Logicality of Conditional Rewrite Systems*

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Abstract

A conditional term rewriting system is called logical if it has the same logical strength as the underlying conditional equational system. In this paper we summarize known logicality results and we present new sufficient conditions for logicality of the important class of oriented conditional term rewriting systems.

1 Introduction

Conditional term rewriting (see [Bergstra and Klop, 1986], [Dershowitz and Okada, 1990], [Kaplan, 1984]) provides a useful framework for the study of a wide range of problems in computation and programming.In this paper we investigate the logical strength of conditional rewrite systems. A conditional rewrite system is called logical if it has the same logical strength as the underlying conditional equational system. Logicality is important because it implies that an equation $s \approx t$ is provable by rewriting ($s \leftrightarrow^* t$) if and only if it is valid in all models of the underlying conditional equational system.

Three main types of conditional rewriting are considered in the literature. In a *natural* system the conditions in the conditional rewrite rules are checked by allowing rewriting in both directions. This is very close to equational reasoning in the underlying conditional equational system and hence it is not surprising that natural systems are logical. However, from a rewriting point of view, natural systems are unnatural because the bidirectional use of rewrite rules in the conditions goes against the spirit of rewriting. In a *join* system the applicability of conditional rewrite rules is determined by joinability of the conditions. Most of the literature on conditional rewriting addresses join systems. Kaplan showed that join systems are logical, provided they are confluent [Kaplan, 1984]. Recently, *oriented* systems emerged as the most natural type of conditional rewriting when modeling logic and functional programming, especially when allowing extra variables in the conditions and right-hand sides of rewrite rules (e.g. [Avenhaus and Loría-Sáenz, 1994],

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[Hanus, 1995], [Suzuki *et al.*, 1995]). In contrast to join systems, confluence is insufficient for ensuring logicality of oriented systems. In this paper we show that under suitable additional conditions logicality is recovered and we argue that these conditions are not too restrictive.

The remainder of this paper is organized as follows. In the next section we briefly recall conditional equational reasoning and we present the basic definitions and properties of conditional term rewriting systems. In Section 3 we give simple proofs of logicality for natural and for confluent join systems. In Section 4 we present two new sufficient conditions (Theorems 14 and 23) for logicality of oriented systems. The usefulness of these conditions is shown in Section 5, where we show that our results cover the classes of conditional rewrite systems considered in [Avenhaus and Loría-Sáenz, 1994] and [Suzuki *et al.*, 1995].

This paper extends and corrects unpublished work [Avenhaus and Loría-Sáenz, 1993] of two of the four authors, cf. the footnotes in Section 4.

2 Preliminaries

We assume the reader is familiar with the basic notions of (unconditional) term rewriting. (See [Dershowitz and Jouannaud, 1990] and [Klop, 1992] for extensive surveys.) We start this preliminary section with a very brief introduction to conditional equational logic.

A conditional equation is a pair $(l \approx r, c)$ consisting of an equation $l \approx r$ and a possibly empty sequence $c = s_1 \approx t_1, \ldots, s_n \approx t_n$ of equations. We write $l \approx r \Leftarrow c$ instead of $(l \approx r, c)$. If the conditional part c is empty we simply write $l \approx r$. A conditional equational system (CES for short) over a signature $\mathcal{F}$ is a set $\mathcal{E}$ of conditional equations over terms in $\mathcal{T}(\mathcal{F}, \mathcal{V})$. We write $s =_{\mathcal{E}} t$ if the equation $s \approx t$ can be deduced from the inference rules of Table 1. Let $\mathcal{F}$ be a signature. An $\mathcal{F}$-algebra $\mathcal{A} = (A, \{f_{\mathcal{A}}\}_{f \in \mathcal{F}})$ consists

Table 1:

reflexivity	$\dfrac{}{t \approx t}$	congruence	$\dfrac{s_1 \approx t_1, \ldots, s_n \approx t_n}{f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)}$ if $f \in \mathcal{F}$ is n-ary
symmetry	$\dfrac{s \approx t}{t \approx s}$	application	$\dfrac{s_1\sigma \approx t_1\sigma, \ldots, s_n\sigma \approx t_n\sigma}{l\sigma \approx r\sigma}$ if $l \approx r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n \in \mathcal{E}$
transitivity	$\dfrac{s \approx t, t \approx u}{s \approx u}$		

of a set A, the carrier of $\mathcal{A}$, and operations $f_{\mathcal{A}}\colon A^n \to A$ for every n-ary function symbol $f \in \mathcal{F}$. An assignment α is a mapping from $\mathcal{V}$ to A. A conditional equation $l \approx r \Leftarrow c$ is valid in $\mathcal{A}$ if $[\alpha](l) = [\alpha](r)$ for every assignment α that satisfies $[\alpha](s) = [\alpha](t)$ for all $s \approx t$ in c. Here $[\alpha]$ denotes the unique homomorphism from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{A}$ that extends α, i.e., $[\alpha](t) = \alpha(t)$ if $t \in \mathcal{V}$ and $[\alpha](t) = f_{\mathcal{A}}([\alpha](t_1), \ldots, [\alpha](t_n))$ if $t = f(t_1, \ldots, t_n)$. In particular, an unconditional equation $l \approx r$ is valid in $\mathcal{A}$ if $[\alpha](l) = [\alpha](r)$ for every assignment α. An algebra $\mathcal{A}$ is a model of a CES $\mathcal{E}$ if every conditional equation in $\mathcal{E}$ is valid in $\mathcal{A}$. Birkhoff's theorem states that $s =_{\mathcal{E}} t$ if and only if the equation $s \approx t$ is valid in every model of $\mathcal{E}$.

