightarrow Y \subset \mathbf{P}^3,$$

$$(s : t : u) \mapsto (\overline{x}(s,t,u) : \overline{y}(s,t,u) : \overline{z}(s,t,u) : \overline{w}(s,t,u))$$

of s surface Y. Then the adjoint length μ of Y is less or equal $\frac{N}{3}$.

Proof: We may assume that the forms $\overline{x}, \overline{y}, \overline{z}, \overline{w}$ do not have a common divisor. We resolve the base locus and get and get an incomplete system Γ of plane divisors with base points, equivalent to $D := NL - m_1 A_1 - \ldots - m_r A_r$.

The map $\sigma' : \mathbf{Y} \rightarrow Y$ is a desingularization of Y. By corollary 8, the adjoint length of μ of Y is the adjoint length of the pair $(\mathbf{Y}, D)$. For this pair, we have the obvious bound $\mu \leq \frac{N}{3}$ (section 2).

Example 8: The homogenization of the parametrization of the sphere is

$$(s : t : u) \mapsto (2su : 2tu : s^2 + t^2 - u^2 : s^2 + t^2 + u^2).$$

The four polynomials have two points in common, namely

$$a_1 = (1 : \sqrt{-1} : 0) \text{ and } a_2 = (1 : -\sqrt{-1} : 0).$$

The divisor D is equivalent to $2L - A_1 - A_2$. Incidentally, $\dim|2L - A_1 - A_2| = 3$, so the pullbacks of plane sections form the complete linear system $|2L - A_1 - A_2|$.

The other inequality about the degree N mentioned before, namely $N^2 \geq d$, is quite obvious from the formula for self-intersection (section 1.4).

2.4 Classification results

Here we classify Italian pairs that arise at the end of adjoint series. This section contains the correctness proofs for the algorithms *deciderationality*, *parametrize*, *adj1dim*, and *adj2dim*.

We start with a proposition about curves occuring as a component in certain linear systems.

Proposition 2.3 *Let (X, D) be an Italian pair, such that either $g(D) \leq 0$ or $D + K \sim 0$. Let C be a prime component of s divisor in $|D|$. Then C is rational smooth, except when C is already in $|D|$ and $D + K \sim 0$. In this case, C is smooth elliptic or rational with a double point.*

Proof: If $g(D) \leq 0$ or if $C < D$, then $|C + K|$ is empty. Hence, $g(C) \leq 0$ (genus in the divisor-sense) by the corollary to Riemann-Roch. But since C is supposed to be prime, we conclude the claim from the corollary to the genus formula.

If $C + K \sim 0$, then $g(C) = 1$, and the claim follows from the genus formula. (The double point makes a difference between the genus in the divisor-sense and the genus in the curve-sense.)

Lemma 2 tells that the adjoint series of a large divisor D with $D^2 > 0$ will contain only large divisors up to a step where the self-intersection becomes zero. Furthermore, the equation

$$\dim|D^{i+1}| = g(D^i) - 1$$

is valid until the mentioned step interferes and makes Grauert-Riemenschneider vanishing inapplicable. Here is a more detailed investigation of this step.

Theorem 2.9 *Let (X, D) be an Italian pair. Let (X', D') be the adjoint pair. Suppose that $D'^2 = 0$, but $D' \not\sim 0$. Then one of the following holds.*

a) $D' \sim nP$ for some rational pencil P. P fulfills $P^2 = 0$, $P \cdot K_{X'} = -2$, $g(P) = 0$ and $\dim|P| = 1$. Every effective divisor of D' has n components, each of them a member of $|P|$. We have $g(D) = n + 1$ and $g(D') = -n + 1$.

b) $g(D') = 1$ and $D^2 = 0$.

Proof: We distinguish two cases. First, let $g(D')>0$. Then $D'+K_{X'}$ is nef and $|D'+K_{X'}|$ is not empty.

D differs from $\pi^*(D'-K_{X'})$ only by some exceptional divisors which are blown down by the minimalization. The difference is orthogonal to $\pi^*\mathrm{Div}X'$. This yields

$$\dim|D| = \dim|\pi^*(D'-K_{X'})| = \dim|D'-K_{X'}|$$

and

$$D^2 \leq (D'-K_{X'})^2.$$

Also, $D'-K_{X'}$ is nef because D is nef. But

$$D'(D'+K_{X'}) + D'(D'-K_{X'}) = 2D'^2 = 0,$$

so we have $D'(D'+K_{X'}) = D'(D'-K_{X'}) = 0$. Consequently, $g(D')=1$.

Now, $|D'-K_{X'}|$ is not empty. Hence, $(D'-K_{X'})^2 \geq 0$. If $(D'-K_{X'})^2 > 0$, then $D' \sim 0$ by the Hodge index theorem. This was excluded, and $(D'-K_{X'})^2=0$ only remains. But $D^2 \leq (D'-K_{X'})^2$, hence $D^2=0$.

Second, let $g(D') \leq 0$. Since D' is nef, $D' \cdot C = 0$ for any component C of an effective divisor in $|D'|$. Let C be a fixed component of $|D'|$. By proposition 1 on rigid curves, $C \cdot K_{X'} \geq -1$. In case of equality, C is exceptional, and $D' \cdot C = 0$ contradicts D'-minimality of X'.

Let F be the fixed part of $|D'|$. Then $F \cdot K_{X'} \geq 0$ and $F^2=0$ (since $D'-F$ is also nef). By Riemann-Roch, applied to $-F$,

$$\dim|-F| + \dim|K+F| = \dim|-F| + (-1) \geq \frac{F \cdot K_{X'}}{2} - 1 \geq -1.$$

Thus $|-F|$ is not empty, which is only possible for $F=0$. $|D'|$ does not have fixed components.

Since all components C of an effective divisor in $|D'|$ move at least in a pencil, they are all nef. This implies that all intersection products $C_i \cdot C_j$ vanish. C_i and C_j must be equivalent, otherwise one could find intersecting curves in their classes. We set $P := C_i, D' \sim nP$. P is smooth rational by lemma 3.

The genera of D and D' can now be easily computed.

Remark: Note that in case (a) we did not use the property that (X',D') is an adjoint pair. (a) holds therefore for all Italian pairs with $D'^2=0$ and $g(D') \leq 0$.

Corollary 2.9 *Let (X,D) be an Italian pair with D large and $D^2>0$. Let $(X^i,D^i)_i$, $i=0,\ldots,\mu$ be its adjoint series. Then the following hold.*

a) D_i is large for $i=0,\ldots,\mu$.

b) $D_i^2>0$ for $i=0,\ldots,\mu-1$.

c) $s(D_i)=0$ for $i=1,\ldots,\mu$.

Proof: The corollary follows by induction from the following claims.

Claim 1: $D^2 > 0$ and $D'^2 = 0$ implies that (X', D') has no adjoint. This follows immediately from the theorem.

Claim 2: $D^2 > 0$ and D is large imply D' is large (lemma 2).

Claim 3: $D^2 > 0$ and D is large implies $s(D') = 0$ (Grauert-Riemenschneider vanishing).

Example 9: Let D be the plane divisor with one base point

$$D = (n+3)L - (n+1)A_1.$$

The corresponding pair $(\mathbf{Y}, D)$ is Italian. Its adjoint pair is $(\mathbf{Y}, D')$, where

$$D' = n(L - A_1).$$

This is an example for (a).

Example 10: Let D be the plane divisor with base points

$$D = 6L - 2A_1 - \ldots - A_9,$$

where the points $a_1, \ldots, a_9$ are supposed to be in generic position. Then the pair $(\mathbf{Y}, D)$ is Italian. Its adjoint pair is $(\mathbf{Y}, D')$, where

$$D' = 3L - A_1, \ldots - A_9.$$

This is an example for (b). Such an example cannot occur in an adjoint series of a nef divisor with positive self-intersection. See also the example 5 for a nef but not large divisor.

Here is the classification of pairs with nonnegative genus and positive self-intersection. Note the similarities with theorem 1.2.

Theorem 2.10 *Let (X, D) be an Italian pair with $g(D) \leq 0$ and $D^2 > 0$. Then $g(D) = 0$, D is very large, and i_D is birational, and* $\dim|D| = D^2 + 1$, *and $s(D) = 0$. Moreover, one of the following holds.*

a) $3D + K \sim 0$, $D^2 = 1$ ($K^2 = 9$). Here, i_D is an isomorphism to P^2.

b) $3D + 2K \sim 0$, $D^2 = 4$ ($K^2 = 9$). Here, i_{2D+K} is an isomorphism to P^2.

c) $2D + K \sim 0$, $D^2 = 2$ ($K^2 = 8$). Here, $i_D(X)$ is a surface of degree 2 in P^3.

d) $2D + K \sim nP$ for some P with $P^2 = 0$, $g(P) = 0$, and $D \cdot P = 1$, $D^2 = n + 2$ ($K^2 = 8$). The image $i_D(P')$ of any divisor $P' \in |P|$ is a line on $i_D(X)$.

Proof: X contains a moving rational curve by proposition 3. In section 5, we will show that this is sufficient for rationality (theorem 12). We need the equality

$$\mathrm{rank}(\mathrm{Num}X) + K^2 = 10$$

(corollary to the theorem 6), mainly for having a good upper bound for K^2.

Let $m > 1$ be the integer defined by the inequality

$$0 \le mD^2 + D{\cdot}K \le D^2 - 1.$$

mD is nef, hence also $mD+K$ is nef and $|mD+K|$ is not empty by lemma 2. Then

$$0 \le (mD+K)^2 = m^2D^2 + mD{\cdot}K + mD{\cdot}K + K^2 \le$$
$$\le mD^2 - m + D{\cdot}K + (m-1)D{\cdot}K + K^2 \le$$
$$\le D^2 - 1 - m + D{\cdot}K + (m-2)D{\cdot}K + K^2 \le$$
$$\le -2 - 1 - m + (m-2)(-3) + K^2 = 3 - 4m + K^2 \le 12 - 4m,$$

hence $m \le 3$.

If $m = 3$, then we must have equality everywhere: $D^2 = 1$, $D{\cdot}K = -3$, $D{\cdot}(3D+K) = 0$, $K^2 = 9$. Hodge index gives $3D+K \sim 0$. Applying Reider's lemma to $R := 4D$, we find that $R + K \sim D$ is very large and i_D is birational. Since $4D$ is obviously large, $s(D) = 0$ by Grauert-Riemenschneider vanishing. Hence $\dim|D| = 2$, and i_D is a birational map to $\mathbf{P}^2$, i.e. a blowing down of a finite number of points. But in absence of exceptional curves normal to D, i_D must be an isomorphism. We have shown (a).

If $m = 2$, then

$$0 \le (2D+K)^2 = 4(D^2 + D{\cdot}K) + K^2 \le -8 + K^2,$$

and on the right hand side we must have an equality.

If $K^2 = 9$, then $(2D+K)^2 = 1$, $D^2 + D{\cdot}K = -2$, hence $D^2 = 4$, $D{\cdot}K = -6$, and $D{\cdot}(3D+2K) = 0$. As in the previous case, one shows that i_{2D+K} is an isomorphism to $\mathbf{P}^2$. Since $D \sim 2(2D+K)$, D is also very large and i_D is birational.

If $m = 2$ and $K^2 = 8$, then $(2D+K)^2 = 0$ and $D^2 + D{\cdot}K = -2$ $(g(D) = 0)$.

If $2D+K \sim 0$, then $D^2 = 2$. Reider's lemma applies to $R := 3D$, showing that D is very large and i_D is birational. By Grauert-Riemenschneider vanishing, $s(D) = 0$ and $\dim|D| = 3$. This shows (c).

Finally, suppose that $(2D+K)^2 = 0$, $2D + K \not\sim 0$, $g(D) = 0$, $K^2 = 8$. Then Num$X$ has rank 2. Thus, X is $2D+K$-minimal, and hence $(X, 2D+K)$ is Italian. By theorem 9, $2D + K \sim nP$ for some P with

$P^2 = 0$ and $P{\cdot}K = -2$. Hence $D^2 = n+2$ and $D{\cdot}K = -n-4$. It follows $D{\cdot}P = 1$.

From a statement on the geometry of a ruled surface (theorem 13), it follows that $s(D) = 0$, D is very large, and i_D is birational. If $P' \in P$, then the generic hyperplane intersects $i_D P'$ in exactly one point, because $D{\cdot}P' = 1$. Hence $i_D(P')$ is a line.

Example 11: Let a_1 be a point in the plane. The divisor

$$D := (r+1)L - rA_1$$

is very large, and $(\mathbf{Y}, D)$ is Italian. The genus is 0, and

$$D^2 = (r+1)^2 - r^2 = 2r+1 > 0.$$

This is an instance of (d), because

$$2D + K = (2r-1)L - (2r-1)A_1 = (2r-1)(L - A_1),$$

and $|L - A_1|$ is a rational pencil (the pencil of lines through a_1).

The following proposition proves the correctness of the algorithm *parametrize* (without sub-algorithms).

Proposition 2.4 *Let (X, D) be an Italian pair that has no adjoint. Suppose that (X, D) appears at the end of an adjoint series of an Italian pair with positive self-intersection. Then D is very large. Moreover, one of the following holds.*

a) $D \sim 0$. The image of i_D is a point.

b) $D \sim nP$ for a rational pencil P with $P^2 = 0$. The image of i_D is a rational normal curve. Also, $|2D+K|$ is empty.

c) $D^2 > 0$ and $g(D) = 0$. The image of i_D is a surface. If $\dim|D| \geq 3$, then $|2D+K|$ is not empty.

Proof: Suppose that $D^2 = 0$. By theorem 9, we have either (a) above, or case (a) in theorem 8: every divisor in $|D|$ has n components, each a divisor in a rational pencil $|P|$. It follows that i_D is $v_n \circ i_P$, where $i_P : X \to \mathbf{P}^1$, and v_n is the embedding of $\mathbf{P}^1$ as a rational normal curve in $\mathbf{P}^n$.

If $D^2 > 0$, then theorem 10 applies. Going through the classification, (c) is checked easily.

Note that theorem 10 and proposition 5 imply theorem 1.2 and therefore the correctness of the algorithm *adj2dim*.

Also the second part of proposition 1.1 is now easy: The polynomials $\overline{x}, \overline{y}, \overline{z}, \overline{w}$ in the parametrizations are equations of divisors in the initial linear system. In case (a) and (b), $X = \mathbf{P}^2$, and the initial divisor has degree $1(2)+3\mu$, modulo the base points that have been blown down during the adjunction process. Case (c) is left to the reader. (It is also an instance of the bound we will establish for (d) in the next section.)

Here is a geometric version of theorem 1.3, implying the correctness of the algorithm *adj1dim*.

Theorem 2.11 *Let (X,D) and (X',D') be the last two Italian pairs in the adjoint series of (X_0,D_0). Suppose that D_0 is very large and i_{D_0} is birational. Suppose that $i'_D(X')$ is a curve in $\mathbf{P}^n$. Then $D+K=nP$ for a rational pencil $|P|$ with $P^2=0$. Moreover, $D{\cdot}P=2$, and the following hold.*

a) $2D+K$ is very large, and i_{2D+K} is birational. If $P'{\in}|P|$, then $i_{2D+K}(P')$ is a conic on $i_{2D+K}(X)$.

b) Suppose that i_D is birational. If $P'{\in}|P|$, then $i_D(P')$ is a conic on $i_D(X)$.

Proof: By proposition 2.4, $D'{\sim}nP'$ for a rational pencil P' with $P'^2=0$. Let $\pi : X{\to}X'$ be the minimalization map in the adjunction. Then we let $P:=\pi^*P'$. P is nef, very large, $P^2=0$, $\dim|P|=2$, and $D+K=nP$. The last implies $D{\cdot}P=2$.

(a): If the length μ of the adjoint series is 1, i.e. $(X_0,D_0)=(X,D)$, then D is very large and i_D is birational. Consequently, $D+nP$ is also very large, because the very large divisors form a sub-semigroup. i_{D+nP} is also birational because i_D factors through it.

Suppose that $\mu{\geq}2$. Then $D-K$ is nef and $(D-K)^2=4D^2-8n>0$, by corollary 9. Hence $D^2>2$ and $(2D)^2{\geq}10$. We apply Reider's lemma to $R:=2D$. A Reider divisor E cannot exist: Since both $R+K$ and $R-K$ are nef, we have the inequality $|E{\cdot}K|{\leq}R{\cdot}E$. Since E^2 and $E{\cdot}K$ have the same parity, the only possible case is $D{\cdot}E=1$, $K{\cdot}E=0$, $E^2=0$. Then $nP{\cdot}E=1$, hence $n=1$ and $P{\cdot}E=1$. By Riemann-Roch, $|E+K|$ is not empty, which is absurd by $P{\cdot}(E+K)=-1$.

In both cases, $2D+K$ is very large and i_{2D+K} is birational. If $P'{\in}|P|$, then the generic hyperplane intersects $i_{2D+K}(P')$ in two points, because $P{\cdot}(2D+K)=2$. Hence $i_{2D+K}(P')$ is a conic.

(b) Let $|D_0|$ and F be the moving part and the fixed part of $|D|$. We show that $D_0{\cdot}P=2$. We may assume that the adjoint length is at least 2, because otherwise $D=D_0$. By corollary 9, this implies that $s(D)=0$.

Since P is nef, we have to exclude two cases: $D_0{\cdot}P=0$ and $D_0{\cdot}P=1$.

$D_0{\cdot}P=0$: Then any divisor in $|D_0|$ contains with any point p a whole component of the unique divisor in $|P|$ passing through p. Hence i_{D_0} is not birational. (With some more effort, one can show that this case cannot occur at all.)

$D_0 \cdot P = 1$: Then any divisor in $|D_0|$ has one component that intersects the pencil transversally in one point, which is therefore rational smooth. Any other component is a component of some curve in $|P|$. The generic divisor is smooth away from the base locus (Bertini's theorem), hence it is irreducible rational smooth. This shows that $g(D_0) = 0$. If $D_0^2 > 0$, then the minimalization of (X, D_0) satisfies the assumption of the classification theorem 10, hence $s(D_0) = 0$. This equation also holds for $D_0^2 = 0$, because then $\dim|D_0| \geq 1$ by Riemann-Roch, and $\dim|D_0| \leq 1$ because a single point determines already the divisor in $|D_0|$.

Now,

$$g(F) = D \cdot F + \frac{D_0^2 - D_0 \cdot K}{2} - \frac{D^2 - D \cdot K}{2} + 1 =$$

$$= D \cdot F + \dim|D_0| - s(D_0) - \dim|D| + s(D) + 1 = D \cdot F + 1 > 0,$$

hence $|F + K|$ is not empty. But $P \cdot (F + K) = -1$ and P is nef, a contradiction.

Example 12: This is the geometric analogue of example 1.7. Let

$$D = 7L - 3A_1 - 2A_2 - \cdots - 2A_{10},$$

$$D' = D + K = 4L - 2A_1 - A_2 - \cdots - A_{10},$$
$$D'' = D' + K = L - A_1,$$

where the points $a_1, \ldots, a_{10}$ are carefully chosen (cf. below) points on a smooth cubic C. We will consider C as equivalent to $3L - A_1 - \ldots A_{10}$.
D'' is a rational pencil with $(D'')^2 = 0$, obviously very large. The base locus of $|D'|$ is contained in C, because we can form divisors in $|D'|$ be adding C divisors in $|D''|$. Doing so, we obtain a pencil contained in $|D'|$. By Riemann-Roch, $\dim|D'| \geq 2$, hence $|D'|$ contains curves not having C as component. Because such a curve D_0 intersects C in exactly one point b, only b can be a base point. Since a single point cannot move on an elliptic curve, b is indeed a base point: any curve in $|D'|$ not containing C intersects C in b. This can be used to show that $\dim|D'| = 2$: For any two curves in $|D'|$, there is a nontrivial linear combination of the equations which is divisible by the equation of C.
We choose the points $a_1, \ldots, a_{10}$ such that there exists a curve in $|D|$ not having C as a component (this can be expressed in an equation of the group of points on C). Then $|D|$ contains a curve disjoint from C, hence D is very large. As above, we see that $\dim|D| = 3$.
With some techniques to be yet described (section 6), we can prove that i_D is birational and $i_{D'}$ is a rational map of degree 2. The last but one adjoint pair does not give the birational map, but theorem 11 tells that

$$2D' + K = 5L - 3A_1 - A_2 - \ldots - A_{10}$$

is very large and gives a birational map.

Remark: Despite the appearance in the given examples, adjoint series where the last but one pair does not give a birational map are hard to find. The restrictions on the intersection product are narrow enough to write down a table of possible cases. However, to decide whether these possible cases really occur is sometimes rather difficult.

2.5 Rulings

This section deals with a special sort of rational surfaces, called ruled surfaces. It contains a rather explicit algebraic description, which is also exploited to improve an algorithm in chapter 1.

Promoting the concept of Italian pairs, I cannot refuse to define a *ruling* as an Italian pair (X,P) with $g(P) = P^2 = 0$. Here is the link to the usual geometrical concept of a ruling.

Proposition 2.5 *Let (X,P) be a ruling. Then any curve in $|P|$ is irreducible, smooth and rational. P is very large, and* $\dim|P| = 1$*.*

Proof: By theorem 9 and the subsequent remark, any curve in $|P|$ is irreducible, smooth and rational. By Riemann-Roch, $\dim|P|\geq 1$. Two curves in $|P|$ do not intersect because $P^2 = 0$. Hence P is very large. Any point on X can only lie on a single curve in $|P|$, hence $\dim|P| = 1$.

Note that the curves in $|P|$ are the fibers of the regular map $i_P : X \to \mathbf{P}^1$. It is sometimes convenient to write $\mathbf{P}^1$ as the union of k, the ground field, and the point ∞. Then we may write

$$X = \bigcup_{t \in \mathbf{P}^1} X_t,$$

where $X_t = (i_D)^{-1}t$ is the fiber over t. Algebraically, the map i_D induces a field embedding $k(t) \hookrightarrow k(X)$ of function fields. It makes sense to consider X as a curve over the ground field $k(t)$. We write X_t for this curve. This time, the subscript is not an index, because t means an indeterminate and not a variable. But the indeterminate can be considered as the generic element of $\mathbf{P}^1$. Hence X_t is the generic fiber of i_D, or the generic element of D. In particular, X_t is a smooth rational curve.

Here is a cornerstone in the theory of rational surfaces.

Theorem 2.12 *Let X be a surface. Let Γ be linear system, not necessarily complete, with dimension 1. Suppose that the generic divisor is rational. Then X is rational.*

Proof: We assume the default assumption on the beginning of this chapter: the ground field of X is algebraically closed.

We are stressing the above ambiguity of the symbol X_t again, considering X_t as generic element of Γ and at the same time as curve over $k(t)$, via the field embedding $k(t) \hookrightarrow k(X)$ induced by i_Γ.

As the generic element of Γ, X_t is a rational curve. Switching viewpoint, we ask whether there is a parametrization over $k(t)$. The ground field of X_t is $k(t)$, which is certainly not algebraically closed. But it is well known that any rational curve over any ground field can be birationally transformed to a conic.

By theorem 1.3, any conic over $k(t)$ has a point defined over $k(t)$. Hence X_t can be properly parametrized with rational functions in s, defined over $k(t)$. This parametrization is also a parametrization of X with rational functions in s, t.

Corollary 2.10 *Let (X, P) be a ruling. Then X is rational.*

Remark: If we regard the above proof as an algorithm, then the most expensive part is the use of theorem 1.3, the computation of a point defined over $k(t)$.

Let us investigate the geometry on a ruled surface more closely.

Lemma 2.4 *Let (X, P) be a ruled surface. Let D be a divisor such that $P \cdot D = 0$. Then $D \sim rP$ for some integer r.*

Proof: Let P be a fiber. Since the restriction of D to P is linearly equivalent to 0, there is a principal divisor D' such that any component (with positive or negative sign) of $D - D'$ is disjoint from P. But any irreducible curve disjoint from P is equivalent to P, because it contains the fiber through one of its points. Hence $D \sim D' + rP \sim rP$.

A point on the generic fiber "is" a point varying with t. Geometrically, this means a curve intersecting each fiber once, also called a cross section. By theorem 1.3, we know that any ruling has a cross section. A cross section is a smooth rational curve, since the restriction of i_D is an isomorphism to the projective line.

Abusing language, we call an effective divisor B a cross section iff $B \cdot P = 1$. The cross section is called fundamental iff $|B - P|$ is empty.

Lemma 2.5 *Fundamental cross sections exist. Two fundamental cross sections are equivalent.*

Proof: Start with an arbitrary cross section B. Let n be the largest number such that $|B-nP|$ is not empty. Then any divisor in $|B-nP|$ is a fundamental cross section. Since all equivalence classes of cross sections differ by a multiple of P, the choice of B has no effect on the class of the result.

We define the *characteristic number* $e := -B^2$. We have $\dim|B|\leq 1$, because otherwise one could assign two base points on a single fiber in the linear system $|B|$ and split of a component P. By Riemann-Roch, this implies $e\geq 0$. In case of equality, (X, B) is another ruling transversal to (X, P). In case of inequality, B is rigid, and there is only one fundamental cross section.

Theorem 2.13 *Let (X, P) be ruling. Let B be a fundamental cross section.*

a) P and B generate Pic$X$.

b) D is nef iff $D{\cdot}P\geq 0$ and $D{\cdot}B\geq 0$.

c) D is very large iff $D{\cdot}P\geq 0$ and $D{\cdot}B\geq 0$.

d) i_D is birational if $D{\cdot}P>0$ and $D{\cdot}B\geq 0$, except when $e=0$. In this case, i_D is birational if $D{\cdot}P>0$ and $D{\cdot}B>0$.

e) $K=-(e+2)P-2B$.

Proof: (a): B generates the cokernel of $\cdot P$: Pic$X\to\mathbf{Z}$, and P generates the kernel.

(b) only if: trivial.

(b) if: Let C be an irreducible curve different from B. Obviously, $C{\cdot}P\geq 0$ and $C{\cdot}B\geq 0$. By (a), $C\sim aP+bB$ for some integers a, b. We know that $b\geq 0$ and $a\geq eb\geq 0$. Hence $D{\cdot}C = aD{\cdot}P + bD{\cdot}B\geq 0$.

(c) only if: trivial.

(c) if: The sum of very large divisors is very large. P is obviously very large. We will show that $eP+B$ is very large. The statement follows from

$$D\sim(D{\cdot}B)P+(D{\cdot}P)(eP+B).$$

(d): We will show that i_{eP+B} is birational for $e>0$. Since a birational divisor plus an effective divisor gives a birational divisor, the statement follows from the equality above. In case $e=0$, we show the birationality of $P+B$, which implies again the statement by the same reason.

(e): follows easily from (a) and $g(B)=g(P)=0$.

Two properties of $eP+B$ were used in the proof above. We state them in a separate proposition, because it will lead to a convenient algebraic description.

Proposition 2.6 *Let (X,P) be a ruling with $e>0$. Then $eP+B$ is very large, and* $\dim|eP+B|=e+1$. *The associated rational map is birational, and the image is the cone over a rational normal curve of degree e in $\mathbf{P}^e$. It contracts B to the vertex of the cone and is an isomorphism everywhere else.*

Proof: Since P is very large, the base locus of $|eP+B|$ is contained in B. We know that $\dim|eP|=e$. By Riemann-Roch, $\dim|eP+B|\geq e+1$. Hence there is a curve in $|eP+B|$ that does not have B as component. This curve is disjoint from B, because $B\cdot(eP+B)=0$. Hence the base locus is empty.

If a curve in $|eP+B|$ contains a point on B, then it contains the whole curve B as a component (since $B\cdot(eP+B)=0$ again). This shows that B is contracted to a point. The residual system is associated to the map followed by projection from this point. It is equal to eP, associated to a map to the rational normal curve of degree e. Hence $i_{eP+B}(X)$ is a variety whose projection from a point is a rational normal scroll, i.e. a cone over the rational normal scroll. This also shows $\dim|eP+B|=e+1$.

One can see that the fibers are the rays of the cone. The map leaves the fibers disjoint except in the vertex.

We can recover X by blowing up the vertex of the cone. Because two rational normal curves of the same degree are isomorphic, we get an easy classification of ruled surfaces.

Corollary 2.11 *Two ruled surfaces are isomorphic iff their characteristic numbers coincide.*

If $e=0$, then $X\cong\mathbf{P}^1\times\mathbf{P}^1$, with i_P and i_B as projections. The map i_{P+B} is an embedding to a smooth quadric in $\mathbf{P}^3$.

Here is a graded coordinate ring for the ruled surface X with characteristic number e. Let $S_r:=k[p,q,u,v]$ be the polynomial ring in 4 variables. We assign bi-degrees from $\mathbf{N}\times\mathbf{N}$ to the generators, as follows.

$$\deg p=\deg q=(1,0), \deg u=(0,1), \deg v=(e,1).$$

Then X is the projective variety of this graded ring. The points of X are all 4-tuples (p,q,u,v) such that $(p,q)\neq(0,0)$ and $(u,v)\neq 0$, modulo the equivalence relation

$$(p,q,u,v)\sim(\lambda p,\lambda q,\mu u,\lambda^e\mu v).$$

All divisors have a global homogeneous equation. Two divisors are equivalent iff their bi-degrees coincides. The class $aP+bB$ has bide-gree (a,b).

For instance, the fibers have equation $\alpha p + \beta q = 0$ for some field elements $s\alpha, \beta$. The equation of B is $u = 0$.

To compute the dimension of $|aP + bB|$ is now a simple combinatorial task. Straightforward calculation shows that $s(D) = 0$ for all nef divisors D.

We apply this analysis to improve the choice of the second parameter in the algorithm *adj2dim*, for the sake of a smaller degree of the parametrization.

The first step is to find two inhomogeneous parameters s, t to express the ratios of homogeneous vectors in s, t. The parameter t is already known, namely $t := \frac{p}{q}$. The parameters should be such that the degree of the result polynomials in s, t is as small as possible. The best choice is $s := \frac{v}{uq^e}$.

