Another Elementary Proof of Kharitonov's Theorem

Andreas Neubacher*
Research Institute for Symbolic Computation
Johannes Kepler University
4040 Linz, Austria
Andreas.Neubacher@risc.uni-linz.ac.at

July 1997

Abstract

In this note yet another proof of Kharitonov's theorem on robust stability of interval polynomials is presented. The proof is based on an analysis of the critical regions where instability may occur; the "bounding property" of the Kharitonov polynomials ensures stability in each case. Only elementary results that have also been used in earlier proofs of the theorem are applied.

1 Introduction

In 1978 Kharitonov presented a theorem that reduces the robust Hurwitz stability problem for an interval polynomial to the Hurwitz stability problem for only four polynomials [5, 6]. As researchers tried to get a better understanding of the underlying structures that make this — on first sight quite astonishing — result possible, several new proofs were found.

One approach is based on looking at the parameter space, which for interval polynomials is identical to the coefficient space. Chapellat and Bhattacharyya present a proof combining this approach with results on the stability of convex combinations of polynomials [3].

Another approach is based on applying the well-known "zero exclusion principle", which implies that to ensure stability zero must not be contained in the value set resulting from evaluating the interval polynomial on the imaginary axis. Yeung and Wang use this approach together with the Hermite-Bieler theorem to prove Kharitonov's theorem [9]. Bose and Shi apply similar ideas to Kharitonov's generalization of his theorem to complex interval coefficients [2]. Minnichelli, Anagnost and Desoer also consider the imaginary axis image and introduce a "tilting argument" based on the increasing argument property of Hurwitz polynomials [7].

*Partially suppored by the Austrian Ministry of Science and Research in the frame of the CEI PACT project.

Barmish gives a comprehensive overview of the theory related to Kharitonov's theorem (with a detailed exposition of the "tilting argument" proof) in [1].

In this note yet another proof of Kharitonov's theorem is presented. It is based on elementary results such as the zero exclusion principle and the increasing argument property of Hurwitz polynomials. Its key idea is the analysis of the critical regions (axis crossings) where instability may occur. By acting as upper and lower bounding polynomials to the interval polynomial, the Kharitonov polynomials ensure stability in each case.

2 Preliminaries

To make this note self-contained we first define Hurwitz stability and then present several well-known results we will need for the proof of Kharitonov's theorem.

Definition 1 (Hurwitz stability) *Let $f \in \mathbb{R}[z]$. f is called **Hurwitz stable** iff*

$$\forall z \in \mathbb{C} : [f(z) = 0 \Longrightarrow \mathrm{Re}\,(z) < 0]\,,$$

where $\mathrm{Re}\,(z)$ denotes the real part of z.

*A polynomial that is Hurwitz stable is also called a **Hurwitz polynomial**.*

This is the only definition of stability that we will use in this note, therefore for brevity we will omit the "Hurwitz" in conjunction with stability from now on.

Lemma 1 (Zero Exclusion Principle) *Let $Q \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[z, q_0, \ldots, q_{n-1}]$. If*

i. Q is connected, and

ii. the degree of $f(z,q)$ is the same for all $q \in Q$, and

iii. $\exists \tilde{q} \in Q : f(z, \tilde{q})$ is stable, and

iv. $\forall w \in \mathbb{R} : \forall q \in Q : f(jw, q) \neq 0$,

then $f(z,q)$ is stable for all $q \in Q$.

The proof can be derived quite naturally from the continuous dependency of the zeros of a polynomial on its coefficients and the fact that the imaginary axis is the boundary between the left and the right halfplane, i.e. the regions where respectively all and no zeros must be located to ensure stability.

Proof: Assume that (i), ... , (iv) hold. Let $Q_s := \{q \in Q |\, f(z,q) \text{ is stable}\}$ and $Q_{ns} := \{q \in Q |\, f(z,q) \text{ is not stable}\}$. Condition (iii) guarantees that Q_s is not empty. If Q_{ns} is empty we are done, therefore assume that Q_{ns} is not empty.

