SYMBOLIC COMPUTATION OF AND WITH OFFSET CURVES

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Chapter 1

Introduction

1.1 What Are Offset Curves - a First Intuitive Description

In geometry and its applications we are sometimes interested in *offsetting* a given generator curve $\mathcal{C}$ by some distance d. That is, given a curve $\mathcal{C}$, we wish to determine a curve $\mathcal{O}_d(\mathcal{C})$ such that, for every point p of $\mathcal{C}$, there is a point q on $\mathcal{O}_d$ such that the distance between p and q is exactly d, and the line $\overline{(p,q)}$ is perpendicular to $\mathcal{C}$ at p and vice versa. Such a curve is called an *offset curve* to $\mathcal{C}$.

I.e., if $\mathcal{C}$ is given parametrically by

$$\begin{aligned} x &= f_1(t), \\ y &= f_2(t), \end{aligned}$$

where $x = f_1(t)$ and $y = f_2(t)$ are functions in t, then $\mathcal{O}_d(\mathcal{C})$ is generated by

$$\begin{aligned} x &= f_1(t) + dn_x(t), \\ y &= f_2(t) + dn_y(t), \end{aligned}$$

where $n(t) = (n_x(t), n_y(t)) = \frac{1}{\sqrt{f_1'(t)^2 + f_2'(t)^2}} \cdot (-f_2'(t), f_1'(t))$ is the unit normal vector to $\mathcal{C}$ at t.

Figure 1.1 illustrates the offset curves to a parabola, parametrized by $x = t$, $y = t^2$, at distance $d = \pm 1$ in $\mathbb{R}^2$.

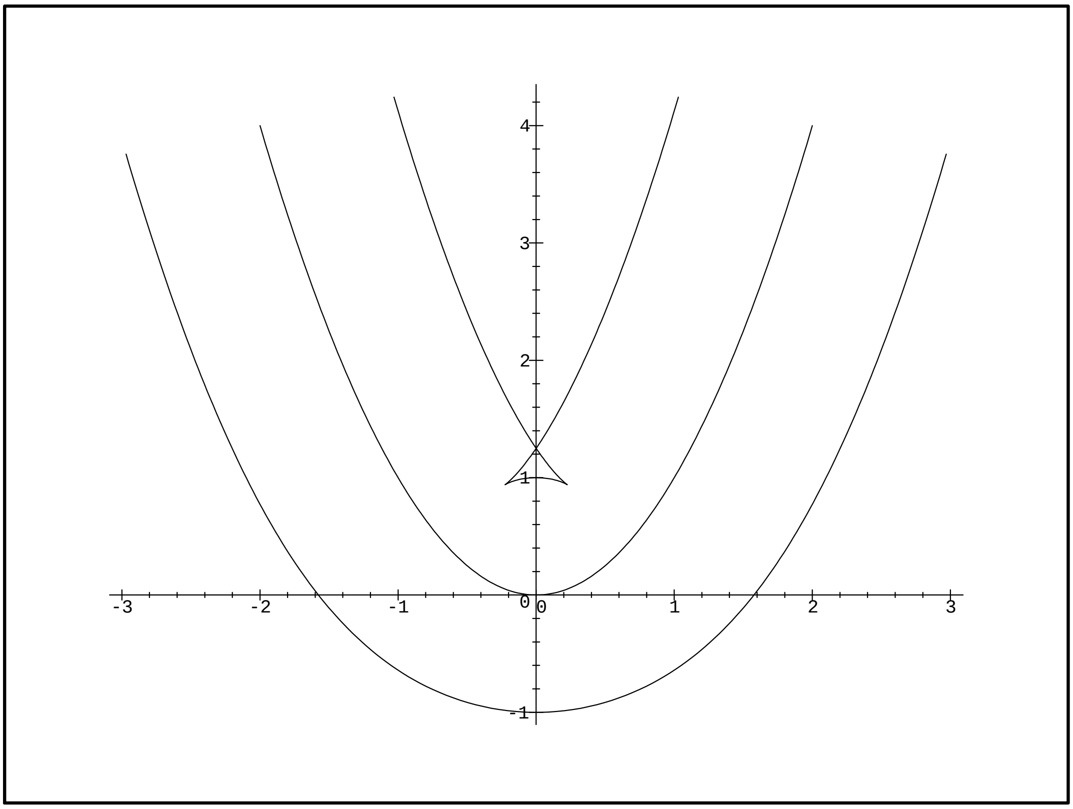

Figure 1.1: Offset to a parabola

1.2 Why Consider Offsets?

Offset curves (and surfaces) or parallel curves play an important rôle in computer-aided geometric design (CAGD). They arise in several practical applications:

- numerical-control-machining:
  describing the path a cylindrical tool must traverse to cut out a prescribed curve.
- tolerance analysis

- geometric control
- robot path planning
- geometric optics
- formulation of geometric procedures, like growing/shrinking, blending, filleting, etc.

1.3 Historical Remarks

The term *parallel curves* was apparently introduced by Leibniz [14] in *Generalia de natura linearum, anguloque contactus et osculi provocationibus aliisque cognatis et eorum usibus nonnullis.* In elementary texts on differential geometry [2], and in some books on algebraic geometry [21], one investigates the concept of curves *parallel* to a given plane curve. However, any detailed analysis of their geometry is usually regarded as being of peripheral interest.

In the field of CAGD, such loci are more commonly known as *offset curves*, and have received considerable attention since they describe the displaced or offset path that the center of a cylindrical tool must traverse to cut out a prescribed curve.

Most of the CAGD literature on offset curves in the 1980s is motivated by the observation that offsets to rational plane curves are not rational themselves (in general), and must therefore be approximated in a piecewise-rational manner for compatibility with the data structures and algorithms of typical computer-aided design systems (Klass [13], Tiller & Hanson [25], Hoschek [10] [11], Pham [17]). So at this time offset curves were only treated numerically.

In 1990 Farouki and Neff published two papers on analytic and on algebraic properties of offset curves ([5] and [6]). This was the starting point for building up a more general theory of offsets. In the algebraic treatment of offset curves there occur several non-trivial problems, such as the determination of the degree and the singularities by the properties of the generator curve. Still in 1994 Farouki and Sederberg wrote a paper [3] dealing only with the properties of the offset to a parabola.

Subsequently several mathematicians investigated the rationality of offset curves. The main difficulty is that the rationality of the original variety is not preserved.

Analyzing the concept of Pythagorean hodograph curves (polynomially parametrized curves whose normal vector has polynomial length), introduced by Farouki and Sakkalis [8], several relevant improvements have been achieved in this context, see [3] [7]. Generalizing to rational normal length, Pottmann gives explicit formulas to produce rational plane curves and surfaces whose offsets have rational components [20]. In [22] Sendra and Sendra present an algorithm for deciding whether the offset to a rational plane curve is rational and, in case of rationality, for computing a rational representation by reparametrizing the generator curve. For deciding the rationality of an offset curve the notion of Pythagorean hodographs (i.e. rationally parametrized curves whose normal vector has rational length) play an important rôle.

By considering higher dimensional spaces and applying a direct isometry to the normal vector, Sendra et al. extend the classical notion of offset to the concept of generalized offset to hypersurfaces [1] and give criteria for the rationality of such offset hypersurfaces. Another possibility to extend the notion of offsets is to consider the pipe surface (constant distance) or the canal surface (rational distance) to a space curve. In [15] and [19] Pottmann et al. prove that pipe surfaces and canal surfaces to rational space curves are rational.