Let D be nef such that $2D + K = nP$. Then $D \sim \frac{n+e+2}{2} P + B$ and $e \leq n + 2$. We write $a := \frac{n+e+2}{2}$ and $b := \frac{n-e+2}{2}$. The vector-space of equations of divisors in $|D|$ is then

$$V = \langle p^a u, \ldots, q^a u, p^b v, \ldots, q^b v \rangle.$$

In terms of s, t, their equations have the following ratio.

$$(p^a u : \ldots : q^a u : p^b v : \ldots : q^b v) = (t^a : \ldots : 1 : st^b : \ldots : s).$$

We can find s as a quotient of two such equations: $s := \frac{q^b v}{q^a u}$. The denominator is the only equation whose product with $\frac{p^a}{q^a}$ is still in V, when $e > 0$. The numerator is such that its product with $\frac{p^b}{q^b}$ is in V. It is not unique, but any element which has this property, and which is not linearly dependant of u over the field $\frac{p}{q}$, is equally good.

When $e = 0$, then there are two equations with the property characterizing suitable denominators, and it does not matter which we take.

Here is the patch for *adj2dim*, to replace the section "computation of the second parameter".

Computation of the second parameter.
Refined version.
$a := n + 2$;
Produce a rational normal curve of degree e**.**
if $n = 1$ then
  $(u_0, u_1, u_2, u_3) := (w_0^3, w_0^2 w_1, w_0 w_1^3, w_1^3)$;
else
  $(u_0, \ldots, u_r) := (w_0^2, \ldots, w_0 w_n, w_1 w_n, w_2, w_n)$;

KER := the zero space;

We de not know e, so we test from the largest possible value on.
while KER = the zero space do
    The denominator condition.
    KER:= $\{v \in V \mid vw_a \in V w_0\}$;
    The kernel has always dimension ≤ 2.
    if dim KER = 2 then
        Found two forms. $(e = 0)$
        (INDEP[1],INDEP[2]) := a basis for KER;
        SFUNCT := INDEP[2]/INDEP[1];
        exit refined version;
    if dim KER = 1 then
        Found denominator.
        INDEP[1]:= a basis for KER;
    $a := a - 1$;

Find a numerator.
$b := n + 2 - a$;
KER := $\{v \in V \mid vw_b \in V w_0\}$;
$(v_1, \ldots, v_r)$:= a basis for KER;
SUB := $V \cdot \langle w_1, \ldots, w_n \rangle + INDEP[1] w_0$;
for i from 1 to r do
    if $v_i w_0 \notin$ SUB then
    INDEP[2]:=v_i;
    SFUNCT := INDEP[2]/INDEP[1];
    exit refined version;

If D is the end of an adjoint tail, then the initial divisor has bi-degree $(a + (e + 2)\mu, 1 + 2\mu)$ on X. The polynomials $\overline{x}, \overline{y}, \overline{z}, \overline{w}$ are elements of the vector-space

$$\langle t^A, \ldots, 1, s^{1+2\mu} t^B, \ldots, s^{1+2\mu} t^B \rangle,$$

where $A := a + (e + 2)\mu$ and $B := A - (1 + 2\mu)e = b + (-e + 2)\mu$. The total degree is less or equal than A if $e = 0$, an less or equal $A + 1 + 2\mu$ if $e = 0$. In both cases, $n + 2 + (n + 4)\mu$ is an upper bound.

We conclude this section by proving a classical result on linear systems.

Theorem 2.14 *Let Γ be a linear system of rational plane curves. Then there is a birational auto-map ρ of the projective plane, transforming Γ to a linear system of curves of degree n with an $n-1$-fold base point, or to a linear system of conics.*

Proof: Let Γ' be the very large linear system obtained by resolving the base points. Let $D'\in\Gamma'$. By a theorem of Bertini (theorem 15), there exists a smooth curve in Γ', hence $g(D)=0$.

Let (X,D) be the minimalization of $(\mathbf{Y},D')$. This is an Italian pair with $g(D)=0$. Then either (X,D) is a ruling, or $D^2>0$.

Suppose that $D^2>0$. Then (X,D) is one of (a), (b), (c), (d) in theorem 10.

In the cases (a) and (b), we set

$$\rho:\mathbf{P}^2\to\mathbf{Y}\to X\cong\mathbf{P}^2.$$

Γ transforms into a system of lines in the case (a), and into a system of conics in case (b).

In case (c), we set

$$\rho:\mathbf{P}^2\to\mathbf{Y}\to X\xrightarrow{\pi}\mathbf{P}^2,$$

where π is the projection of the quadric surface X from a smooth point. It transforms Γ to a system of conics passing through two plane points (if the quadric surface is smooth) or through two points where one is infinitely near to the other (if the quadric is a cone).

In case (d), X is ruled by a rational pencil P, and $2D+K\sim mP$ for some integer m, and $P\cdot D=1$. Suppose that X is a ruled surface with characteristic number e. Then $D\sim aP+B$, where $a=\frac{n+e+2}{2}$ and B is the fundamental section $u=0$. We set

$$\rho:\mathbf{P}^2\to\mathbf{Y}\to X\xrightarrow{\pi_e}\mathbf{P}^2,$$

where $\pi_e: X\to\mathbf{P}^2$ is the birational map

$$(p,q,u,v)\mapsto(pq^{e-1}u:v:q^eu),$$

$$(x:y:z)\overset{\pi^{-1}}{\mapsto}(x,z,1,yz^{e-1})\text{ if }e>0,$$

$$(x:y:z)\overset{\pi^{-1}}{\mapsto}(x,z,z,y)\text{ if }e=0.$$

This transforms Γ to a linear system of divisors with equation in the vectorspace

$$\langle x^a,\ldots,z^a,x^{a-e}z^{e-1}y,\ldots,z^{a-1}y\rangle\text{ if }e>0,$$

$$\langle x^az,\ldots,z^{a+1},x^ay,\ldots,z^ay\rangle\text{ if }e=0.$$

These divisors have an equation that is linear in y, hence the degree is 1 plus the multiplicity at $(0:1:0)$.

Finally, suppose that (X,D) is a ruling. Let e be the characteristic number. We set

$$\rho:\mathbf{P}^2\to\mathbf{Y}\to X\xrightarrow{\pi_e}\mathbf{P}^2,$$

with π_e as above. This transforms Γ to the pencil of lines through $(0:1:0)$.

2.6 Historical and philosophical remarks

Here we give the main methods of different proofs of Castelnuovo's criterion. We discuss them from a computational point of view. (I hope that these discussions reveal why the notion of Italian pairs and their adjoints had to be introduced.)

The strongest influence in the theory of algebraic surfaces came from the Italian school of geometry, which was founded by Cremona, Segre, and Bertini, in the second half of the last century. At that time, algebraic curves had already been investigated systematically. Also, a lot of special properties of particular surfaces were known, especially for the cubic surface in $\mathbf{P}^3$. But for a long time, these results were not combined by deeper ideas.

For the Italians, the main tool for the study of surfaces were linear (and nonlinear) systems of divisors. In the investigation of linear systems, the crucial idea was to consider the restriction of the system to a smooth curve on the surface. This is a linear system of divisors on a curve, and for those linear systems they had a well developed theory.

The following remarks assume that the reader is familiar with the theory of linear systems on a curve.

The Italians had no difficulties to find a smooth curve, because they can be found be random pick in a linear system.

Theorem 2.15 (Bertini) *Let X be a smooth surface. Let Γ be a linear system (complete or not) of divisors on X. Then the generic curve is smooth away from the base locus.*

Especially, if Γ has no base points, then the generic curve is smooth.

Moreover, if Γ has no base points and $\Gamma^2 > 0$, then the generic curve of Γ is irreducible.

Let D be smooth irreducible. Then the restriction of $|D|$ to D (i.e. the pull-back along the inclusion) is a linear system on D, called the *self-intersection system* $|D|_D$. Its dimension is $\dim|D| - 1$ (one degree of freedom is lost, since we factor out the equation of D). Its degree is the number of intersections of D with another curve in $|D|$, that is D^2. The base locus of $|D|_D$ is the base locus of D intersected with D. i_D is birational iff $i_{|D|_D}$ is birational.

The above statements follow from elementary geometric arguments. Here are some deeper facts.

If $q(X) = 0$, then the self-intersection is complete.

If $q(X) = p_g(X) = 0$, then the specialty index of the self-intersection system is equal to the superabundance.

Example 12 continued: Recall the divisors in example 12:

$$D = 7L - 3A_1 - 2A_2 - \cdots - 2A_{10},$$

$$D' = D + K = 4L - 2A_1 - A_2 - \cdots - A_{10},$$

with base points $a_1, \ldots, a_{10}$ chosen such that $\dim|D| = 3$. We already saw that D and D' are both very large. By Bertini's theorem, we conclude that both system contain smooth irreducible curves. Assume wlog that D an D' are smooth irreducible.
D has genus 3. The self-intersection system is a g_4^2 with specialty index 1, hence the canonical system. Assume "by genericness" that D is non-hyperelliptic. The canonical embedding is birational, hence i_D is birational.
D' has genus 2. The self-intersection system is a g_3^1 which is non-special. It has one fixed point, namely the base point b of $|D'|$. The moving part is a g_2^1, hence the canonical system. Its rational map is not an embedding, hence i'_D is not birational. More precisely, the canonical map is a 2-1-morphism to $\mathbf{P}^1$. Hence $i_{D'}$ is also a 2-1-morphism.

The cornerstone in the classical Italian geometry was the theorem of adjunction: a divisor equivalent to $D + K$ intersects D in a canonical divisor. If $q(X) = 0$, then $|D + K|_D$ is complete. If also $p_g(X) = 0$, then $\dim|D+K| = \dim|D+K|_D = g(D)-1$. If $g(D) > 0$, the canonical system of D is very large. Hence the base locus of $|D + K|$ is disjoint from D.

Example 12 continued: Let D, D' be as in example 11. $|D'|_D$ is the canonical g_4^2. From section 4, it is clear that $|D|_D = |D'|_D$, because a linear subsystem of codimension 1 contains the cubic C as a component, which is disjoint from D. This agrees with the former observation that $|D|_D$ is canonical.
The base point b of $|D'|$ lies on C and does therefore not lie on D.

In order to repeat adjunction (i.e.adding K), one has to control the base locus. A big problem are the curves C with $D{\cdot}C = 0$, so-called *fundamental curves*. If D is very large, then the fundamental curves are exactly those curves that get contracted by i_D, because a the image of a curve C has degree $D{\cdot}C$.

If D is very ample, then i_D is an isomorphism, and there are no fundamental curves. Consequently, $|D + K|$ does not have fixed components. The worst that can happen are isolated base points, but they show up in the self-intersection system.

Using the concept of repeated adjunction, Enriques gave an algorithm for the parametrization of rational surfaces ([14]), and Castelnuovo discovered his criterion ([8]).

Let us discuss Enriques' and Castelnuovo's results from a computational point of view. Enriques' algorithm is not too specific when it comes to explicit calculations. He merely sketches the outline in a geometric language.

But this does not mean that it is impossible to make an efficient algorithm from this sketch. The same holds for Castelnuovo's proof, which implicitly contains also an algorithm.

The main drawback of the method of Castelnuovo and Enriques is that they work with very ample divisors. Especially, the initial divisor has to be very ample, and the input surface must be smooth.

Comparing with the algorithm presented here, it must be remarked that we do compute a desingularization, to compute the adjoint ideals. But we resolve the singularities locally by a divide and conquer procedure. It is not necessary to know an embedding of the whole desingularization. We use it only to do the theory (meaning geometry) there, and to find the adjoint ideals. It would be much more complicated to compute a global desingularization, which would be a necessary preparation for Castelnuovo's and Enriques' algorithm. The desingularization would also be an intermediate result which is much bigger than the input. I also cannot imagine any control of the degree of the parametrization.

Note that the input specification is also the main difference of Enriques' and Castelnuovo's algorithm: Enriques assumes rationality, but Castelnuovo assumes only $p_g = p_a = P_2 = 0$.

Half a century later, the ideas of Castelnuovo and Enriques were revisited. Zariski brought much light into the theory of minimal models ([43]) and also gave a "readable" proof of Castelnuovo's criterion ([43] and [42]) (it should be noted that the papers of Castelnuovo and Zariski are far from being self-contained; they assume a lot of things that must have been commonly known at that time).

One year earlier, Serre published a proof due to Kodaira ([36], also in [39] and [18]). It is probably the first proof in modern language. The proofs of Zariski and Kodaira are similar. Kodaira's proof uses a quick topological argument at a place where Zariski solves the problem with explicit calculation.

In the Zariski-Kodaira variant of adjunction, we assume wlog that X is minimal. To demonstrate the usefulness of this assumption, we write down a fact revealing the usefulness of repeated adjunction. It says basically that "nef" can be replaced by "ample" in the case of minimal surfaces.

Proposition 2.7 *Let X be a minimal surface with $q(X) = p_a(X) = P_2(X) = 0$. Let D be ample and $g(D) > 0$. Then $D + K$ is ample or*

a) $D + K \sim nP$ with $P^2 = g(P) = 0$, $D \cdot P = 2$ or

b) $D + K = 0$.

Proof: By lemma 2, $D + K$ is nef and $|D + K|$ is not empty, hence $(D + K)^2 \geq 0$.

Assume that there exists a curve C such that $(D+K)\cdot C = 0$. Since D is ample, $D\cdot C > 0$. Hence $C\cdot K = -D\cdot C < 0$. In the absence of exceptional curves and by proposition 1, C cannot be rigid, hence $C^2 \geq 0$. By Hodge index, $(D+K)^2 = 0$. Then either (a) or (b) by theorem 9.

If $(D+K)\cdot C > 0$ for all curves C, then $(D+K)^2 > 0$ because there are divisors with components in $|D+K|$. Then $D+K$ is ample by the Nakai-Moishezon criterion.

From a computational point of view, the assumption that X is minimal is extremely annoying. The problem of constructing a minimal model is probably equally hard than the parametrization problem itself, because it is hard to find exceptional lines.

A reasonable way to make the proof constructive is a dynamical minimalization, blowing down exceptional lines whenever they lead to an error in the algorithm. But still, the earlier remark on the usage of ample instead of nef divisors applies also here.

Chapter 3

Del Pezzo surfaces

Italian pairs with $D + K \sim 0$ must be treated in a special way. In this chapter, we discuss this case. It is the most difficult (theoretically), but also the most interesting one.

Del Pezzo surfaces can be nicely classified. They form a finite number of algebraic families of surfaces. Much emphasize has been spent on a uniform treatment of the various families. Nevertheless, some case distinction could not be avoided.

Here is a summary of the particular sections.

1: Geometry on a Del Pezzo surface. We show some general properties of the divisor D. We also establish a birational connection between different families and study possible configurations of rigid curves.

2: Del Pezzo surfaces with $D^2 \geq 5$. Del Pezzo surfaces of this type are easy cases. We give a simple and efficient parametrization algorithm. We also give the classification.

3: Del Pezzo surfaces with $D^2 \leq 4$. With D^2 decreasing, the situation gets more difficult. We analyze the algebraic families in an explicit way and give parametrization algorithms for them.

4: The crop. We collect the fruits. At that time, we have finished the proof of Castelnuovos criterion. Moreover, we have a classification for Del Pezzo surfaces and for minimal rational surfaces. A side result classifying linear systems of elliptic curves is also given.

5: Historical and aesthetical remarks. We give some account on the fascination that attracted mathematicians of two centuries to that topic. We also discuss variants of the proof for this case in Castelnuovo's criterion.

3.1 Geometry on a Del Pezzo surface

We study general properties of Del Pezzo surfaces and the linear systems $|mD|$. Moreover, we will study exceptional curves and -2-curves.

A *Del Pezzo pair* is an Italian pair (X, D) with $D + K \sim 0$. Obviously, D is determined up to equivalence by the surface.

Example 1: Let $X \in \mathbf{P}^3$ be a smooth cubic surface. Let D be a smooth hyperplane section (which always exists by Bertini). Then $D^2 = 3$ and $g(D) = 1$, hence $D + K \sim 0$ by 1 (cf also example 1.2).

Example 2: Let $r \leq 8$. Let $a_1, \ldots, a_r$ be base points such that no four points are collinear, no seven points are on a conic, and no base point has more than one other base point in the infinitely near. Let

$$D = 3L - A_1 - \ldots - A_r.$$

By Riemann-Roch, $\dim|D| \geq 9 - r$. Any curve C with $D \cdot C < 0$ must be a fixed component of $|D|$, especially of degree ≤ 2. By the choice of the base points, D is nef.

Theorem 3.1 *Let (X, D) be a Del Pezzo pair. Then the generic curve of $|D|$ is smooth irreducible elliptic. $s(mD) = 0$ for all $m \in \mathbf{N}$.*

If $D^2 \geq 3$, then D is very large, and i_D is birational.

If $D^2 = 2$, then D is very large, and i_{2D} is birational.

If $D^2 = 1$, then $2D$ and $3D$ are very large, and i_{3D} is birational.

Proof: $s(mD) = 0$ follows from Grauert-Riemenschneider vanishing.

Using that the parity of E^2 and $E \cdot D$ must be the same (genus formula), one easily checks that no Reider divisor exists for $R = mD$, $m \geq 2$. By Reider's lemma, mD is very large for $(m+1)^2 D^2 \geq 5$, and i_{mD} is birational for $(m+1)^2 D^2 \geq 10$. This includes all claimed statements on very largeness and birationality.

If $D^2 \geq 2$, then the generic curve of $|D|$ is smooth and irreducible (hence elliptic by the genus formula) by Bertini.

If $D^2 = 1$ and $|D|$ has no fixed components, then D can only have one simple base point, and again the generic curve is smooth and irreducible by Bertini.

We show indirectly that $|D|$ does not have fixed components. Suppose that the fixed part F is not zero. Let $|D_0|$ be the moving part. Then $|D_0 + K|$ is empty, hence $g(D_0)\leq 0$. Then $g(D_0) = D_0^2 = 0$ (think of a minimalization of (X, D_0) and apply the classification theorems 2.9 and 2.10). Then $D_0 \cdot D = 2$ and $F \cdot D = -1$, contradicting nef-ness of D.

Corollary 3.1 *Let (X, D) be a Del Pezzo pair.*

If $D^2 = 3$, then $i_D(X)$ is a cubic surface in $\mathbf{P}^3$ (not necessarily smooth).

If $D^2 = 4$, then $i_D(X)$ is a complete intersection of two quadrics in $\mathbf{P}^4$.

Proof: In case $D^2 = 3$, we only have to note that $\dim|D| = 3$. In case $D^2 = 4$, $\dim|D| = 4$ and $i_D(X)$ is a quartic surface in $\mathbf{P}^4$. The linear system of quadric 3-folds in $\mathbf{P}^4$ has dimension 14, but $\dim|2D| = 12$. Hence there are two different quadrics passing through X. The intersect in a surface of degree 4, which can only be X.

The integer D^2, called the *degree*, is a sort of characteristic number for Del Pezzo surfaces. By corollary 1, the Del Pezzo surfaces with $D^2 = 3$ and 4 form two algebraic families of surfaces (surfaces that may be algebraically given by a finite number of parameters). This holds also for the other values of D^2, as we will see.

We turn to internal geometry on the Del Pezzo surface. Let C be a rigid curve. By proposition 2.1, $C \cdot D\leq 1$ and $C^2 \leq -C \cdot D$. By corollary 2.1 (genus formula), $C^2 \leq -C \cdot D$. Since $C \cdot D \geq 0$, we have only three possibilities.

$C^2 = -1$ and $C \cdot D = 1$. This is called a *-1-curve.*

$C^2 = -2$ and $C \cdot D = 2$. This is called a *-2-curve.*

$C^2 = 0$ and $C \cdot D = 0$. This is impossible, because it contradicts the Hodge index theorem.

The -1-curves are exactly the exceptional curves, as defined in section 2.3. The technique of blowing up points and blowing down exceptional curves is crucial for inductive proofs on the degree. However, we need to know that a blowing down of a Del Pezzo surface is again Del Pezzo.

Lemma 3.1 *Let (X, D) be a Del Pezzo surface.*

a) Let E be a -1-curve, and let $\pi : X \to X'$ be the blowing down of E. Let $D' := (\pi^*)^{-1}(D + E)$. Then (X', D') is again Del Pezzo, of degree $D'^2 = D^2 + 1$. The collapse point $\pi(E) \in X'$ does not lie on a -2-curve.

b) Suppose that $D^2 \geq 2$. Let p be a point that does not lie on a -2-curve. Let $\pi : \tilde{X} \to X$ be the blowing up at p. Let $E \subset \tilde{X}$ be the exceptional divisor. Let $\tilde{D} := \pi^* D - E$. Then $(\tilde{X}, \tilde{D})$ is again Del Pezzo, of degree $\tilde{D}^2 = D^2 - 1$.

Proof: In both parts (a) and (b), the divisor is inverse to the canonical class. The linear systems are non-empty by Riemann-Roch. Any surface is K-minimal, so it suffices to show nef-ness.

a) Let $C \subset X'$ be a curve. Then

$$C \cdot D' = \pi^* C \cdot (D + E) = \pi^* C \cdot D \geq 0.$$

b) Assume, indirectly, that there is a curve $C \subset \tilde{X}$ such that $\tilde{D} \cdot C < 0$. C is not E because $\tilde{D} \cdot E = 1$. Hence C is the proper transform of its set-theoretic image C'. The total transform of C' is $C - rE$, where r is the multiplicity of C' at p. We have

$$\tilde{D} \cdot C = D \cdot C' - r < 0,$$

hence $r > 0$.

C must be a fixed component of $|\pi^* D - E|$. The set-theoretic images of the divisors of this system are in $|D|$. Hence C' is a component of some divisors in $|D|$. By proposition 2.3, C' is rational smooth. In particular, the multiplicity at p is 1. This implies $D \cdot C' = 0$. But then, C' is a -2-curve, in contradiction to the assumption that p does not lie on a -2-curve.

The -2-curves are the fundamental curves of i_D (or i_{mD}). Since they are orthogonal to D, any divisor in $|D|$ or $|mD|$ that contains one point contains the whole -2-curve (as a component). These means that the -2-curves are collapsed to a single point by i_D (or i_{mD}).

Let m be such that i_{mD} is birational (for instance, $m = 1$ and $D^2 \geq 3$). Let $Z := i_{mD}(X)$. Then $i_{mD} : X \to Z$ is a desingularization of Z. Since X is D-minimal, no exceptional divisors are contracted. This implies that the desingularization is minimal, i.e. any other desingularization factors through it. The images of -2-curves must therefore be singular points. The inverse image of such a singular point is a connected component of the configuration of -2-curves on X.

By Hodge index, intersection product is negative definite on the space generated by the -2-curves. Hence, the -2-curves form the base of a root system. This has many interesting consequences, but we will not use this fact.

What we will use is bound for the intersection product of two -2-curves.

Lemma 3.2 *Let (X, D) be a Del Pezzo pair. Let E, E_1, E_2 be divisors with self-intersection -1 and genus 0 (for instance, -1-curves). Let F, F_1, F_2 be divisors with $F^2 = -2$ and $g(F) = 0$ (for instance, -2-curves). Then the following hold.*

a) If $F_1 + F_2 \not\sim 0$, then $F_1 \cdot F_2 \leq 1$.

b) If $D^2 \geq 3$, then $E_1 \cdot E_2 \leq 1$.

c) If $D^2 \geq 2$, then $E \cdot F \leq 1$.

Proof: a) We have $D{\cdot}(F_1 + F_2) = 0$. By Hodge index,

$$(F_1 + F_2)^2 = 2(F_1{\cdot}F_2) - 2) < 0,$$

hence (a).

b) Let $d := D^2$. Then $D{\cdot}(2D - dE_1 - dE_2) = 0$. By Hodge index again,

$$(2D - dE_1 - dE_2)^2 = 2d^2(E_1{\cdot}E_2 - 1) - 4d \leq 0.$$

If $d{\geq}3$, then this implies $E_1{\cdot}E_2{\leq}1$.

c) Let $d := D^2$. Then $D{\cdot}(D - dE - dF) = 0$. By Hodge index again,

$$(D - dE - dF)^2 = 2d^2(E{\cdot}F - 1) - d^2 - d \leq 0.$$

If $d{\geq}2$, then this implies $E{\cdot}F{\leq}1$.

Lemma 2 does not only give restrictions for the intersection number of -2-curves and -1-curves, but it also puts strong restriction on the configuration of rigid curves on X. Its power lies in the fact that it can be applied to composite curves, for instance to $E + F$, where, E is a -1-curve and F is a -2-curve intersecting E.

We ask the question: how does an effective divisor with such intersection multiplicities look like? Here is an answer.

Lemma 3.3 *Let (X, D) be a Del Pezzo surface.*

a) Any effective divisor orthogonal to D is a sum of -2-curves.

b) Any effective divisor B with $B{\cdot}D = 1$ is a sum of one -1-curves and some number of -2-curves, or is numerically equivalent to D (only if $D^2 = 1$).

Proof: (a): Let C be a component of a divisor that is orthogonal to D. Since D is nef, we have $D{\cdot}C = 0$. By Hodge index, $C^2 < 0$, and C must be rigid. Hence C is a -2-curve.

(b) Let C be the one component of B with $C{\cdot}D = 1$. Let $d := D^2$. Then $D{\cdot}(D - dC) = 0$. By Hodge index,

$$(D - dC)^2 = d^2C^2 - d{\leq}0.$$

We also know that C^2 and $C{\cdot}D$ must have the same parity (odd), hence $C{\cdot}D = -1$ and C is a -1-curve, or $d = 1$ and $(D - C)^2 = 0$. In the second case, we conclude that $D - C$ is numerically zero, again by Hodge index.

Example 2 continued: Let $0{\leq}r{\leq}8$. Let $a_1, \ldots, a_r$ be base points meeting the assumptions in example 2. The following divisors have self-intersection -1 and genus 0.

$$A_i, 0{\leq}i{\leq}r,$$

$$L - A_{i_1} - A_{i_2}, 0 \leq i_1 < i_2 \leq r,$$
$$2L - A_{i_1} - \ldots - A_{i_5}, 0 \leq i_1 < \ldots < i_5 \leq r,$$
$$3L - 2A_{i_1} - A_{i_2} - A_{i_7}, 0 \leq i_1 < \ldots < i_7 \leq r.$$

If $r \leq 7$, then it is not hard to show that the above are the only divisor classes with self-intersection -1 and genus 0. If $r = 8$, then there are also the divisors

$$4L - 2A_{i_1} - 2A_{i_2} - 2A_{i_3} - A_{i_4} - \ldots - A_{i_8},$$
$$5L - 2A_{i_1} - \ldots - 2A_{i_6} - A_{i_7} - A_{i_8},$$
$$6L - 3A_{i_1} - 2A_{i_2} - \ldots - 2A_{i_8},$$

where $(i_1, \ldots, i_8)$ is a permutation of $(1, \ldots, 8)$.

Example 3: Let $0 \leq r \leq 8$. Let $a_1, \ldots, a_r$ be base points meeting the following assumptions, which are stronger than those in example 2: All base points are plane points, no three points are on a line, no six points lie on a conic, and no cubic through all 8 points is singular at one of the base points (only if $r = 8$). Then $(\mathbf{Y}, D := 3L - A_1 - \ldots - A_r)$ is Del Pezzo by example 2. The additional assumptions imply that there are no -2-curves. (Note that any -2-curve must be a component of a divisor in $|D|$.)

Any of the divisor classes in the list in example 2 contains effective divisors by Riemann-Roch. By lemma 3 and by the absence of -2-curves, any such class contains a -1-curve. Now, -1-curves are rigid, and we have a 1-1 correspondence between -1-curves and the classes above.

This allows to calculate the number of -1-curves on Del Pezzo surfaces of this type.

r	0	1	2	3	4	5	6	7	8
D^2	9	8	7	6	5	4	3	2	1
-1-curves	0	1	3	6	10	16	27	56	120

3.2 Del Pezzo surfaces with $D^2 \geq 5$

Here we prove the degree bound $D^2 \leq 9$ and parametrize Del Pezzo surfaces with $D^2 \geq 5$. In the algorithms in this section, it is not necessary to compute the equations of $i_D(X)$ explicitly.

To parametrize Del Pezzo surfaces, we look for linear systems of divisors of genus 0. Recall [theorem 2.12] that the existence of a moving rational curve already implies that X is rational. If the rational curve moves at least in a system of dimension 2, then parametrization is very easy.

The essential idea is to obtain rational curves either as components of divisors in $|D|$ or as curves in $|D|$ with a singular point. The generic curve of $|D|$ is elliptic, so we need special divisors. Depending on what is known, we can do various things. Here is what can be achieved if a -1-curve or a -2-curve is known.

Lemma 3.4 *Let (X, D) be a Del Pezzo pair.*

a) Suppose that $D^2 \geq 3$. Let $E \subset X$ be a -1-curve. Let $|D_0|$ be the moving part of $|D - E|$. Then $g(D_0) = 0$ and $D_0^2 = D^2 - 3$.

b) Suppose that $D^2 \geq 2$. Let $F \subset X$ be a -2-curve. Let $|D_0|$ be the moving part of $|D - F|$. Then $g(D_0) = 0$ and $D_0^2 = D^2 - 2$.

Proof: a) Let $E_0 := E$. Easy calculation shows that $g(D - E_0) = 0$ and $(D - E_0)^2 = D^2 - 3$. If $D - E_0$ is nef, then it is also very large by theorem 2.10 for $D^2 > 3$ and by proposition 2.5 for $D^2 = 3$. Hence $D_0 \sim D - E_0$, and the statement is true.

If $D - E_0$ is not nef, then let C be a curve such that $(D - E_0) \cdot C < 0$. C must be rigid, and lemma 2 leaves only $D \cdot C = 0$ (i.e. C is a -2-curve) and $E_0 \cdot C = 1$. C is a fixed component of $|D - E_0|$. In other words, the moving part of $|D - E_0|$ coincides with the moving part of $|D - E_0 - C|$. Then we replace E_0 by $E_0 + C$ and repeat the argument.