Conditional rewrite rules are conditional equations $l \approx r \Leftarrow c$ that are used to rewrite terms by replacing an instance of the left-hand side l with the corresponding instance of the right-hand side r provided the corresponding instance of the conditional part c is satisfied. To express this directed use of conditional equations we denote conditional rewrite rules by $l \to r \Leftarrow c$ and CESs consisting of conditional rewrite rules are called conditional term rewriting systems (CTRSs for short). Depending on the interpretation of the equality sign $\approx$ in the conditional part of conditional rewrite rules, different rewrite relations can be associated with a given CTRS. The most common interpretations are convertibility ($\leftrightarrow^*$), joinability ($\downarrow$), and reduction ($\to^*$).

The rewrite relation $\to_{\mathcal{R}}$ of a *natural* CTRS $\mathcal{R}$ is defined as follows: $s \to_{\mathcal{R}} t$ if and only if $s \to_{\mathcal{R}_n} t$ for some $n \geqslant 0$. The minimum such n is called the *depth* of $s \to_{\mathcal{R}} t$. Here the relations $\to_{\mathcal{R}_n}$ are inductively defined as follows:

$$\begin{array}{lcl} \to_{\mathcal{R}_0} & = & \varnothing, \\ \to_{\mathcal{R}_{n+1}} & = & \{(C[l\sigma], C[r\sigma]) \mid l \to r \Leftarrow c \in \mathcal{R} \text{ with } c\sigma \subseteq \leftrightarrow^*_{\mathcal{R}_n}\}. \end{array}$$

Here $c\sigma$ denotes the set $\{(s\sigma, t\sigma) \mid s \approx t \text{ belongs to } c\}$, so $c\sigma \subseteq \leftrightarrow^*_{\mathcal{R}_n}$ with $c = s_1 \approx t_1, \ldots, s_n \approx t_n$ is a shorthand for $s_1\sigma \leftrightarrow^*_{\mathcal{R}_n} t_1\sigma, \ldots, s_n\sigma \leftrightarrow^*_{\mathcal{R}_n} t_n\sigma$. If we replace $c\sigma \subseteq \leftrightarrow^*_{\mathcal{R}_n}$ by $c\sigma \subseteq \downarrow_{\mathcal{R}_n}$ we obtain the rewrite relation of a *join* CTRS and if we replace $c\sigma \subseteq \leftrightarrow^*_{\mathcal{R}_n}$ by $c\sigma \subseteq \to^*_{\mathcal{R}_n}$ we obtain the rewrite relation of an *oriented* CTRS. This classification of CTRSs goes back to [Bergstra and Klop, 1986], where the terminology type I, II, and III are used. Natural CTRSs are also called semi-equational in the literature and join CTRSs are sometimes called standard. Note that we don't put any restrictions on the distribution of variables among the different parts of conditional rewrite rules. In particular, we allow extra variables in the right-hand sides as well as in the conditions of conditional rewrite rules.

In the following we frequently compare different types of CTRSs associated with the same CES. Hence it is convenient to make the explicit notational convention of writing $\mathcal{R}^{\mathrm{n}}$ ($\mathcal{R}^{\mathrm{j}}$, $\mathcal{R}^{\mathrm{o}}$) if the $\mathcal{R}$ is considered as a natural (join, oriented) CTRS. Furthermore we abbreviate $\to_{\mathcal{R}^{\mathrm{n}}}$ to $\to_{\mathrm{n}}$ ($\downarrow_{\mathcal{R}^{\mathrm{o}}}$ to $\downarrow_{\mathrm{o}}$, $\leftrightarrow^*_{\mathcal{R}^{\mathrm{j}}}$ to $\leftrightarrow^*_{\mathrm{j}}$, etc.). We write $\mathcal{R}$ and $\to_{\mathcal{R}}$ if something applies to all three kinds of CTRSs (e.g., when defining properties of CTRSs).

The following basic fact is easily proved by induction on the depth of conditional rewrite steps.

Lemma 1 *The relation $\to_{\mathcal{R}}$ of a CTRS $\mathcal{R}$ is closed under contexts and substitutions.* $\square$

The following well-known result provides a useful characterization of the rewrite relation $\to_{\mathrm{n}}$ of a natural CTRS $\mathcal{R}^{\mathrm{n}}$. A similar statement holds for join (oriented) CTRSs by replacing $\to_{\mathrm{n}}$ by $\to_{\mathrm{j}}$ ($\to_{\mathrm{o}}$) and $\leftrightarrow^*_{\mathrm{n}}$ by $\downarrow_{\mathrm{j}}$ ($\to^*_{\mathrm{o}}$).

Lemma 2 *Let $\mathcal{R}^{\mathrm{n}}$ be a natural CTRS. The relation $\to_{\mathrm{n}}$ is the smallest relation that satisfies the following two properties:*

1. *$\to_{\mathrm{n}}$ is closed under contexts, and*
2. *$l\sigma \to_{\mathrm{n}} r\sigma$ for all $l \to r \Leftarrow c \in \mathcal{R}$ and σ with $c\sigma \subseteq \leftrightarrow^*_{\mathrm{n}}$.*

$\square$

Due to the above lemma we can avoid proofs by induction on the depth of conditional rewrite steps in the sequel. The following lemmata are easy consequences of the previous lemma.