Chapter 2

Basic Theory of Offset Curves

2.1 A Formal Definition

From an algebraic point of view our first description of offset curves has several disadvantages:

- It is not very general since it considers only parametrized generator curves, i.e. curves of genus 0. Since we want to deal with offset curves to any algebraic curve, we should define them by an implicit representation.
- At singular points the normal vector is not defined (the derivatives of f_1 and f_2 vanish simultaneously). So our first description cannot handle singularities.
- There appears a unit vector in this formulation, i.e. it involves a square root. Hence the described curve may be not rational (later we will see that it is a part of an algebraic curve which may be not rational).

So let us try a different approach: given a plane algebraic curve $\mathcal{C}$ in implicit form, i.e. as the set of zeros of a bivariate polynomial $f(u_1, u_2) = 0$, we consider a family of circles c of radius d, each of whose centers is constrained to lie on the curve $\mathcal{C}$. The *envelope* of this family contains the set of points whose distance from f is d. Figure 2.1. illustrates this idea of constructing the envelope.

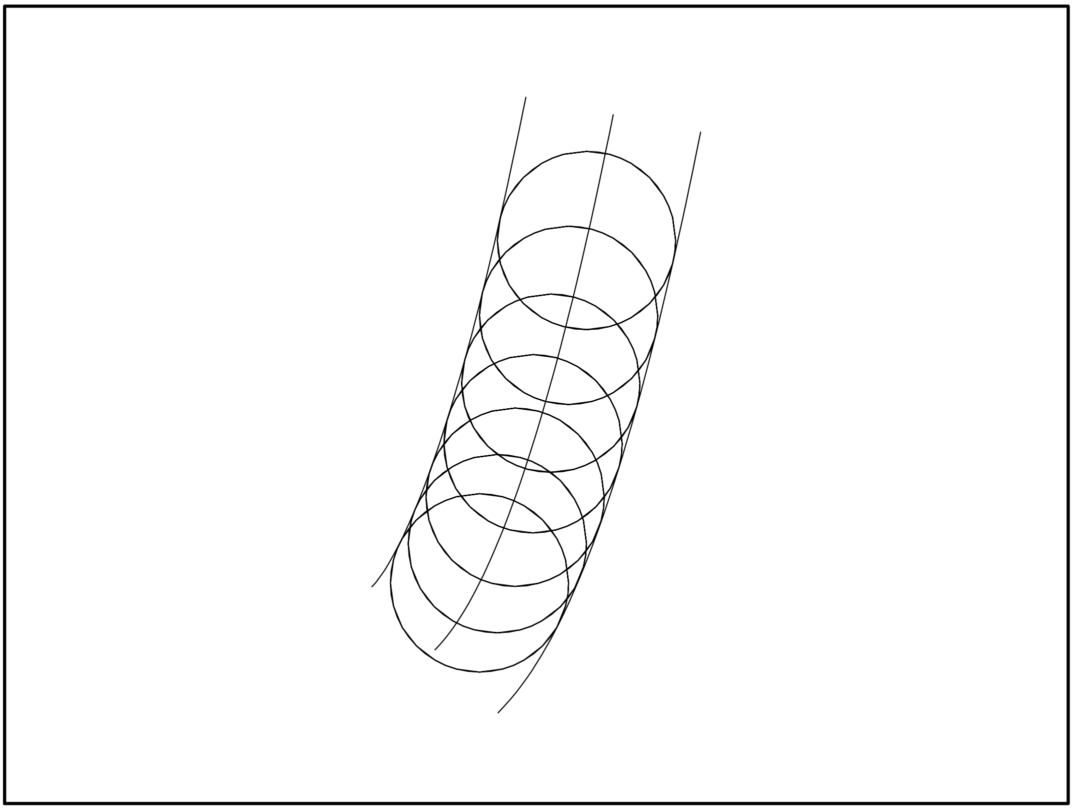

Figure 2.1: The envelope of a curve

From now on let K be a field of characteristic 0, $\bar{K}$ the algebraic closure of K and $\mathbb{A}^2$ the affine plane over $\bar{K}$.

The previous considerations lead to the following definition.

Definition 2.1 *Given an irreducible algebraic curve $\mathcal{C}$ defined by $f(u_1, u_2) = 0$ in $\mathbb{A}^2$, called the* generator *or* original *curve, a distance d and the system*

$$f(u_1, u_2) = 0 \tag{2.1}$$

$$(x - u_1)^2 + (y - u_2)^2 - d^2 = 0 \tag{2.2}$$

$$-\frac{\partial f}{\partial u_2}(u_1, u_2)(x - u_1) + \frac{\partial f}{\partial u_1}(u_1, u_2)(y - u_2) = 0 \tag{2.3}$$

defining the algebraic set

$$\mathcal{A}_d = \{(x, y, u_1, u_2) | (x, y, u_1, u_2) \textit{ satisfy } (2.1), (2.2) \textit{ and } (2.3)\} \subseteq \mathbb{A}^4$$

The envelope $\mathcal{E}_d$ *of* $\mathcal{C}$ *at distance* d *is the result of eliminating* u_1 *and* u_2 *from the system above.* □

Remark. Definition 2.1 has an obvious geometric interpretation. (2.1) constrains (u_1, u_2) to lie on the curve. Then (2.2) generates a circle of distance d around this point, which is intersected with the normal line of $\mathcal{C}$ in (u_1, u_2) described by (2.3).

If $\mathcal{C}$ is given in parametrized form $u_1 = f_1(t), u_2 = f_2(t)$, the system (2.1)-(2.3) is equivalent to

$$(x - f_1(t))^2 + (y - f_2(t))^2 - d^2 = 0 \tag{2.4}$$

$$\frac{\partial f_1}{\partial t}(t)(x - f_1(t)) + \frac{\partial f_2}{\partial t}(t)(y - f_2(t)) = 0 \tag{2.5}$$

and the envelope is derived by eliminating t.

Remark. Since only the square of the distance d occurs in the equation (2.4), the envelope is obviously not influencend by the sign of d. This means we consider the envelope to the curve $\mathcal{C}$ at distance d on both sides simultaneously and in fact the parallel curve on both sides does not factor into an interior and exterior part in general.

Lemma 2.1 *Let* $\mathcal{C}$ *be an irreducible algebraic curve in* $\mathbb{A}^2$*, defined by* $f(u_1, u_2) = 0$*. Let* $g(x, y)$ *be the defining polynomial of the envelope* $\mathcal{E}_d$ *of* $\mathcal{C}$ *at distance* d*, and* (a_1, a_2) *be a singular point of* $\mathcal{C}$*. Then* $(x - a_1)^2 + (y - a_2)^2 - d^2$ *is a factor of* $g(x, y)$ *(the circle defined by this quadratic polynomial is called* extraneous circle*).*

Proof. If (a_1, a_2) is a singular point of $\mathcal{C}$ then

$$\frac{\partial f}{\partial u_1}(a_1, a_2) = \frac{\partial f}{\partial u_2}(a_1, a_2) = 0$$

and equation (2.3) is satisfied for any (x, y). Equation (2.1) is also satisfied. Hence only equation (2.2) remains and we get the desired result. $\square$

Remark. So from Lemma 2.1 we see that the *envelope* is a product of a polynomial and a series of extranous circles around the singular points. For practical applications these circles may be useful but for algebraic treatment we want to eliminate them to be able to talk about rationality, irreducibility, etc.