Of course, this replacement cannot be done infinitely often, because the fixed part of $|D - E|$ has only finitely many components.

b) Analogously.

Note that under the assumption of lemma 4 above, X is rational. If $D^2 \geq 4$ (with a -1-curve known) or $D^2 \geq 3$ (with a -2-curve known), then we have an efficient parametrization method similar to *adj0dim*.

The key fact for the algorithm *adj0dim* is theorem 2.10. There, we have 4 cases of Italian pairs. The largest case is (d), when $2D + K$ is a multiple of a rational pencil.

In the Del Pezzo case, we can give a very elegant construction of the parametrization via this rational pencil.

Lemma 3.5 *Let (X, D) be a Del Pezzo pair of degree $D^2 \geq 6$. Let $E \subset X$ be a -1-curve. Let $|D_1|$ be the moving part of $|D - 2E|$. Then i_{D_1} is birational onto the plane and $D^2 = 7$, or the following holds: $D_1 = nP$ for a rational pencil P with $P^2 = 0$, $n := D^2 - 5$. $P + E$ is very large, and its associated map is birational to the plane.*

Proof: Let $|D_0|$ be the moving part of $|D - E|$. By lemma 4, $g(D_0) = 0$ and $D_0^2 = n + 2$. Let (X', D') be the minimalization of (X, D_0), with minimalization map $\pi : X \to X'$. By theorem 2.10, $D_1' := 2D' + K_{X'}$ either gives a birational map to the plane and $n = 2$, or $D_1' = nP'$ for a rational pencil P'. We show that $|D_1| = |\pi^* D_1'|$.

Let E_0 be such that $|D - E_0| = |D_0|$. In the proof of the lemma 4, E_0 is constructed by starting with $E_0 := E$ and then adding a -2-curve F with $E_0 \cdot F = 1$, until no such curve exists. But such a curve is at the same time doubly fixed in $|D - 2E_0|$, hence the moving part of $|D - 2E_0|$ coincides with the moving part of $|D - 2E|$.

If $C{\cdot}(D - 2E_0) < 0$, then C is a -1-curve with $C{\cdot}E_0 = 1$ by lemma 2. But this is an exceptional curve orthogonal to D_0. Hence, $D - 2E_0$ is nef iff X is D_0-minimal. In this case, we have

$$D_1' = 2D_0 + K \sim 2(D - E_0) - D = D - 2E_0,$$

which shows the claim. In the general case, the moving part of $|2D_0 + K|$ and $|\pi^*|D_1'|$, because their associated adjoint maps are the same by proposition 2.2 (the functorial property of adjoints).

It remains to show the statement about $P + E$, in the case where $D_1 = nP$ for a rational pencil P. The fixed part of $|D - 2E_0|$, namely

$$B := D - 2E_0 - nP = K_X - \pi^* K_{X'},$$

is the sum of exceptional divisors or their total transforms. All these summands are blown down during the minimalization of (X, D_0). Hence B is orthogonal to $D - E_0$:

$$0 = (D - E_0){\cdot}(D - 2E_0 - nP) = D^2 - 5 - nP{\cdot}D + nP{\cdot}E_0 = -n + nP{\cdot}E_0.$$

Thus, $P{\cdot}E_0 = 1$. Since P is nef, $P{\cdot}E = 0$ or 1. $P{\cdot}F = 1$ is not possible for a -2-curve F, by Hodge index, applied to F and $2P + F$. Hence $P{\cdot}E = 1$. This implies that we have the right algebraic properties $P + E$, namely $(P + E)^2 = 1$ and $g(P + E) = 0$.

Only E can be a fixed component of $|P{+}E|$, and $E{\cdot}(P{+}E) = 0$. Hence, $P + E$ is nef, and the statement follows from theorem 2.10.

Remark: If $D^2 = 5$, then lemma 5 holds with $n = 0$: $D_1{\sim}0$, and there exists a rational pencil P with $P{+}E$ very large, $g(P{+}E) = 0$, $(P{+}E)^2 = 1$, $(P + E){\cdot}E = 0$. This can be seen by considering case (c) in theorem 2.10 as a degenerate case of (d) with $n = 0$. If X' is a ruled surface with characteristic number $e = 0$, then there are two choices for P.

The main idea for the parametrization is to produce Del Pezzo surfaces with -1-curves by blowing up a point p not lying on a -2-curve.

Theorem 3.2 *Let (X, D) be a Del Pezzo pair. If $D^2{\geq}4$, then X is rational. Moreover, $D^2{\leq}9$.*

Proof: By theorem 1, i_D is birational. Hence, there exists a point $p{\in}X$ that does not lie on a fundamental curve.

The blowing up $(\tilde{X}, \tilde{D})$ is Del Pezzo of degree $D^2 - 1{\geq}3$ by lemma 1. The exceptional divisor E is a -1-curve. Let $|D_0|$ be the moving part of

$|\tilde{D} - E| = |\pi^* D - 2E|$. The minimalization of $(\tilde{X}, D_0)$ is an Italian pair with $g(D_0) = 0$. Hence $\tilde{X}$ and in turn X is rational by theorem 2.12.

The inequality $D^2 \leq 9$ follows from the equation

$$\mathrm{rk}(\mathrm{Num}X) + K^2 = 10$$

(corollary 2.5) for rational surfaces (note that $K^2 = D^2$ in this case).

The equation $\mathrm{rk}(\mathrm{Num}X) + D^2 = 10$ indicates a somewhat paradoxical fact: The higher the degree, the simpler is the surface, and the easier is the parametrization problem.

Here is the classification theorem for Del Pezzo surfaces of degree $D^2 \geq 5$. The proof of this theorem will give the idea for the parametrization algorithm.

Theorem 3.3 *Let (X, D) be a Del Pezzo pair.*

If $D^2 = 9$, then $X \cong \mathrm{P}^2$, and $D \sim 3L$.

If $D^2 = 8$, then X is a ruled surface with characteristic number e =0,1, or 2, and $D \sim (e+2)P + 2B$.

If $5 \leq D^2 \leq 7$, then X is the blowing up of the plane at $r := 9 - D^2$ base points, and $D \sim 3L - A_1 - \ldots - A_r$.

Proof: Let $p \in X$ be a point not on a -2-curve. Let $(\tilde{X}, \tilde{D})$ be the Del Pezzo pair constructed by blowing up at ps, i.e. $\tilde{D} \sim \pi^* D - E$. If D^2 =9, 7, or 6, then lemma 5 gives a rational pencil $P \in \mathrm{Div}\tilde{X}$, such that $P + E$ is very large and its associated map is a blowing down map to the plane. (If $D^2 = 6$, we probably have two such pencils, by the remark following lemma 5.) Since $E \cdot (P + E) = 0$, the curve E is fundamental and gets blown down. Hence i_{P+E} factors through π:

$$\begin{tikzcd}
 & \tilde{X} \arrow[dl, "\pi"'] \arrow[dr, "i_{P+E}"] & \\
\tilde{X} \arrow[rr, "f"] & & \mathbf{P}^2
\end{tikzcd}$$

f is a birational regular map between smooth surfaces, hence a composition of blowing downs. Consequently, X is a blowing up of the plane. If $a_1, \ldots, a_r$ are the base points, then $D \sim -K_X \sim 3L - A_1 - \ldots - A_r$, and $D^2 = 9 - r$.

If $D^2 = 8$, then there are two possibilities. If D_1 in lemma 5 is twice a rational pencil P, then we may proceed as above. X is then a blowing up of the plane at one point a_1, i.e. a ruled surface with characteristic number $e = 1$. We have

$$D \sim 3L - A_1 \sim 3(L - A_1) + 2A_1 \sim 3P + 2B,$$

as desired.

Otherwise, D_1 induces a blowing down map $i_{D_1} : \tilde{X} \to \mathbf{P}^2$. Since $K_{\tilde{X}}^2 = 7$ and $K_{\mathbf{P}^2}^2 = 9$, $\tilde{X}$ is the blowing up of the plane at 2 base points a_1, a_2. To get X, we blow down the exceptional line E, which is the line $|L - A_1 - A_2|$. We get a ruled surface. Its characteristic number is 0, iff a_1 and a_2 are plane points. It is 2, if one point is infinitely near to the other. The formula for D follows from the formula for the canonical divisor on a ruled surface (theorem 2.13).

Finally, let $D^2 = 5$. Lemma 5 gives a blowing down map $i_{\tilde{D}-E} : \tilde{X} \to \mathbf{P}^2$. Comparing the values of K^2, we see that $\tilde{X}$ is the blowing up of the plane at 5 base points $a_1, \ldots, a_5$. The exceptional divisor E is a conic (since $(\tilde{D} - E)\cdot E = 2$), necessarily the curve $|2L - A_1 - \ldots - A_5|$. E must be irreducible, and so no three points among $a_1, \ldots, a_5$ are collinear. Then we can find a -1-curve E' disjoint from E, namely $|L - E_1 - E_2|$. Blowing down E yields X. Subsequent blowing down of E' yields a Del Pezzo surface of degree 6, which is the blowing up of the plane at 3 points. Hence X is the blowing up at 4 points. The formula for D follows as above.

To parametrize a Del Pezzo surface of degree $D^2 \geq 5$, we use the birational map to the plane constructed in the proof above. We compute a proper parametrization by inverting this map.

Algorithm *delpezzo*:

Input:

A Del Pezzo pair (X, D), with $D^2 \geq 5$.
A point $p \in X$ not lying on a -2-curve.

Output:

A proper parametrization of X.

Compute $\pi : \tilde{X} \to X$, the blowing up at p;
E:= the exceptional divisor;
if $D^2 = 5$ then
    $|L|$:= the moving part of $|\pi^* D - 2E|$;
if $D^2 = 6$ then
    $Q := |\pi^* D - 2E|$;
    Compute the quadratic equation of $i_Q(X)$;
    P:= the inverse image of a line;
    $L := P + E$;
if $D^2 \geq 7$ then
    $|D_1|$:= the moving part of $|\pi^* D - 3E|$;
    if $\dim(i_{D_1}(\tilde{X})) = 1$ then

P:= such that $D_1 \sim nP$;
$L := P + E$;
else ($D^2 = 8$)
$L := D_1$;
return *inverse*(i_L);
exit;

The degree of this parametrization is $L \cdot \pi^* D$. This is 3 for $D^2 = 9, 7, 6$, and $D^2 = 8$ when the characteristic number of X is 1. It is 4 for $D^2 = 8$ in the two other cases. When $D^2 = 5$, there exists a parametrization of degree $N = 3$, but the parametrization computed by *delpezzo* has degree $N = 5$. In all other cases, the degree of the computed parametrization is the smallest possible.

The algorithm may be called be *adj0dim*. The required operations are easy enough so that it is not necessary to compute the equations of X or even of $i_D(X)$.

Suppose that $Y \in \mathbf{P}^3$ is a surface satisfying Castelnuovos criterion. Assume that the adjoint length μ is positive, and that $v_{1,\mu} = 1$. The algorithm *parametrize* wants to call the sub-algorithm *adj0dim* (which we partially give below). Let $\pi : \tilde{X} \to X$ be a desingularization of Y. Let $\tilde{D} := \pi^* H$ be the pullback of a hyperplane in $\mathrm{Div}\tilde{X}$. By corollary 2.8, we have the commutative diagram

$$\begin{tikzcd}
 & Y \arrow[dr, "i_{1,\mu-1}"] & \\
\tilde{X} \arrow[ur, "\pi"] \arrow[rr, "i_{\tilde{D}+(\mu-1)K}"] \arrow[dr, "f"'] & & \mathbf{P}^d \\
 & X \arrow[ur, "i_D"'] &
\end{tikzcd}$$

where (X, D) is Del Pezzo of degree $D^2 = v_{1,\mu-1} - 1$, and f is the minimalization map in the adjunction of $(\tilde{X}, \tilde{D})$.

We turn to the question of finding a point $p \in X$ not on a -2-curve.

Lemma 3.6 *In the diagram above, assume that i_D is birational. Let $q \in Y$ be a smooth point. Then $p := f(\pi^{-1}(q))$ is not on a -2-curve.*

Proof: We may assume that π is an isomorphism around q. Especially, $p' := \pi^{-1}(q)$ and hence p is defined. Moreover, $|\tilde{D}|$ does not have a fundamental curve through p'.

Assume, indirectly, that p lies on a -2-curve F. Then p' lies on f^*F. But $f^*F\cdot\tilde{D} = F\cdot(\mu D) = 0$, hence f^*F is a fundamental curve of $|\tilde{D}|$, a contradiction.

Let $V := V_{1,\mu-1}$. It corresponds to the linear system $|D|$ on X. Let $\pi : \tilde{X} \rightarrow X$ be the blowing up of X at p.

The linear system $|\pi^*D - mE|$ on $\tilde{X}$ corresponds to $V{\downarrow}q^m$, the vector-space of forms with assigned m-fold point at q. To calculate $V{\downarrow}q^m$, it is convenient to make an analytic coordinate transformation that transforms p to the origin and Y to the xy-plane (cf section 4.3). However, if at least one form in V does not vanish at p, then we may also proceed by intersecting V with the maximal ideal at q to the power of m.

To find the linear system $|P+E|$, we compute two forms $v_1, v_2 \in V$ such that $|P|$ is the varying part of the divisor with equation $\lambda_1 v_1 + \lambda_2 v_2 = 0$. Then we assign the greatest common divisor as fixed component to V.

The case $\mu = 1$ plays a special role. We have here $v_{1,0} = 4$, although we maybe still have to do with a Del Pezzo surface of degree ≥ 5. In this case, we work with $V_{2,1}$ instead of $V_{1,0}$, Doing this, we get the complete linear system $|D|$ (again by theorem 2.7), i.e. $v_{2,1} = d+1$ $(= \deg Y)$. Here we set $n := 2$, $m := 1$.

Example 4: Let $Y \subset \mathbf{P}^3$ be the surface with the equation

$$F(x,y,z,w) = xy^2z^3 - w^6 = 0.$$

The first adjoint ideal is $A_1 = (yz^2, yzw^2, zw^3, w^4)$. Hence

$$V_{1,1} = \langle yz^2 \rangle, v_{1,1} = 1,$$

and we are in the Del Pezzo case with $\mu = 1$. $|D|$ corresponds to

$$V = V_{2,1} = \langle xyz^2, y^2z^2, yz^3, yz^2w, yzw^2, zw^3, w^4 \rangle.$$

Let us denote the base forms with $v_1, \ldots, v_7$. The image $i_D(X) = i_{2,1}Y$ is a surface of degree 6 in $\mathbf{P}^6$. It has the following equations.

$$v_5v_6 - v_4v_7 = v_5^2 - v_2v_7 = v_4v_6 - v_3v_7 = v_4v_5 - v_2v_6 =$$
$$= v_4^2 - v_3v_5 = v_2v_6^2 - v_3v_5v_7 = 0.$$

The projection map

$$(v_1 : v_2 : v_3 : v_4 : v_5 : v_6 : v_7) \mapsto (x : y : z : w) = (v_1 : v_2 : v_3 : v_4)$$

gives back the original surface Y.

The point $q := (1:1:1:1)$ is smooth. The form $v_1 = xyz^2$ does not vanish at q. This has to checked, because it might be the case that all forms vanish at q, and in this case we would have to divide out common factors in the local ring.

To find $|\pi^* D - 3E|$, we compute

$$V{\downarrow}q^3 = \{v \in V \mid v \in (x-w, y-w, z-w)^3 + (F)\} = \langle v_1 - v_2 + 3v_5 - 3v_7 \rangle.$$

The linear system has only one effective divisor. Since $yz(v_1 - v_2 + 3v_5 - 3v_7) + (yz - w^2)^3 = F$, this divisor has a triple curve $C : yz - w^2 = x - y = 0$, and maybe some other components that do not pass through q. (It is hard to tell something about the other components without an analysis of the desingularization.)
To find $|\pi^* D - 2E|$, we compute

$$W := V{\downarrow}q^2 = \langle w_1, w_2, w_3, w_4 \rangle =$$

$$= \langle 2v_1 + v_2 - 3v_7, v_1 + v_3 - 2v_6, v_1 + v_4 - v_6 - v_7, v_1 + v_5 - 2v_7 \rangle.$$

The equation of the birational image of i_W is

$$w_3^2 - w_2 * w_4 = 0.$$

This is a singular conic surface. The ruling P is cut out by $\langle s, t \rangle = \langle w_3, w_4 \rangle$. The common divisor of v_1 and v_2 has two components, the line $x = w = 0$ and the curve C that was computed above. If we assign these components as fixed, we get

$$W' = \langle s, t, u \rangle, u := v_1 - v_7.$$

We obtain a parametrization of X by inverting $i_{W'}$:

$$(x : y : z : w) = (u^3 : -(t-u)^3 : (s-u)^2 u : (s-u)(t-u)u) =$$

$$= (u^3 : -t'^3 : s'^2 u : s't'u).$$

(The last equation is just a linear change of the parameters.)

Here is the version of *delpezzo* that avoids to compute the equations of $i_D X$.

Algorithm *delpezzo*:
Version 2.

Input:
F, the equation of a rational surface $Y \subset \mathbf{P}^3$.
q, a smooth point on Y.
V, an adjoint vector-space corresponding to a Del Pezzo pair. It is assumed that $\dim V \geq 6$.

Output: 4 polynomials $\overline{x}, \overline{y}, \overline{z}, \overline{w} \in k[s,t]$, representing a parametrization of Y.

DEGREE := $\dim V - 1$;
if DEGREE = 5 then
Assign a double base point.
$W := V{\downarrow}p^2$;

(v_0, v_1, v_2) := a basis for W;
BIRAT:= $(\frac{v_1}{v_0}, \frac{v_2}{,} v_0)$;
else
if DEGREE = 6 then
Birational map to a conic.
$W := V{\downarrow}p^2$;
Compute linear relation in W^2;
Find a line on the image;
(s, t):= the forms passing through this line.
else
Assign a triple point.
$W := V{\downarrow}p^2$;
if $C := i_W(Y)$ is a curve then
find $(s, t){\in}W$ such that $\frac{s}{t}$ is a parameter;
else
In this case, DEGREE **must be 8.**
(v_0, v_1, v_2) := a basis for W;
BIRAT:= $(\frac{v_1}{v_0}, \frac{v_2}{,} v_0)$;
goto $*$;
FIX:= one-dimensional part of (s, t, F);
(s, t, u):= $V\cap$ FIX;
BIRAT:= $(\frac{s}{u}, \frac{t}{u})$;
$*$: return *inverse*(BIRAT);
exit;
C := one varying component of a divisor in W;
Assign all other things as fixed.
return *inverse*(BIRAT);
exit;

The degree of the parametrization computed by the algorithm *delpezzo* is

$$N = \pi^* L{\cdot}D_0 = \mu L{\cdot}D.$$

We have reached the theoretical lower bound $N = 3\mu$ in several cases ($D^2 = 9, 7, 6$, and $D^2 = 8$ if the linear system $|D_1|$ splits).

3.3 Del Pezzo surfaces with $D^2{\leq}4$

In this section we extend the theorems in the previous section to the case $D^2{\leq}4$. We give algorithms to compute improper parametrizations for $D^2 = 4, 3, 2$. We also outline algorithms to compute proper parametrizations and thereby finish the last gap in the proof of Castelnuovo's criterion.

In contrast to the previous section, we compute the equation of $i_D(X)$ (or $i_{mD}(X)$) explicitly. This surface is birational equivalent to the input surface Y. The parametrization of Y is reduced to the parametrization of X or $i_D(X)$.

The cases $D^2 = 4, 3, 2$ have much in common. The algorithms for improper parametrization are very simple. A disadvantage is that the computed parametrization has rather high degree. Proper parametrization of smallest possible degree may alternatively be computed, but here one has to find some geometrical object (line on the surface, bitangent, ...). For the improper parametrization, it suffices to know a smooth point on the input surface.

3.3.1 $D^2 = 4$

$Z := i_D(X)$ is a complete intersection of two quadrics in $\mathbf{P}^4$ (corollary 1). Since i_D is regular and does not blow down exceptional lines, X may be constructed from Z as its minimal desingularization. By abuse of language, we will call Z a Del Pezzo surface of degree 4.

We know by theorem 2 that X is rational. Recall the proof of theorem 2: we blew up a point p not on a -2-curve, obtaining a Del Pezzo surface of degree $D^2 - 1 = 3$. By corollary 1, $\tilde{Z} := i_{\tilde{D}}(\tilde{X})$ is a cubic surface in $\mathbf{P}^3$. We know a -1-curve $E \subset \tilde{X}$, the exceptional divisor of the blowing up. It is a line L on $\tilde{Z}$. Then we assigned the line as fixed component to get $|\tilde{D} - E|$. The moving part is a rational pencil, proving the rationality of $\tilde{X}$ by theorem 2.12. The pencil can be seen: it is the pencil of the plane sections of the planes through L. Any plane intersects $\tilde{Z}$ - besides in L - in a conic.

The following idea gives an improper parametrization: First, parametrize the line with the first parameter t. For a particular t, compute the tangent plane to $\tilde{Z}$ at $p_t \in L$. In intersects $\tilde{Z}$ in a conic through p_t. We use the second parameter to parametrize the conic.

The so computed parametrization is not proper: The generic point on Z determined determines a plane H through L. H intersects Z in a conic C. C intersects L in two points p_1, p_2. These two points correspond to two values t_1, t_2 of the first parameter. The parametrization is therefore a 2:1-parametrization.

Example 5: Let $Z \subset \mathbf{P}^4$ be the surface given by the equations

$$Q1 := x_0x_1 + x_2^2 + x_4^2, Q_2 := x_0x_2 + x_1^2 + x_3^2.$$

The point $p := (1:0:0:0:0)$ is smooth. The tangent plane at p is given by $x_1 = x_2 = 0$. By projection from p, we obtain the cubic surface $\tilde{Z} \in \mathbf{P}^3$ with equation

$$x_1Q_2 - x_2Q_1 = x_1R_2 - x_2R_1 = x_1^3 + x_1x_3^2 - x_2^2 - x_2x_4^2.$$

Here, we introduced the symbols R_1, R_2 for the tails of Q_1, Q_2 in the variables x_1, x_2, x_3, x_4. The blowing down map $\pi : \tilde{Z} \to Z$ is given by

$$(x_1 : x_2 : x_3 : x_4) \mapsto (-R_1 : x_1^2 : x_1x_2 : x_1x_3 : x_1x_4) =$$

$$= (-R_2 : x_1x_2 : x_2^2 : x_2x_3 : x_2x_4).$$

The inverse image of p is the line $L : x_1 = x_2 = 0$.
We parametrize the line: $p_t := (0 : 0 : 1 : t)$. The tangent plane H_t at p_t has equation $x_1 - t^2x_2 = 0$. The intersection of H_t and $\tilde{Z}$ splits into L and the conic curve

$$C_t : t^6x_2^2 + t^2x_3^2 - x_2^2 - x_4^2 = 0.$$

We still know the point p_t, with coordinates $(x_2 : x_3 : x_4) = (0 : 1 : t)$. The general line through p_t has the equation

$$L_{s,t} : sx_2 + tx_3 - x_4 = 0.$$

It intersects C_t in the point

$$p_{s,t} : (x_2 : x_3 : x_4) = (2st : t^6 - s^2 - 1 : t^7 + s^2t - t).$$

We compute the remaining coordinates

$$x_1 = t^2x_2, x_0 = \frac{x_2^2 + x_4^2}{x_1}$$

and get, after clearing denominators, the following improper parametrization of Z:

$$(x_0 : x_1 : x_2 : x_3 : x_4) =$$

$$= (2s^2 + s^4 + 2t^6s^2 + t^12 - 2t^6 + 1 :$$

$$: 4t^4s^2 : 4s^2t^2 : 2st^7 - 2st - 2s^3t : 2s^3t^2 + 2st^8 - 2st^2).$$

Algorithm *improper4.*

Input:

Q_1, Q_2, two quadratic forms in 5 variables $x_0, \ldots, x_4$. The solution Z of Q_1, Q_2 is a Del Pezzo surface of degree 4.

A smooth point $p \in Z$.

Output: 5 polynomials $\overline{x_0}, \ldots \overline{x_4}$, representing an improper parametrization of Z.

translate p to $(1:0:0:0:0)$;
rotate the tangent plane to $x_1 = x_2 = 0$;
basetrafo of $\langle Q_1, Q_2 \rangle$;
    the leading coefficient of Q_i with regard to x_0
    must be x_i.
$R_1 :=$ constant coefficient of Q_1 with regard to x_0;
$R_2 :=$ constant coefficient of Q_2 with regard to x_0;
eliminate x_0, using $x_0 x_1 + R_1 = 0$;
$r_1 := R_1(0,0,1,t)$;
$r_2 := R_2(0,0,1,t)$;
Equation of the tangent plane.
eliminate x_1, using $r_2 x_1 - r_1 x_2 = 0$;
Equation of the conic.
$C_t := r_2 R_1 - r_2 R_1$;
Equation of the line through p.
eliminate x_4, using $s x_2 + t x_3 - x_4$;
The second intersection point.
$p := \frac{C_t}{x_3}$;
$\overline{x_2}, \overline{x_3} := solve(\overline{p})$;
The rest is easy.
find $\overline{x_0}, \overline{x_1}, \overline{x_5}$ be plugging in;
clear denominators in $(\overline{x_0}, \ldots, \overline{x_5})$;
return $(\overline{x_0}, \ldots, \overline{x_5})$;
exit;

If the point p happens to lie on one or more -1-curves, then these lines get blown down by the projection map. The inverse map is then not regular any more. However, this does not disturb the functionality of the above algorithm.

A proper parametrization of degree $N = 3$ (which is the smallest possible value) can be computed by lemma 4, if we have a -1-curve.

Lemma 3.7 *Any Del Pezzo surface of degree 4 has a -1-curve.*

Proof: We give a dimension argument. Let Q be the set of all quadric divisors in $\mathbf{P}^4$ (classes of nonzero forms of degree 2 in 5 variables modulo scalar multiplication). Let L be the set of all lines in $\mathbf{P}^4$. Let

$$\Gamma := \{(q_1, q_2, l) \in Q \times Q \times L \mid l \in q_1, l \in q_2\}.$$

We show that the projection map

$$\pi_{12} : \Gamma \rightarrow Q \times Q$$

is surjective. This implies then that any complete intersection of two quadric hypersurfaces contains a line. The pullback of this line is a divisor that has intersection multiplicity 1 with D. It contains a -1-curve by lemma 3.

The sets Q and L are irreducible projective varieties. Their dimensions are $\dim Q = 14$ and $\dim L = 6$. The set Γ is projective, because the conditions $l \in q_i$ can be expressed as equations. Let $\pi_3 : \Gamma \rightarrow L$ be the projection on the third This map has isomorphic fibers. Each fiber is irreducible and has dimension 22. This implies that Γ is irreducible and $\dim\Gamma = 22 + 6 = 28$.

Let $D \subset Q \times Q$ be the image of π_{12}. As an image of a projective variety, it is again projective. π_{12} has zero-dimensional fibers (for instance, the Del Pezzo surface in example 3 has exactly 16 lines). The dimension of the fiber may jump but not rise on a closed subset. Hence the generic fiber has dimension 0. This dimension is the difference of the dimensions of the domain and the codomain: $\dim\Gamma - \dim D = 0$. Hence $\dim D = 28$ and $D = Q \times Q$.

Corollary 3.2 *If (X, D) is a Del Pezzo pair of degree $D^2 = 4$, then X is the blowing up of the plane at 5 base points, and $L \sim 3L - A_1 - \ldots - A_5$.*

Proof: Since a -1-curve exists, we can blow it down and get a Del Pezzo (X', D') of degree 5. By theorem 3, X' is the blowing up of the plane at 4 points. And X is the blowing up of X' at one point.

The existence proof for lines was not constructive: the dimension argument does not give an idea for a reasonable algorithm to find a line. In the generic case, Z is smooth, i.e. $X \cong Z$ and there are no -2-curves. By example 3, there are exactly 16 lines. One can show that these lines are algebraically not distinguishable: A Galois group is acting transitively on them ([Ma]). This means that one has at least to solve a univariate equation of degree 16 to find a single line.

This illustrates another paradoxical fact in algebraic geometry: The difficulty of the problem of finding some object increases with the number of existing objects of the same type.

3.3.2 $D^2 = 3$ (cubic surfaces)

$Z := i_D(X)$ is a cubic surface in $\mathbf{P}^3$. As in the previous case, X may be recovered as the minimal resolution of X. Z is smooth iff there are no -2-curves. Otherwise, i_D contracts the -2-curves to isolated double points.

Here is a converse that makes it easy to construct examples.

Proposition 3.1 *Let $Z\mathbf{P}^3$ be a cubic surface without curve singularities. Suppose that Z is not the cone over a plane cubic. Then Z is a Del Pezzo surface.*

Proof: If Z is smooth, then Z is the Del Pezzo surface in example 1. Hence, we may assume that Z has a singular point p. Since p is not the vertex of the cone over a plane cubic, p can only be a double point.

The projection from p is a birational map to the plane. Hence, Z is rational.

If the adjoint ideal A_1 is not (1), then the adjoint length is $\mu = 0$. Then the generic plane section must be rational. But the generic plane section is a smooth cubic, and so the adjoint ideal A_1 must be (1).

Then $\mu = 1$ and $v_{1,1} = 1$. The algorithm *parametrize* forks to *adj0dim*, and the minimal resolution is a Del Pezzo surface as desired.

Let $p \in X$ be a point not on a -2-curve. Then $q := i_D(p)$ is a smooth point on Z. The process of assigning a double base point at p has a very intuitive meaning: A plane section will have a singular point at p iff the plane is the tangent plane at Z through q. Hence, $|\pi^* D - 2E_p|$ contains always exactly one effective divisor, namely the inverse image of the tangent plane at q.