Lemma 3 *For every CTRS $\mathcal{R}$ we have $\rightarrow_{\rm o} \subseteq \rightarrow_{\rm j} \subseteq \rightarrow_{\rm n}$.* □

Lemma 4 *Let $\mathcal{R}^{\rm n}$ be a natural CTRS over a signature $\mathcal{F}$ and $\sim$ an equivalence relation on $\mathcal{T}(\mathcal{F},\mathcal{V})$ that is closed under contexts. If $l\sigma \sim r\sigma$ for all $l \rightarrow r \Leftarrow c \in \mathcal{R}^{\rm n}$ and σ with $c\sigma \subseteq \sim$ then $\leftrightarrow_{\rm n}^* \subseteq \sim$.*

Proof. The relation $\sim$ satisfies the two properties expressed in Lemma 2 because the equivalence closure of $\sim$ (i.e., convertibility with respect to $\sim$) is $\sim$ itself. Hence $\rightarrow_{\rm n} \subseteq \sim$ and thus also $\leftrightarrow_{\rm n}^* \subseteq \sim$, again because the equivalence closure of $\sim$ is $\sim$. □

The above lemma also holds for join and oriented CTRSs, with a small change in the proof.

3 Logicality

Definition 5 A CTRS $\mathcal{R}$ is called *logical* if the relations $=_{\mathcal{R}}$ and $\leftrightarrow_{\mathcal{R}}^*$ coincide. Here $=_{\mathcal{R}}$ denotes the relation defined via the inference system of Table 1 for the underlying CES $\mathcal{R}$.

The terminology logicality stems from [Bertling and Ganzinger, 1994] although the study of the concept dates back to [Kaplan, 1984]. Logicality is an important property because it entails that (bidirectional) rewriting is sound and complete with respect to the underlying equational logic.

Thoerem 6 *Every natural CTRS is logical.*

Proof. Let $\mathcal{R}^{\rm n}$ be a natural CTRS. We have to show that $=_{\rm n}$ and $\leftrightarrow_{\rm n}^*$ coincide. The inclusion $=_{\rm n} \subseteq \leftrightarrow_{\rm n}^*$ is easily proved by induction on the structure of proofs of equations in the inference system of Table 1, using closure under contexts of $\leftrightarrow_{\rm n}^*$ if the last step of the proof is an application of the congruence rule. According to Lemma 4, for the reverse inclusion $\leftrightarrow_{\rm n}^* \subseteq =_{\rm n}$ it is sufficient to show that

1. $=_{\rm n}$ is an equivalence relation,
2. $=_{\rm n}$ is closed under contexts, and
3. $l\sigma =_{\rm n} r\sigma$ for all $l \rightarrow r \Leftarrow c \in \mathcal{R}^{\rm n}$ and σ with $c\sigma \subseteq =_{\rm n}$.

Property 1 is obvious due to the presence of the reflexivity, symmetry, and transitivity inference rules in the inference system of Table 1. Closure under contexts is easily proved by induction on the structure of contexts, using the congruence and reflexivity inference rules. Finally, property 3 is an immediate consequence of the application inference rule. □

An immediate consequence of Theorem 6 is that a join (oriented) CTRS $\mathcal{R}^{\rm j}$ ($\mathcal{R}^{\rm o}$) is logical if and only if the relations $\leftrightarrow_{\rm j}^*$ ($\leftrightarrow_{\rm o}^*$) and $\leftrightarrow_{\rm n}^*$ coincide.

Join CTRSs need not be logical, as shown in the following example.

Example 7 Consider the CTRS $\mathcal{R} = \{a \rightarrow b, a \rightarrow c, d \rightarrow e \Leftarrow b \approx c\}$. We have $d \rightarrow_{\rm n} e$ since $b \;{}_{\rm n}\!\!\leftarrow a \rightarrow_{\rm n} c$. However, $d \rightarrow_{\rm j} e$ doesn't hold because the condition $b \downarrow_{\rm j} c$ is not satisfied. Hence $d \leftrightarrow_{\rm j}^* e$ doesn't hold either.

Note that the above $\mathcal{R}^{\rm j}$ lacks confluence. Kaplan observed that this is essential.

Thoerem 8 ([Kaplan, 1984]) *Every confluent join CTRS is logical.*

Proof. Let $\mathcal{R}^{\mathrm{j}}$ be a confluent join CTRS. We claim that $\rightarrow_{\mathrm{j}} = \rightarrow_{\mathrm{n}}$, implying the desired $\leftrightarrow_{\mathrm{j}}^{*} = \leftrightarrow_{\mathrm{n}}^{*}$. We already know that $\rightarrow_{\mathrm{j}} \subseteq \rightarrow_{\mathrm{n}}$. For the reverse inclusion we use Lemma 2. To this end we have to show that

1. $\rightarrow_{\mathrm{j}}$ is closed under contexts, and
2. $l\sigma \rightarrow_{\mathrm{j}} r\sigma$ for all $l \rightarrow r \Leftarrow c \in \mathcal{R}^{\mathrm{j}}$ and σ with $c\sigma \subseteq \leftrightarrow_{\mathrm{j}}^{*}$.

Closure under contexts is expressed in Lemma 1. For property 2 we note that $\leftrightarrow_{\mathrm{j}}^{*} \subseteq \downarrow_{\mathrm{j}}$ by confluence and thus $l\sigma \rightarrow_{\mathrm{j}} r\sigma$ follows from $c\sigma \subseteq \leftrightarrow_{\mathrm{j}}^{*}$. ☐

4 Oriented CTRSs

For oriented CTRSs confluence is not sufficient for ensuring logicality, as shown by the following example.