From (i) we know that the intersection of the closures of Q_s and Q_{ns} must contain at least one element $r \in Q$. Condition (ii) guarantees that all polynomials in Q have the same number of roots; by definition the polynomials related to Q_s have all roots in the open left halfplane and the polynomials related to Q_{ns} have at least one root in the closed right halfplane; and we know that the roots depend continuously on the parameters. Therefore, $f(z,r)$ must have at least one root on the imaginary axis. This is a contradiction to (iv).

□

Lemma 2 (Argument derivative) *Let $f \in \mathbb{C}[z]$, $f_R, f_I \in \mathbb{R}[z]$ with $f(z) = f_R(z) + jf_I(z)$. Then*

$$\frac{d}{dw}\arg(f(z)) = \mathrm{Im}\left(\frac{f'(z)}{f(z)}\right) = \frac{f_R(z)f_I'(z) - f_R'(z)f_I(z)}{f_R^2(z) + f_I^2(z)}.$$

This lemma follows immediately from $\arg(f(z)) = \mathrm{Im}\,(\log(f(z)))$.

Lemma 3 (Positive argument derivative) *Let $f \in \mathbb{R}[z]$. If f is stable, then*

$$\forall \tilde{w} \in \mathbb{R} : \left[\frac{d}{dw}\arg(f(jw))\right]_{w=\tilde{w}} > 0.$$

Zero is a sharp lower bound only for complex polynomials, but for our purposes it will suffice. In [8] a sharp lower bound for real polynomials is given.

Proof: Let $f(z) = \prod(z - z_i)$. For the zeros z_i of f stability of f implies $\mathrm{Re}\,(z_i) < 0$. Furthermore $\arg(f(z)) = \sum \arg(z - z_i)$ and $\arg'(f(z)) = \sum \arg'(z - z_i)$. Using Lemma 2 we have for any z_i

$$\begin{aligned}
\frac{d}{dw}\arg(jw - z_i) &= \mathrm{Im}\left(\frac{j}{jw - z_i}\right) \\
&= \mathrm{Re}\left(-\frac{jw + \overline{z_i}}{w^2 + |z_i|^2}\right) \\
&= -\frac{1}{w^2 + |z_i|^2}\mathrm{Re}\,(z_i) \\
&> 0,
\end{aligned}$$

and thus $0 < \sum \frac{d}{dw}\arg(jw - z_i) = \frac{d}{dw}\arg(f(jw))$.

□

3 Kharitonov's Theorem

Theorem 1 (Kharitonov [5]) *Let $Q \subseteq \mathbb{R}^n$ be of the form $Q = [\underline{q_0}; \overline{q_0}] \times \ldots \times [\underline{q_{n-1}}; \overline{q_{n-1}}]$ and let $f \in \mathbb{R}[z, q_0, \ldots, q_{n-1}]$ be of the form $f = q_0 + q_1 z + q_2 z^2 + \ldots + q_{n-1}z^{n-1} + z^n$.*

Then $f(z, q)$ is stable for all $q \in Q$ if and only if the four ***Kharitonov polynomials***

$$\begin{aligned}
k_1(z) &= \underline{q_0} + \underline{q_1}z + \overline{q_2}z^2 + \overline{q_3}z^3 + \underline{q_4}z^4 + \underline{q_5}z^5 + \overline{q_6}z^6 + \ldots \\
k_2(z) &= \overline{q_0} + \overline{q_1}z + \underline{q_2}z^2 + \underline{q_3}z^3 + \overline{q_4}z^4 + \overline{q_5}z^5 + \underline{q_6}z^6 + \ldots \\
k_3(z) &= \overline{q_0} + \underline{q_1}z + \underline{q_2}z^2 + \overline{q_3}z^3 + \overline{q_4}z^4 + \underline{q_5}z^5 + \underline{q_6}z^6 + \ldots \\
k_4(z) &= \underline{q_0} + \overline{q_1}z + \overline{q_2}z^2 + \underline{q_3}z^3 + \underline{q_4}z^4 + \overline{q_5}z^5 + \overline{q_6}z^6 + \ldots
\end{aligned}$$

are stable.