If q lies on a line on Z, then this line is contained in the tangent plane. By lemma 3, the pullback has a -1-curve as a component. Since p does not lie on a -2-curve, the -1-curve passes through p. By the previous case, these data allow to produce an improper parametrization (2:1).

If q does not lie on a line on Z, then the tangent plane H intersects Z in a irreducible cubic curve that has a double point at q.

This can be exploited to construct an improper parametrization of X. We use the first parameter t to parametrize the irreducible cubic curve $|\pi^* D - 2E_p|$. The generic point q_t on this curve does not lie on a -1-curve: If there is a -1-curve, then X is the blowing up at 6 base points, and there are only finitely many -1-curves by example 2.

We use the second parameter s to parametrize the singular cubic $C_t = |\pi^* D - 2E_{q_t}|$.

In the generic case, C_t has a tangent which is not contained in H. Otherwise, C_t would always be contained in H, which is absurd. When the parameter slightly changes, the tangent plane sweeps out a two-dimensional infinitesimal piece of the surface. This shows that the image of the parametrization is two-dimensional. Since it is an algebraic map, the image is dense in Z.

To analyze the type of this parametrization, we take a generic point $p \in Z$ and ask on how many curves C_t it lies. The points with tangent plane through p form a curve, the shape curve of Z (imagine how the surface would look when an eye is placed on p). We have to compute the degree of this curve. This number coincides with the number of intersections with H, the number that tells on how many curves C_t the point p lies.

Let $p := (0:0:0:1)$. Then the equation of Z may be written as

$$f_1 w^2 + 2f_2 w + f_3 = 0,$$

where f_i is a form in x, y, z of degree i, $i = 1, 2, 3$. The shape curve has the equations

$$f_1 w + f_2 = f_2 w + f_3 = 0.$$

This is a curve of degree 6. The parametrization is therefore a 6:1 parametrization.

Example 6: Let $Z \in \mathbf{P}^3$ be the surface with equation

$$F(x, y, z, w) = xw^2 + x^3 + y^2 w - z^3 = 0.$$

The point $q = (0:0:0:1)$ is smooth. The tangent plane is $H : x = 0$. The intersection of H and Z is a cubic curve with the parametrization

$$(y : z : w) = (t^3 : t^2 : 1).$$

The tangent plane at p_t on Z is:

$$H_t : x + 2t^3 y - 3t^4 z + t^6 w = 0.$$

To intersect with Z, we eliminate w and get the cubic curve

$$C_t : x(x + 2t^3 y - 3t^4 z)^2 + t^1 2x^3 - t^6 y^2 (x + 2t^3 y - 3t^4 z) - t^1 2z^3 = 0.$$

We know the singular point $q = (0 : t^3 : t^2) \in C_t$. The generic line through q is

$$L_{st} : sx - y + tz = 0.$$

This line intersects C_t doubly in q_t and in the varying point

$$p_{st} = (2t^4 + 6t^7 s + 3s^2 t^{10} : 6st^4 + 9t^7 s^2 + 3s^3 t^{10} + t + t^{13} :$$

$$: 1 + 4t^3 s - 2t^9 s^3 + 3t^6 s^2 + t^{12}).$$

The w-coordinate is easily obtained by the equation of H_t. After clearing denominators, we get the following parametrization of Z.

$$(x : y : z : w) = ((2t^7 + 6t^{10} s + 3s^2 t^{13} : 6st^7 + 9t^{a10} s^2 + 3s^3 t^{13} + t^4 + t^{16} :$$

$$: t^3 + 4t^6 s - 2t^{12} s^3 + 3t^9 s^2 + t^{15} : -t - 6st^4 - 12t^7 s^2 - 8t^{10} s^3 + t^{13}).$$

Algorithm *improper3*:

Input:

F, a cubic form in 4 variables x, y, z, w, representing a Del Pezzo surface Z of degree 4.

A smooth point $q \in Z$.

Output: 4 polynomials $\overline{x}, \overline{y}, \overline{z}, \overline{w}$, representing an improper parametrization of Z.

translate q to $(0:0:0:1)$;
rotate the tangent plane to $x = 0$;
Intersection with H;
$C :=$ set $x = 0$ in $\overline{F}$.
$y_t, z_t, w_t := parametrize(C)$;
Intersection with the the tangent plane.
compute H_t;
eliminate w, using the equation $H_t = 0$;
Equation of the generic line.
$L_{s,t} = sx + z_t y - y_t z$;
Compute variable intersection. eliminate z, using the equation $L_{s,t} = 0$;
$p = \frac{F}{x^2}$;
$\overline{x}, \overline{y} := solve(\mathrm{p})$;
The rest is easy.
find $\overline{z}, \overline{w}$ be plugging in;
clear denominators in $(\overline{x}, \ldots, \overline{w})$;
return $(\overline{x}, \ldots, \overline{w})$;
exit;

The above algorithm proves that a Del Pezzo surface of degree 3 is unirational. To prove rationality, we have to produce a -1-curve.

Lemma 3.8 *Any Del Pezzo surface of degree 3 has a -1-curve.*

Proof: We give a dimension argument, analogously to the proof of lemma 7. Let Q be the set of all cubic divisors in $\mathbf{P}^3$ (classes of nonzero forms of degree 3 in 4 variables modulo scalar multiplication). Let L be the set of all lines in $\mathbf{P}^3$. Let

$$\Gamma := \{(q, l) \in Q \times Q \times L \mid l \in q\}.$$

As in the proof of lemma 7, we show that the projection map

$$\pi_2 : \Gamma \rightarrow Q \times Q$$

is surjective. This implies then that any cubic surface contains a line. The pullback of this line contains a -1-curve by lemma 3.

The sets Q and L are again irreducible projective varieties. Their dimensions are $\dim Q = 19$ and $\dim L = 4$. The set Γ is projective, because the conditions $l \in q$ can be expressed as equations. The map $\pi_2 : \Gamma \to L$ has isomorphic fibers. Each fiber is irreducible and has dimension 15. This implies that Γ is irreducible and $\dim\Gamma = 15 + 4 = 19$.

Let $D \subset Q$ be the image of π_1. As an image of a projective variety, it is again projective. π_1 has zero-dimensional fibers (for instance the Del Pezzo surface in example 3 has 27 lines). As above, we conclude that $\dim D = 19$ and $D = Q$.

Corollary 3.3 *If (X, D) is a Del Pezzo pair of degree $D^2 = 3$, then X is the blowing up of the plane at 6 base points, and $L \sim 3L - A_1 - \ldots - A_6$.*

Proof: As in the proof of corollary 2, we may blow down the exceptional curve and reduce the statement to the case $D^2 = 4$.

Recall that in the previous case the knowledge of a line allowed a very easy parametrization algorithm, computing a proper parametrization of smallest possible degree $N = 3$. In the case $D^2 = 3$, the situation is more difficult. Giving the task to compute a proper parametrization of degree $N = 3$, we distinguish two cases.

If Z has a double point (say the image of the -2-curve F), then projection from the double point gives a proper parametrization. Its degree is $(D - F) \cdot F = 3$.

If Z is smooth, then we have an instance of example 3. There are exactly 27 lines. They form an interesting symmetric configuration. Each line intersects exactly 10 other lines. These ten lines form 5 pairs of intersecting lines. For instance, the -1-curve $L := A_1$ (cf example 3) intersects the lines

$$|L - A_1 - A_i|, i = 2, \ldots, 6,$$

$$|2L - A_1 - \ldots - A_6 + A_i|, i = 2, \ldots, 6,$$

the line $|L - A_1 - A_i|$ forming a pair with $|2L - A_1 - \ldots - A_6 + A_i|$. The pencil of conics cut out by the planes through L splits exactly 5 times into two lines. We have to compute two lines L_1, L_2, intersecting L but disjoint from each other.

Then the quadratic forms through L, L_1, L_2 give a birational map onto the plane. Its inverse is a parametrization as desired.

To see this, we just have to verify that $B := 2D - L - L_1 - L_2$ is nef and has the right intersection properties (self-intersection 1, genus 0, product with D is 3). This is easy: B is nef because only -1-curves may have negative intersection product with B. But no line can intersect all 3 lines L, L_1, L_2. Moreover

$$B{\cdot}D = 2D^2 - D{\cdot}L - D{\cdot}L_1 - D{\cdot}L_2 = 3,$$

$$B{\cdot}L = 2D{\cdot}L - L^2 - L{\cdot}L_1 - L{\cdot}L_2 = 1,$$

$$B{\cdot}L_1 = 2D{\cdot}L_1 - L{\cdot}L_1 - L_1^2 - L_1{\cdot}L_2 = 2,$$

$$B^2 = 2B{\cdot}D - B{\cdot}L - B{\cdot}L_1 - B{\cdot}L_2 = 6 - 1 - 2 - 2 = 1.$$

Lemma 2.10 shows that i_B is birational onto the plane. The inverse is a parametrization of degree $N = B{\cdot}D = 3$.

Here is a geometric interpretation of this parametrization (figure 4). Choose a coordinate system such that L is $x = z = 0$, L_1 is $x = y = 0$, and L_2 is $z = w = 0$. The map i_B is then given by

$$(x : y : z : w) \mapsto (s : t : 1) = (xw : yz : xz).$$

To invert this map, we have to find the point $(x : y : z : w)$ when $s = \frac{w}{z}$ and $t = \frac{y}{x}$ is known. We parametrize the lines L_1 and L_2 using s and t as parameters:

$$L_1 = (0 : 0 : 1 : s), L_2 = (1 : t : 0 : 0).$$

The set of points $(x : y : z : w)$ such that $s = \frac{w}{z}$ and $t = \frac{y}{x}$ is the line L_{st} through $(0 : 0 : 1 : s) \in L_1$ and $(1 : t : 0 : 0) \in L_2$. Generically, L_{st} intersects Z in 3 points. Two of them are known. The third point gives the parametrization.

Example 7: Let $Z \subset \mathbf{P}^3$ be the smooth cubic surface with equation

$$x^2 w + xw^2 + y^2 z + yz^2 = 0.$$

It contains the lines $L : x = z = 0$, $L_1 : x = y = 0$, and $L_2 : z = w = 0$. Inverting the birational map

$$(x : y : z : w) \mapsto (s : t : 1) = (xw : yz : xz),$$

we get the parametrization

$$(x : y : z : w) = (s^2 + t : s^2 t + t^2 : -s - t^2 : -s^2 - st^2).$$

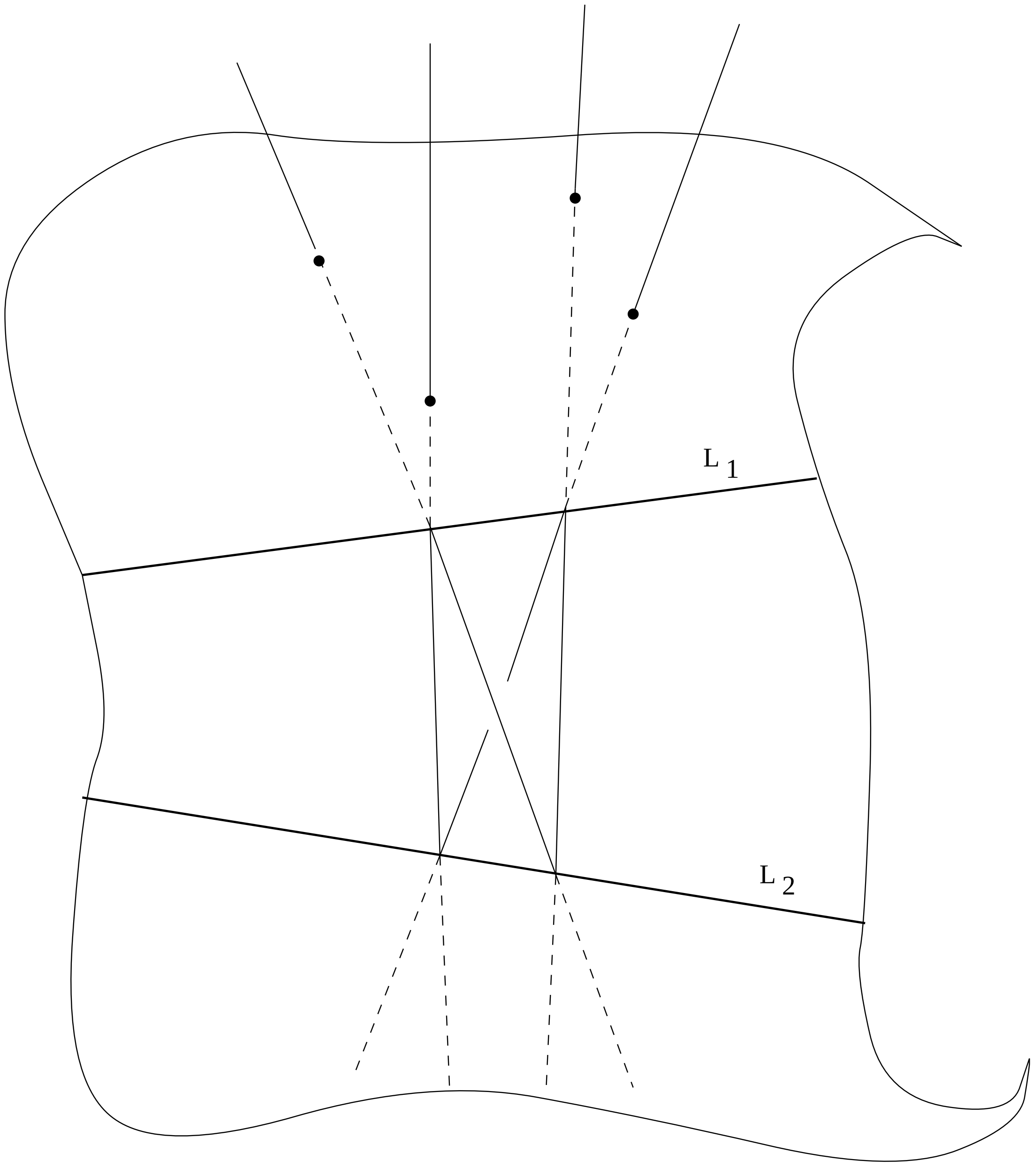

Figure 3.1: Parametrization of a cubic surface

3.3.3 $D^2 = 2$

We know that i_D is regular onto the plane, but not birational. Since $D^2 = 2$, i_D is a 2:1 map. The map i_{2D} is regular and birational. We start with constructing homogeneous coordinates for $Z := i_{2D}(X)$.

The plane $i_D(X)$ is given the usual coordinates x, y, z. Since $s(2D) = 0$, we have $\dim|2D| = 6$, and the associated vector-space S_2 has dimension 7. (In the algorithm *parametrize*, S_2 appears as the adjoint vector-space $V_{2,2\mu-2}$.) The vector-space of the quadratic forms in x, y, z may be regarded as subspace of S_2. Its dimension is 6. Let $w \in S_2$ be an element that is not in this subspace. Then we have a basis for S_2, consisting of w and of the quadratic monomials in x, y, z. We consider x, y, z, w as weighted homogeneous coordinates. The degree of x, y, z is 1, and the degree of w is 2.

We will use the notation $(x : y : z; w)$ for a point, given by weighted homogeneous coordinates. Set-theoretically, p is the point of all 4-tuples without $(0 : 0 : 0; 0)$, modulo weighted scalar multiplication, i.e. $(x : y : z; w) = (\lambda x : \lambda y : \lambda z; \lambda^2 w)$.

4 homogeneous variables of a 2-dimensional variety have to satisfy a relation. Indeed, we have $\dim|4D| = 20$, and the vector-space S_4 of forms of total degree 4 has dimension 21. The number of monomials of total degree 4 is 22. Therefore, we have a relation

$$cw^2 + wf_2(x, y, z) + f_4(x, y, z) = 0.$$

where c is a constant and f_i is a quadratic form of degree i for $i = 2, 4$. The constant c is not zero, because otherwise w could be expressed as a rational function in x, y, z and i_D would be birational. We may assume that $c = 1$. Next, we replace w by $w + \frac{1}{2} f_2$ to make the linear term vanish. Then Z has the homogeneous equation

$$w^2 + f_4(x, y, z) = 0.$$

Z appears as a two-sheeted covering of the plane, branched at a quartic Δ (with equation f_4).

i_{2D} contracts the -2-curves to isolated double points. They coincide with the singular points of the branching curve Δ.

For any -1-curve E, $|D - E|$ is not empty. Hence any -1-curve is a component of some divisor in $|D|$. Any line that intersects Δ either doubly at 2 points or fourfold at one point splits on the covering Z (figure 5). These lines are called bitangents, although they may also be flexes of multiplicity four. The inverse image of such a line is a pair of conjugate -1-curves, plus maybe some -2-curves arising from singular points of Δ. It may also happen

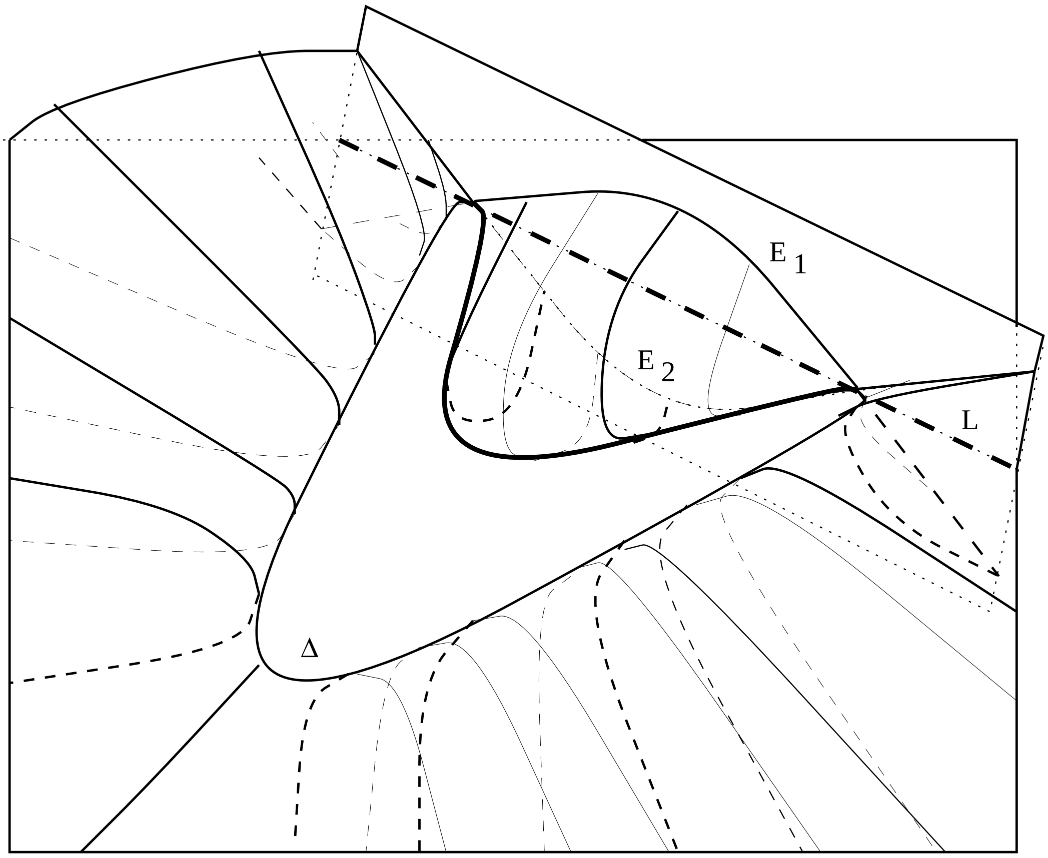

Figure 3.2: A bitangent over the branch curve Δ

that the inverse image is a -1-curve with multiplicity 2. This is the case iff Δ has the line as a component.

Example 8: Let $Y \subset \mathbf{P}^3$ be the surface with equation

$$XY^3 + X^3Z - Z^2W^2 = 0.$$

The first two adjoint ideals of this surface are

$$A_1 = (X, Y, Z), A_2 = (X^2, XY, Y^2, Z).$$

Hence $\mu = 2$ and $v_{1,2} = 1$, and we have a Del Pezzo surface. Its degree is $v_{1,1} - 1 = 2$.

To find the equation of the surface Z, we compute

$$V_{1,1} = \langle X, Y, Z \rangle,$$

$$V_{2,2} = \langle X^2, XY, XZ, Y^2, YZ, Z^2, ZW \rangle.$$

With the new coordinates

$$x := X, y := Z, z := Y, w := ZW,$$

we have the equation

$$w^2 - xz^3 - x^3y = 0.$$

The branch curve splits into the line $L : x = 0$ and the cubic $C : z^3 + x^2y = 0$. The only bitangent is the line L: any other line must intersect L transversally. Therefore, C has to go through the intersection point. L and C meet only at $p = (0 : 0 : 1)$, and any line through p except L intersects C transversally.
The inverse image of L is the curve $x = w = 0$ with multiplicity 2, plus some -2-curves over the singular point $(x : y : z; w) = (0 : 0 : 1; 0)$.

Example 9: Let Z be the Del Pezzo surface given by

$$w^2 + f_4(x, y, z) = w^2 - z^4 + x^3y = 0.$$

The branch curve is irreducible. It has exactly two bitangents, $L_1 : x = 0$ and $L_2 : y = 0$. The inverse image of both lines have two -1-curves as components, given by the additional equation $w \pm z^2 = 0$.

Let $p \in X$ be a point not on a -2-curve, and $q := i_{2D}(p) \in Z$. In general, we do not succeed when trying to assign a double point to $|D|$ at p, since $\dim|D| = 2$ and $\dim|\pi^*D - 2E_p|$ might be empty. Indeed, there is a curve in $|D|$ with a double point at p iff $q \in \Delta$, namely the inverse image of the tangent to q at Δ.

We may, however, assign a triple point on $|2D|$. By Riemann-Roch, $\dim|\pi^*(2D) - 3E_p| \geq 0$, hence there is at least one divisor in $|2D|$ with a triple point at q. If $q \notin \Delta$, then there is exactly one such divisor, for which we will compute the equation explicitly.

We may assume that q has the coordinates $(x : y : z; w) = (0 : 0 : 1; 1)$. The constant coefficient of f_4 is then -1, and we may write

$$f_4 = -z^4 + a_1z^3 + a_2z^2 + a_3z + a_4,$$

where a_i is a form of degree i, $i = 1, 2, 3, 4$. The equation of the curve C in $|D|$ with a triple point in $|D|$ is

$$w + g_2(x, y, z) := w - z^2 + \frac{a_1}{2}z + \frac{a_1^2}{8} + \frac{a_2}{2}.$$

The map i_D maps C birationally to a plane curve of degree 4 with a triple point in $i_D(p) = (0:0:1)$. Its equation is $f_4 - g_2^2$, which is linear in z.

Note that $f_4 - g_2^2$ does not identically vanish, because otherwise the equation $w^2 - f_4$ could not be irreducible.

The curve C is rational: it can be parametrized by the lines through the triple point, which intersects the plane quartic in exactly one moving point. If $f_4 - g_2^2$ is reducible, the components are necessarily rational.

This gives the idea for a parametrization algorithm. Specify a point p_t on C with the first parameter. Then assign a triple base point on p_t to $|2D|$, and parametrize the curve C_t using the second parameter.

To show that we have indeed a parametrization, we have to show that the generic curve C_t has a tangent at $i_D(p_t)$ that does not coincide with the tangent of C at $i_D(p_t)$. Suppose that this is the case. With t moving slightly, this tangent sweeps out some two-dimensional infinitesimal piece of the plane.

If all tangents at $i_D(p_t)$ coincide with the tangent to C at p_t, then C must be a triple component of the divisor in $|2D|$ that was obtained by assigning the triple point. However, this divisor has always components that are not triple, for instance because $(2D)^2 = 8$ is not divisible by 9.

If p lies on the branch curve, we assign not a triple point to $|2D|$ but a double point to $|D|$.

As in the case $D^2 = 3$, the parametrization computed by this method is improper. In [Se], it is shown that it is a 24:1 parametrization in the generic case.

Example 9 continued: Let Z be the Del Pezzo surface of example 9. The point $q = (0:0:1;1)$ is not on Δ. The divisor in $|2D|$ with a triple point on q is given by $w - z^2 = x^3y = 0$. It splits into $E_1 + 3E_2$, where E_1 is the -1-curve $y = w - z^2 = 0$ and E_2 is the -1-curve $x = w - z^2 = 0$.
It is not a good idea to go along E_2 with the first parameter. The unique divisor in $|2D|$ with a triple point on $q' \in E_2 - \Delta$ is again $E_1 + 3E_2$, so that we would not get a parametrization of the whole surface.
A parametrization of E_1 is $q_t = (t:0:1;1)$. The equation of the divisor with triple point at q_t is

$$w + g_2 = w - z^2 - y^3tz + \frac{1}{8}t^6y^2 + \frac{3}{2}t^2xy.$$

The projection to the plane (equation $g^2 - f_4$) can be parametrized by the lines through q_t. The equation of the generic line is $x - sy - tz = 0$. The moving intersection point is

$$q_{st} = (16st^9 + 48s^2t^5 - 128s^3t + t^13 : -8t^9 - 96t^5s - 192ts^2 :$$
$$: 144t^4s^2 + 64s^3 + t^12 + 24t^8s).$$

Together with the 4-th coordinate

$$w = -g_2 = -19200t^8s^4 - 30720t^4s^5 - 1152t^{1}6s^2 -$$
$$-6528t^{1}2s^3 + 4096s^6 - 3t^{2}4 - 96t^{2}0s,$$

we have a parametrization of Z.

Algorithm *improper2*:

Input:

f_4, a quartic form in 3 variables x, y, z, the equation of the branch curve Δ of a Del Pezzo surface Z of degree 2. (Z has equation $w^2 - f_4$.)

A smooth point $q \in Z$.

Output: 4 polynomials $\overline{x}, \overline{y}, \overline{z}, \overline{w}$, representing an improper parametrization of Z.

Trace a rational curve with the first parameter.
if q lies on Δ then
  translate q to $(0:0:1;0)$;
  C is the inverse image of the tangent to Δ;
  q_t:= *parametrize*(C);
else
  translate q to $(0:0:1;1)$;
  $g(x,y,z)$:= as in the above formula;
  $h(x,y,z) = g^2 - f_4$;
  if h is irreducible then
    C is the curve $h = 0, w = g(x,y,z)$;
  else
    h':= a factor of h not triple;
    C is the curve $h' = 0, w = g(x,y,z)$;
  q_t:= *parametrize*(C);
assign triple point at q_t.
translate q_t to $(0:0:1;1)$;
$g(x,y,z)$:= as in the above formula;
translate back;
$h(x,y,z) = g^2 - f_4$;
intersect with general line through q_t.
L_{st}:= the equation of the general line through q_t;
eliminate z, using $L_{st} = 0$;
$p(x,y) := \frac{h}{(y_t x - x_t y)^3}$;
$\overline{x}, \overline{y}$:= *solve*$(p)$;
The rest is easy.
find $\overline{z}, \overline{w}$ be plugging in;

clear denominators in $(\overline{x}, \ldots, \overline{w})$;
return $(\overline{x}, \ldots, \overline{w})$;
exit;

To show that Del Pezzo surfaces of degree 2 are rational, we have to verify that -1-curves always exist, or that the quartic Δ always has a bitangent.

Lemma 3.9 *Any Del Pezzo surface of degree 2 has a -1-curve.*

Proof: Let Q be the set of all of all quartic divisors in the plane (forms of degree 4 modulo scalar multiplication). Let L be the set of lines in the plane. Let

$$\Gamma = \{(q, l) \mid l \text{ is a bitangent to } q\}.$$

We have $\dim Q = 14$, $\dim L = 2$. We have to show that the fiber of π_2 (of course, all fibers are isomorphic) is irreducible, projective, and has dimension 12. Then $\dim\Gamma = 14$. The projection $\pi_2 : \Gamma \to Q$ has zero-dimensional fibers (in fact, no square-free quartic equation can have infinitely many bitangents). Then $\dim(\pi_2(\Gamma)) = 12$ and $\pi_2(\Gamma) = Q$, as in the proofs of lemmata 7 and 8.

Let l be the line $z = 0$. The fiber $\pi_2^{-1}(l)$ is the set of all curves with the property that

$$c_{400}x^4 + c_{310}x^3y + c_{220}x^2y^2 + c_{130}xy^3 + c_{040}y^4$$

is a complete square, where c_{ijk} is the coefficient of $x^iy^jz^k$ in f_4. This set is projective and irreducible, because it is the cone over the image of the map

$$(x_1 : x : 2 : x_3) \mapsto (x_1^2 : 2x_1x_2 : 2x_1x_3 + x_2^2 : 2x_2x_3 : x_3^2)$$

from $\mathbf{P}^2$ to $\mathbf{P}^4$. This image is a surface, and the center of the cone is a 9-dimensional linear subspace, hence the cone has dimension 12.

Corollary 3.4 *If (X, D) is a Del Pezzo pair of degree $D^2 = 2$, then X is the blowing up of the plane at 7 base points, and $L \sim 3L - A_1 - \ldots - A_7$.*

Proof: Analogous to the proofs of the corollaries 2 and 3.

We turn to the task of computing a proper parametrization of minimal degree. This is a parametrization with $\overline{x}, \overline{y}, \overline{z}$ of degree ≤ 3, and $\overline{w}$ of degree ≤ 6, and at least one polynomial reaches the bound.

Suppose that $L : x = 0$ is a bitangent. Then we may write

$$f_4(x, y, z) = xf_3(x, y, z) - (f_2(y, z))^2$$

for suitable forms f_3, f_2, of degree 3 and 2. The inverse image of L contains two -1-curves E_1, E_2: $x = w \pm f_2(y,z) = 0$. The moving part of $|2D - E_1|$ is $|D + E_2|$ (as in the proof of lemma 4). The minimalization of $(X, D + E_2)$ is a Del Pezzo surface X' of degree 3 by lemma 1. Its equation is

$$XW^2 - 2Wf_2(Y,Z) + f_3(X,Y,Z) = 0,$$

$$(X : Y : Z : W) = (x^2 : xy : xz : w + f_2(y,z),$$

$$(x : y : z; w) = (X : Y : Z; XW - f_2(Y,Z).$$

A parametrization of X' of degree 3 gives rise to a parametrization of X as desired.