Definition 2.2 *Given an irreducible algebraic curve $\mathcal{C}$ defined by $f(u_1, u_2) = 0$ in $\mathbb{A}^2$, a distance d and the envelope $\mathcal{E}_d$. If $(a_1, b_1), \ldots, (a_n, b_n)$ are the singular points of $\mathcal{C}$ then let $c_1, \ldots, c_n$ describe the circles with radius d and centers in the singular points, respectively. By Lemma 2.1 the defining polynomial of $\mathcal{E}_d$ is of the form $f \cdot c_1 \cdot \ldots \cdot c_n$. The* offset curve *$\mathcal{O}_d$ is defined by the polynomial f.* $\square$

The following picture shows the curve, the offset curve and an extranous circle in a specific example:

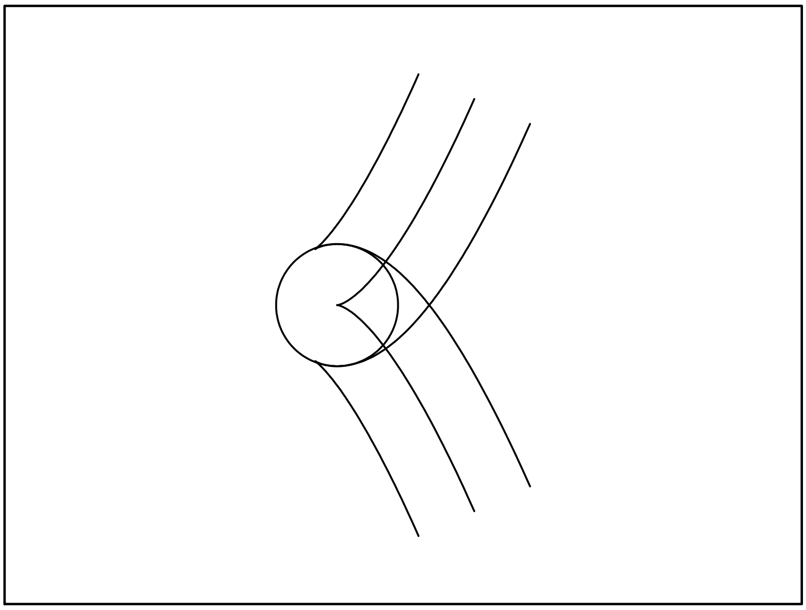

Figure 2.2: The Envelope of (t^2, t^3)

Now in the literature we can also find other approaches to introduce offset curves. One of them is by using constructible sets (see Sendra [1]). We can take the system $\mathcal{A}_d$, subtract the algebraic set generated by the singular points of the curve and project this constructible set to the xy-plane and take the Zariski closure of this set as definition for the offset. This motivates the following definitions.

Definition 2.3 *The algebraic set defined by the following polynomials:*

$$f(u_1, u_2) = 0 \tag{2.6}$$

$$(x - u_1)^2 + (y - u_2)^2 - d^2 = 0 \tag{2.7}$$

$$\frac{\partial f}{\partial u_1}(u_1, u_2) = 0 \tag{2.8}$$

$$\frac{\partial f}{\partial u_1}(u_1, u_2) = 0 \tag{2.9}$$

is denoted by $\mathcal{S}_d$, i.e.

$$\mathcal{S}_d = \{(x, y, u_1, u_2) | (x, y, u_1, u_2) \textit{ satisfy } (2.6), (2.7), (2.8) \textit{ and } (2.9)\}$$

and we and define $\mathcal{B}_d$ by $\mathcal{A}_d \setminus \mathcal{S}_d$ which is a constructible set. □

Since projections map constructible sets onto constructible sets we get:

Definition 2.4 *Let π_1 be the projection*

$$\begin{array}{rccc} \pi_1 : & \mathbb{A}^2 \times \mathbb{A}^2 & \longrightarrow & \mathbb{A}^2 \\ & (x, y, u_1, u_2) & \longrightarrow & (x, y) \end{array}$$

We call the constructible set $\mathcal{Q}_d$ defined by $\pi_1(\mathcal{B}_d)$ the open offset. □

Lemma 2.2 *There exists a one-to-one correspondence between the points of $\mathcal{O}_d$ and the Zariski closure of $\mathcal{Q}_d$.*

Proof: Let (x, y) be a point of $\mathcal{Q}_d$, then for some non singular point (u_1, u_2) of the curve $\mathcal{C}$ the point $(x, y, u_1, u_2) \in \mathcal{A}_d$. Hence $(x, y) \in \mathcal{O}_d$.
On the other hand every point (x, y) of $\mathcal{O}_d$ generated by a non singular point of $\mathcal{C}$ is also a point of $\mathcal{Q}_d$. Since $\mathcal{C}$ has only finitely many singular points, $\mathcal{O}_d$ must be the Zariski closure of $\mathcal{Q}_d$. □

At this point we should think about the correspondence to our first description of offset curves.

Definition 2.5 *Given a rational curve $\mathcal{C}$ defined by $x = f_1(t), y = f_2(t)$ ($f_1(t)$ and $f_2(t)$ rational functions over K) we define the* two-sided parallel curve $\mathcal{P}_d$ *as subset of $\mathbb{A}^2$ generated by:*

$$\mathcal{P}_d(t) := f(t) \pm \frac{d}{m(t)}\mathcal{N}(t) \tag{2.10}$$

where $f(t) = (f_1(t), f_2(t))$, $\mathcal{N}(t) = (-f_2'(t), f_1'(t))$ and $m(t) = \sqrt{f'^2_1(t) + f'^2_2(t)}$ for all except the singular points of $\mathcal{C}$. $\square$

Remark. We have to be careful with this definition. $\mathcal{P}_d(t)$ looks like a rational parametrization of a curve, but the term $m(t)$ is defined by a square root. If there does not exist a rational function $r(t)$ such that $r(t)^2 = m(t)$, the representation given by (2.10) is no rational function. Furthermore this formula gives the impression that the two-sided parallel curve always consists of 2 components, which is not true in general (it also can be irreducibel).

Lemma 2.3 *Given a rationally parametrized generator curve $\mathcal{C}$ defined by $x = f_1(t)$ and $y = f_2(t)$ and denote its normal vector by $\mathcal{N}(t) = (n_1(t), n_2(t))$. If the set $\mathcal{A} = \{t | n_1(t)^2 + n_2(t)^2 = 0\}$ is finite then the Zariski closure of $\mathcal{P}_d$ is the offset curve $\mathcal{O}_d$ to $\mathcal{C}$.*