Example 9 Consider the CTRS $\mathcal{R} = \{a \rightarrow c, b \rightarrow c \Leftarrow c \approx a\}$. We have $b \rightarrow_{\mathrm{n}} c$ since $c \,{}_{\mathrm{n}}\!\leftarrow a$. However, $b \rightarrow_{\mathrm{o}}^{*} c$ doesn't hold because the condition $c \rightarrow_{\mathrm{o}}^{*} a$ is not satisfied. Hence $b \leftrightarrow_{\mathrm{o}}^{*} c$ doesn't hold either. Note that $\mathcal{R}^{\mathrm{o}}$ is confluent.

The CTRS $\mathcal{R}^{\mathrm{o}}$ in the above example is not a so-called normal CTRS.

Definition 10 Let $\mathcal{R}$ be a CTRS. A term t is called *normal* if it is ground and doesn't encompass the left-hand side l of a conditional rewrite rule $l \rightarrow r \Leftarrow c$ in $\mathcal{R}$. The latter requirement means that t is irreducible with respect to the unconditional TRS obtained from $\mathcal{R}$ by dropping all conditions. We say that the oriented CTRS $\mathcal{R}^{\mathrm{o}}$ is *normal* if every right-hand side t of an equation $s \approx t$ in the conditional part c of a conditional rewrite rule $l \rightarrow r \Leftarrow c$ in $\mathcal{R}^{\mathrm{o}}$ is normal.

Note that normality is a decidable property of finite oriented CTRSs.

Thoerem 11 *Every confluent normal CTRS is logical.*

Proof. Let $\mathcal{R}^{\mathrm{o}}$ be a confluent normal CTRS. According to Lemma 3 we have $\rightarrow_{\mathrm{o}} \subseteq \rightarrow_{\mathrm{j}}$. The reverse inclusion $\rightarrow_{\mathrm{j}} \subseteq \rightarrow_{\mathrm{o}}$ is an easy consequence of the join version of Lemma 2, cf. the proof of Theorem 8, and the normality assumption. Hence $\rightarrow_{\mathrm{o}} = \rightarrow_{\mathrm{j}}$ and thus also $\leftrightarrow_{\mathrm{o}}^{*} = \leftrightarrow_{\mathrm{j}}^{*}$. According to Theorem 8 $\leftrightarrow_{\mathrm{j}}^{*} = \leftrightarrow_{\mathrm{n}}^{*}$. Therefore $\mathcal{R}^{\mathrm{o}}$ is logical. ☐

In the presence of extra variables in the right-hand sides of the conditional rewrite rules, normality is too strong a requirement. Such extra variables appear naturally in applications of conditional rewriting (e.g. [Avenhaus and Loría-Sáenz, 1994] [Bertling and Ganzinger, 1994] [Hanus, 1995] [Suzuki *et al.*, 1995]). Below we present other, more useful, sufficient conditions for the logicality of oriented CTRSs. These sufficient conditions are derived from the following key lemma.

Lemma 12 *Let $\mathcal{R}^{\mathrm{o}}$ be a confluent oriented CTRS. If for every $l \rightarrow r \Leftarrow c \in \mathcal{R}^{\mathrm{o}}$ and every substitution σ with $c\sigma \subseteq \downarrow_{\mathrm{o}}$ there exists a substitution τ such that*

1. *$\sigma(x) \rightarrow_{\mathrm{o}}^{*} \tau(x)$ for all $x \in \mathcal{V}$, and*
2. *$c\tau \subseteq \rightarrow_{\mathrm{o}}^{*}$*

then $\mathcal{R}^{\mathrm{o}}$ is logical.

Proof. The inclusion $\leftrightarrow_{\mathrm{o}}^* \subseteq \leftrightarrow_{\mathrm{n}}^*$ follows from Lemma 3. For the reverse inclusion we use Lemma 4 with $\sim\; = \;\leftrightarrow_{\mathrm{o}}^*$. So suppose that $l \to r \Leftarrow c \in \mathcal{R}$ with $c\sigma \subseteq \leftrightarrow_{\mathrm{o}}^*$. We have to show that $l\sigma \leftrightarrow_{\mathrm{o}}^* r\sigma$. Confluence of $\mathcal{R}^{\mathrm{o}}$ yields $c\sigma \subseteq \downarrow_{\mathrm{o}}$. By assumption there exists a substitution τ such that $\sigma(x) \to_{\mathrm{o}}^* \tau(x)$ for all $x \in \mathcal{V}$ and $c\tau \subseteq \to_{\mathrm{o}}^*$. The latter statement implies $l\tau \to_{\mathrm{o}} r\tau$. The first statement implies $l\sigma \to_{\mathrm{o}}^* l\tau$ and $r\sigma \to_{\mathrm{o}}^* r\tau$. Therefore $l\sigma \leftrightarrow_{\mathrm{o}}^* r\sigma$. □

Definition 13 Let $\mathcal{R}$ be a CTRS. A term t is called *strongly irreducible* if $t\sigma$ is irreducible for every irreducible substitution σ. We say that $\mathcal{R}$ is strongly irreducible if every right-hand side t of an equation $s \approx t$ in the conditional part c of a conditional rewrite rule $l \to r \Leftarrow c$ in $\mathcal{R}$ is strongly irreducible.

Note that irreducibility depends on the rewrite relation associated with $\mathcal{R}$, so it is possible that an oriented CTRS $\mathcal{R}^{\mathrm{o}}$ is strongly irreducible whereas the corresponding join CTRS $\mathcal{R}^{\mathrm{j}}$ is not. Because it is undecidable whether a term is irreducible with respect to a CTRS [Kaplan, 1984], strong irreducibility is undecidable in general. A sufficient condition is presented in Definition 16 below.