Proof: Assume that the four Kharitonov polynomials are stable. We will base the proof on Lemma 1, therefore we need to show that the four conditions of this lemma hold. Conditions (i) and (ii) follow immediately from the definition of Q

and f; condition (iii) follows from the stability of the Kharitonov polynomials. Thus it remains to show condition (iv), i.e. $\forall w \in \mathbb{R} : \forall q \in Q : f(jw, q) \neq 0$.

We can write $f(jw, q)$ as $f_R(w, q) + jf_I(w, q)$ with

$$
\begin{aligned}
f_R(w, q) &= q_0 - q_2 w^2 + q_4 w^4 - q_6 w^6 + \dots, \\
f_I(w, q) &= q_1 w - q_3 w^3 + q_5 w^5 - q_7 w^7 + \dots
\end{aligned}
$$

and we see that $f(j\tilde{w}, \tilde{q}) = 0$ is equivalent to both $f_R(\tilde{w}, \tilde{q}) = 0$ and $f_I(\tilde{w}, \tilde{q}) = 0$. Therefore we will show that if one of these polynomials is zero, the other must be non-zero. To do this we consider the **lower** and **upper bound polynomials** of f_R and f_I: Defining

$$
\begin{aligned}
\underline{f_R}(w) &:= \underline{q_0} - \overline{q_2} w^2 + \underline{q_4} w^4 - \overline{q_6} w^6 + \dots, \\
\overline{f_R}(w) &:= \overline{q_0} - \underline{q_2} w^2 + \overline{q_4} w^4 - \underline{q_6} w^6 + \dots, \\
\underline{f_I}(w) &:= \underline{q_1} w - \overline{q_3} w^3 + \underline{q_5} w^5 - \overline{q_7} w^7 + \dots, \\
\overline{f_I}(w) &:= \overline{q_1} w - \underline{q_3} w^3 + \overline{q_5} w^5 - \underline{q_7} w^7 + \dots
\end{aligned}
$$

we have for all $q \in Q$

$$
\begin{array}{ccccl}
\underline{f_R}(w) & \leq & f_R(w, q) & \leq & \overline{f_R}(w), \\
\underline{f_I}(w) & \leq & f_I(w, q) & \leq & \overline{f_I}(w) \text{ for } 0 \leq w, \\
\overline{f_I}(w) & \leq & f_I(w, q) & \leq & \underline{f_I}(w) \text{ for } w \leq 0.
\end{array}
$$

These lower and upper bound polynomials are related to the Kharitonov polynomials by

$$
\begin{aligned}
k_1(jw) &= \underline{f_R}(w) + j\underline{f_I}(w) \\
k_2(jw) &= \overline{f_R}(w) + j\overline{f_I}(w) \\
k_3(jw) &= \overline{f_R}(w) + j\underline{f_I}(w) \\
k_4(jw) &= \underline{f_R}(w) + j\overline{f_I}(w).
\end{aligned}
$$

We see that for arbitrary but fixed w the set $\{f(jw, q) | \, q \in Q\}$ can be regarded as a rectangle parallel to the axes in the complex plane with the Kharitonov polynomials at its corners. This view was first presented by Dasgupta in [4]. Our goal is to show that due to this "bounding property" the Kharitonov polynomials force $f(jw, q)$ away from the origin. See Figure 1 for an illustration of this idea.

According to Lemma 3 stability of the Kharitonov polynomials $k_i(z)$ implies that $\frac{d}{dw} \arg(k_i(jw)) > 0$. Applying Lemma 2 yields

$$
\underline{f_R}(w)\underline{f_I}'(w) > \underline{f_R}'(w)\underline{f_I}(w), \tag{1}
$$
$$
\overline{f_R}(w)\overline{f_I}'(w) > \overline{f_R}'(w)\overline{f_I}(w), \tag{2}
$$
$$
\overline{f_R}(w)\underline{f_I}'(w) > \overline{f_R}'(w)\underline{f_I}(w), \tag{3}
$$
$$
\underline{f_R}(w)\overline{f_I}'(w) > \underline{f_R}'(w)\overline{f_I}(w). \tag{4}
$$

Now a case distinction provides the key argument of the proof:

- $\underline{f_R}(w) = 0$: (1) and (4) imply $\underline{f_R}'(w) \neq 0$.