Example 9 continued: In example 9, $x = 0$ is a bitangent, and

$$f_2(y,z) = z^2, f_3(x,y,z) = x^2 y.$$

The cubic surface X' has the equation

$$XW^2 - 2WZ^2 + X^2 Y = 0.$$

It has a double point $p = (0 : 1 : 0 : 0)$. We invert the projection map

$$(X : Y : Z : W) \mapsto (s : t : 1) = (X : Z : W)$$

and get the parametrization

$$(X : Y : Z : W) = (s^3 : 2t^2 - s : s^2 t : s^2)$$

for X'. This gives the parametrization

$$(x : y : z : w) = (s^3 : 2t^2 - s : s^2 t; s^5 - s^4 t^2)$$

for Z, which is indeed much simpler than the one computed by *improper2*.

Example 8 continued: In example 8, $x = 0$ is the only bitangent, and

$$f_2(y,z) = 0, f_3(x,y,z) = -z^3 - x^2 y.$$

The cubic surface X' has the equation

$$XW^2 - Z^3 - X^2 Y = 0.$$

It has a double point $p = (0 : 1 : 0 : 0)$. We invert again the projection map and get the parametrization

$$(X : Y : Z : W) = (s^3 : s - t^3 : s^2 t : s^2)$$

for X'. This gives the parametrization

$$(x : y : z; w) = (s^3 : s - t^3 : s^2 t; s^5)$$

for X.

Here is a proposition that makes it easy to construct examples.

Proposition 3.2 *Let f_4 be a square-free form of degree 4 in x, y, z. Suppose that the zero set Δ is not the union of 4 lines through a common point. Then the twofold covering*

$$w^2 + f_4(x, y, z) = 0$$

is a Del Pezzo surface of degree 2.

Proof: In the proof of lemma 9 we showed that any degree 4 form has a bitangent. We may therefore apply the birational correspondence as above. It is easy to show that the cubic surface has curve singularities iff Δ has double components, and that the cubic surface has a triple point iff Δ has a fourfold point. This reduces the statement to proposition 1.

As a nice application, we prove a proposition on plane quartic curves. The generic case of the statement follows from the Plücker formulas, but the proof below is valid for all cases.

Proposition 3.3 *Any smooth quartic plane curve has exactly 28 bitangents.*

Let Δ be the smooth quartic. Let Z be the twofold covering of the plane with branching divisor Δ (with equation $w^2 + f_4(x, y, z) = 0$). By proposition 2, Z is a Del Pezzo surface of degree 2. By corollary 4, Z is the blowing up of the plane at 7 points. Since Δ is smooth, Z is smooth, and there are no -2-curves. Hence all base points are plane points, and no three points are on a line, and no six points are on a conic.

The table in example 3 says that Z has exactly 56 -1-curves. Δ contains no line, so no -1-curve is conjugate to itself. Hence we have exactly 28 pairs of conjugate -1-curves, which means 28 bitangents.

3.3.4 $D^2 = 1$

This is the most difficult case. Our program is to find a -1-curve and reduce to the case $D^2 = 2$. This done, one may either blow down another -1-curve, reducing the problem to $D^2 = 3$, or compute an improper parametrization.

By theorem 1, we know that $|D|$ does not have fixed component, $2D$ and $3D$ are very large, and i_{3D} is birational. We start by constructing convenient coordinates for $Z := i_{3D}(X)$.

We choose degree 1 coordinates x, y generating the vector-space S_1 associated to the pencil D. The vector-space S_2 associated to $|2D|$ has dimension $\dim|2D| + 1 = 4$. We already know three independent vectors

x^2, xy, y^2. A vector z, independent from these three, serves as the third coordinate (of degree 2).

The map i_{2D} is given by the vector-space $\langle x^2, xy, y^2, zrangle$. Since $(x^2)(y^2) - (xy)^2 = 0$, the image $i_D(X)$ is a quadric cone. The inverse image of a plane splits into two elements in $|D|$ iff the plane goes through the vertex of the cone. The inverse image of the lines through the vertex are the divisors in $|D|$.

The degree of the cone is 2, but $D^2 = 4$. Therefore, i_{2D} is a 2:1 map.

The vector-space S_3 associated to $|3D|$ has dimension $\dim|3D| + 1 = 7$. We know 6 linear independent vectors: $x^3, x^2y, xy^2, y^3, xz, yz$. A vector w, independent of these, serves as the fourth coordinate. Its degree is 3.

The algebraic relation between x, y, z, w has degree 6, because there are $\dim S_4 = \dim|4D| + 1 = 22$, and there are 23 monomials of degree 6. Because i_{2D} is not birational, the coefficient of w^2 is not zero. Like in the previous subsection, we replace w by a suitable vector in S_3 so that the linear term in w vanishes. The equation of Z has then the form

$$-w^2 + cz^3 + f_2(x,y)z^2 + f_4(x,y)z + f_6(x,y) = 0,$$

where c is a constant and f_i is a form of degree i for $i = 2, 4, 6$.

We can also prove that $c \neq 0$: Assume, indirectly, that $c = 0$. Then the generic fiber of i_D has equation

$$-w^2 + f_2(1,t)x^2z^2 + f_4(1,t)x^4z + f_6(1,t)x^6 = 0.$$

Dehomogenizing at x shows that this curve is rational (since it is quadratic in $(\frac{w}{x^3}, \frac{z}{x^2})$). This contradicts theorem 1, where we said that the generic divisor of $|D|$ is elliptic. We may assume wlog that $c = 1$.

Note that we still have some freedom for the choice of z, it just has to be an element in S_2 linearly independent of x^2, xy, y^2. For instance, we could make the quadratic term f_2z^2 vanish (Tschirnhausen transformation). Other preferences will play a role later.

The surface z appears as a twofold covering of the quadric cone. The branch locus has two components. The first is the curve Δ with equation $z^3 + f_2z^2 + f_4z + f_6 = 0$, which is a complete intersection of the cone with a cubic in 3-space (and therefore a curve of degree 6).

The other component is an isolated branch point at the vertex of the cone. To see this, we dehomogenize at z and analyze the situation in a neighborhood of the vertex. With affine coordinates

$$x_1 := \frac{x^2}{z}, x_2 := \frac{xy}{z}, x_3 := \frac{y^2}{z},$$

$$y_1 := \frac{xw}{z^2}, y_2 := \frac{yw}{z^2}, y_3 := \frac{w^2}{z^3},$$

we see that y_3 is close to 1, while the other coordinates are close to 0. We approximate the equations

$$x_1x_3 = x_2^2, y_1y_2 = y_3x_2, y_1^2 = y_3x_1, y_2^2 = y_3x_3$$

by setting $y_3 = 1$. Then x_1, x_2, x_3 can be expressed in terms of y_1, y_2. Z is therefore smooth at the point $(x : y; z; w) = (0 : 0; 1; 1)$, with local coordinates y_1, y_2. The regular map $i_{2D} \circ i_{3D}^{-1}$ from Z to the quadric cone looks locally like

$$(y_1, y_2) \mapsto (x_1, x_2, x_3) = (y_1^2, y_1y_2, y_2^2).$$

The fiber over any point of the cone except the vertex has two points, and the vertex is indeed an isolated branch point.

To imagine this situation, take a piece of paper and cut along the negative x-axis. Then bend the paper around the origin, until the two cut parts meet again (figure 6). The result is a cone consisting of two sheets of paper.

Any -1-curve E is a component of some divisor in $|2D|$, because $\dim|2D_E|\geq 0$. We distinguish two possible types of -1-curves.

Type 1: There exists a divisor in $|D|$ containing E. Then $i_{2D}(E)$ is a line through the vertex. The line passes through a singular point of Z, the image of the -2-curves summing up to $|D-E|$. Conversely, any line through the vertex and a singular point is the image of a -1-curve, by lemma 3.

Type 2: $i_{2D}(E)$ is a smooth conic, not passing through the vertex. The divisor in $|2D|$ corresponding to this plane splits into two conjugate -1-curves, plus maybe some -2-curves. Such a plane is called a tritangent, because it is tangential at all 3 points of intersections with the branch curve Δ. Conversely, the inverse image of a tritangent plane not passing through the vertex splits into two conjugate curves on Z. By lemma 3, these curves are the images of two conjugate -1-curves.

Note that we have also tritangent planes passing through the vertex: such a plane is tritangent iff it is tangent to the quadric cone.

Suppose that $p = (0 : 1; 0; 0)$ is a double point. (Any point can on the branch curve can be translates to this position.) Then

$$f_4(x, y) = xg_3(x, y), f_6(x, y) = x^2g_4(x, y)$$

for some forms g_3, g_4 of degree $3, 4$.

If $E+F$ is the inverse image of the line $x = 0$, then we want to minimalize $(X, D + E) = (X, 2D - F)$. This leads to the consideration of the divisors in $|2D|$ passing through the double point p.

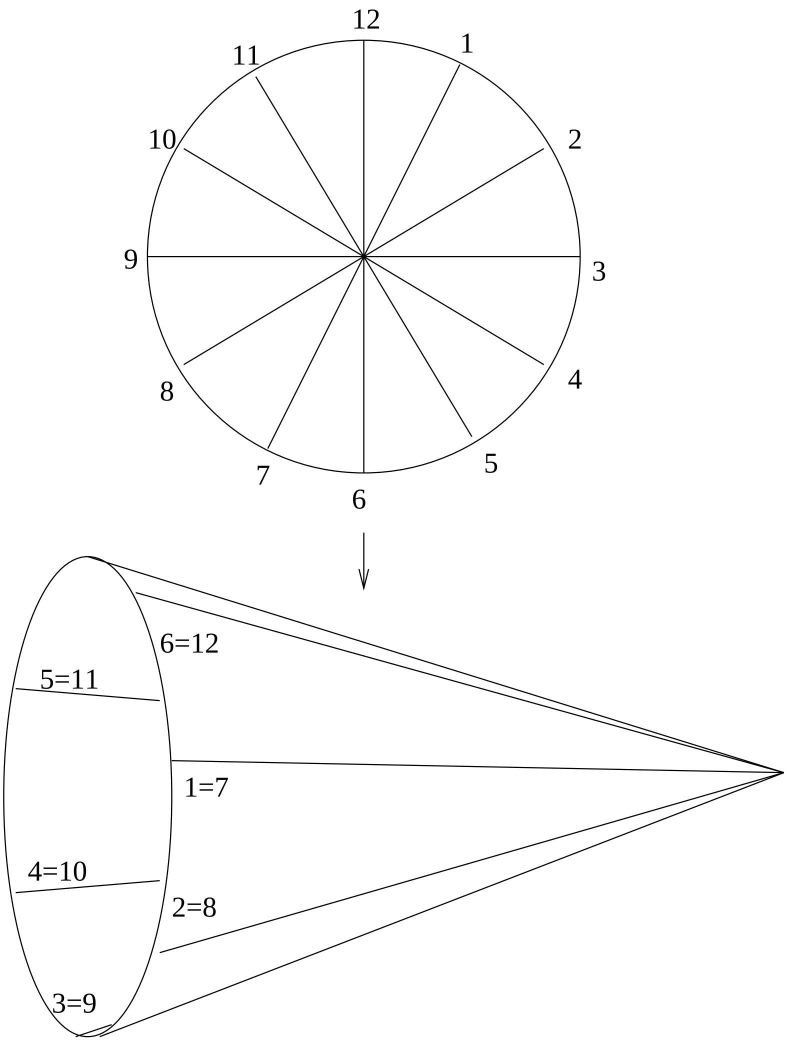

Figure 3.3: An isolated branch point on a cone

The vector-space of the of degree 2 forms vanishing at p is $\langle x^2, xy, z\rangle$. The vector-space of degree 4 forms vanishing doubly at p is

$$\langle x^4, x^3y, x^2y^2, zx^2, zxy, z^2, wz\rangle.$$

We have a birational equivalence

$$(X : Y : Z; W) = (x^2 : xy : z; xw),$$

$$(x : y; z; w) = (X : Y; XZ; XW)$$

to the Del Pezzo surface Z' of degree 2, with equation

$$-W^2 + XZ^3 + Z^2 f_2(X, Y) + Zg_3(X, Y) + g_4(X, Y) = 0.$$

We also have the point $q : (X : Y : Z; W) = (0 : 0 : 1; 0)$ on Z'. By lemma 1, it is a smooth point, and we have the required input for *improper2*. We plug the parametrization into the birational map and get a parametrization for Z.

Note that if we plug in a proper parametrization of minimal degree for Z', then we get a proper parametrization of minimal degree for Z. This is a parametrization with $\overline{x}, \overline{y}$ of degree ≤ 3, $\overline{z}$ of degree ≤ 6, $\overline{w}$ of degree ≤ 9, and at least one polynomial reaches the bound.

Example 10: (Cf also example 1.9.) Let Z be the surface given by

$$-w^2 + z^3 + x^5 y = 0.$$

The point $(x : y; z; w) = (0 : 1; 0; 0)$ is a double point. The equivalent Del Pezzo surface Z' of degree 2 has equation

$$-W^2 + XZ^3 + X^3Y = 0,$$

which we already saw in example 8. There, we computed the parametrization

$$(X : Y : Z; W) = (s^3 : s - t^3 : s^2 t; s^5).$$

By plugging in, we get the parametrization

$$(x : y; z; w) = (s^3 : s - t^3; s^5 t; s^8).$$

Now, suppose that $z = 0$ is a tritangent to Δ. (If there is a tritangent, then it can always be transformed to this position, by a suitable choice of the z-coordinate). Then we may write

$$f_6(x, y) = g_3(x, y)^2$$

for some form g_3 of degree 3.

If the inverse image of i_{2D} of the tritangent splits into two conjugate -1-curves E_1, E_2, then we are interested in the system $|D+E_1| = |3D-E_2|$. E_3 has the equations $w - g_3(x,y) = z = 0$. The degree 3 forms vanishing at E_2 are $\langle zx, zy, w-g_3(x,y)\rangle$. The degree 6 forms vanishing doubly at E_1 are

$$\langle z^2x^2, z^2xy, z^2y^2, (w-g_3)^2, zx(w-g_3), zy(w-g_3), z^3\rangle.$$

This gives the birational equivalence

$$(X:Y:Z;W) = (xz:yz:w-g_3(x,y);z^3),$$

$$(x:y;z;w) = (X:Y;W;ZW+g_3(X,Y)$$

to the Del Pezzo surface of degree 2 with equation

$$W^2 + Wf_2(X,Y) + f_4(X,Y) - Z^2W - 2Zg_3(X,Y) = 0.$$

To make the linear term in W vanish, we set

$$W' := W + \frac{f_2(X,Y) - Z^2}{2}$$

and get the equation of the branch curve

$$f_4(X,Y,Z) = 2Zg_3(X,Y) - f_4(X,Y) + (\frac{f_2(X,Y) - Z^2}{2})^2.$$

We also have the smooth point $(X:Y:Z;W) = (0:0:1;0)$. Again, we have the required input for *improper2*. The above remark on the minimal parametrization applies also here.

The singular points of Z are the solutions of the equations

$$w = 0, 3z^2 + 2zf_2(x,y) + f_4(x,y) = 0,$$

$$\frac{\partial f_2}{\partial x}z^2 + \frac{\partial f_4}{\partial x}z + \frac{\partial f_6}{\partial x}z^2 = 0,$$

$$\frac{\partial f_2}{\partial y}z^2 + \frac{\partial f_4}{\partial y}z + \frac{\partial f_6}{\partial y}z^2 = 0.$$

It is rather easy to solve this system of equations. But in the generic set, the solution set is empty.

To find tritangent planes not passing through the vertex, one has to find a polynomial solution of

$$-\overline{w}^2 + f_2\overline{z}^2 + f_4\overline{z} + f_6 = 0,$$

where $\overline{w}$ must be a form of degree 3 and $\overline{z}$ must be a form of degree 2 in x, y. This is a system of 7 equations in 7 unknowns (the coefficients of $\overline{w}$ and $\overline{z}$). Dimension counting suggests that in general there is a finite number of solutions. However, the search space $\mathbf{A}^7$ is not projective in this case, and this means that we cannot prove by dimension counting that there is always a solution. In fact, one can show that all tritangents of the the branch curve in example 10 pass through the vertex of the quadric cone ([RoSch]).

The clue is to replace the condition "the plane does not pass through the vertex" by a condition which is satisfied on a closed subset.

Lemma 3.10 *Let $Q \cong \mathrm{P}^5$ be the set of quadric divisors of the plane. Let $C \cong \mathrm{P}^9$ be the set of cubic divisors of the plane. Let $C_0 \subset C$ be the open subset of nonsingular cubic curves. Let $\Gamma_0 \subset Q \times C_0$ be the set of all pairs (q, c), such that the divisor on c cut out by q is divisible by two, but its half is not cut out by a line.*

Then Γ_0 is closed in $Q \times C_0$, and its dimension is 11.

Proof: Let $\Gamma_1 \subset Q \times C_0$ be the set of all pairs (q, c), such that q is a double line. Then $\Gamma := \Gamma_0 \cap \Gamma_1$ is closed in $Q \times C_0$, because it is the set of all pairs such that the divisor on c cut out by q is divisible by 2, and this condition can be expressed by equations. The same is true for Γ_1.

To show that both Γ_0 and Γ_1 are clopen in Γ, we consider the classes of the divisors on c cut out by q. For a fixed elliptic curve c, there are exactly 4 such classes if (q, c) is in Γ, corresponding to the 4 elements of order 2 in the Jacobean of c. One class is distinguished, namely the class of line sections. This suggests that Γ_0 and Γ_1 are not connected with each other.

In making this argument precise, we encounter the following difficulty. We would like to say that "the Jacobean varies continuously with c". But it is rather hard to construct the Jacobean in an algebraic way when we know only the equation but no point on the curve. This makes the following roundabout necessary.

Let $\Delta \subset \mathbf{P}^2 \times C_0$ be the set of all pairs (p, c), such that $p \in c$ and $6p \sim 2l$ in the class group of c, where l is a line section ($\deg l = 3$). Any curve c has exactly 36 points with this property, corresponding to the elements of order 6 in the Jacobean of c. Hence Δ is an unbranched covering of C_0, of dimension $\dim \Delta = \dim C_0 = 9$.

Let $\Delta_1 \subset \Delta$ be the set of all pairs (p, c), such that $3p \sim l$ in the class group of c (p is a flex of c). Any curve c has exactly 9 such points, corresponding to the elements of order 3 in the Jacobean. Hence Δ_1 and $\Delta_0 := \Delta - \Delta_1$ are also unbranched coverings. Especially, Δ_0 and Δ_1 are clopen in Δ.

For any point (q, c) in Γ, there are exactly 9 points $(p, c) \in \Delta$, such that $3p$ is equivalent to half of the divisor cut out by q. If $(q, c) \in \Gamma_0$, then all nine points are in Δ_0, and if $(q,c)\in\Gamma_1$, then all nine points are in Δ_1. Since the points move continuously with q and c, both Γ_0 and Γ_1 are clopen in Γ.

Lemma 3.11 *Any Del Pezzo surface of degree 1 has a -1-curve.*

Proof: We have to show that any complete intersection of a quadric cone and a cubic divisor in $\mathbf{P}^3$ has either a tritangent not passing through the vertex of the quadric cone or a singular point.

Let E, Q', and C' be the set of all linear, quadric, and cubic divisors in $\mathbf{P}^3$. For any triple (e,q,c), we consider the equations of q and c modulo the linear equation e, and regard them as quadric and cubic form in $\mathbf{P}^2$. We define $\Gamma\subset E\times Q\times C$ as the set of triples (e,q,c), such that (q,c) modulo e is in the closure of the variety Γ_0 defined in lemma 10 above, or one of q, e vanishes on e. Here, the closure is meant to be the closure in $Q\times C$. The fiber over the first projection has dimension 11+4+10=25, namely 11 degrees of freedom for the choice of a point in the closure of Γ_0, 4 for the choice of a representative q in the class module e, and 10 for the choice of a representative c in the class modulo e. Since $\dim E = 3$, we have $\dim\Gamma = 25 + 3 = 28$.

Now, $\dim(Q'\times C') = 9 + 19 = 28$, and the map

$$(\pi_2, \pi_3) : \Gamma\rightarrow Q'\times C'$$

has finite fibers. By the usual argument, the map is onto (note that Γ is projective!).

Let $y^2 - xz = 0$ be the equation of the quadric cone, and let $C(x,y,z,w)0$ be a nonzero cubic form that does not vanish at the vertex $(0:0:0:1)$. Then there exists a plane such that the sections with the quadric cone and with the cubic are in Γ_0, or the cubic contains the plane. This plane is necessarily a tritangent plane. We distinguish two cases:

Case 1: The plane does not pass through the vertex $(0:0:0:1)$. Then we have the desired tritangent plane not passing through the vertex.

Case 2: The plane does pass through the vertex. Since the plane is tritangent, it is a tangent plane to the quadric cone. Also, we may assume wlog that C is linear in y, since we may reduce C be the equation $y^2 - xz$ without changing the intersection. Modulo $z = 0$, we have then a quadric $y^2 = 0$ and a cubic $f_3(x,w) + f_2(x,w)y = 0$, and this pair is in the closure of Γ_0. Note that $f_3\neq 0$ because the cubic does not vanish at the vertex $(x:y:w) = (0:0:1)$.

By the definition of Γ_0, and by lemma 10, the cubic cannot be smooth. But then there is a singular point on the line $y = 0$ (if f_2 is not identically zero, all singular points are on that line). This is then a singular point of the branch curve.

Corollary 3.5 *If (X, D) is a Del Pezzo pair of degree $D^2 = 1$, then X is the blowing up of the plane at 9 base points, and $L \sim 3L - A_1 - \ldots - A_9$.*

Proof: Analogous to corollaries 2 to 4.

Finally, here is the result that allows to construct examples.

Proposition 3.4 *Let $z^3 + f_2(x, y)z^2 + f_4(x, y)z + f_6(x, y)$ be a square-free homogeneous form of weighted degree 3 in $x, y; z$. Let Δ be its zero set on the quadric cone. Then the twofold covering*

$$-w^2 + z^3 + f_2(x, y)z^2 + f_4(x, y)z + f_6(x, y) = 0$$

is a Del Pezzo surface of degree 1, unless Δ is the union of three conics passing through a common point p and having the same tangent at p.

Proof: Analogous to proposition 2.

3.4 The crop

It is time to collect the fruits: Castelnuovo's criterion is now established, as well as a classification for Del Pezzo surfaces and minimal rational surfaces, and the classification of linear systems of elliptic curves in the plane.

We have improved the theorems 2 and 3: After having investigated the cases $D^2 = 4, 3, 2, 1$ separately, we may drop the assumption $D^2 \geq 4$ in theorem 2 and $D^2 \geq 5$ in theorem 3.

Since the Del Pezzo case was the last remaining case in the algorithm *parametrize*, theorem 2 concludes the proof of the following theorem.

Theorem 3.4 *Let (X, D) be an Italian pair. Then X is rational.*

As a corollary, we obtain theorem 1.1 (Castelnuovo's rationality criterion).

The improvement of theorem 3 gives the classification for Del Pezzo surfaces.

Theorem 3.5 *Let (X, D) be a Del Pezzo pair. Then $D^2 \leq 9$, and X is either the blowing up of $\mathbf{P}^2$ at $r := 9 - D^2$ points, or a ruled surface with characteristic number $e = 0$ or 2.*

Another remarkable result is the classification theorem for minimal rational surfaces.

Theorem 3.6 *Let X be a minimal rational surface. Then X is the plane or a ruled surface with characteristic number $e{\neq}1$.*

Proof: Let D be some nef divisor, such that (X, D) is Italian. Let (X, D_n) be the last adjoint in the adjoint sequence. (Note that the adjunction does not change the surface, since X is minimal.) By proposition 2.4, three cases may occur.

1) $D_n{\sim}nP$ for a rational pencil P. Then (X, P) is a ruling.

2) $g(D_n) = 0$ and $D_n^2 > 0$. Then X is the plane or a ruled surface by theorem 2.10.

3) $D_n{\sim}0$. Then X is a Del Pezzo surface without -1-curves. By theorem 6, X is either the plane or a ruled surface with characteristic number 0 or 2.

In (1) and (2), the characteristic number of the ruling cannot be 1, because this ruled surface is not minimal.

Remark: The theory of minimal surfaces is even more elegant for non-rational surfaces. We have unique minimal models in all birational equivalence classes with $\kappa{\geq}0$. Every minimal surface with $\kappa = -\infty$ is either rational or *irrationally ruled* (cf [Za3]).

In the language of linear systems, we have a classification of systems of elliptic curves.

Theorem 3.7 *Let Γ be a linear system of elliptic plane curves of dimension at least 2. Then there is a birational auto-map ρ of the projective plane, transforming Γ to a linear system of curves of cubics or to a linear system of quartics with two double base points.*

Proof: Let Γ' be the very large linear system obtained by resolving the base points. Let $D'{\in}\Gamma'$. By Bertini (theorem 15), there exists a smooth curve in Γ', hence $g(D) = 1$.

Let (X, D) be the minimalization of $(\mathbf{Y}, D')$. This is an Italian pair with $g(D) = 1$. Since $\dim|D|{\geq}2$, there exist two curves in $|D|$ meeting at the generic point. Hence either $D^2{\geq}1$. By lemma 2.2, (X, D) is a Del Pezzo pair.

By theorem 5, X is either a blowing up of the plane, or X is a ruling with characteristic number 0 or 2.

If X is a blowing up of the plane, then $D{\sim}3L - Ai_1 - \ldots - A_r$. Define ρ as

$$\mathbf{P}^2 \to \mathbf{Y} \to X \to \mathbf{P}^2.$$

This transforms Γ to a linear system of cubics.

If X is a ruling of characteristic number 0, then $D{\sim}2P_1 + 2P_2$, where (X, P_1) and (X, P_2) are the two rulings. The map $i_{P_1+P_2}$ embeds X into $\mathbf{P}^3$ as a smooth quadric. Let $\pi : X{\to}\mathbf{P}^2$ be the composition of $i_{P_1+P_2}$ with the projection from a point p on the quadric. Define ρ as

$$\mathbf{P}^2 \to \mathbf{Y} \to X \xrightarrow{\pi} \mathbf{P}^2.$$

This transforms Γ to a linear system of quartics with two double points (namely the images of the two lines on the quadric passing through p).

If X is a ruling of characteristic number 2, then $D{\sim}4P + 2B$, where (X, P) is the ruling and B is the fundamental curve. The map i_{2P+B} is birational onto a quadric cone. The fundamental curve is contracted to the vertex of the cone. Let $\pi : X{\to}\mathbf{P}^2$ be the composition of i_{B+2P} with the projection from a smooth point p on the quadric. Define ρ as

$$\mathbf{P}^2 \to \mathbf{Y} \to X \xrightarrow{\pi} \mathbf{P}^2.$$

This transforms Γ to a linear system of quartics with a tacnode (the image of the line through p). The tacnode resolves to two double base points, one in the infinitely near.

Here is the completion of the above classification.

Theorem 3.8 *Let Γ be a pencil of elliptic plane curves. Then there is a birational auto-map ρ of the projective plane, transforming Γ to a linear system of curves of degree $3n$ with an 9 n-fold base points (Halphen pencil).*

Proof: Let (X, D) be the Italian pair obtained by resolution of base points and subsequent minimalization. Since the intersection of two different curves is already the base locus, and the base locus is resolved, we have $D^2 = 0$. By the same argument as in theorem 7 (and theorem 2.14), it suffices to show thar X is a blowing up of the plane at 9 points, and $D{\sim}3nL - nA_1 - \ldots - nA_9$.

Claim 1: Let (X, D) be a Del Pezzo pair with $D^2 = 0$ and $g(D) = 1$. Let μ be the adjoint length of (X, D). Then $D{\sim} - \mu D$.

We prove claim 1 by induction on μ. If $\mu = 0$, then $|D + K|$ is empty, and $D{\sim}0$ by lemma 2.2.

If $\mu > 0$, then $D + K$ is nef, and $|D + K|$ is not empty (lemma 2.2 again). Since $D^2 = D{\cdot}K = 0$, we have $(D + K)^2 = K^2{\geq}0$. By Riemann-Roch, $|-K|$ is not empty, hence $(D+K){\cdot}(-K) = -K^2{\geq}0$. Hence $K^2 = 0$. Let (X', D') be the minimalization of $(X, D+K)$. By induction hypothesis, $D'{\sim}(1 - \mu)K_{X'}$.

If $\pi : X{\to}X'$, then $\pi^*(D' - K_{X'})$ and D differ by exceptional divisors orthogonal to $\pi^*(\mathrm{Div}X')$. Hence $D^2{\leq}(D' - K_{X'})^2 = K_{X'}^2$. Now, we have two cases.

If $\mu > 1$, then $K_{X'}^2 = 0$ by the same argument as in the proof of $K_X^2 = 0$. Then $X = X'$, because π does not blow down exceptional curves. Then $D \sim D - K \sim -\mu K$.

If $\mu = 1$, then $D' \sim 0$ implies $D \sim -K \sim 0$, no matter what π does.

Claim 2: Let (X, D) be an Italian pair with $D^2 = 0$ and $D \sim -K$. Then X is the blowing up of the plane at 9 points, and $D \sim 3L - A_1 - \ldots - A_9$.

Proof of claim 2: Since $K^2 = 0$, X cannot be a minimal surface, because we have $K^2 = 8$ or 9 for minimal rational surfaces. Let E be an exceptional curve. Then the blowing down of E is a Del Pezzo surface of degree 1 (cf the proof of lemma 1). This must be a plane blown up at 8 points, by theorem 5. Hence, X is the blowing up of the plane at 9 points.

Claim 1 and claim 2 together imply the theorem.