Thoerem 14 *Every strongly irreducible weakly normalizing confluent oriented CTRS is logical.*[1]

Proof. Let $\mathcal{R}^{\mathrm{o}}$ be a strongly irreducible weakly normalizing confluent oriented CTRS. We use Lemma 12. So let $l \to r \Leftarrow c$ be a conditional rewrite rule of $\mathcal{R}^{\mathrm{o}}$ and σ a substitution with $c\sigma \subseteq \downarrow_{\mathrm{o}}$. We have to define a substitution τ such that

1. $\sigma(x) \to_{\mathrm{o}}^* \tau(x)$ for all $x \in \mathcal{V}$, and
2. $c\tau \subseteq \to_{\mathrm{o}}^*$.

Because $\mathcal{R}^{\mathrm{o}}$ is confluent and weakly normalizing, every term t has a unique normal form $t\!\downarrow_{\mathrm{o}}$ and hence we can define τ as $\tau(x) = \sigma(x)\!\downarrow_{\mathrm{o}}$ for all $x \in \mathcal{V}$. Property 1 is clearly satisfied. Let $s \approx t$ be an equation in c. We have $s\sigma \downarrow_{\mathrm{o}} t\sigma$. From 1 we infer that $s\sigma \to_{\mathrm{o}}^* s\tau$ and $t\sigma \to_{\mathrm{o}}^* t\tau$ and thus $s\tau \leftrightarrow_{\mathrm{o}}^* t\tau$. Since τ is irreducible by construction, $t\tau$ is irreducible by the strong irreducibility assumption. Confluence of $\mathcal{R}^{\mathrm{o}}$ yields $s\tau \to_{\mathrm{o}}^* t\tau$. We conclude that property 2 holds. □

Example 9 shows that Theorem 14 cannot be strengthened by dropping the strong irreducibility requirement. The following example shows the necessity of weak normalization.

Example 15 Consider the CTRS

$$\mathcal{R} = \left\{ \begin{array}{rcl} a & \to & a \\ f(a) & \to & a \\ g(x) & \to & b \quad \Leftarrow \quad a \approx f(x) \end{array} \right.$$

We have $a\ {}_{\mathrm{n}}\!\leftarrow f(a)$ and thus $g(a) \to_{\mathrm{n}} b$. However, since there is no term t such that $a \to_{\mathrm{o}}^* f(t)$, the relation $\to_{\mathrm{o}}$ coincides with the rewrite relation induced by the unconditional TRS $\mathcal{S} = \{a \to a, f(a) \to a\}$. Hence $g(a) \leftrightarrow_{\mathrm{o}}^* b$ doesn't hold and hence $\mathcal{R}^{\mathrm{o}}$ is not logical. Clearly the TRS $\mathcal{S}$ and thus $\mathcal{R}^{\mathrm{o}}$ is confluent. Furthermore, $\mathcal{R}^{\mathrm{o}}$ is strongly irreducible because there is no irreducible term t such that $f(t)$ is reducible.

[1]This result originates from [Avenhaus and Loría-Sáenz, 1993].

Definition 16 Let $\mathcal{R}$ be a CTRS. A term t is called *absolutely irreducible* if no non-variable subterm of t unifies (after variable renaming) with the left-hand side l of a conditional rewrite rule $l \to r \Leftarrow c$ in $\mathcal{R}$. We say that $\mathcal{R}$ is absolutely irreducible if every right-hand side t of an equation $s \approx t$ in the conditional part c of a conditional rewrite rule $l \to r \Leftarrow c$ in $\mathcal{R}$ is absolutely irreducible.

Unlike strong irreducibility, absolute irreducibility doesn't depend on the rewrite relation associated with $\mathcal{R}$. (That is to say, absolute irreducibility is a property of CESs.) Note that every normal CTRS is absolutely irreducible but not vice-versa.

Note that the CTRS $\mathcal{R}^{\mathrm{o}}$ of Example 15 is not absolutely irreducible since the right-hand side $f(x)$ of the condition $a \approx f(x)$ in the rule $g(x) \to b \Leftarrow a \approx f(x)$ is unifiable with the left-hand side $f(a)$ of the rule $f(a) \to a$. Nevertheless, even if we strengthen strong irreducibility to absolute irreducibility, we cannot dispense with weak normalization in Theorem 14 as shown by the following example.

Example 17 Consider the CTRS [2]

$$\mathcal{R} = \left\{ \begin{array}{rcll} a & \to & b & \\ b & \to & a & \\ f(a,b) & \to & c & \\ g(x) & \to & d & \Leftarrow \quad c \approx f(x,x) \end{array} \right.$$

We have $c \;{}_{\mathrm{n}}\!\leftarrow f(a,b) \;{}_{\mathrm{n}}\!\leftarrow f(a,a)$ and thus $g(a) \to_{\mathrm{n}} d$. However, since there is no term t such that $c \to_{\mathrm{o}}^{*} f(t,t)$, the relation $\to_{\mathrm{o}}$ coincides with the rewrite relation induced by the unconditional TRS $\mathcal{S} = \{a \to b, b \to a, f(a,b) \to c\}$. Clearly $g(a) \leftrightarrow_{\mathcal{S}}^{*} d$ doesn't hold. Hence $\mathcal{R}^{\mathrm{o}}$ is not logical. Note that $\mathcal{S}$ and thus $\mathcal{R}^{\mathrm{o}}$ is confluent. Furthermore, $\mathcal{R}^{\mathrm{o}}$ is absolutely irreducible because the term $f(x,x)$ doesn't unify with $f(a,b)$.