Proof: Let $\bar{t}$ be a parameter value s.t. the two-sided parallel curve $\mathcal{P}_d(t) = (p_1(t), p_2(t))$ is defined. By setting $\bar{x} = p_1(\bar{t})$ and $\bar{y} = p_2(\bar{t})$ the triple $(\bar{x}, \bar{y}, \bar{t})$ satisfies the equations (2.4) and (2.5). Since $(f_1(\bar{t}), f_2(\bar{t}))$ is no singular point on $\mathcal{C}$ ($\mathcal{P}_d$ is not defined in singularities of the generator curve) $(\bar{x}, \bar{y})$ has to be a point on the offset curve $\mathcal{O}_d$. Hence $\mathcal{P}_d \subseteq \mathcal{O}_d$.
On the other hand let $(\bar{x}, \bar{y})$ be a point on $\mathcal{O}_d$. Then by definition of $\mathcal{O}_d$ there must exist a parameter value $\bar{t}$ s.t. the triple $(\bar{x}, \bar{y}, \bar{t})$ satisfies the equations (2.4) and (2.5). If $\bar{t} \notin \mathcal{A}$, by plugging in $\bar{x}$ for $p_1(\bar{t})$ and $\bar{y}$ for $p_2(\bar{t})$ in (2.10) we see that $(\bar{x}, \bar{y})$ is a point on $\mathcal{P}_d$. $\mathcal{A}$ is finite and, in case we do not compute the offset $\mathcal{O}_d$ to a circle with radius d, there correspond at most 2 pairs (x, y) to each t. Hence we miss only finitely many points on $\mathcal{P}_d$. $\mathcal{O}_d$ is an algebraic curve, hence $\bar{\mathcal{P}}_d = \mathcal{O}_d$. The offset $\mathcal{O}_d$ to a circle with radius d is given by the center of the generator circle and a circle with radius $2d$. Computing the two-sided offset curve $\mathcal{P}_d$ to this generator we see that $\mathcal{O}_d$ is the algebraic closure of $\mathcal{P}_d$ in this case, too. $\square$

Remark. In fact, we loose only finitly many points of the offset curve (corresponding to the singular points). So we will not distinguish between offset curves and two-sided parallel curves.

2.2 Construction of Offset Curves

The obvious way to compute the polynomial describing the offset curve $\mathcal{O}_d$ of a curve $\mathcal{C}$ would be to eliminate all variables except x and y from the defining equations by some method of elimination theory (e.g. by Gröbner bases, resultants, characteristic sets, ...). Then we divide the resulting polynomial by the polynomials describing circles of radius d around the singular points and get the desired offset curve.

However, from a computational point of view this is not very efficient. If the curve $\mathcal{C}$ is given in parametrized form, we have two defining equations and we have to eliminate the variable t. Hence we can use resultants with respect to t. Also there is a possibility to avoid the extraneous circles.

First we consider polynomial parametrizations. We already noted that the extraneous circles are caused by the singular points, because of the vanishing of the derivatives. This defect can be avoided as follows.

Theorem 2.4 *Let the algebraic curve $\mathcal{C}$ defined by $f(t) = (f_1(t), f_2(t))$ be a polynomially parametrized generator curve,*

$$P(t,x,y) = (x - f_1(t))^2 + (y - f_2(t))^2 - d^2$$

$$r_1(t) = \frac{f_1'(t)}{\gcd(f_1'(t), f_2'(t))}, \quad r_2(t) = \frac{f_2'(t)}{\gcd(f_1'(t), f_2'(t))}$$

$$Q(t,x,y) = r_1(t) \cdot (x - f_1(t)) + r_2(t)(y - f_2(t))$$

then

$$\mathcal{O}_d(x,y) = \mathrm{res}_t(P(t,x,y), Q(t,x,y)) \tag{2.11}$$

Proof: We start from the equations 2.4 and 2.5. We already proved that the extraneous circles contained in the envelope of these equations are caused by the singular points of the original curve, i.e. by the simultaneous vanishing of f_1' and f_2', i.e. by common factors of f_1' and f_2'. Hence by dividing by the gcd of f_1' and f_2' we avoid the extraneous factors without changing the other solutions. By the resultant computation we can eliminate t and we get the desired offset curve. □

By a detailed analysis of the structure of the Sylvester determinant we obtain:

Corollary 2.5 *The offset curve $\mathcal{O}_d$ given by (2.11) is of degree $n_O = 4n - 2v$, where n is the degree of the polynomial generator curve $f(t) = (f_1(t), f_2(t))$ and v is the degree of* $\gcd(f_1'(t), f_2'(t))$.

Proof: See Farouki and Neff [6]. □

For rationally given generator curves, avoiding the circles is rather subtle. We just state the result and refer the reader to Farouki and Neff [6] for a detailed discussion.

Theorem 2.6 *Let $\mathcal{C}$ be a rational generator curve given by the rational parametrization*

$$r(t) = (\frac{f_1(t)}{g(t)}, \frac{f_2(t)}{g(t)})$$

with $gcd(f_1, f_2, g) = 1$.
Let

$$\begin{aligned} R(t) &= f_1^2(t) + f_2^2(t) & S(t) &= \gcd(g(t), R(t)) \\ G_1(t) &= \frac{g(t)}{S(t)} & R_1(t) &= \frac{R(t)}{S(t)} \\ U(t) &= g(t) \cdot f_1'(t) - g'(t) \cdot f_1(t) & V(t) &= g(t) \cdot f_2'(t) - g'(t) \cdot f_2(t) \\ U_1(t) &= \frac{U(t)}{\gcd(U(t), V(t)} & V_1(t) &= \frac{V(t)}{\gcd(U(t), V(t)} \end{aligned}$$

$$\begin{aligned} P(t,x,y,w) &= g(t)G_1(t)(x^2+y^2) - 2G_1(t)(f_1(t)x+f_2(t)y)w + \\ &\quad (R_1(t) - d^2 g(t)G_1(t))w^2 = 0 \\ Q(t,x,y,w) &= G_1(t)(U_1(t)x+V_1(t)y) - (U_1(t)f_1(t)+V_1(t)f_2(t))w/S(t) = 0 \end{aligned}$$

There are N distinct parameter values $t_1 \ldots t_N$ for which the conditions $g(t) = 0$ and $G_1(t) \neq 0$ are satisfied. For each t_i let

$$e_i(x,y,w) = f_1(t_i)x + f_2(t_i)y - \frac{R_1(t_i)}{2G_1(t_i)}w$$

then

$$\mathcal{O}_d = \frac{\mathrm{res}_t(P(t,x,y,w), Q(t,x,y,w))}{w^{2(n-k)} \prod_{i=1}^{N} e_i(x,y,w)} \tag{2.12}$$

Proof: See Farouki and Neff [6]. □

Corollary 2.7 *The offset curve $\mathcal{O}_d$ given by (2.12) is of degree $n_O = 6n - 4 - k - 2l - 2m - 2s - N$, where n is the degree of the generator curve, k is the degree of* $\gcd(g, f_1^2 + f_2^2)$*, $l + m$ is the degree of* $\gcd(gf_1' - g'f_1, gf_2' - g'f_2)$*, $s =$* $\max(0, |\max(\deg(f_1), \deg(f_2)) - \deg(g)| - 1)$ *and N is the number of distinct parameter values satisfying the conditions above.*

Proof: See Farouki and Neff [6]. □

2.3 Geometric Properties of Offset Curves

Already in the first example (Figure 1.1) we have seen that the offset curve to a simple generator curve can have interesting geometric properties. There occur some special points like cusps and self-intersections on the offset. Now we want to investigate which properties of the generator curve cause this behavior.

These geometric properties correspond mainly to the analytic structure of the generator. Here we state only the main results, for a detailed analysis we refer to Farouki and Neff [5].

We start from the generator curve $\mathcal{C}$ parametrized by the rational functions $f(t) = (f_1(t), f_2(t))$. Then by Lemma 2.1.3 $f_o(t) = f(t) \pm \frac{d}{m(t)}\mathcal{N}(t)$ generates almost all (i.e. all except finitely many) points of the offset curve to $\mathcal{C}$. By $\kappa(t)$ and $\mathcal{N}_o(t)$ we denote the curvature of $f(t)$ and the normal vector of $f_o(t)$ at parameter value t, respectively. Now $\kappa(t)$ and $\mathcal{N}_o(t)$ can be expressed as

$$\kappa(t) = \frac{f_1'(t)f_2''(t) - f_1''(t)f_2'(t)}{(f_1'(t)^2 + f_2'(t)^2)^{\frac{3}{2}}}$$

and

$$\mathcal{N}_o(t) = \frac{1+\kappa d}{|1+\kappa d|}\mathcal{N}(t).$$

From this equation we obtain immediatly that the normal vector at the corresponding points of the offset and the generetor curve differ at most by the sign, when $\frac{1+\kappa d}{|1+\kappa d|}$ is defined.