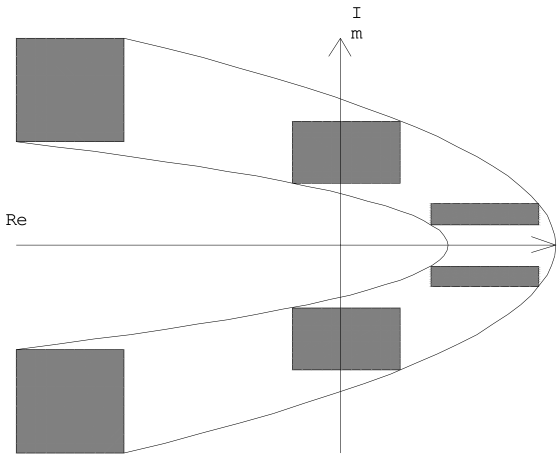

Figure 1: Intuition underlying Kharitonov's Theorem. *This figure shows* $\{f(jw, q_0, q_1) \,|\, (q_0, q_1) \in Q\}$ *for various values of* w *with* $f(z, q_0, q_1) = q_0 + q_1 z + z^2$, $Q = [1, 2]^2$. *The curves passing through the corners of these rectangles show the images of the imaginary axis generated by the Kharitonov polynomials* k_1 *and* k_2.

- $\underline{f_R}'(w) < 0$: (1) implies $0 < \underline{f_I}(w)$, (4) implies $0 < \overline{f_I}(w)$, and therefore $f_I(w, q) \neq 0$.
- $0 < \underline{f_R}'(w)$: (1) implies $\underline{f_I}(w) < 0$, (4) implies $\overline{f_I}(w) < 0$, and therefore $f_I(w, q) \neq 0$.

- $\overline{f_R}(w) = 0$: (2) and (3) imply $\overline{f_R}'(w) \neq 0$.
  - $\overline{f_R}'(w) < 0$: (3) implies $0 < \underline{f_I}(w)$, (2) implies $0 < \overline{f_I}(w)$, and therefore $f_I(w, q) \neq 0$.
  - $0 < \overline{f_R}'(w)$: (3) implies $\underline{f_I}(w) < 0$, (2) implies $\overline{f_I}(w) < 0$, and therefore $f_I(w, q) \neq 0$.

We can apply a similar case distinction to the imaginary lower and upper bounds and thus show that $f(jw, q)$ cannot be zero if either f_R or f_I is equal to a lower or upper bound polynomial.

To conclude the proof we must show that $f(jw, q)$ cannot be zero for arbitrary f_R and f_I. We do this by reducing these "interior" cases to the boundary case: The leading coefficient of f is 1, therefore there exists a $\tilde{w} > 0$ such that $f(j\tilde{w}, q)$ is non-zero for all $q \in Q$. Continuity of f implies that if $f_R(w, q)$ and $f_I(w, q)$ have a common zero then there must exist a w such that $f_R(w, q)$ and $f_I(w, q)$ have a common zero and one of them is equal to a lower or upper bound polynomial.

$\square$

4 Conclusions

The proof of Kharitonov's theorem presented in this note applies the zero exclusion principle and the increasing argument property of Hurwitz polynomials. While in [7] the same elementary results are used to prove Kharitonov's theorem by constructing a contradiction (the "tilting argument"), the proof presented in this note directly shows how the "bounding property" of the Kharitonov polynomials guarantees that the conditions of the zero exclusion principle hold.

It is conceivable that such an analysis of the regions where stability may fail can be applied together with a "bounding property" to conjectures similar to Kharitonov's theorem that do not support a "tilting argument".

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