The non-linearity of the term $f(x,x)$ in the above example is essential, as we will see below.

Since in applications of conditional rewriting weak normalization is often a severe restriction, e.g. CTRSs that model (lazy) functional programs are not weakly normalizing in general, we are especially interested in a sufficient condition for logicality of oriented CTRSs that doesn't rely on weak normalization. The above examples show that the problem with strong and absolute irreducibility is that the structure of the right-hand sides of equations in the conditional parts are not preserved under rewriting. For instance, in Example 15 we have $f(a) \to_{\mathrm{o}} a$ destroying the structure $f(\cdot)$. Absolute irreducibility guarantees that the structure of the right-hand sides of equations in the conditional parts is preserved by one-step rewriting but not by many-step rewriting: in Example 17 we have $f(a,a) \to_{\mathrm{o}} f(a,b) \to_{\mathrm{o}} c$ destroying $f(\cdot,\cdot)$.

The condition defined below guarantees that the structure of the right-hand sides of equations in the conditional parts is preserved by many-step rewriting.

Definition 18 Let $\mathcal{R}$ be a CTRS. A term s is called *stable* if $p \notin \mathcal{P}\mathrm{os}_{\mathcal{F}}(s)$ whenever $s\sigma \to_{\mathcal{R}}^{*} t \xrightarrow{p}_{\mathcal{R}} u$, for all substitutions σ, terms t and u, and positions p. We say that $\mathcal{R}$ is stable if every right-hand side t of an equation $s \approx t$ in the conditional part c of a conditional rewrite rule $l \to r \Leftarrow c$ in $\mathcal{R}$ is stable.

The structure preservation of stable terms is formally expressed in the following lemma.

Lemma 19 *Let $\mathcal{R}$ be a CTRS. If s is a stable term and $s\sigma \to_{\mathcal{R}}^{*} t$ then*

[2] This example refutes Theorem 5.2 in [Avenhaus and Loría-Sáenz, 1993].

1. $\mathrm{root}(s\sigma_{|p}) = \mathrm{root}(t_{|p})$ *for all* $p \in \mathcal{P}\mathrm{os}_{\mathcal{F}}(s)$*, and*

2. $s\sigma_{|p} \to_{\mathcal{R}}^* t_{|p}$ *for all* $p \in \mathcal{P}\mathrm{os}_{\mathcal{V}}(s)$*.*

Proof. Both properties are easily proved by induction on the length of the reduction $s\sigma \to_{\mathcal{R}}^* t$. □

The next lemma expresses the fact that for confluent CTRSs the substitution part of an instance of a stable term can be consistently reduced. This property plays a crucial role in the proof of our main result (Theorem 23 below).

Lemma 20 *Let* $\mathcal{R}$ *be a confluent CTRS. If* s *is a stable term and* $s\sigma \to_{\mathcal{R}}^* t$ *then there exists a substitution* τ *such that*

1. $\sigma(x) \to_{\mathcal{R}}^* \tau(x)$ *for all* $x \in \mathcal{V}$*, and*

2. $t \to_{\mathcal{R}}^* s\tau$*.*

Proof. If s is a ground term then it must be irreducible and hence any substitution τ satisfies both requirements. Suppose s is not ground. Let x be an arbitrary variable in s and define $A_x = \{t_{|p} \mid s_{|p} = x\}$. Since $\sigma(x) \to_{\mathcal{R}}^* u$ for every $u \in A_x$ by part 2 of Lemma 19, the set A_x consists of pairwise convertible terms. Since it is finite and non-empty, confluence yields a term u_x such that $u \to_{\mathcal{R}}^* u_x$ for all $u \in A_x$. Now define τ as follows: $\tau(x) = u_x$ if $x \in \mathcal{V}\mathrm{ar}(s)$ and $\tau(x) = \sigma(x)$ otherwise. It is easy to see that this τ satisfies both requirements. □

Stability alone is not enough for ensuring the logicality of confluent, not necessarily weakly normalizing, oriented CTRSs. This is shown in the next example.

Example 21 Consider the CTRS

$$\mathcal{R} = \left\{ \begin{array}{rcll} a & \to & f(a) & \\ g(x) & \to & b & \Leftarrow \quad f(x) \approx x \end{array} \right.$$

We have $g(a) \to_{\mathrm{n}} b$ since $f(a)\ {}_{\mathrm{n}}\!\leftarrow a$. Since there is no term t such that $f(t) \to_{\mathrm{o}}^* t$, the relation $\to_{\mathrm{o}}$ coincides with the rewrite relation induced by the single rewrite rule $a \to f(a)$. Hence $\mathcal{R}^{\mathrm{o}}$ is confluent and $g(a) \leftrightarrow_{\mathrm{o}}^* b$ doesn't hold. Note that $\mathcal{R}^{\mathrm{o}}$ is stable since variables are trivially stable.

Definition 22 A CTRS $\mathcal{R}$ is *well-directed* if every conditional rewrite rule $l \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n$ of $\mathcal{R}$ satisfies $\mathcal{V}\mathrm{ar}(s_j) \cap \mathcal{V}\mathrm{ar}(t_i) = \varnothing$ for all $1 \leqslant j \leqslant i \leqslant n$.

All example CTRSs introduced above except the one of Example 21 are well-directed. Normal CTRSs are trivially well-directed. We are now ready for the main theorem of the paper.