When $\kappa(\bar{t}) = -1/d$ the normal vector of the offset curve is not defined at parameter value $\bar{t}$ which leads to a singular point at the offset curve. Here we distiguish 2 cases. If $\kappa'(\bar{t}) \neq 0$ the tangent of the offset curve suffer a sudden inversion as we traverse such a point and we obtain a *cusp* on the offset curve. On the other hand if $\kappa'(\bar{t}) = 0$ the tangent is not inverted on passing through $\bar{t}$. In this case we will call this singular point an *extraordinary point.*

In the exceptional case that several consecutive derivatives of the generator curvature vanish at some parameter value $\bar{t}$ for which $\kappa(\bar{t}) = -1/d$, $\kappa'(\bar{t}) = \kappa''(\bar{t}) = \cdots = \kappa^{(m)}(\bar{t}) = 0$ say, then $\bar{t}$ will be a cusp or extraordinary point on the offset according to whether m is even or odd, respectively.

Simple examples which illustrate the above concepts are the offsets to the parabola $f(t) = (t, t^2)$ at various distances d. The unit normal vector n and the curvature κ are given by

$$n(t) = (2t, -1)/\sqrt{1+4t^2} \qquad \text{and} \qquad \kappa(t) = 2(1+4t^2)^{-3/2}$$

Thus κ increases uniformly with t at first, attaining the maximal value $\kappa_{max} = 2$ at $t = 0$, and then decreases uniformly. When $0 > d > -1/2$, the offset has no singular points in $\mathbb{R}^2$ since $\kappa(t) < -1/d$ for all t. When $d < -1/2$, there are two symmetric values $\pm t_c$ at which $\kappa = -1/d$ and $\kappa' \neq 0$, which induce cusps on the offset. Finally, if $d = -1/2$ we have $\kappa(0) = -1/d$ and $\kappa'(0) = 0$, so $t = 0$ is an extraordinary point on the offset. These cases are illustrated in Figure 2.3.

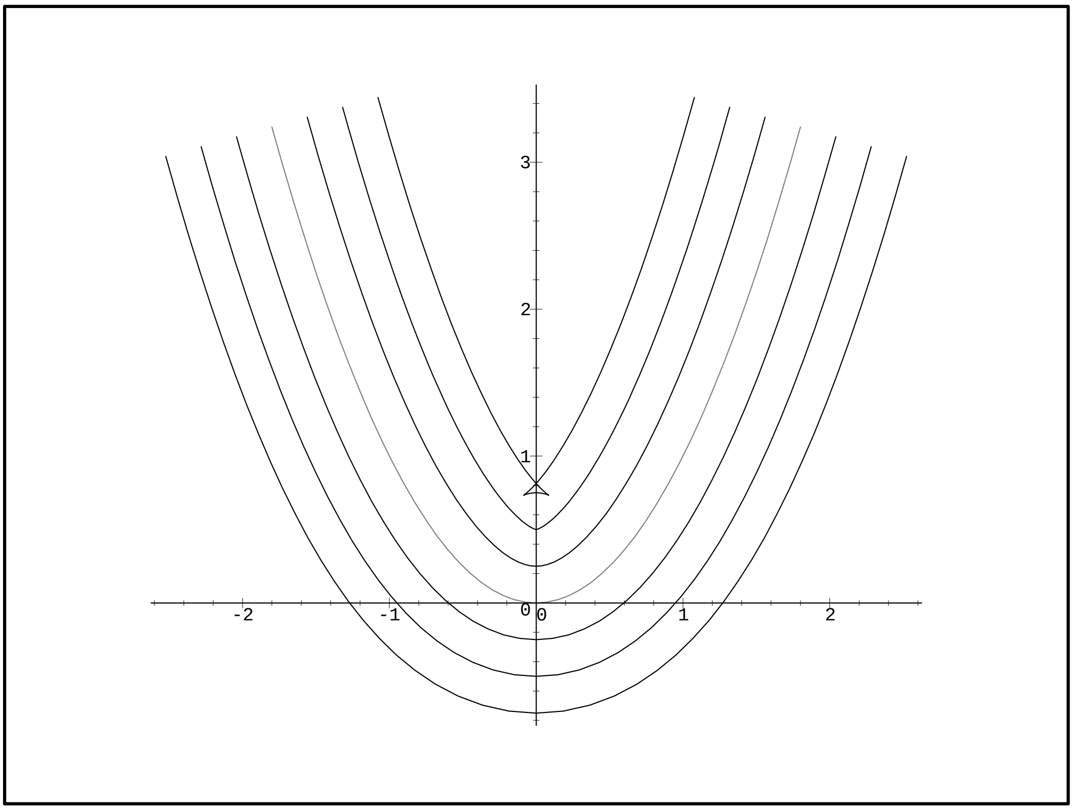

Figure 2.3: Offsets to the parabola at distances 1/4, 1/2 and 3/4

An interesting feature of offsets is also the occurence of *self-intersections*. If the generator curve $\mathcal{C}$ can be parametrized by $f(t)$, and $n(t)$ denotes the unit normal vector to the generator curve, then such self-intersections arise whenever there exist distinct parameter values u and v such that the equation

$$f(u) + dn(u) = f(v) + dn(v)$$

holds. In [6] Farouki and Neff present an algorithm, based on resultant and real-root isolation techniques, for detecting the self-intersections of an offset curve.

The methods described in this section are important for applications like CAGD since they provide the possibility to compute the singular points of offset curves in the real plane exactly without the knowledge of an algebraic formula of the offset. These singular points are needed for a stable approximation of offsets. But if the offset curve is given by the implicit equation $f_o(x, y) = 0$ the set of all singular points (self-intersections, cusps, ...) obviously is given by the solutions of

$$f_o = \frac{\partial f_o}{\partial x} = \frac{\partial f_o}{\partial y} = 0.$$

In the next section we will give a detailed analysis of the singular points of the offset to the parabola and we will see that there is more structure contained in the offset curve than one can observe by just looking at the real plane.