3.5 Historical and aesthetical remarks

We give an impression on the historical rule of Del Pezzo surfaces, especially the smooth cubic. Moreover, we discuss other approaches to the Del Pezzo case in Castelnuovos criterion. Finally, we give a justification for our somewhat unusual definition, referring to root systems.

One of the most prominent surfaces ever is the smooth cubic in $\mathbf{P}^3$. It has been a benchmark for the theory of algebraic varieties of dimension higher than 1 in the early days of algebraic geometry ([Sa2]). The amount of symmetry of the configuration of the 27 lines has attracted many geometers. A sub-configuration is known as "Schlaefli's double-six", namely two six-tuples of pairwise disjoint lines, each line of one tuple intersecting exactly five of the other. Schlaefli's double-six can be found exactly 36 times as a subset of the 27 lines.

The 27 lines were discovered in 1849 by Salmon and Cayley. Here are some fragments of [He], cited from [Ma]:

"Indeed Sylvester once remarked, in his characteristical florid style: 'Surely with as good reason as Archimedes to have the cylinder, cone and sphere engraved on his tombstone might our distinguished countrymen leave testamentary directions for the cubic eikosiheptagram to be engraved on theirs.' "

"If Cayley and Salmon had wished to follow Sylvester's advice and to insert a clause in their wills, they would have had no certainty of the correct fulfillment of their directions until the year 1869, when Christian Wiener made a model of a cubic surface showing twenty-seven real lines lying upon it. This achievement of Wiener, Sylvester once remarked, is one of the discoveries 'which must for ever make 1869 stand out in the Annals of Science'."

The symmetry is appealing from an aesthetic point of view. But it is also responsible for the fact that one has to solve algebraic equations of high degree to find, for instance, one line. The reason is that the symmetry group (in this case the Weil group W_8) acts as a possible Galois group transitively on the lines and on all other configurations. Therefore, the computation of a single line cannot be cheaper in general than the solution of a univariate equation of degree 27.

Del Pezzo surfaces over algebraic number fields have been studied from a number theoretic viewpoint. The multitude of techniques to approach the problem of finding a rational point on the surface is impressive. Here, we could not even touch this question. An excellent reference for this matter is [Ma].

We are especially interested in one way in which Del Pezzo surfaces pop up: as the most difficult case in the proof of Castelnuovos criterion. Although there are quite different proofs for this theorem, the Del Pezzo case appears everywhere as a difficulty (sometimes not so explicit). In the logic of most of the proofs, it is requested either to prove the existence of a -1-curve or to prove rationality.

This task is getting harder with D^2 decreasing. In the most difficult case $D^2 = 1$, one has to show the existence of a tritangent plane of a complete intersection of a quadric cone and a cubic, which does not pass through the vertex. This must have been obvious to the geometers of the last century, they do not lose many words on this matter. The answer to all sort of problems in enumerative algebraic geometry must have been common knowledge those days.

The matter gets extremely complicated in positive characteristic, especially in characteristic 2 and 3. (Note that the proof given here fails exactly for these two cases.) When looking for a variant of the parametrization algorithm in this case, I studied some of the proofs very carefully, only to find that the proof in [Za4] has a gap and the proof in [Ku] has an error. A proof that I did not study is M. Artin's idea sketched in [BoHu].

In principle, the question could be settled by a computer. The statement that any Del Pezzo pair with $D^2 = 1$ is rational is equivalent to the following statement ([Za4]).

Lemma 3.12 *For $i = 1, 2, 3, 4, 6$, let f_i be a homogeneous polynomial of degree i in two variables x, y. Suppose that the partial derivatives of*

$$F(x, y, z, w) := w^2 + w(f_1 z + f_3) + z^3 + f_2 z^2 + f_4 z + f_6$$

do not have a common nontrivial solution.

Then there exist homogeneous polynomials $w_3(x,y)$, $\deg w_3 = 3$, *and* $z_2(x,y)$, $\deg z_2 = 2$, *such that*

$$w_3^2 + w_3(f_1 z_2 + f_3) + z_2^3 + f_2 z_2^2 + f_4 z_2 + f_6 = 0.$$

This is a quantifier elimination problem for algebraically closed fields, which can be solved algorithmically ([Wei]).

An elegant variant for the field of complex numbers is due to Kodaira ([Ko]). The difficult Del Pezzo case can be avoided by choosing a different starting divisor D, which is not a multiple of $-K$ (up to equivalence). Therefore, it suffices to prove the following.

Lemma 3.13 *There exists no minimal surface* X *with* $p_a = p_g = P_2 = 0$, $K^2 > 0$, *such that* K *generates* $\mathrm{Pic}X$.

The impossibility of such a surface follows immediately from some general formulas involving the topological invariants p_a, K^2, χ (Euler characteristics).

Unfortunately, it is extremely hard to make this argument constructive, in the sense of using it as a base for an algorithm that finds divisors that are not multiples of K. Alone the explicit computation of a triangulization would be a rather cumbersome task.

The definition of Del Pezzo surfaces differs from the definition given in [Ma], [Na], and [Ha]. It seems not be common to allow -2-curves. I do not understand why; but there are good reason to allow them.

Although the occurrence of -2-curves is a singular case (there are no -2-curves in the generic case), they occured quite often in examples I picked accidentally. Of course, I used to pick sparse equations with low coefficients. It seems that such equations tend to be singular.

In the material presented here, one cannot say that -2-curves are a hazard. They do not make the proofs substantially more complicated, neither the algorithms. On the contrary: a cubic with a double point is more easy to parametrize than than a smooth one.

What was not presented here is the beautiful analysis of the class group of a Del Pezzo surface with root systems ([Ma]). The circle is closed with the observation that -2-curves form the base of a root system (cf section 1). This root system is a subsystem of the root system corresponding to the class group. Hence one can classify the configurations of -2-curves on Del Pezzo surfaces by morphisms of root systems, where the smaller system has a distinguished base. The generic case without -2-curves corresponds to the case where the domain is the zero root system (of rank 0).

Chapter 4

Resolution of the singularities

To compute the adjoint ideals of a singular surface in $\mathbf{P}^3$, it is necessary to analyze the singularities. This chapter is concerned with such an analysis. Its goal is to provide an algorithm for the computation of the adjoint ideals.

Here is a summary of the particular sections.

1: Subdivision into local cells. We describe some techniques for splitting problems into local ones. We split the projective space into affine cells and split the affine cells further to separate the singularities. The "local-global principle" (theorem 1) allows us to join local information into a global result.

2: Blowing up the singularities. In chapter 1, we defined the adjoint ideal A_m as the ideal of all forms having a certain order at the singularities, including infinitely near singularities. Here, we introduce infinitely near singularities by blowing up, and we make it more clear what it means to have a certain order there.

We also give an algorithm for the resolution of the singularities by repeated blowing up, and an algorithm for the computation of the adjoint ideals.

3: Nonlinear transformations. This section closes a gap in the algorithm described in the previous section. There, we gave an efficient algorithm for the blowing up of lines, but we need to blow up arbitrary smooth curves in space. Here, we describe a useful trick that allows us to transform an arbitrary smooth curve to something like a line.

4: Adjoint sheafs. Sticking to the didactic method of postponing abstract theory, we give here a theoretically clean introduction of the adjoint sheaves (especially adjoint ideals). We prove the correctness of the algorithms in section 2 and theorem 2.7 which played a fundamental role in chapter 2.

5: Computational aspects. We discuss the algorithms and mention some further applications beyond parametrization.

4.1 Subdivision into local cells

Singularity analysis can easily be split into local problems. Here, we explain some techniques for this splitting, and also for the reverse process of collecting local information to global results.

Let $X \subset \mathbf{P}^3$ be a surface, given by an equation $F(X,Y,Z,W)=0$ of degree d. We are interested in certain homogeneous ideals of the graded coordinate ring $S := k[X,Y,Z,W]/(F)$. Or equivalently, we are interested in homogeneous ideals of $k[X,Y,Z,W]$ that contain F.

The open affine subset $U_w \subset \mathbf{P}^3$ is the complement of the "infinite" plane $W=0$. The functions

$$x := \frac{X}{W}, y := \frac{Y}{W}, z := \frac{Z}{W}$$

are affine coordinates for U_w. We have the *dehomogenization homomorphism*

$$d_w : k[X,Y,Z,W] \to k[x,y,z],$$
$$X \mapsto x, Y \mapsto y, Z \mapsto z, W \mapsto 1$$

forgetting the grading.

Let $I < k[X,Y,Z,W]$ be a homogeneous ideal. In general, the ideal $I_w := d_w^{-1}(d_w I k[x,y,z])$ will be bigger than I. One reason for this is geometrically clear: by the process of dehomogenization and subsequent homogenization we lose the information on the infinite plane.

Example 1: Let $I := (X,Y,W)$. This is the maximal ideal corresponding to the point $(0:0:1:0)$. This point is on the infinite plane. Indeed, $d_w^{-1}(d_w I k[x,y,z])$ is the trivial ideal (1).

There are two different workarounds for this loss of information. One approach is to shake the coordinates. When the infinite hyperplane is generic, nothing interesting is going on there, and this is fine (this is going on in [2]). We have, however, payed a lot of attention to sparse equations. Unfortunately, random rotations have a rather bad effect on sparse equations, turning them to huge expressions.

The other approach takes into account other open affine subsets. The four subsets U_w, U_z, U_y, U_x cover $\mathbf{P}^3$. More precisely, we can cover $\mathbf{P}^3$ by open sets of the following kind: U_w, a neighborhood of the plane $\frac{W}{Z} = 0$ in U_z, a neighborhood of the line $\frac{W}{Y} = \frac{Z}{Y} = 0$ in U_y, and a neighborhood of the origin $(\frac{W}{X}, \frac{Z}{X}, \frac{Y}{X}) = (0,0,0)$ in U_x.

The naive hope is that we can recover I by intersecting the ideals I_w, I_z, I_y, I_x. But this fails already for the *irrelevant ideal* $I := S^+ = (X, Y, Z, W)$. We take the viewpoint "the irrelevant ideal has no geometric meaning". This leads to the notion of saturated ideals.

The saturation of I is defined as

$$\mathrm{Sat}(I) := \{a \mid a(S^+)^m \subset I \text{ for } m \gg 0\}.$$

This is a closure operation, i.e. $I \subset \mathrm{Sat}(I)$ and $\mathrm{Sat}(\mathrm{Sat}(I)) = \mathrm{Sat}(I)$. The saturation may be computed by dehomogenization and homogenization:

$$\mathrm{Sat}(I) = I_w \cap I_z \cap I_y \cap I_x.$$

The saturated ideals are exactly those that carry only local information.

The adjoint ideals A_m are saturated ideals containing F. To compute them, it suffices to know the affine adjoint ideals.

Example 2.1: (cf. also example 1.9.) Let X be the surface given by

$$F(X, Y, Z, W) = X^2W^3 + Y^3W^2 + Z^5 = 0.$$

It has a singular line $L : z = w = 0$ and an isolated double point $(0 : 0 : 0 : 1)$. By dehomogenization with respect to X, we get the affine surface X_x with equation

$$f(y, z, w) = w^3 + y^3w^2 + z^5 = 0.$$

The singularities of X_x are analyzed below. The first adjoint ideal is

$$A_1(X_x) = (yw, z^2, zw, w^2),$$

and the homogenization is

$$(A_1)_x = (YW, Z^2, ZW, W^2).$$

Dehomogenization at Y gives the surface equation $x^2w^3 + w^2 + z^5 = 0$. The adjoint ideal is

$$A_1(X_y) = (w, z^2),$$

and its homogenization is

$$(A_1)_y = (W, Z^2).$$

X_z is a smooth surface, hence the adjoint ideal and its homogenization are both (1). X_w has equation $x^2 + y^3 + z^5 = 0$, with an isolated double point at the origin. Its adjoint ideal is again (1).
To find A_1, we compute

$$A_1 = (A_1)_x \cap (A_1)_y \cap (A_1)_z \cap (A_1)_w =$$
$$= (A_1)_x = (YW, Z^2, ZW, W^2).$$

It can be said that only the following singularities contributed something to A_1: the singular line L and the triple point $(1:0:0:0)$ on L (the origin of U_x). The singular line is covered by U_x and U_y. This explains why $A_1(X_y)$ was not trivial but did not give something new to the intersection.

We continue subdividing the open affine sets. A *cell* U of $\mathbf{A}^3$ is defined by a finite set of equations and one inequation. We are interested not only in the points in the cell (i.e. those that satisfy the equations and the inequation), but also in the points near the cell. For instance, the cell could have only a single point, and we are interested in the surface singularity at this point.

The function ring $k[U]$ is therefore by definition not the function ring of U as a quasiprojective variety, but the ring of all rational functions defined on some sufficiently small neighborhood of U. In the extreme case, when the cell has only one inner point, then $k[U]$ is the local ring at the point.

Let $X \subset \mathbf{A}^3$ be a surface, given by the equation $f(x, y, z) = 0$. Let U be a cell of X (all the points of the cell must lie in X). Then we consider the local adjoint ideals $A_m(X|_U) := A_m(X)k[U]$. These are $k[U]$-ideals, containing f and generated by polynomials. They depend only on the local behavior the cell: we will give methods for calculating these ideals for certain primitive types of cells, by analyzing the singularity in the cell.

Suppose that we have a decomposition of $\mathbf{A}^3$ into cells $U_1, \ldots, U_r$. Suppose that we know $A_m(X|_{U_i})$ for each U_i. Then we may recover $A_m(X)$ as follows. Let

$$(A_m(X))_i := A_m(X|_{U_i}) \cap k[x, y, z]$$

be the $k[x, y, z]$-ideal of the polynomials in $A_m(X|_{U_i})$. Then

$$A_m(X) = (A_m(X))_1 \cap \ldots \cap (A_m(X))_r.$$

This follows from a more general principle, saying essentially "affine ideals carry only local information". Here is the precise statement.

Theorem 4.1 (local-global principle) *Let U be an affine variety. Let $I \subset k[U]$ be an ideal of the function ring $k[U]$. Let $U_1, \ldots, U_r$ be a cell decomposition of U. For $i = 1, \ldots, r$, let*

$$I_i := Ik[U_i] \cap k[U]$$

be the restriction of the extension to the local function ring $k[U_i]$. *Then*

$$I = I_1 \cap \ldots \cap I_r.$$

This principle is in contrast to the homogeneous case. As we have seen already, homogeneous ideals cannot be fully described by local information only. However, we have an analogous principle if we restrict to saturated ideals.

As a first application of the concept of cell decomposition, we improve our method for reducing the computation of the homogeneous adjoint ideals to the affine case. We use the decomposition of $\mathbf{P}^3$ into the following 4 cells. The inequation is always the trivial inequation $1 \neq 0$.

1. The full cell U_w.
2. The cell $w = 0$ in the open affine set U_z.
3. The cell $z = w = 0$ in the open affine set U_y.
4. The cell $y = z = w = 0$ in the open affine set U_x.

Algorithm *homadjideal*:

Input:

$F(x, y, z, w)$, a homogeneous form in 4 variables. It is the equation of a surface $X \subset \mathbf{P}^3$.

m, a positive integer.

Output:

A basis for $A_m(X)$.

VARLIST := $[w, z, y, x]$;
A := (1);
for VAR in VARLIST do
  compute cell.
  EQLIST := empty list;
  VAR2 := w;
  while VAR2 $\neq$ VAR do
    append VAR2 to EQLIST;
    VAR2 := *next*(VAR2,VARLIST);
  INEQ := 1;
  dehomogenize.
  f := *substitute*(VAR=1,F);
  compute local adjoint ideal.

I := *adjideal*(f,EQLIST,INEQ);
homogenize.
B := *groebnerbase*(I,"total degree order");
B := *homogenize*(B,VAR);
intersect.
A := *intersect*(A, (B));
return (A);
exit;

The 4 cells occuring in *homadjideal* are subdivided further.

Let $f(x, y, z) = 0$ is the equation of the surface X. Since the adjoint ideals contain f, it is trivial on any cell with inequality $fg \neq 0$ (because in these cells, f is invertible).

Any cell U may be divided into $U_{in} \cup U_{out}$. Here U_{in} is obtained by adding f to the equation list, and U_{out} is obtained by multiplying the inequality by f. As we just saw, the cells U_{out} do not contribute to the intersection. Therefore, we may restrict our attention to cells inside X.

By our intuitive definition of the adjoint ideals, the adjoint ideals are trivial outside the singular locus $\mathrm{Sing}(X)$. The singular locus is the solution set of

$$f = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0.$$

It consists of isolated points and curves. The curves may themselves be singular.

We may divide U into a cell U_{in} containing only singularities, and three cells U_x, U_y, U_z, containing no singularities.

U_{in} is obtained by adding $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$, $\frac{\partial f}{\partial z}$ to the equations.

U_x is obtained by multiplying the inequation by $\frac{\partial f}{\partial x}$.

U_y is obtained by adding $\frac{\partial f}{\partial x}$ to the equations and multiplying the inequation by $\frac{\partial f}{\partial y}$.

U_z is obtained by adding $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ to the equations, and multiplying the inequation by $\frac{\partial f}{\partial z}$.

The adjoint ideals are trivial in U_x, U_y, U_z. Hence we may restrict our attention to U_{in}.

This subdivision trick is worth remembering; we put in the form of an algorithm.

Algorithm *inout*:

Input:

A cell U, given by a list E of equations and an inequation u.
A list I of equations.

Output:
A cell U_{in}, such that its inner points points are the points in U that satisfy all equations in I.

A list O of cells, such that the points in the union are the points in U that do not satisfy all equations in I.

$U_{in} := (E + I, u)$;
return (U_{in});
$O :=$ empty list;
for a in I do
    $V := (E, u{\cdot}a)$;
    append V to O;
    append a to E;
return (O);
exit;

The analysis of the singularities consists of two tasks. One task is the subdivision of $\mathrm{Sing}(X)$ into simple cells (e.g. points, smooth branches of a curve). The other task is the resolution of these singularities by blowing up.

4.2 Blowing up the singularities

Recall the intuitive definition of the adjoint ideals in section 1.2: "A_m is the ideal of all forms that have an $m(n-1)$-fold curve on each n-fold singularity of X, and an $m(n-2)$-fold point on each n-fold singular point of X. 'Each singularity' means including infinitely near singularities."

We introduce infinitely near singularities by blowing up. If one has a proper strategy, then repeating blowing up eventually leads to a smooth surface. We use this to give an algorithm for the computation of adjoint ideals.

In this section, we do not care about the irreducibility of X. The reason is that irreducibility is a global property, and we are studying local cells of a surface.

We already saw the blowing up of a smooth surface at a point (subsection 2.1.3). Now, we want to blow up singular surfaces. We do this by blowing up the ambient 3-space (figure 7). In section 2, we did not bother about algebraic realizations of the blowing up. Here, we do.

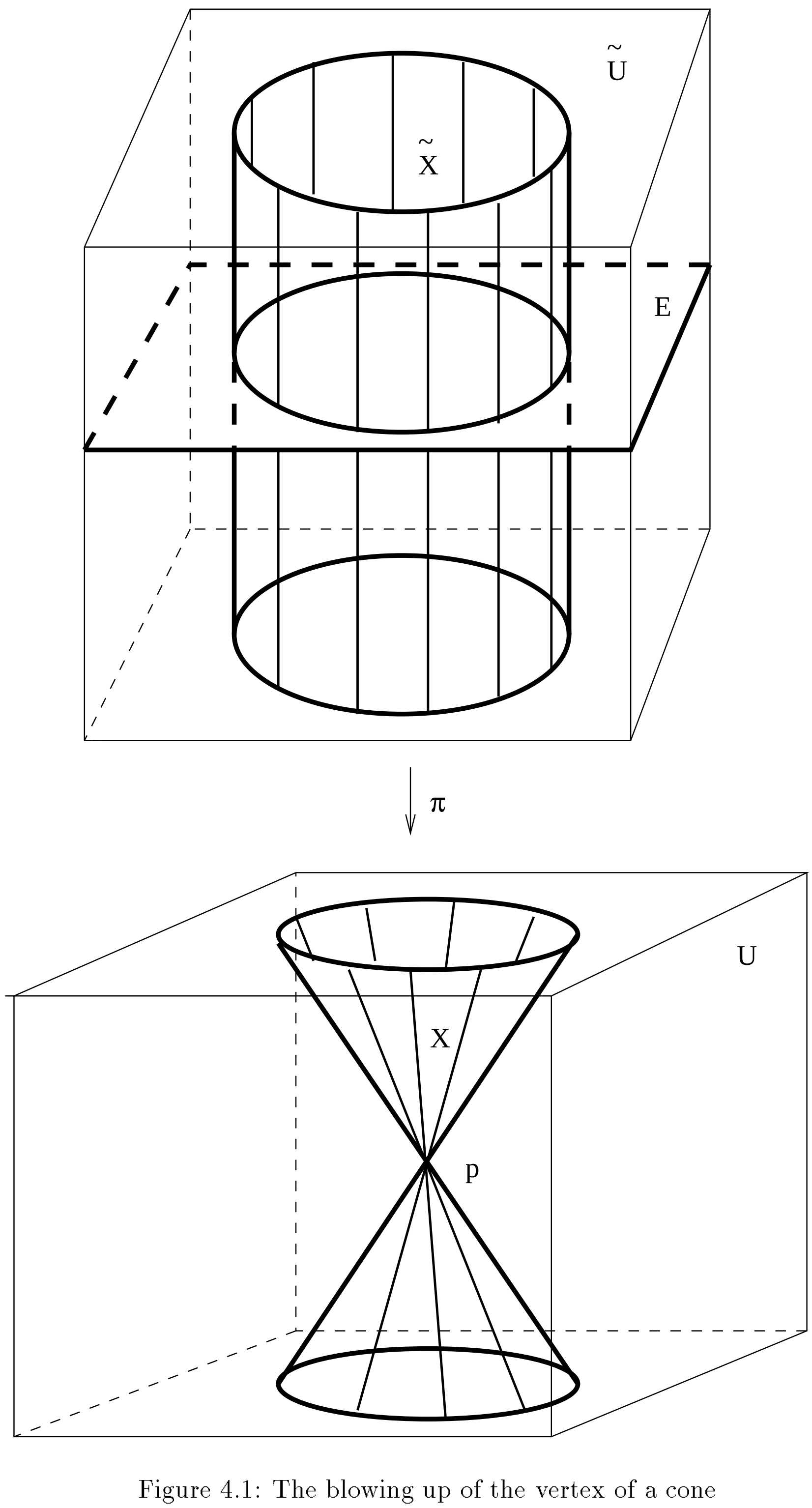

Figure 4.1: The blowing up of the vertex of a cone

It is easier to realize blowing up locally. Let $U = \mathbf{A^2}$ with coordinate ring $k[x, y]$. The blowing up of U at the origin may be covered by two affine open subsets, as follows. The first affine subset is $\tilde{U}_x$ with coordinate ring $k[x, \tilde{y}]$ (the overload of x is on purpose). The blowing down map is defined by

$$\pi_x : \tilde{U}_x \to U,$$
$$(x, \tilde{y}) \mapsto (x, y) = (x, x\tilde{y}).$$

The second affine subset is $\tilde{U}_y$ with coordinate ring $k[\tilde{x}, y]$. There, we define the blowing down map

$$\pi_y : \tilde{U}_y \to U,$$
$$(\tilde{x}, y) \mapsto (x, y) = (\tilde{x}y, y).$$

For $\tilde{y}\neq 0$, we identify the points $p_x := (x, \tilde{y}) \in \tilde{U}_x$ and $p_y := (\frac{1}{\tilde{y}}, x\tilde{y}) \in \tilde{U}_y$. It is easily checked that $\pi_x(p_x) = \pi_y(p_y)$. This identification *defines* a smooth manifold, covered by two overlapping charts; and a blowing down map π.

Here is a subdivision of $\tilde{U}$ into cells.

1. The full cell $\tilde{U}_y$.

2. The cell $\tilde{y} = 0$ in $\tilde{U}_x$.

The exceptional line is the line $y = 0$ in the first cell plus the point $(0, 0)$ in the second cell.

Let $X \in U = \mathbf{A}^3$ be a surface, given by the equation $f(x, y, z) = 0$. Suppose that the point $p = (0, 0, 0)$ is in $\mathrm{Sing}(X)$. To analyze the singular point, we blow up U at p. The algebraic realization of this blowing up is analogous to the blowing up of the plane. Here is a subdivision into cells.

1. The full cell $\tilde{U}_z$, with coordinate ring $k[\tilde{x}, \tilde{y}, z]$. The blowing down map is defined by $(\tilde{x}, \tilde{y}, z) \mapsto (\tilde{x}z, \tilde{x}y, z)$. The exceptional divisor is the cell $z = 0$.

2. The cell $\tilde{z} = 0$ in $\tilde{U}_y$. The exceptional divisor is the subcell $y = \tilde{z} = 0$.

3. The cell $\tilde{z} = \tilde{y} = 0$ in $\tilde{U}_x$. The exceptional divisor is the subcell $x = \tilde{y} = \tilde{z} = 0$ (the origin).

The divisor π^*X in $\mathrm{Div}\tilde{U}$ splits has two components. One of them is E, the exceptional divisor. Its multiplicity is equal to the multiplicity of f at p. The second component $\tilde{X}$ (proper transform) is simple. The map π restricts to a birational regular map $\pi : \tilde{X} \to X$. This is the blowing up of X at p.

Example 2.2: Let X be the surface given by

$$f(x,y,z) = x^2 + y^3 + z^5 = 0.$$

(This is the dehomogenization at w of example 2.1.) The singular locus is the single point $p = (0,0,0)$, and this is a double point.
We blow up and have the following situation in the first cell. The equation of the pullback

$$f(\tilde{x}z, \tilde{y}z, z) = (\tilde{x^2} + \tilde{y}^3 z + z^3)z^2$$

splits as expected. The proper transform with equation

$$\tilde{x}^2 + \tilde{y}^3 z + z^3 = 0$$

has a double point $p_1 : (\tilde{x}, \tilde{y}, z) = (0,0,0)$ infinitely near to p. (Note that it suffices to look for singularities in the exceptional plane $z = 0$). In the second cell ($\tilde{z} = 0$), the proper transform with equation

$$\tilde{x}^2 + y + \tilde{z}^5 y^3 = 0$$

is smooth. The third cell ($\tilde{z} = \tilde{y} = 0$) is disjoint from the proper transform

$$1 + \tilde{y}^3 x + \tilde{z}^5 x^3 = 0$$

anyway.

As the example shows, we do not necessarily obtain a desingularization by blowing up a singular point, not even if the point is the only singular point. It is also very hard to find some means by which the new singularities are less severe than the blown up point. Here is a lemma, telling at least that the new singularities are not more severe (in some sense) than the blown up point.

Lemma 4.1 *Let $X \subset U = \mathbb{A}^3$ be a surface. Let p be an n-fold point of the surface. Let $\tilde{X}$ be the blowing up at p. Then no point infinitely near to p has multiplicity $> n$.*

Proof: Let p_1 be a point infinitely near to p. We choose the coordinates as nice as possible: $p = (0,0,0)$, $p_1 = (0,0,0) \in \tilde{U_z}$. Let

$$f(x,y,z) = \sum_{i,j,k} a_{ijk} x^i y^j z^k$$

be the equation of X. Since p has multiplicity n, we have $i + j + k \geq n$ for all nonzero summands, and there is a nonzero summand where this inequality is sharp: $i_0 + j_0 + k_0 = n$. The equation of the proper transform is

$$\sum_{i,j,k} a_{ijk} \tilde{x}^i \tilde{y}^j z^{i+j+k-n},$$

and the multiplicity at p_1 is less than or equal to

$$i_0 + j_0 + (i_0 + j_0 + k_0 - n) = i_0 + j_0 \leq n.$$

The analysis is recursive: we repeat blowing up the infinitely near singularities until the proper transform is smooth.

Example 2.3 (Exercise): Blow up the double point p_1 of the surface $\tilde{X}$

$$\tilde{x}^2 + \tilde{y}^3 z + z^3 = 0$$

in example 2.2. Translate all double points in the infinitely near to the origin and repeat.
Result: one finds 6 new isolated double points, until the proper transforms is smooth.
Isolated double points do not impose conditions on the adjoint ideals. Hence, the adjoint ideals of $\tilde{X}$ and also of X in example 2 with equation

$$x^2 + y^3 + z^5 = 0$$

are all trivial.

Example 3: Let X be the Fermat surface

$$f(x, y, z) = x^n + y^n + z^n = 0.$$

It has an isolated n-fold point at the origin. The proper transform $\tilde{X} \subset \tilde{U}_z$ has equation

$$\tilde{x}^n + \tilde{y}^n + 1 = 0,$$

and there are no singularities in the infinitely near lying in $\tilde{U}_z$. By symmetry, there is no infinitely near singularity.
The adjoint ideal A_m is

$$A_m = (x^i y^j z^k \mid i + j + k = m(n-2)) + (f).$$

Equivalently, we may write

$$A_m = \{g \mid z^{m(n-2)} \text{ divides } g(\tilde{x}z, \tilde{y}z, z)\} + (f).$$

Example 4: What would have happened if $\tilde{X} \subset \tilde{U}_z$ had had singularities? The following formula is true for example 3, and extends also to surfaces that have infinitely near singularities, lying in $\tilde{U}_z$.

$$A_m = \{g \mid g(\tilde{x}z, \tilde{y}z, z) \in z^{m(n-2)} A_m(\tilde{X})\} + (f).$$

Here is what makes our intuitive definition of the adjoint ideals exact.

Theorem 4.2 *Let $X \subset U = \mathbf{A}^3$ be a surface with equation $f(x,y,z) = 0$. Suppose that $p = (0,0,0)$ is a point of multiplicity n. Let $\tilde{X}_z \subset \tilde{U}_z$ be the surface in the first cell of the blowing up of X. Let*

$$I_z := \{g \mid g(\tilde{x}z, \tilde{y}z, z) \in z^{m(n-2)} A_m(\tilde{X}_z)\}.$$

Let I_y, I_x be the analogous ideals with regard to the other cells. Then

$$A_m(X) = (I_x \cap I_y \cap I_z) + (f)$$

in the cell p.