Thoerem 23 *Every stable well-directed confluent oriented CTRS is logical.*

Proof. Let $\mathcal{R}^{\mathrm{o}}$ be a stable well-directed confluent oriented CTRS. We use Lemma 12. So let $l \to r \Leftarrow c$ be a conditional rewrite rule of $\mathcal{R}^{\mathrm{o}}$ and σ a substitution with $c\sigma \subseteq \downarrow_{\mathrm{o}}$. Let $c = s_1 \approx t_1, \ldots, s_n \approx t_n$. We have to define a substitution τ such that

1. $\sigma(x) \to_{\mathrm{o}}^* \tau(x)$ for all $x \in \mathcal{V}$, and

2. $c\tau \subseteq \to_{\mathrm{o}}^*$.

To this end we inductively define substitutions $\tau_0, \ldots, \tau_n$ such that for all $0 \leqslant i \leqslant n$

3. $\sigma(x) \to_{\mathrm{o}}^* \tau_i(x)$ for all $x \in \mathcal{V}$, and
4. $s_j\tau_i \to_{\mathrm{o}}^* t_j\tau_i$ for all $1 \leqslant j \leqslant i$.

Letting $\tau_0 = \sigma$, properties 3 and 4 are trivially satisfied for $i = 0$. Let $i \geqslant 1$. From the induction hypothesis, confluence and stability of $\mathcal{R}^{\mathrm{o}}$, and Lemma 20 we infer the existence of a substitution θ_i such that $s_i\tau_{i-1} \to_{\mathrm{o}}^* t_i\theta_i$ and $\sigma(x) \to_{\mathrm{o}}^* \theta_i(x)$ for all $x \in \mathcal{V}$, see Fig. 1. From the induction hypothesis we obtain $\sigma(x) \to_{\mathrm{o}}^* \tau_{i-1}(x)$ for all $x \in \mathcal{V}$. Hence confluence

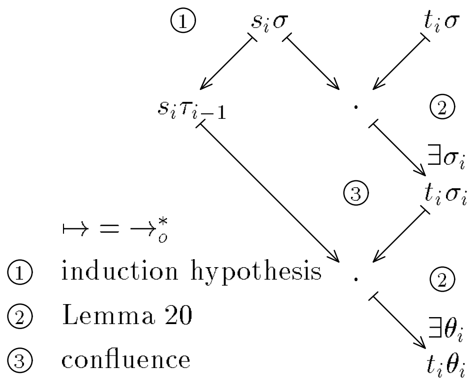

Figure 1:

yields terms u_x for $x \in \mathcal{V}$ such that $\tau_{i-1}(x) \to_{\mathrm{o}}^* u_x \;{}_{\mathrm{o}}^*\!\leftarrow \theta_i(x)$. Partition the set of variables $\mathcal{V}$ into $V_1 = \mathcal{V}\mathrm{ar}(t_i) \cap \bigcup_{1 \leqslant j < i} \mathcal{V}\mathrm{ar}(t_j)$, $V_2 = \mathcal{V}\mathrm{ar}(t_i) \setminus \bigcup_{1 \leqslant j < i} \mathcal{V}\mathrm{ar}(t_j)$, and $V_3 = \mathcal{V} \setminus \mathcal{V}\mathrm{ar}(t_i)$. Now define τ_i as follows: $\tau_i(x) = u_x$ if $x \in V_1$, $\tau_i(x) = \theta_i(x)$ if $x \in V_2$, and $\tau_i(x) = \tau_{i-1}(x)$ if $x \in V_3$. We claim that τ_i has properties 3 and 4. For property 3 we distinguish three cases. If $x \in V_1$ then $\sigma(x) \to_{\mathrm{o}}^* \tau_{i-1}(x)$ by the induction hypothesis, $\tau_{i-1}(x) \to_{\mathrm{o}}^* u_x$ by construction of u_x, and $u_x = \tau_i(x)$ by definition of τ_i. If $x \in V_2$ then $\sigma(x) \to_{\mathrm{o}}^* \theta_i(x)$ by construction of θ_i and $\theta_i(x) = \tau_i(x)$ by definition of τ_i. If $x \in V_3$ then $\sigma(x) \to_{\mathrm{o}}^* \tau_{i-1}(x)$ by the induction hypothesis and $\tau_{i-1}(x) = \tau_i(x)$ by definition of τ_i. Hence in all cases we obtain the desired $\sigma(x) \to_{\mathrm{o}}^* \tau_i(x)$. For property 4 we reason as follows. Let $1 \leqslant j \leqslant i$. By well-directedness $\mathcal{V}\mathrm{ar}(s_j) \cap \mathcal{V}\mathrm{ar}(t_i) = \varnothing$ and thus $\mathcal{V}\mathrm{ar}(s_j) \subseteq V_3$. Consequently $s_j\tau_i = s_j\tau_{i-1}$ by definition of τ_i. So it remains to show that $s_j\tau_{i-1} \to_{\mathrm{o}}^* t_j\tau_i$. We distinguish two cases. If $1 \leqslant j < i$ then $s_j\tau_{i-1} \to_{\mathrm{o}}^* t_j\tau_{i-1}$ by the induction hypothesis and $t_j\tau_{i-1} \to_{\mathrm{o}}^* t_j\tau_i$ because $\mathcal{V}\mathrm{ar}(t_j) \subseteq V_1 \cup V_3$ and $\tau_{i-1}(x) \to_{\mathrm{o}}^* u_x = \tau_i(x)$ for $x \in V_1$ and $\tau_{i-1}(x) = \tau_i(x)$ for $x \in V_3$. If $j = i$ then $s_j\tau_{i-1} \to_{\mathrm{o}}^* t_j\theta_i$ by construction of θ_i and $t_j\theta_i \to_{\mathrm{o}}^* t_j\tau_i$ because $\mathcal{V}\mathrm{ar}(t_j) \subseteq V_1 \cup V_2$ and $\theta_i(x) \to_{\mathrm{o}}^* u_x = \tau_i(x)$ for $x \in V_1$ and $\theta_i(x) = \tau_i(x)$ for $x \in V_2$. This concludes the induction step.