2.4 Examples and Special Cases

Example 2.1 Consider the cubic curve $f(t) = (t^2, t^3)$. Then $f_1'(t) = 2t$, $f_2'(t) = 3t^2$ and $\gcd(f_1', f_2') = t$ (corresponding to the singular point at the origin). We get:

$$P(t, x, y) := t^6 + t^4 - 2\,y\,t^3 - 2\,x\,t^2 + x^2 + y^2 - d^2$$

$$Q(t, x, y) := -3\,t^4 - 2\,t^2 + 3\,y\,t + 2\,x$$

Computing their resultant we obtain:

$$\begin{aligned}\mathcal{O}_d(x, y) := {} & 729\,d^8 + (\,-2916\,x^2 - 216 - 729\,y^2 - 1944\,x\,)\,d^6 + (6318\,y^2\,x \\ & + 2187\,y^2\,x^2 + 4104\,x^3 + 16 + 288\,x + 1728\,x^2 + 1188\,y^2 \\ & + 4374\,x^4)d^4 + (-32\,y^2 - 4374\,y^4\,x - 32\,x^3 - 2916\,x^6 - 504\,y^2\,x \\ & - 2376\,x^5 - 2484\,y^2\,x^2 - 2187\,y^2\,x^4 - 4860\,y^2\,x^3 - 504\,x^4 - 1701\,y^4) \\ & d^2 + 16\,x^6 + 16\,y^4 - 1458\,y^2\,x^5 + 729\,y^4\,x^2 + 216\,y^4\,x + 729\,y^2\,x^6 \\ & + 729\,y^6 + 729\,x^8 - 1458\,y^4\,x^3 - 32\,y^2\,x^3 - 432\,y^2\,x^4 + 216\,x^7\end{aligned}$$

We observe that the offset curve is of degree 8 only, whereas the offset to a general cubic, with $\gcd(f_1'(t), f_2'(t)) = 1$, would be of degree 10. In fact if we neglected to divide by $\gcd(f_1'(t), f_2'(t))$ we would obtain the envelope curve which is the offset curve multiplied by the extraneous factor $x_2+y_2-d_2$ representing a circle of radius d centered on the cusp of the generator.

This offset curve is irreducible. The real locus of the curve is illustrated by the first picture of the following figure, which is deceptive in certain respects. First, the cusps of the offsets are quite unrelated to the cusp of the generator. They are induced by the generator having a certain critical curvature. Second, if we plot the exterior and interior parallel curves seperately it is apparent that these loci are not even continuous.

Furthermore we observe that only even powers of d occur in the offset formula. Thats clear since we consider the offset curves on both sides simultaneously.

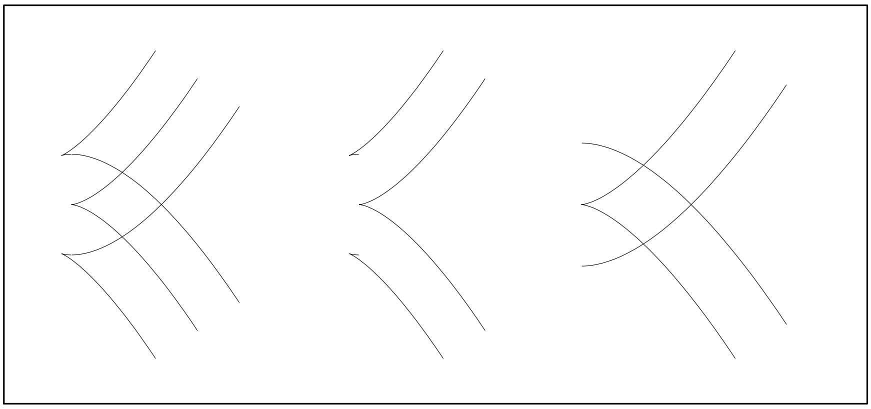

Figure 2.4: The offset, the exterior and the interior parallel curve

One further comment on this example: if we choose the offset distance $d = 2/3\sqrt{3}$, the constant term of the defining polynomial of the offset curve vanishes, indicating that the offset passes through the origin (in fact, it has a double point there). Referring to the figure above it is not clear how the offset to any point on the generator can end up at the origin if $d \neq 0$. To explain this we must bear in mind that the generator, regarded as an algebraic curve, is not limited merely to a real locus. Ordinarily, real points of the generator curve correspond to real points of the offset, but it is conceivable also that two complex conjugate points of the generator might yield a real isolated point on the offset.

Example 2.2 Now we want to determine the offset to the unit circle at distance $d \neq 1$ by application of Theorem 2.6. For the familiar rational parametrization of the unit circle:

$$f_1(t) := 1 - t^2$$

$$f_2(t) := 2\,t$$

$$g(t) := 1 + t^2$$

We get the polynomials:

$$P(t,x,y,w) := \left((\, x + w\,)^2 + y^2 - d^2\, w^2 \right)\, t^2 - 4\, w\, y\, t + (\, x - w\,)^2 + y^2 - d^2\, w^2$$

$$Q(t,x,y,w) := -y\, t^2 - 2\, x\, t + y$$

and on eliminating t we obtain, apart from a constant,

$$\begin{aligned} Res_t(P,Q) &:= (\, x^2 + y^2\,)\,(\, x^2 + y^2 - d^2\, w^2 + 2\, d\, w^2 - w^2\,) \\ &\quad (\, x^2 + y^2 - 2\, d\, w^2 - d^2\, w^2 - w^2\,) \end{aligned}$$

The second and the third factors above certainly agree with expectation, but the first is rather enigmatic. Indeed, since $g(t) = 0$ and $G_1(t) = 1$ for $t_1 = +i$ and $t_2 = -i$ by Theorem 2.6 we get some extraneous factors. In particular, this leads to the factors $e_1 = 2(x + yi)$ and $e_2 = 2(x - yi)$, and dividing them out this leads to the expected result:

$$\mathcal{O}_d(x,y) := \left(x^2 + y^2 - (\,1 + d\,)^2\right)\, \left(x^2 + y^2 - (\,1 - d\,)^2\right)$$

on setting $w = 1$ for the inhomogeneous form. Furthermore we are in the exceptional case that this offset curve factors into two components representing the interior and the exterior offsets individually.

Now consider the case that $d = 1$. By using the above formula we would obtain:

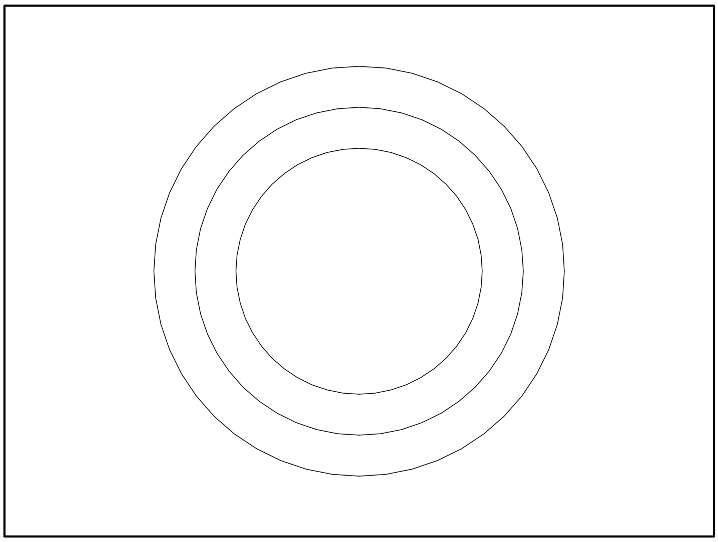

Figure 2.5: The offset to the circle

$$(x^2 + y^2 + 4)(x^2 + y^2)$$

defining the offset curve. But this factors into

$$(x^2 + y^2 + 4)(x + iy)(x - iy)$$

Hence we obtain an algebraic set consisting of 3 components, namely one circle and 2 lines. By intersecting the normal to any point of the circle with the line $x + iy$ or $x - iy$ we always get only the origin. Plugging in the offset distance 1 in the defining equations and computing the offset curve at distance 1 leads to the correct solution:

$$\mathcal{O}_1 = \{(0,0)\} \cup \{(x,y) | x^2 + y^2 - 4 = 0\}$$

Remark. The problem arising from setting the distance $d = 1$ in the example above stems from the following fact. We compute the offset to a given generator curve for the transcendental distance d. When plugging in special values for d this may change the structure of the solution and therefore lead to a wrong result. However, this can happen only for a finite number of values for d.