Example 5: Isolated double points do not impose conditions on the adjoint ideals. By lemma 1, we know that the multiplicity will not increase by blowing up. Does this mean that we can always forget about isolated double points? Here is an example showing that this is not so.
Let X be the surface given by the equation

$$f(x,y,z) = x^2 + y^4 + z^4 = 0.$$

Its only singularity is an isolated double point at $p = (0,0,0)$. The proper transform $\tilde{X}_z$,

$$\tilde{f}_z = \tilde{x}^2 + \tilde{y}^4 z^2 + z^2 = 0,$$

has a double line $\tilde{x} = z = 0$. The first adjoint ideal is

$$A_1(\tilde{X}_z) = (\tilde{x}, z),$$

and we compute $I_z = (x,y,z)$.
By symmetry, $I_y = (x,y,z)$, too. The third cell does not intersect the surface. Hence theorem 2 gives

$$A_1(X) = (x,y,z).$$

Although the adjoints do not have an assigned multiplicity at p, they have to pass through p in order to fulfill the assigned multiplicity at the infinitely near double line!

Blowing up a line in $U = \mathbf{A}^3$ is rather similar to blowing up points. Let $L : x = y = 0$ be a line. Then we cover the blowing up $\tilde{U}$ by two cells.

1. The full cell $\tilde{U}_y$, with coordinate ring $k[\tilde{x}, y, z]$. The blowing down map is defined by $(\tilde{x}, y, z) \mapsto (\tilde{x}y, y, z)$. The exceptional divisor is the subcell $y = 0$.

2. The cell $\tilde{y} = 0$ in $\tilde{U}_x$. The exceptional divisor is the subcell $x = 0$.

We have the following analogue to theorem 2.

Theorem 4.3 *Let $X \subset U = \mathbf{A}^3$ be a surface with equation $f(x,y,z) = 0$. Suppose that $L : x = y = 0$ is a line of multiplicity n. Let $\tilde{X}_y \subset \tilde{U}_y$ be the surface in the first cell of the blowing up of X. Let*

$$I_y := \{g \mid g(\tilde{x}y, y, z) \in y^{m(n-1)} A_m(\tilde{X}_y)\}.$$

Let I_x be the analogue with regard to the other cell. Then

$$A_m(X) = (I_x \cap I_y) + (f)$$

in the cell L.

The theorems 2 and 3 give a recursive algorithm for the computation of adjoint ideals. We are currently only able to handle point and line singularities. This will improve in the next section.

Example 2.4: Let X be the surface X with equation

$$f(y,z,w) = w^3 + y^3 w^2 + z^5 = 0$$

from example 2.1. It has a double line $L : z = w = 0$, and an embedded triple point $(0,0,0)$ on L.

Below is a tree of infinitely near singularities, showing the repeated blowing ups. Each node is the equation of the surface in some affine cell. A line above shows the singularity which is blown up, and its multiplicity. If there is no line above, then the surface is smooth. Each arrow is labeled with the exceptional divisor of the source node.

The blowing down map belonging to an arrow can be read off in the following way: every variable which is a label of a *different* arrow with the same domain is multiplied by the label.

(As always, the equation of the proper transform is obtained as follows: Substitute the map to the old surface. Then cancel the highest power of the exceptional divisor.)

Of course, this example is lucky in the sense that we never have to translate anything to the origin.

$$
\begin{tikzcd}
 & \overset{z=w=0(2)}{w^3+y^3w^2+z^5} & & \\
w+y^3+z^5w^3 \arrow[ur, "w"] & & \overset{z=w=0(2)}{w^3z+y^3w^2+z^3} \arrow[ul, "z"'] & \\
 & \overset{(0,0,0)(3)}{zw^3+y^3w^2+z^3} \arrow[ur, "w"] & & z^2w^3+y^3w^2+z \arrow[ul, "z"'] \\
z+y^3+z^3w \arrow[ur, "w"] & \overset{(0,0,0)(2)}{w^3+y^3+zw} \arrow[u, "z"] & w^2z+1+yz^3w \arrow[ul, "y"'] & \\
w+y^3w+z \arrow[ur, "w"] & w^2+y+zw \arrow[u, "z"] & w^2+y^3z+w \arrow[ul, "y"'] &
\end{tikzcd}
$$

To compute $A_1(X)$, we start with trivial ideals in the leafs and work backward to the root of the graph. We use theorems 2 and 3: for each branch, we multiply the adjoint ideal with the exceptional divisor to the power of 0 (double point) or 1 (triple point and double line).

$$
\begin{tikzcd}
 & (yw,z^2,zw,w^2) & & \\
(1) \arrow[ur, "w"] & & (yw,z,w^2) \arrow[ul, "z"'] & \\
 & (x,y,z) \arrow[ur, "w"] & & (1) \arrow[ul, "z"'] \\
(1) \arrow[ur, "w"] & (1) \arrow[u, "z"] & (1) \arrow[ul, "y"'] & \\
(1) \arrow[ur, "w"] & (1) \arrow[u, "z"] & (1) \arrow[ul, "y"'] &
\end{tikzcd}
$$

We did not yet mention the termination problem. Do we arrive at a desingularization by repeating blowing ups?

Example 6: Let X be the surface with equation

$$f(x,y,z) = y^2 + x^2z^2 = 0.$$

It has two singular double lines meeting at a double point $p = (0,0,0)$. If we blow up p, then the surface $\tilde{X}_z$ has equation

$$\tilde{y}^2 + \tilde{x}^2z^2 = 0.$$

This is exactly the same situation, and we can blow up forever in an infinite loop.

As already indicated by lemma 1, the most important tool for the control of the resolution by blowing up is multiplicity. The idea is that multiplicity should not rise by blowing up. This restricts the number of possible blowing ups.

Example 7: Let X be the surface with equation

$$f(x,y,z) = y^3 + x^2z^4 + y^6 = 0.$$

It has a double line $L : x = y = 0$, and the point $p = (0,0,0)$ is an embedded triple point. Blowing up the line L, we get in $\tilde{U}_x$

$$\tilde{y}^3x + z^4 + x^4 = 0.$$

We got rid of the double line, but there is a new isolated 4-fold point.

As the example shows, we have to avoid blowing up of lines with embedded points of higher multiplicity, in order to keep the multiplicity non-increasing. (In example 2.4, we have just been lucky!) By the following, this is the only condition that we need to care about.

Lemma 4.2 *Let $X \subset U = \mathrm{A}^3$ be a surface. Let L be an n-fold line without points of higher multiplicity. Then no point infinitely near to L has multiplicity $> n$.*

Proof: (This is analogous to the proof of lemma 1.) Let p be a point on L, and let p_1 be a point infinitely near to L, lying over p. We choose coordinates as nice as possible: $p = (0,0,0)$, $p_1 = (0,0,0) \in \tilde{U}_y$. Let

$$f(x,y,z) = \sum_{i,j,k} a_{ijk}x^iy^jz^k$$

be the equation of X. Since L has multiplicity n, we have $i + j \geq n$ for all nonzero summands. Since p has multiplicity n, there is a nonzero summand $i_0 + j_0 + k_0 = n$. The equation of the proper transform is

$$\sum_{i,j,k} a_{ijk}\tilde{x}^iy^{i+j-n}z^k,$$

and the multiplicity at p_1 is smaller or equal

$$i_0 + (i_0 + j_0 - n) + k_0 = i_0 \leq n.$$

The next problem is that there might be curve singularities infinitely near to points (e.g. example 5). This is controlled by the following lemma.

Lemma 4.3 *Let $X \subset U = \mathbb{A}^3$ be a surface. Let p be an n-fold point. If there are n-fold curves infinitely near to p, then there is only one, and it is a line.*

Proof: We assume that $p = (0,0,0)$, and that all infinitely near n-fold curves have an intersection with $\tilde{U}_z$. The points infinitely near to p satisfy the equations

$$z = 0, g(\tilde{x}, \tilde{y}) := \tilde{f}(\tilde{x}, \tilde{y}, 0) = 0.$$

If these points have multiplicity n on $\tilde{X}$, then the partial derivatives of $\tilde{f}$ vanish. This implies that the n-fold points infinitely near are n-fold points of the bivariate equation g.

But g has total degree n. If g has more than two n-fold points, then it is a linear equation raised to the n-th power. Hence all infinitely near points lie on a line in the exceptional plane.

Here is another brick in the wall, completing the information on curve singularities in the infinitely near.

Lemma 4.4 *Let $X \subset U = \mathbb{A}^3$ be a surface. Let L be an n-fold line without points of higher multiplicity. If there are n-fold curves infinitely near to L, then there is only one, and it is smooth.*

Proof: (This is analogous to the proof of lemma 3.) We assume that L is $x = y = 0$, and that all infinitely near n-fold curves have an intersection with $\tilde{U}_z$, and that all their singular points are also in $\tilde{U}_z$. The points infinitely near to L satisfy the equations

$$y = 0, g(\tilde{x}, z) := \tilde{f}(\tilde{x}, 0, z) = 0.$$

If these points have multiplicity n on $\tilde{X}$, then the partial derivatives of $\tilde{f}$ vanish. This implies that the n-fold points infinitely near are n-fold points of the bivariate equation g.

But g has degree n with regard to $\tilde{x}$. The zero set of g cannot have vertical components $z = a$, since then one can easily show that (x, y, a) would be have multiplicity at least $n+1$. If there is an n-fold curve in the infinitely near, then its equation has the form $\tilde{x} + h(z)$, and $g = (\tilde{x} + h(z))^n$.

This curve is obviously smooth, and there is no other curve (in fact, there are no other points in the infinitely near).

In the next section, we will reduce the blowing up along a smooth curve to the blowing up along a line. In this section, we will afford the luxury of assuming that any smooth curve can be "transformed" to the line $x = y = 0$. (This is okay, if we take a local analytic viewpoint, and allow analytic coordinate transformations.)

Our recursive algorithm for the resolution of surface singularities is based on the lemmata 1 to 4. We have to avoid blowing ups of lines with embedded points of higher multiplicity. When two (or more) lines intersect in a point of the same multiplicity, we have to proceed by blowing up one of the lines and not the point, in order to avoid infinite loops.

This leads to the following strategy: as long as we still have singularities, do one of the following actions.

1. Decompose the singular locus into isolated points and curves.
2. Blow up isolated point singularities.
3. Blow up points on a curve singularity with higher multiplicity.
4. Blow up singular points on a curve singularity.
5. Blow up smooth equimultiple curve singularities.

For a convenient organization of this strategy, we use the cell concept to divide and conquer. We introduce two objects.

An *egg* is a record of:

1. A point or a curve, called the singularity of the egg.
2. A cell, such that the inner points lie in the singularity.
3. f, the equation of a surface.
4. An integer $n > 1$, which is the multiplicity of f at all points in the cell, and also of the singularity.

Valid eggs fulfill the following additional requirements:

1. If the singularity is a curve, then the cell contains only smooth points of this curve.
2. If the singularity is a point, then it is either isolated in the n-fold locus, or a singular point on an n-fold curve.

A *hen* is a record of:

1. A cell.
2. f, the equation of a surface.
3. An integer $n > 1$, which is an upper bound for the multiplicity of f at the points in the cell.

The upper bound for the multiplicity is only here because it is so easy to provide it, by lemma 1 and lemma 2.

The algorithm for the resolution of a hen does the following: It finds the singularities and subdivides into eggs. Then it resolves the eggs.

The algorithm for the resolution of an egg blows up the egg, yielding the standard decomposition of the blowing up into 2 or 3 hens. These are then resolved.

The output, a resolution, will have a recursive structure, too. Here is a definition. A resolution of a hen is:

1. A list of eggs, such that the union of their cells is the singular locus in the cell of the hen.
2. For each egg in the list, a resolution.

And a resolution of an egg A is:

1. A pair (if the singularity is a curve) or a triple (if the singularity is a point) of hens, forming a cell decomposition of the blowing up of the cell along the singularity.
2. For each hen, the blowing down map, given by three polynomials.
3. For each hen, a variable, which is the equation of the exceptional divisor.
4. For each hen, a resolution.

Here is an "algorithm" for the computation of resolutions of hens and eggs. The quote mark is there because termination is not clear.

The algorithm consists of two sub-algorithms, called *findsing* and *blowup*, calling each other recursively.

Algorithm *findsing*:

Input:
A hen $A = (U, f, n)$.

Output:
A resolution for A.

```
EGGLIST := empty list;
add derivatives of f to equations of U;
CELLS := [U];
while there exists U in CELLS do
    remove U from CELLS;
    if U contains a curve C then
        Separate the curve.
        (U,OUT) := inout(U,equations of C);
        add OUT to CELLS;
        Separate singularities.
        (V,SMOOTH) := inout(U,2x2-Jacobians of C);
        append V to CELLS;
        Separate embedded points of higher multiplicity.
        m:= multiplicity of f along C;
        while there is U in SMOOTH do
            remove U from SMOOTH;
            if m < n then
                (V,NOEMB) := inout(U,derivatives of f of order m);
                append V to CELLS;
            else
                NOEMB := [U];
            Now we have valid eggs.
            for U in NOEMB do
                EGG := (C,U,m);
                RES := blowup(EGG);
                append (EGG,RES) to EGGLIST;
    else if U contains a point p then
        Separate the point.
        (U,OUT) := inout(U,equations of C);
        add OUT to CELLS;
        n:= multiplicity of f at p;
        Find a proper singularity through p.
        I:= derivatives of F of order up to n − 1 near p;
        if there is a smooth curve C fulfilling I then
            SING := C;
        else
            SING := p;
        EGG := (SING,p,n);
        RES := blowup(EGG);
        append (EGG,RES) to EGGLIST;
    Else, U is empty.
return(EGGLIST);
```

exit;

It is clear that *findsing* subdivides into valid eggs. A weakness is that it could compute the same singular curve more than once, namely first for making eggs from it, and then as a possible singularity for point-like eggs. Probably it pays to maintain a global list of curve singularities, but this is difficult to program.

We make the following observation about curve-like eggs produced by *findsing*: There is always a 2x2-Jacobian that does not vanish in the cell. This fact will be crucial for the "transformation" to a line.

Algorithm *blowup*:

Input:
An egg $A = (Y, U, f, n)$.

Output:
A resolution for A.

HENLIST := empty list;
if Y is a point then
    transform Y to $(0,0,0)$;
    VARS := $[x, y, z]$;
else (Y is smooth piece of a curve)
    transform Y to $x = y = 0$;
    transform inequality u of U;
    Simplification of u.
    substitute($x = 0, y = 0$) in u;
    VARS := $[x, y]$;
transform f;
DONE := empty list;
for EXC in VARS do
    Compute a hen.
    V := (DONE,u);
    MAP := any variable in VARS except EXC is multiplied by EXC;
    g := substitute MAP in f;
    $g := g\cdot$ EXC^{-n};
    HEN := (V,g,n);
    RES := *findsing*(HEN);
    transform MAP back;
    append (HEN,MAP,EXC,RES) to HENLIST;
    append EXC to DONE;

We have not been too specific about the transformation process here. If the singularity is a point, then we may make a translation. If it is a line, then we may take an affine transformation. Other singularities are treated in the next section.

As it stands, the algorithm above computes only local resolutions of singularities, meaning that each blowing up is subdivided and the parts are then treated independently. Such a "local uniformization" ([41]) is in fact sufficient for our application, the computation of adjoint ideals.

However, it is not difficult to show that the local resolutions patch together in an essentially unique way. Here is an informal argument.

By the glueing lemma ([19]), it suffices to show compatibility on any three overlapping eggs. Suppose that we have a point-like egg, and two curve-like eggs near this point. Any blowing up operation on the point-like egg is compatible with isomorphisms on the other eggs, except the two blowups along the two curves. But such a blowup is compatible with the only operation on the corresponding curve-like egg, and with an isomorphism on the third egg. Hence, the local resolution defines a global blowing up, when isomorphisms are filled in properly.

Other neighboring situations can be treated analogously.

Therefore, a resolution of the hen (U, f, n) defines a birational map $\pi : \tilde{U} \to U$, such that the proper transform of f is the equation of a smooth surface in the threefold $\tilde{U}$. Furthermore, $\tilde{U}$ is obtained by blowing up points and smooth curves. This is called an *embedded resolution* of the singularities of f in U.

The termination of the algorithm is a consequence of two lemmata.

Lemma 4.5 (Resolution of space curves) *By blowing up singular points, any space curve is eventually transformed into a smooth curve.*

Proof: Cf. [27].

Lemma 4.6 *Let $n \geq 2$ be an integer. Then there is no infinite sequence of triples $(Y_i \subset X_i \subset U_i)$, such that:*

X_i is a surface, U_i is a smooth threefold;

Y_i is either an isolated n-fold point of X_i, or a smooth n-fold curve;

U_{i+1} is an open subset of the blowing up of U_i at Y_i, and X_{i+1} is the proper transform of X_i;

Y_{i+1} is infinitely near to Y_i.

Proof: Cf. [22].

Theorem 4.4 (Resolution of surface singularities) *The given algorithm findsing for the resolution of hens terminates.*

Proof: Any hen produces only finitely many eggs, where *findsing* calls *blowup*. Any egg produces only finitely many (two or three) hens, where *blowup* calls *findsing*. By lemmata 5 and 6, the recursive depth of the algorithm *findsing* is finite. This proves the theorem.

Once we have a resolution, we may compute the adjoint ideals by theorems 1 (the local-global principle), and theorems 2 and 3.

Algorithm *adjideal*:
for hens.

Input:
A hen $A = (U, f, n)$.
A resolution SINGS for A.
An integer $m \geq 1$.

Output:
The adjoint ideal $A_m(X)$, where X is the surface with equation f in U.

$I := (1)$;
for (EGG,RES) in SINGS do
  $J :=$ *adjideal*(EGG,RES,m);
  $I := I \cap J$;
return I;
exit;

Algorithm *adjideal*:
for eggs.

Input:
An egg $A = (Y, U, f, n)$.
A resolution INFNEAR for A.
An integer $m \geq 1$.

Output:
The adjoint ideal $A_m(X)$, where X is the surface with equation f in U.

$I := (1)$;
if Y is a curve then
  EXP $:= n - 1$;
else (Y is a point)
  EXP $:= n - 2$;

for (HEN,MAP,EXC,RES) in INFNEAR do
  J := *adjideal*(HEN,RES,m);
  $J := J \cdot \mathrm{EXC}^{\mathrm{EXP}}$;
  J := *inverseimage*(MAP,J);
  $I := I \cap J$;
$I := I + (f)$;
return I;
exit;

The subtask of computing the intersection of two ideals, and the subtask of computing the inverse image of an ideal under a ring homomorphism, can be done by Gröbner bases ([4]).

Here is an alternative. When we do not need to know the whole adjoint ideal, but only finitely many adjoint vector-spaces $V_{m,n}$, one might replace *adjideal* by *adjlinear*. Instead of Gröbner bases, we only need linear algebra methods.

Algorithm *adjlinear*:

Input:
A hen or an egg, with cell U and surface equation f.
A resolution.
An integer $m \geq 1$.
A finite dimensional vector-space V of regular functions in $k[U]/(f)$.

Output:
The subspace $V_m := V \cap A_m(X)$.

Computation as in *adjideal*,
just replace ideal operations by vector-space operations.

In the algorithms *deciderationality* and *parametrize*, we only have to know a finite number of adjoint vector-spaces. Therefore, it is possible to use *adjlinear* instead of *adjideal*.

4.3 Nonlinear transformations

In the previous section, we treated smooth space curves as if they were lines. This was convenient for giving an efficient blowing up algorithm. This chapter is concerned with the problem of transforming smooth curves into something like a line.

Let $C \subset U = \mathbf{A}^3$ be a space curve. Suppose that U has coordinates x, y, z. We wish to have an isomorphism $U \xrightarrow{\tau} U'$ to an $\mathbf{A}^3$ with coordinates x', y', z' (coordinate transformation), such that $\tau(C)$ is the line $x' = y' = 0$.

Any affine transformation is of the form

$$(x, y, z) \mapsto (x', y', z') = (x, y, z)A + (x_0, y_0, z_0),$$

$$(x', y', z') \mapsto (x, y, z) = (x', y', z')A^{-1} - (x_0, y_0, z_0)A^{-1},$$

where A is an invertible 3x3-matrix. Clearly, lines transform into lines. If C is not a line, then our desire cannot be met by an affine transformation.

More becomes possible with polynomial transformations.

Example 8: Let C be the "twisted cubic curve" with equations

$$x - z^3 = y - z^2 = 0.$$

The following transformation transforms C into the line $x' = y' = 0$:

$$(x, y, z) \mapsto (x', y', z) = (x - z^3, y - z^2, z),$$

$$(x', y', z) \mapsto (x, y, z) = (x' + z^3, y' + z^2, z).$$

We are still far away from being able to transform arbitrary smooth curves to lines, but example 8 shows that it is not completely hopeless. The first wish was probably too strong. Let us formulate a wish in terms of cells, which is more what we really need.

Recall that a cell is given by some equalities and one inequality; points fulfilling these are points in the cell, but we are also interested in the neighborhood of inner points. Consequently, a map $\phi : U \rightarrow V$ of two cells is represented by a map from some neighborhood of U to some neighborhood of V, mapping inner points to inner points. Two representations define the same cell map iff there is a neighborhood W of U contained in both definition domains, such that the two restrictions to W coincide.

Let $C \subset \mathbf{A}^3$ be a space curve. Let $u, v \in k[x, y, z]$ be two polynomials vanishing at C. Let U be a cell such that all inner points are in C, and the 2x2-Jacobian

$$\frac{\partial(u, v)}{\partial x, y} = \det \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}$$

does not vanish on U. Especially, the inner points are smooth points of C.

We wish to have an isomorphisms of cells $U \xrightarrow{\tau} U'$, such that $\tau(C)$ is a subset of the line $x' = y' = 0$.

Example 8 continued: Let C be the twisted cubic as in example 8. Let $u := x - z^2$, $v := y - z^2$. Let U be the cell defined by $u = v = 0, 1 \neq 0$. Then the isomorphism in example 8 gives an isomorphism of the cell U to the cell $U' : x' = y' = 0, 1 \neq 0$, mapping C into the line $x' = y' = 0$.

Unfortunately, our wish is far too strong. Such an isomorphism would give a parametrization of C, and there are many smooth space curves that do not have a parametrization.

In example 8, we could easily give a parametrization because it was possible to compute x and y in terms of z. We have to try an example where such a coincidence does not occur.

Example 9: Let C be the circle $x = y^2 + z^2 - 1 = 0$. Let $u = x$, $v = y^2 + z^2 - 1$. Let U be the cell $x = y^2 + z^2 - 1 = 0, z^2 - 1 \neq 0$. (The inequality guarantuees that the Jacobian does not vanish.)
C can be parametrized in terms of algebraic functions, by $x = 0$, $y = \sqrt{1 - z^2}$. Imitating example 8, we get the transformation

$$(x, y, z) \mapsto (x', y', z) = (x, y - \sqrt{1 - z^2}, z),$$

$$(x', y', z) \mapsto (x, y, z) = (x', y' + \sqrt{1 - z^2}, z).$$

A similar transformation can be written down for arbitrary smooth curves, whether they have a parametrization or not:

Let $C \subset U \subset \mathbf{A}^3$ be a curve. Let $u, v \in k[x, y, z]$ be two polynomials in the ideal of C, such that the Jacobian $\frac{\partial(u,v)}{\partial(x,y)}$ does not vanish in U. (Especially, C is smooth inside U.)

The algebraic functions $\alpha(z)$ and $\beta(z)$ are defined implicitly, by the functional equations

$$g(\alpha(z), \beta(z), z) = 0$$

for all g in the ideal of C (especially for u and v). It suffices to take generators for the ideal of C.

Now we can give the transformation

$$(x, y, z) \mapsto (x', y', z) = (x - \alpha(z), y - \beta(z), z),$$

$$(x', y', z) \mapsto (x, y, z) = (x' + \alpha(z), y' + \beta(z), z),$$

transforming the curve C into the line $x' = y' = 0$. Restricting to C, we have obtained a parametrization of C in terms of algebraic functions, in a trivial way.

The transformation makes it possible to proceed further with the algorithms *blowup* and *findsing*. Examples convince.

Example 10: Let X be the surface given by the equation

$$f(x,y,z)=x(y^2+z^2-1).$$

(It is the union of a circular cylinder with a plane.) The singular locus is a curve C, with equations u,v:

$$u=x, v=y^2+z^2-1.$$

C is a double curve of the surface. The Jacobian is $\frac{\partial(u,v)}{\partial(x,y)}=2y$, so we choose U as the set defined by the inequation $y\neq 0$.
The functional equations are

$$\alpha(z)=0, \beta(z)^2+z^2-1=0.$$

We substitute

$$(x,y)=(x',y'+\beta(z))$$

into f and get

$$f'(x',y',z)=x'(y'^2+2y'\beta(z)+\beta(z)^2+z^2-1)=x'y'(y'+2\beta(z)).$$

Now, C is transformed to the line $C':x'=y'=0$. As we are only interested in a neighborhood of C', we simplify the inequality $y'+\beta(z)\neq 0$ to $\beta(z)\neq 0$.
We blow up the line $x'=y'=0$. For computing the cell in $\tilde{U}_{y'}$, we substitute $x'=\tilde{x}y'$ in f' and get the total transform

$$\tilde{x}y'^2(y'+2\beta(z)).$$

It is divisible by y'^2. The proper transform is

$$\tilde{f}(\tilde{x},y',z)=\tilde{x}(y'+2\beta(z)).$$

To compute the singularities in the infinitely near, we compute the derivatives:

$$\frac{\partial\tilde{f}}{\partial\tilde{x}}=y'+2\beta(z),$$

$$\frac{\partial\tilde{f}}{\partial y'}=\tilde{x},$$

$$\frac{\partial\tilde{f}}{\partial z}=2\tilde{x}\beta'(z)=-\frac{-2\tilde{x}z}{\beta(z)}.$$

There are no singular points in the exceptional plane $y'=0$.
Analogously, one computes that the the proper transform is also smooth in the second cell. The resolution algorithm stops here.

In the *blowup* step, the only difficulty is the division by the exceptional divisor to the n-th power. The solution is to reduce the surface equation by the functional equations.

Let $I < k[U]$ be the ideal of C. Then

$$J := \{g(\alpha, \beta, z) \mid g(x, y, z) \in I\}$$

is the ideal of all relations. Obviously,

$$I < J + (x - \alpha, y - \beta) = J + (x', y').$$

Since f has multiplicity n along C, we have $f \in I^n$ and consequently

$$f \in (J + (x', y'))^n \subset J + (x', y')^n.$$

After the substitution $x' = \tilde{x}y'$, the surface equation is divisible by y'^n.

In the *findsing* step, the only problem is to compute derivatives of equations that contain α and β, and maybe also algebraic functions introduced at some earlier step. The solution is to apply the chain rule and the formula for the derivatives of implicit functions. In our case, we have to compute $\alpha'(z)$ and $\beta'(z)$ from the linear equation system

$$\alpha'(z)\frac{\partial u}{\partial x}(\alpha(z), \beta(z), z) + \beta'(z)\frac{\partial u}{\partial y}(\alpha(z), \beta(z), z) + \frac{\partial u}{\partial z} = 0,$$

$$\alpha'(z)\frac{\partial v}{\partial x}(\alpha(z), \beta(z), z) + \beta'(z)\frac{\partial v}{\partial y}(\alpha(z), \beta(z), z) + \frac{\partial v}{\partial z} = 0.$$

The inequality of U' implies that the determinant does not vanish.

It is about time to mention a theoretical problem: Algebraic functions are not functions in the literal sense, they are many-valued. One could try to split them into several uni-valued functions. But this leads necessarily into a deeper topological discussion, which I feel does not belong here.

The problem is that we have to split into differentiable, especially continuous functions. This is impossible in the non-Hausdorff Zariski topology, because here any two open sets overlap. In the usual metric topology on complex 3-space, we would have to further subdivide into simply connected subcells.

A more adequate answer to this theoretical problem is to interpret the algebraic functions using local analytic isomorphisms.

Let $U \subset \mathbf{A}^3$. Let $C \subset U$ be a smooth curve with tangents not in the xy-plane. Let $\phi : C \to L$ be the restriction of the projection π_z on the z-axis. For any point $p \in C$ with tangent not in the xy-plane, ϕ is an analytic isomorphism of some analytic neighborhood of p onto some analytic neighborhood of $\phi(p)$.

This statement should be made precise without explicit reference to some specific topology. We do this by saying that ϕ induces an isomorphism of the completions of the local ring at $\phi(p)$ to the local ring at p.

Our nonlinear transformation turns out to be a local analytic isomorphism Φ from some threefold V to U together with a curve $D \subset V$ mapped isomorphically to C, such that $V = \mathbf{A}^2 \times D$. We get V as the pullback of the diagram

$$\begin{tikzcd} & V_{(x,y,z,\alpha,\beta)} \arrow[dl, "\pi"'] \arrow[dr, "\Phi"] & \\ C_{(z,\alpha,\beta)} \arrow[dr, "\phi"'] & & U_{(x,y,z)} \arrow[dl, "\pi_z"] \\ & L_{(z)} & \end{tikzcd}.$$

As the pullback of a local analytic isomorphism, Φ is also a local analytic isomorphism.

Since $U = L \times \mathbf{A}^2$, we have $V \cong C \times \mathbf{A}^2$. The projection $\pi : W \to C$ has a cross section $\sigma : C \hookrightarrow V$,

$$(\alpha, \beta, z) \mapsto (\alpha, \beta, z, \alpha, \beta),$$

such that $\Phi \circ \sigma$ is the embedding $C \hookrightarrow U$. We let $D := \sigma(C)$. Then Φ maps D isomorphically to C, which is what we claimed.

After the nonlinear transformation, V has coordinates $(x', y', z, \alpha, \beta)$, and D is the origin times C. Thanks to this situation, D can be blown up as if it were a line.