Now we define $\tau = \tau_n$. Since properties 3 and 4 for $i = n$ are equivalent to properties 1 and 2, we are done. □

In the remainder of this section we present sufficient syntactic criteria for stability.

Definition 24 Let $\mathcal{R}$ be a CTRS over a signature $\mathcal{F}$. A function symbol $f \in \mathcal{F}$ is called a *constructor* if for every conditional rewrite rule $l \to r \Leftarrow c \in \mathcal{R}$ neither $l \in \mathcal{V}$ nor $\mathrm{root}(l) = f$. A *constructor term* is built from constructors and variables.

Definition 25 A term s is called a *linearization* of t if s is linear and $s\sigma = t$ for some variable substitution σ. (A substitution σ is a variable substitution if $\sigma(x) \in \mathcal{V}$ for all $x \in \mathcal{V}$.) Let $\mathcal{R}$ be a CTRS. A term t is called *strongly stable* if every linearization of t is absolutely irreducible.

Note that it is sufficient to test one (arbitrary) linearization for absolute irreducibility when checking strong stability. Note also that every linear absolutely irreducible term is strongly stable, hence stable according to the following lemma. Since the CTRS $\mathcal{R}$ in Example 17 is well-directed, this shows that the non-linearity of $f(x,x)$ is essential for the non-logicality of $\mathcal{R}$.

Lemma 26 *Let $\mathcal{R}$ be a CTRS.*

1. *Every strongly stable term is stable.*
2. *Every constructor term is stable.*
3. *Every normal term is stable.*

Proof. The proof of statement 1 is routine. Statements 2 and 3 follows from 1 because constructor and normal terms are always strongly stable. $\square$

Since normal CTRSs are trivially well-directed Theorem 11 is a special case of Theorem 23.

5 Concluding Remarks

In this paper we studied logicality of CTRSs. The main results are summarized in Table 2. We illustrate the usefulness of the last result, Theorem 23, by showing that the class of CTRSs proposed in [Suzuki *et al.*, 1995] falls within its scope. This class can be viewed as a computational model for functional logic programming languages with local definitions such as let-expressions and where-constructs.

Table 2:

type	requirements		Theorem
natural			6
join	confluence		8
oriented	confluence +	normality	11
		weak normalization + strong irreducibility	14
		stability + well-directedness	23

Definition 27 An oriented CTRS is called *properly* oriented if every conditional rewrite rule $l \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n$ with $\mathcal{V}\mathrm{ar}(r) \not\subseteq \mathcal{V}\mathrm{ar}(l)$ satisfies $\mathcal{V}\mathrm{ar}(s_i) \subseteq \mathcal{V}\mathrm{ar}(l) \cup \bigcup_{j=1}^{i-1} \mathcal{V}\mathrm{ar}(s_j \approx t_j)$ for all $1 \leqslant i \leqslant n$. An oriented CTRS is called *right-stable* if every conditional rewrite rule $l \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n$ satisfies $(\mathcal{V}\mathrm{ar}(l) \cup \bigcup_{j=1}^{i-1} \mathcal{V}\mathrm{ar}(s_j \approx t_j) \cup \mathcal{V}\mathrm{ar}(s_i)) \cap \mathcal{V}\mathrm{ar}(t_i) = \varnothing$ and t_i is either a linear constructor term or a normal term, for all $1 \leqslant i \leqslant n$.

In [Suzuki *et al.*, 1995] it is shown that orthogonal properly oriented right-stable CTRSs are *level-confluent*. A CTRS $\mathcal{R}$ is called level-confluent if the relations $\to_{\mathcal{R}_n}$ for $n \geqslant 0$ are confluent.

Thoerem 28 *Every orthogonal properly oriented right-stable CTRS is logical.*

Proof. The first requirement of right-stability implies well-directedness, the second requirement implies stability due to Lemma 26. Since level-confluence implies confluence, logicality follows from Theorem 23. □

Theorem 14, the other new sufficient condition for the logicality of oriented TRSs, covers the class of quasi-reductive strongly deterministic confluent CTRSs, which is studied in [Avenhaus and Loría-Sáenz, 1994]. This class is useful for studying the (unique) termination behaviour of well-moded Horn clause programs. Quasi-reductivity is a criterion guaranteeing termination. Strong determinism is defined as follows.

Definition 29 An oriented CTRS is called *strongly deterministic* if every conditional rewrite rule $l \to r \Leftarrow s_1 \approx t_1, \ldots, s_n \approx t_n$ satisfies $l \notin \mathcal{V}$ and, for all $1 \leqslant i \leqslant n$, $\mathcal{V}\mathrm{ar}(s_i) \subseteq \mathcal{V}\mathrm{ar}(l) \cup \bigcup_{j=1}^{i-1} \mathcal{V}\mathrm{ar}(s_j \approx t_j)$ and t_i is absolutely irreducible.

In [Avenhaus and Loría-Sáenz, 1994] a critical pair criterion is presented for proving confluence of quasi-reductive strongly deterministic CTRSs.

Thoerem 30 *Every quasi-reductive strongly deterministic confluent CTRS is logical.*

Proof. Quasi-reductivity implies termination hence weak normalization and strong determinism implies absolute irreducibility hence strong irreducibility. Hence the conditions of Theorem 14 are fulfilled. □

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