Example 2.3 We want to compute the offset at distance 2 of the line $y - ix$ which can be parametrized by

$$f_1(t) = t \qquad f_2(t) = it$$

with the normal vector

$$n_1(t) = -i \qquad n_2(t) = 1$$

Using the method described in Theorem 2.4 we obtain the polynomials

$$P = (x - t)^2 + (y - it)^2 - 1 \qquad Q = x + iy$$

and as result

$$\mathcal{O}_d = x + iy$$

Interestingly enough the offset curve is again the same curve as the generator curve. But now consider the two-sided parallel curve $\mathcal{P}_1$ to this line. We observe that $m(t) = n_1(t)^2 + n_2(t)^2 = (-i)^2 + 1^2 = 0$ for all parameter values $t \in \bar{K}$, i.e. $\mathcal{P}_d$ is not defined anywhere.

Remark. The example above is an example for the exceptional case of Lemma 2.3, i.e. $\{t | n_1(t)^2 + n_2(t)^2 = 0\}$ is infinite. Fortunately the following lemma tells us that such exceptional cases are restricted to a very small family of lines, such that we do not have to worry about the discrepancy of these definitions.

Lemma 2.8 *The lines parametrizable by $f_1(t) = t$ and $f_2(t) = \pm it + c$ with $c \in \bar{K}$ are all curves such that $\mathcal{A} = \{t | (-f_2'(t))^2 + f_1'(t)^2 = 0\}$ is infinite.*

Proof: Let $\mathcal{C}$ be a curve parametrized by $(f_1(t), f_2(t))$. Its normal vector is given by $\mathcal{N}(t) = (-f_2'(t), f_1'(t))$. Assume that $(-f_2'(t))^2 + f_2'(t)^2 = 0$ for infinitely many $t \in \bar{K}$, i.e. $f_2'(t) = \pm i f_1'(t)$ for infintely many $t \in \bar{K}$. Since we are dealing with rational functions and they agree on infinitely many values they have to be the same. By integrating both sides of the equation we obtain:

$$f_2(t) = \pm i f_1(t) + c$$

i.e. the curve $\mathcal{C}$ can be parametrized by $(f_1(t), \pm i f_1(t) + c)$ which parametrizes the same locus as $(t, \pm it + c)$, the desired form of the curve. $\square$

Example 2.4 Now we give a detailed analysis of the offset to the parabola $f(t) = (t, kt^2)$ which was analyzed by Farouki and Sederberg [4]. With $f_1'(t) = 1$, $f_2'(t) = 2kt$ and $\gcd(f_1', f_2') = 1$ we obtain:

$$P := x^2 - 2\,x\,t + t^2 + y^2 - 2\,y\,k\,t^2 + k^2\,t^4 - d$$

$$Q := x - t + 2\,k\,t\,y - 2\,k^2\,t^3$$

and taking their resultant with respect to t gives:

$$\begin{aligned}\mathcal{O}_d(x,y) := {} & -k^4\,d + k^4\,y^2 + k^6\,x^4 - 8\,k^6\,d^2 - 8\,k^5\,y^3 + 16\,k^8\,x^6 - 16\,k^8\,d^3 \\ & + 16\,k^6\,y^4 + 32\,k^6\,x^2\,y^2 - 8\,k^6\,d\,y^2 - 2\,k^5\,y\,x^2 - 20\,k^6\,x^2\,d + 8\,k^5\,y\,d \\ & + 16\,k^8\,x^4\,y^2 - 48\,k^8\,x^4\,d + 48\,k^8\,x^2\,d^2 - 32\,k^7\,x^2\,y^3 - 40\,k^7\,x^4\,y \\ & + 16\,k^8\,d^2\,y^2 - 32\,k^7\,d\,y^3 + 32\,k^7\,d^2\,y - 32\,k^8\,x^2\,d\,y^2 + 8\,k^7\,x^2\,y\,d\end{aligned}$$

Since the parabola is a rational curve it is natural to ask if the offset to the parabola is also rational. The attempt to use the definition of the two-sided parallel curve will fail, since the occuring $m(t)$ therein is in our case $\sqrt{1+4k^2t^2}$ and therefore we will not obtain a rational parametrization. But if we reparametrize the given parametrization of the parabola by the quadratic reparametrizing function

$$t(s) = \frac{s^2 - 16k^2}{16k^2 s}$$

we obtain the improper parametrization

$$x = \frac{t^2 - 16k^2}{16k^2 t}$$

$$y = \frac{(t^2 - 16k^2)^2}{256k^3t^2}.$$

Now each point on the parabola corresponds to two parameter values t. In fact, the parabola is doubly traced by this parametrization, it is traversed in opposite directions as t increases from $-\infty$ to 0 and from 0 to $+\infty$. Furthermore,

$$m(t) = \frac{(s^2 + 16k^2)^2}{k^3 s^3}$$

is rational. Using these facts, we obtain, by applying the definition of the twosided parallel curve, the following rational parametrization of the offset curve.

$$x_O := \frac{1}{16}\,\frac{(\,-s^2 - 16\,k^2 + 16\,d\,k^2\,s\,)\,(\,-s^2 + 16\,k^2\,)}{s\,k^2\,(\,s^2 + 16\,k^2\,)}$$

$$y_O := \frac{1}{256} \frac{s^6 - 16\,k^2\,s^4 - 256\,k^4\,s^2 + 4096\,k^6 + 2048\,d\,k^4\,s^3}{(\,s^2 + 16\,k^2\,)\,k^3\,s^2}$$

This example gives us a first idea of how to solve the problem of parametrizing an offset curve. In Chapter 3 we will give a general procedure of parametrizing an offset curve to a rational generator curve if such a parametrization exists.

Now by section 2.3 affine cusps arise when the curature $\kappa(t) = -1/d$ and $\kappa' \neq 0$. The curvature of the parabola is given by $\kappa(t) = \frac{2k}{(1+4k^2t^2)^{\frac{3}{2}}}$. This leads to the equation

$$64k^6t^6 + 48k^4t^4 + 12k^2t^2 - 4k^2d^2 + 1 = 0$$

which has two real roots if $d > \frac{1}{2k}$ and none if $d < \frac{1}{2k}$ (see figure 2.3). Thus, the offset has either two real and four complex cusps or six complex cusps.

Affine nodes of the offset arise when the normal lines at two distinct curve points intersect at distance d from each those points. In the case of the parabola this leads to

$$4k^2t^2 - 4k^2t^2 + 1 = 0.$$

From this equation we can deduce that the offset to the parabola has a self-intersection of the real locus if $d > \frac{1}{2k}$, and an isolated real point where conjugate branches cross if $d < \frac{1}{2k}$. The coordinates of this node are $(0, kd^2 + \frac{1}{4}k)$.

Now in the exceptional case $d = \frac{1}{2k}$ we obtain a triple point at $(0, \frac{1}{2k})$ (two cusps and the node meet in this point) with three coincident tangents. By quadratic transformations it can be verified that this non-ordinary triple point has only regular points in its first neighbourhood, and thus counts for just three ordinary double points. Hence there are always (the equivalent of) seven ordinary double points in the affine plane.

Since the offset to the parabola is rational, its genus g is zero. The genus of the offset curve is

$$g = \frac{1}{2}[(n-1)(n-2) - \sum_i m_i(m_i - 1)]$$

where n is the degree of the offset curve and m_i are the multiplicities of its singular points. The offset to the parabola is of degree 6 and there exist 7 affine double double points. Thus there must exist singular points at infinity.