After having settled the theoretical difficulty, let us return to the naive viewpoint again: we think of $\alpha(z)$, $\beta(z)$ as of some algebraic functions that must be introduced to translate C to the origin. We still have to solve a more practical problem: How do we get rid of the algebraic functions in the adjoint ideals?

The answer is similar to the answer on the question: If I have to introduce algebraic numbers in order to translate a point or a line to the origin, how do I get rid of them in the adjoint ideal? They disappear in the "transforming back" step.

The following two examples show the analogy.

Example 11: Suppose that we are initially working over the ground field $\mathbf{Q}$. Let X be the surface with equation

$$f(x, y) = y(x^2 + 1) = 0.$$

The singular locus is the solution set of $x = y^2 + 1 = 0$ It consists of two double lines. We introduce the algebraic number α with equation $\alpha^2 + 1 = 0$. Then we make the translation

$$(x', y', z) = (x - \alpha, y),$$
$$(x, y, z) = (x' + \alpha, y').$$

The transformed equation is

$$f'(x', y') = y'(x'^2 + 2\alpha x' + \alpha^2 + 1) = x'y'(x' + 2\alpha).$$

One of the two lines has been transformed to the z-axis.
By blowing up, one finds $A_1(X') = (x', y')$ in the cell $x' = y' = 0$. In the old coordinates, we have $A_1 = (x-\alpha, y)$. Since we know that $A_1(X)$ can be generated by polynomials in $\mathbf{Q}[x, y, z]$, we compute

$$A_1(X) = (x - \alpha, y) \cap \mathbf{Q}[x, y, z] = (x^2 + 1, y).$$

As an extra present, the two polynomials $x^2 + 1$, y also generate the adjoint ideal in the cell of the second double line, because the computation is the same.

Example 10 continued: We already resolved the surface

$$f(x, y, z) = x(y^2 + z^2 - 1)$$

in the cell $y \neq 0$. The transformation of the double curve $C : x = y^2 + z^2 - 1 = 0$ to the z-axis is

$$(x, y, z) = (x', y' + \beta(z), z),$$
$$(x', y', z) = (x, y - \beta(z), z),$$

where $\beta(z)$ was the algebraic function defined by

$$\beta(z)^2 + z^2 - 1 = 0.$$

The transformed equation is

$$f'(x', y', z) = x'(y'^2 + 2y'\beta(z) + \beta(z)^2 + z^2 - 1) = x'y'(y' + 2\beta(z)).$$

There are no singularities in the infinitely near. Hence, the adjoint ideal A_1 is (x', y') in the cell $x' = y' = 0$, $\beta(z) \neq 0$.
We transform back and intersect with $k[x, y, z]$ (knowing that A_1 can be generated by polynomials):

$$A_1(X) = (x, y - \beta(z)) \cap k[x, y, z] = (x, y^2 + z^2 - 1).$$

To make sure that the method always works, we have to state and prove a statement in the theoretically correct context.

Lemma 4.7 *Let U be a smooth 3-fold. Let $C \subset U$ be a curve. Let $\Phi : V \to U$ be a local analytic isomorphism. Let $D \subset V$ be a curve that is mapped isomorphically onto C. Let A be the cell of C in U. Let B be the cell of D in V. Let X be a surface in U.*

Then the adjoint ideals $I := A_m(X|_A)$ and $J := A_m(\Phi^* X|_B)$ are related in the following way:

$$I = J \cap k[A], J = Ik[B],$$

where the operations intersection and multiplication are defined via the ring inclusion $\Phi^* : k[A] \hookrightarrow k[B]$. ($\Phi^*$ should be called "the introduction of algebraic functions".)

Proof: Let $q \in D$, $p := \Phi(p) \in C$. Φ^* induces an inclusion of local rings $k[U]_p \hookrightarrow k[V]_q$. Let A_p and A_q the local adjoint ideals at p and q, respectively. By the local-global principle, it suffices to show

$$A_p = A_q \cap k[U]_p, A_q = A_p k[V]_q.$$

Let R be the completion of A_p and A_q (since Φ is a local analytic isomorphism, the two completions are the same. It makes perfect sense to compute adjoint ideals on power series rings, because blowup operations and multiplicity are analytic concepts. The completions of the two adjoint ideals are the same and coincide with the analytic adjoint ideal I. The rest of the proof is quite easy.

$A_p = A_q k[U]_p$: For any $f \in k[U]_p$,

$$f \in A_p \leftrightarrow f \in I \leftrightarrow f \in A_q$$

because ideal membership statements (positive and negative) are preserved by completion.

$A_q = A_p k[V]_q$: The ideals on both side have the same completion

$$A_q R = A_p k[V]_q R = I,$$

hence they coincide.

Of course we need only the first equation of the lemma. The second is here because it gives some insight in the relation between the two adjoint ideals.

4.4 Adjoint sheaves

To get rid of the inconvenient distinction between local and global objects, we introduce adjoint ideal sheaves and adjoint sheaves. Then we define the adjoint divisor and prove the correctness of the algorithms *adjideal* and *homadjideal*, as well as the fundamental theorem 2.7.

The simplest way to verify all the statements previously made is to reintroduce adjoint sheaves in an axiomatic way.

For any surface X, we define the adjoint sheaf $\mathcal{B}(X)$ to satisfy the following conditions.

Functoriality: If $f : X \to Y$ is a proper birational regular map, then

$$f_* \mathcal{B}_m(X) \cong \mathcal{B}_m(Y).$$

Pluricanonicity: If X is smooth, then

$$\mathcal{B}_m(X) \cong \omega^{\otimes m}(X).$$

The two axioms determine the adjoint sheaves up to isomorphism: any surface X has a desingularization $\pi : \tilde{X} \to X$. The two axioms allow only

$$\mathcal{B}_m(X) \cong \pi_* \omega^{\otimes m}(\tilde{X}).$$

To show that the axioms are consistent, it suffices to show the following.

Let $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$ be two desingularizations of Y. Then

$$(f_1)_* \omega^{\otimes m}(X_1) \cong (f_2)_* \omega^{\otimes m}(X_2).$$

There exists a common desingularization X_3 of f_1 and f_2:

$$\begin{tikzcd} & X_3 \arrow[dl, "g_1"'] \arrow[dr, "g_2"] & \\ X_1 \arrow[dr, "f_1"'] & & X_2 \arrow[dl, "f_2"] \\ & Y & \end{tikzcd}.$$

Thus, it suffices to show

$$f_* \omega^{\otimes m}(X) \cong \omega^{\otimes m}(Y)$$

for any proper birational regular map between smooth surfaces. This is a known theorem ([23]).

As a consequence of the definition, the sections of the sheafs $\mathcal{B}_m(X)$ restrict to rational differential forms of symmetric degree m, i.e. elements of the one-dimensional $k(X)$-vector-space $\omega(k(X))^{\otimes m}$. The reason is that this is so for smooth surfaces, and a birational map restricts to an isomorphism at the generic point.

Moreover, direct images of torsion-free sheafs along surjective maps are torsion-free. Hence, all the sheafs $\mathcal{B}_m(X)$ are torsion-free, and the restriction to the generic point is injective. This allows us to say that the $\mathcal{B}_m(X)$ is a sheaf of rational differential forms of symmetric degree m.

Suppose that $X \subset \mathbf{A}^3$ is a surface in space, defined by an equation $F(x, y, z) = 0$. Then we have a distinguished rational differential form

$$w := \frac{dx \wedge dy}{\frac{\partial F}{\partial z}} = \frac{dy \wedge dz}{\frac{\partial F}{\partial x}} = \frac{dz \wedge dx}{\frac{\partial F}{\partial y}},$$

generating the $k(X)$-vector-space of rational differential forms of symmetric degree m. By tensoring the adjoint sheaf with the dual of the invertible sheaf generated by the rational differential form w^m of symmetric degree m, we get a sheaf $\mathcal{A}_m(X)$ of rational functions. Below, we will show that the sections of this sheaf are regular, in other words that $\mathcal{A}_m(X)$ is an ideal sheaf. We will call it the sheaf of adjoint ideals.

We have to define these concepts a little bit more generally. Let $i : X \hookrightarrow U$ be a surface in a smooth 3-fold. We define the sheaves $\mathcal{D}_m(X)$ as the invertible sheaves on X locally generated by $dx \wedge dy / \frac{\partial F}{\partial z}$, where x, y, z are local equations for U and F is a local equation of X.

The sheaf $\mathcal{D}_1(X)$ is better known as the "dualizing sheaf", but we do not want to use this terminology because we will not use duality.

The invertible sheaf $\mathcal{D}_m(X)$ is the restriction of some invertible sheaf on U, which we denote by $\mathcal{D}_m(X, U)$:

$$\mathcal{D}_m(X) \cong i^* \mathcal{D}_m(X, U),$$

$$\mathcal{D}_m(X, U) := \omega^{\otimes m} \otimes \mathcal{L}(mX).$$

This is well-known for the dualizing sheaf ([19]), and the general case follows easily from the case $m = 1$.

We define the sheafs

$$\mathcal{A}_m(X) := \mathcal{B}_m(X) \otimes \mathcal{D}_m(X)^{\vee}.$$

A priori, these are sheafs of rational functions. But we will show that they are sheafs of ideals, and call $\mathcal{A}_m(X)$ the sheaf of m-th adjoint ideals. For smooth surfaces, we obviously have $\mathcal{A}_m(X) = \mathcal{O}_X$.

To push the sheafs $\mathcal{A}_m(X)$ forward along a proper birational regular map $f : X \to Y$, we introduce the correction sheafs

$$\mathcal{C}_m(f) := \mathcal{D}_m(X) \otimes f^* \mathcal{D}_m(Y)^\vee .$$

Then we have the rule

$$\mathcal{A}_m(Y) = f_*(\mathcal{A}_m(X) \otimes \mathcal{C}_m(f))$$

by the projection formula ([19]).

Let

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle j} \\ U & \xrightarrow[a]{} & V \end{array}$$

be a commuting diagram, where a is a proper regular birational morphism between smooth 3-folds, Y is a surface in V, and X is the proper transform. (In the following, we will only consider blowing ups along points and smooth curves and their composites.)

Then the correction sheafs are restrictions of the the invertible sheafs $\mathcal{C}_m(a)$, defined by

$$\mathcal{C}_m(A) := \mathcal{D}_m(X, U) \otimes a^* \mathcal{D}_m(Y, V)^\vee \cong$$

$$\cong (\omega(U) \otimes a^* \omega(V)^\vee)^{\otimes m} \otimes \mathcal{L}(X - a^* Y).$$

We compute the correction sheaf explicitly for blowing ups.

Let $p \in V$ be an n-fold point of Y. Let $a : U \to V$ be the blowing up at p. Let $E \in \mathrm{Div} U$ be the exceptional divisor. Then

$$\omega(U) \otimes a^* \omega(V)^\vee \cong \mathcal{L}(2E),$$

$$X - a^* Y = -nE,$$

hence

$$\mathcal{C}_m(A) \cong \mathcal{L}(m(2-n)E).$$

Let $C \in V$ be a smooth n-fold line of Y. Let $a : U \to V$ be the blowing up at C. Let $E \in \mathrm{Div} U$ be the exceptional divisor. Then

$$\omega(U) \otimes a^* \omega(V)^\vee \cong \mathcal{L}(E),$$

$$X - a^* Y = -nE,$$

hence

$$\mathcal{C}_m(A) \cong \mathcal{L}(m(1-n)E).$$

At this point, we could design a method for the calculation of the sheafs $\mathcal{A}_m$ of a singular surface in 3-space, using a resolution of singularities: We know how to compute the sheafs on smooth surfaces, and we can compute $\mathcal{A}_m(Y)$ if we know $\mathcal{A}_m(X)$ on the blowing up.

However, we have assigned ourselves the task of proving theorem 2 and theorem 3, and thereby proving that the more easy algorithm *adjideal* is correct. Not to mention, we still have to prove that the functions in $\mathcal{A}_m(X)$ are regular. To achieve all this, we use cohomological techniques.

Suppose that $a : U \to V$ is either a blowing up of a point of multiplicity $n \geq 2$ on Y or a blowing up of a smooth curve C of multiplicity $n \geq 1$ on Y. Assuming that $\mathcal{A}_m(X)$ is an ideal sheaf, we will prove that $\mathcal{A}_m(Y)$ is an ideal sheaf. This shows that the sheaf $\mathcal{A}_m(X)$ is always an ideal sheaf, because any surface can be desingularized by blowing ups of points with multiplicity ≥ 2 and smooth curves.

We represent the sheaf $\mathcal{A}_m(X) \otimes \mathcal{C}_m(f)$ as a quotient sheaf of two ideal sheafs on U. The advantage is that pushforwards of ideal sheafs on U are ideal sheafs on V, because a_* is left exact and $a_*\mathcal{O}_U = \mathcal{O}_V$.

Let $\mathcal{A}(U, X)$ be the ideal sheaf of all functions on U restricting to functions in $\mathcal{A}_m(X)$. The sheaf $\mathcal{C}_m(a)$ is an invertible ideal sheaf. The sheaf

$$\mathcal{A}_m(U, X) \otimes \mathcal{C}_m(a) = \mathcal{A}(U, X) \cdot \mathcal{C}_m(a)$$

is therefore again an ideal sheaf. Here, the multiplication of ideal sheaves is defined locally as the multiplication of ideals.

Any function in $\mathcal{A}_m(X) \otimes \mathcal{C}_m(f)$ is the restriction of a function in $\mathcal{A}_m(U, X) \cdot \mathcal{C}_m(a)$ defined in some suitable neighborhood. A function restricts to zero iff it is in the subsheaf $(\mathcal{A}_m(U, X) \cdot \mathcal{C}_m(a)) \cap \mathcal{L}(-X)$, where the operation $\cap$ has again to be understood locally as an intersection of ideals.

We claim:

$$(\mathcal{A}_m(U, X) \cdot \mathcal{C}_m(a)) \cap \mathcal{L}(-X) = \mathcal{C}_m(a) \cdot \mathcal{L}(-X) =$$

$$= \begin{cases} \mathcal{L}(-X + m(2-n)E) & \text{for blowing up at a point,} \\ \mathcal{L}(-X + m(1-n)E) & \text{for blowing up at a curve .} \end{cases}$$

Clearly, the left hand side is a subsheaf of the right hand side. Vice versa, $\mathcal{A}_m(X, U)$ contains $\mathcal{L}(-X)$, hence the left hand side contains

$$(\mathcal{L}(-X) \cdot \mathcal{C}_m(a)) \cap \mathcal{L}(-X) = \mathcal{L}(-X) \cdot \mathcal{C}_m(a).$$

Suppose for a moment that U is the blowing up at a point. Then, we have an exact sequence

$$0 \to \mathcal{L}(-X + m(2-n)E) \to \mathcal{A}_m(U,X)\cdot\mathcal{C}_m(a) \to \mathcal{A}_m(X)\cdot\mathcal{C}_m(f) \to 0$$

of sheafs on U, where the quotient sheaf may be regarded as a sheaf on X because it is 0 outside X (this abuse of language is quite common, cf. [19]). As a sheaf on X, this is $\mathcal{A}_m(X)\otimes\mathcal{C}_m(f)$, since the second factor is an invertible ideal sheaf.

If U is the blowing up at a line, then the number 2 above has to be replaced by 1.

We have to show that the sequence remains exact after pushing forward:

$$0 \to \mathcal{K} \to a_*(\mathcal{A}_m(U,X)\cdot\mathcal{C}_m(a)) \to \mathcal{A}_m(Y) \to 0$$

is an exact sequence of sheaves on U. Here, $\mathcal{K}$ is a sheaf of ideals on U which is not of further interest (it might be described as the sheaf of all functions having some multiplicity at p or C and vanishing on Y). The sheaf in the middle is the ideal of all functions with pullback in $\mathcal{A}_m(X,U)\cdot\mathcal{C}_m(a)$. Especially, this implies that the functions in $\mathcal{A}_m(Y)$ are regular, and that theorems 2 and 3 hold: the algorithm *adjideal* is correct.

Since the functor a_* is not right exact, we study the long exact sequence of "cohomology sheafs". It suffices to show that the sheafs $R^1a_*\mathcal{L}(-X + m(2-n)E)$ (for blowing up a point) and $R^1a_*\mathcal{L}(-X + m(1-n)E)$ (for blowing up a curve) vanish.

For any divisors Y and D, we have the projection formula

$$R^1a_*\mathcal{L}(a^*Y + D)\cong\mathcal{L}(Y)\otimes R^1a_*\mathcal{L}(D).$$

Because we may write $-X$ as $-a^*Y + nE$, we only have to calculate the sheafs

$$R^1a_*\mathcal{L}((n + m(2-n))E) \text{ for points,}$$
$$R^1a_*\mathcal{L}((n + m(1-n))E) \text{ for curves.}$$

Here is the calculation that shows that these sheafs are zero.

Lemma 4.8 *a) Let $a : U \to V$ be the blowing up of a smooth 3-fold V at a point. Then*

$$R^1a_*\mathcal{L}(rE) = 0$$

for arbitrary integers r.

b) Let $a : U \to V$ be the blowing up of a smooth 3-fold V at a smooth curve. Then

$$R^1a_*\mathcal{L}(rE) = 0$$

for $r\leq 1$.

Proof: The statement is local in V, and invariant under nonlinear transformations. Therefore, we may assume that $V = \mathbf{A}^3$ and the blowup center is the origin or the line $x = y = 0$.

The sheaf in question is the coherent sheaf corresponding to the $k[x, y, z]$-module $H^1(U, \mathcal{L}(rE))$. We will use Cech cohomology to compute this module. This is rather technical, but I have no better idea.

(a): We cover $\tilde{U}$ by the affine subsets

$$\tilde{U} = \tilde{U}_x \cup \tilde{U}_y \cup \tilde{U}_z.$$

We write down a k-base for the sections on the subsets in terms of monomials with integer exponents:

$$\Gamma(\tilde{U}_x, \mathcal{L}(rE)) = \langle x^a y^b z^c \mid a + b + c + r \geq 0, b \geq 0, c \geq 0 \rangle = M_x,$$

$$\Gamma(\tilde{U}_y, \mathcal{L}(rE)) = M_y,$$

$$\Gamma(\tilde{U}_z, \mathcal{L}(rE)) = M_z,$$

$$\Gamma(\tilde{U}_x \cap \tilde{U}_y, \mathcal{L}(rE)) = \langle x^a y^b z^c \mid a + b + c + r \geq 0, c \geq 0 \rangle = M_{xy},$$

$$\Gamma(\tilde{U}_x \cap \tilde{U}_z, \mathcal{L}(rE)) = M_{xz},$$

$$\Gamma(\tilde{U}_y \cap \tilde{U}_z, \mathcal{L}(rE)) = M_{yz},$$

$$\Gamma(\tilde{U}_x \cap \tilde{U}_y \cap \tilde{U}_z, \mathcal{L}(rE)) = \langle x^a y^b z^c \mid a + b + c + r \geq 0 \rangle = M_{xyz}.$$

The Cech complex is

$$M_x \times M_y \times M_z \xrightarrow{d_0} M_{xy} \times M_{xz} \times M_{yz} \xrightarrow{d_1} M_{xyz} \to 0.$$

The maps are defined by

$$d_0(f_1, f_2, f_3) = (f_1 - f_2, f_1 - f_3, f_2 - f_3),$$

$$d_1(f_1, f_2, f_3) = f_1 - f_2 + f_3.$$

Let $f = (f_1, f_2, f_3)$ be a closed 1-cochain. Let (c_1, c_2, c_3) be a triple of coefficients of $x^a y^b z^c$ in (f_1, f_2, f_3), which is different from $(0, 0, 0)$. Since $d_1(f) = 0$, we have $c_1 - c_2 + c_3 = 0$. Hence, at least two of the three numbers c_1, c_2, c_3 does not vanish. We discuss the case $c_1 = c_2 \neq 0, c_3 = 0$. In this case, it follows

$$a + b + c + r \geq 0, c \geq 0, b \geq 0,$$

$$(c_1, c_2, c_3) x^a y^b z^c = d_0((c_1, 0, 0) x^a y^b z^c).$$

The case where c_1 or c_2 vanishes is analogous. If none of the c_i vanishes, then

$$a+b+c+r \geq 0, a \geq 0, b \geq 0, c \geq 0,$$
$$(c_1, c_2, c_3)x^a y^b z^c = d_0((0, -c_1, -c_2)x^a y^b z^c).$$

Hence f is exact.

(b): We cover $\tilde{U}$ by the affine subsets

$$\tilde{U} = \tilde{U}_x \cup \tilde{U}_y.$$

Similar to above, we write

$$\Gamma(\tilde{U}_x, \mathcal{L}(rE)) = \langle x^a y^b z^c \mid a+b+r \geq 0, b \geq 0 \rangle = M_x,$$
$$\Gamma(\tilde{U}_y, \mathcal{L}(rE)) = M_y,$$
$$\Gamma(\tilde{U}_x \cap \tilde{U}_y, \mathcal{L}(rE)) = \langle x^a y^b z^c \mid a+b+r \geq 0 \rangle = M_{xy}.$$

The Cech complex is

$$M_x \times M_y \xrightarrow{d_0} M_{xy} \to 0,$$

where $d_0(f_1, f_2) = f_1 - f_2$.

Let f be a 1-cochain (all 1-cochains are closed). Let $cx^a y^b z^c$ be a monomial occuring in f. Since $a+b+1 \geq a+b+r \geq 0$, we have either $a \geq 0$ or $b \geq 0$. In both cases, the monomial is exact. Hence, f is exact.

Remark: The definitions of $\mathcal{B}_m(X)$ and $\mathcal{A}_m(X)$ make sense also for $m=0$. One finds that $\mathcal{A}_0(X) = \mathcal{B}_0(X)$ is the sheaf of integral functions ([23]). This is not an ideal sheaf, and cannot be computed by *adjideal*. The above calculation gives an explanation why.

Let us return to the projective case. Let $Y \in \mathbf{P}^3$ be a surface given by an equation of degree d.

The dualizing sheaf is the twisted sheaf $\mathcal{O}_Y(d-4)$ ([19]). Hence the adjoint sheaf is the adjoint ideal sheaf with a degree shift of $m(d-4)$. The adjoint vector-space is the vector-space

$$V_{n,m} = \Gamma(Y, \mathcal{A}_m(X)(n + m(d-4))) = \Gamma(Y, \mathcal{B}_m(X)(n)).$$

This follows from the following general fact.

Lemma 4.9 *Let $Y \subset \mathbf{P}^3$ be a surface with equation $F(x,y,z,w) = 0$. Let $\mathcal{A}$ be a sheaf of ideals on X. Let $A < k[x,y,z,w]$ be the ideal of homogeneous forms restricting to global sections of $\mathcal{A}$. Let n be an integer.*

Then $\Gamma(Y, \mathcal{A})$ is the vector-space of all homogeneous forms in A of degree n, modulo multiples of F.

Proof: Clearly, any form modulo F is a global section. The nontrivial part is the converse.

The homogeneous ideal A is obviously saturated. Let $\mathcal{B}$ be the sheaf on $\mathbf{P}^3$ associated to A. We have to show that the surjective map of sheafs in the exact sequence

$$0 \to \mathcal{L}(-Y)(n) \to \mathcal{B}(n) \to \mathcal{A}(n) \to 0$$

induces a surjective map on global sections.

Taking cohomology, we see that it suffices to show that

$$H^1(\mathbf{P}^3, \mathcal{L}(-Y)(n)) = H^1(\mathbf{P}^3, \mathcal{O}(-d+n)) = 0.$$

But this is well-known ([19]).

Let $\pi : X \to Y$ be a desingularization of the surface $Y \subset \mathbf{P}^3$. Let $D \in \mathrm{Div}(X)$ be a divisor in the linear system whose associated map is π. Then we have

$$\pi^* \mathcal{O}_Y(n) = \mathcal{L}(nD).$$

Let K be a canonical divisor on X. By the projection formula again, we have

$$\pi_* \mathcal{L}(nD + mK) = \mathcal{O}_Y(n) \otimes \pi_* \mathcal{L}(mK) \cong \mathcal{B}_m(Y)(n).$$

Taking global sections, we see that $V_{n,m}$ is isomorphic to $\Gamma(X, \mathcal{L}(nD + mK))$. The isomorphism induces an isomorphism of the restrictions to the generic point, which means that the two vector-spaces have the same associated map. This was stated and used before (corollary 2.8).

For a proof of the fundamental theorem 2.7, knowing that there is an isomorphism is not enough. Let $\Delta \in \mathrm{Div}(X)$ be the divisor defined by the equation

$$\mathcal{C}_1(\pi) = \mathcal{L}(\Delta).$$

Then the following are true.

1. $(d-4)D - \Delta$ is a canonical divisor on X.
2. A nonzero form G modulo F is in A_m iff $(\pi^* G) \geq m\Delta$.
3. The linear system

$$\{(\pi^* G) - m\Delta \mid G \in V_{n,m}\}$$

is equal to the complete linear system $|nD + mK|$.

Proof: 1: We show that the associated invertible sheaf is isomorphic to the canonical sheaf:

$$\mathcal{L}((d-4)D-\Delta) \cong \mathcal{D}_1(X) \cong \omega(X).$$

2: Rewriting the statement in terms of sheafs, we get

$$\pi_* \mathcal{L}(m\Delta) = \mathcal{A}_m(Y).$$

This follows from $\mathcal{A}_m(X) = \mathcal{O}_X$ and $\mathcal{C}_m(\pi) = \mathcal{L}(m\Delta)$.

3: Because of

$$(\pi^* G) \sim (n + m(d-4))D$$

and (1), the given linear system is a subsystem of $|nD + mK|$. But we gave an isomorphism of defining vector-spaces $V_{n,m}$ and $\mathcal{L}(nD+mK)$, hence the subsystem is all of the complete system.

(1) and (3) are the fundamental theorem, and (2) is a description of the adjoint ideals in geometric terms.

4.5 Computational aspects

We discuss computational aspects of the algorithms in this chapter. Furthermore, we mention other applications of them beyond parametrization.

The algorithms described in this chapter are much more difficult than the algorithms described in the other chapters. It seems almost impossible to come up with a complexity analysis. Recall that the termination proof in [22] is so cumbersome that it was not reproduced here.

If we do not focus on the worst-case complexity, but on the average-case complexity (whatever that means), then the situation is much better. Really bad singularities are rare, and the generic case is always much better than the worst case. (This statement can be made precise, in more than one sense.)

On the other hand, we are rather often engaged with non-generic examples. For humans, it is sometimes difficult to come up with a generic example of an algebraic specification. The reason is that one tends to invent sparse equations with low coefficients, thereby increasing the probability that something degenerate is happening.

In an algorithm for the analysis of singularities - which is intrinsically something degenerate - it is not a good idea to take the generic case as a measure. On the other hand, taking the worst case as a measure is frustrating, because it is hard to tell something positive here.

So, what I did was to design an algorithm for examples that came to my mind: with sparse equations, low coefficients, and lots of degenerate coincidences that make it interesting.

To compute with these algorithms, I hardly needed a computer, since I can manipulate sparse equations on a medium-sized piece of paper. The fact that this lead to results proves that the algorithm is of some practical use. Of course, I admit that it is far from being satisfactory.

One consequence of this design is that the algorithm avoids all kinds of perturbation techniques. Many algorithms in algebraic geometry use such a technique to get rid of degenerate coincidences (for some instances, see [2]). The most common one is "shaking the coordinates": a random linear transformation of the variables. One gains generic position with respect to the coordinate frame, and this makes many problems easier. For instance, one may neglect all other cells except the first largest cell when looking for infinitely near points.

Unfortunately, these techniques have a bad effect on sparse representations: after the shaking, one suddenly has rather large expressions.

The emphasize on sparse representations is also responsible for the preference of *adjideal* to *adjlinear*. The representation of the adjoint ideals by generators is much more condensed than the representation of the corresponding adjoint vector-spaces by bases.

For the treatment of more random-like examples, the algorithm *adjlinear* should be preferred. It uses only linear algebra in those tasks where *adjideal* has to use Gröbner bases (e.g., the intersection of ideals).

The intersection step in *adjideal* should be improved anyway. Our intersection problems are rather specific, and the intersection algorithm for arbitrary ideals does not take advantage of this. For instance, if the closures of the sets of inner points are disjoint, then it suffices to multiply the local adjoint ideals.

In this context, we showed that adjoint ideals can be used to compute several numerical invariants and to parametrize rational surfaces. But this is only the beginning of a whole program of computational birational geometry. I think that adjoint ideals could be the workinghorse in such a program. For instance, I consider them to be helpful in the following problems:

- Given a surface with $\kappa = 2$.
  Find a canonical birational map.

- Given a surface with $\kappa = 1$.
  Find a rational map with elliptic fibers.

In both problems, we know the existence of the required map by Enriques' classification of rational surfaces ([39]).

Some material in this chapter generalizes to varieties of arbitrary dimension: the concept of adjoint ideals and adjoint sheafs and almost all of section 4. To compute the adjoint ideals, one has to resolve the singularities. The resolution algorithm that works for surfaces does not work for arbitrary dimensions, but there is a resolution algorithm ([21]). Such an algorithm could be used to compute adjoints for varieties of higher dimensions.

A method for the computation of adjoints for 3-folds would be very useful: one could calculate the numerical invariants p_a, p_g and the plurigeni.

We still do not have a cohomological criterion for unirationality and rationality of 3-folds. But one could try to use adjoints to compute a parametrization for a given unirational surface. Recall that Enriques had an analogous algorithm for surfaces ([14]) before Castelnuovo came up with his famous criterion ([8]).

It is not hard to think of applications of the resolution algorithm beyond computation of adjoint sheaves. Some global applications would be the computation of numerical birational invariants, or the computation of the Albanese map. Most applications would be local, i.e. dealing with one singularity at a time. An example is to compute power series expansions at a singular point.

In [26], one can find another application of nonlinear transformations in differential algebra. It is a small trick worth remembering.

The local-global principle or some variant is applicable whenever many local problems are easier to treat than one global problem. The key idea is to use sheaf theoretic knowledge for merging the local results.

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