Consider the homogenized form of the implicit equation of the offset to the parabola

$$
\begin{aligned}
\mathcal{O}_d(x,y,w) := &-k^4\,d\,w^6 + k^4\,y^2\,w^4 + k^6\,x^4\,w^2 - 8\,k^6\,d^2\,w^6 - 8\,k^5\,y^3\,w^3 \\
&+ 16\,k^8\,x^6 - 16\,k^8\,d^3\,w^6 + 16\,k^6\,y^4\,w^2 + 32\,k^6\,x^2\,y^2\,w^2 - 8\,k^6\,d\,y^2\,w^4 \\
&- 2\,k^5\,y\,x^2\,w^3 - 20\,k^6\,x^2\,d\,w^4 + 8\,k^5\,y\,d\,w^5 + 16\,k^8\,x^4\,y^2 \\
&- 48\,k^8\,x^4\,d\,w^2 + 48\,k^8\,x^2\,d^2\,w^4 - 32\,k^7\,x^2\,y^3\,w - 40\,k^7\,x^4\,y\,w \\
&+ 16\,k^8\,d^2\,y^2\,w^4 - 32\,k^7\,d\,y^3\,w^3 + 32\,k^7\,d^2\,y\,w^5 - 32\,k^8\,x^2\,d\,y^2\,w^2 \\
&+ 8\,k^7\,x^2\,y\,d\,w^3
\end{aligned}
$$

At $(x,y,w) = (0,1,0)$ all first derivatives vanish whereas the second derivative with respect to w does not vanish, i.e. it is a double point. Since all other second derivatives vanish, the quadratic form whose factors define the two tangent lines at $(0,1,0)$ is a perfect square, and this double point has the line at infinity, $w = 0$, as both its tangents. Hence it is a non-ordinary double point. By applying standard quadratic transformations we observe that this double point becomes ordinary in the second neighborhood. Thus, the point $(0,1,0)$ is equivalent to three ordinary double points, and the offset to the parabola has the equivalent of a total of 10 double points. So the equation for the genus (2.4) indeed is satisfied.

Chapter 3

Rationality of Offset Curves

3.1 Motivation

We have already provided an algorithmic approach, based on gcd computations and resultants, to compute the implicit equation of an offset to a parametrized generator curve. In CAGD it is common to use data structures which represent the objects in parametric form and many algorithms implicitly assume that the objects are given parametrically. For this reason we investigate the rational parametrizability of offset curves. The main difficulty is that the rationality of the original curve is not preserved (in general). This chapter is mainly based on results of Sendra et al. and we refer to [22] and [1] for more details.

Which rational curves have offsets with rational components? If we consider the defining equation of the two-sided parallel curve

$$\mathcal{P}_d(t) := f(t) \pm \frac{d}{m(t)} \mathcal{N}(t) \tag{3.1}$$

where $f(t) = (f_1(t), f_2(t))$, $\mathcal{N}(t) = (-f_2'(t), f_1'(t))$ and $m(t) = \sqrt{f'^2_1(t) + f'^2_2(t)}$, we observe that the rationality of $m(t)$ will play an important role. Curves with rational $m(t)$ will be called RPH (Rational Pythagorean Hodographs). We will prove that all components of the offset curve are rational if and only if there exists a paramatrization of the original curve which is RPH.

Next we will investigate the structure of offsets to a rational curve $\mathcal{C}$. We will prove that the offsets to a rational curve have some special properties. They are either reducible with two rational components (this can be fastly detected), or rational, or irreducible but not rational. So they cannot factor into two irreducible non-rational components.

When computing the offset curve we do not want to specify the distance in advance. Fortunately neither the rationality nor the reducibility of the offset curve depends on the distance. We will be able to compute rational parametrizations of the offset which hold for any distance (with finitely many exceptions).

The theorems of this chapter are constructive and an algorithm can be derived from them. But in this algorithm we have to compute the implicit offset curve at distance 1 and we have to parametrize this curve which can be a pretty hard computational problem. By introducing a so-called reparametrizing hypersurface we can usually simplify the algorithm.

3.2 Rationality and Rational Pythagorean Hodographs

We start with the characterization of the rationality of the components of the offset by means of the existence of an RPH parametrization of the generator curve. Furthermore we show that if the offset has rational components, then there exists always a rational reparametrization function of degree at most two, that determines a suitable parametrization of the generator curve to paramatrize the offset immediately.

In this chapter let the generator curve always be a rational curve (except in section 5, there we will investigate the rationality of offsets to non-rational generator curves).

Definition 3.1 *Let $\mathcal{P}(t) = (p_1(t), p_2(t)) \in K(t)$ be a rational parametrization of $\mathcal{C}$. Then $\mathcal{P}(t)$ is* RPH (a Rational Pythagorean Hodograph) *iff its normal vector $\mathcal{N}(t) = (n_1(t), n_2(t))$ satisfies $n_1(t)^2 + n_2(t)^2 = m(t)^2$ for some $m(t) \in \bar{K}(t)$, i.e. the normal vector has rational length $m(t)$.* $\square$

Remark. If $\mathcal{P}(t)$ is an RPH parametrization and $\mathcal{Q}(t) = \mathcal{P}(\varphi(t))$ is a reparametrization of $\mathcal{P}(t)$, then the normal vector of $\mathcal{Q}(t)$ is $\varphi'(t)\mathcal{N}(t)$. Therefore $\mathcal{Q}(t)$ is also RPH.

Theorem 3.1 *If $\mathcal{C}$ is rational and there exists an RPH parametrization of $\mathcal{C}$ then all the components of the offset curve $\mathcal{O}_d$ are rational.*

Proof: If there exists an RPH paramatrization $\mathcal{P}(t) = (p_1(t), p_2(t))$ of $\mathcal{C}$,
i.e. $p_1'(t)^2 + p_2'(t)^2 = m(t)^2$, for some $m(t) \in \bar{K}(t)$, then

$$\mathcal{O}_d^-(t) = \mathcal{P}(t) + \frac{d}{m(t)}(-p_2'(t), p_1'(t))$$

$$\mathcal{O}_d^+(t) = \mathcal{P}(t) - \frac{d}{m(t)}(-p_2'(t), p_1'(t))$$

are rational parametrizations of the components of the offset curve. $\square$

Remark. If the offset curve is irreducible then $\mathcal{O}_d^+(t)$ and $\mathcal{O}_d^-(t)$ parametrize the same curve. Obviously this can only happen if $\mathcal{P}(t)$ is a non-proper parametrization of the generator curve (every point on $\mathcal{P}(t)$ has to generate two points on the offset curve). Later we will see that in this case $\mathcal{P}(t)$ is a 2-degree reparametrization of a proper parametrization of the generator curve.

If we can prove also the converse of Theorem 3.1, then we can characterize the rationality of the offset by the RPH property of $\mathcal{C}$. First we should have a look at the construction of the offset curves. Obviously almost every point of the generator curve generates 2 points on the offset curve, resulting from the two intersections of the corresponding circle and normal line. On the other hand it seems that almost every point of the offset is generated by exactly 1 point of the generator curve. But this is not always true. If one computes the offset to the offset of $\mathcal{C}$, the situation is different. Note that, by construction, $\mathcal{C}$ is a component of $\mathcal{O}_d(\mathcal{O}_d)$. Furthermore, almost all points of $\mathcal{O}_d$ are generated by two points of $\mathcal{O}_d$.

In Figure 3.1 we take the offset to the parabola as generator curve $\mathcal{C}$. Now we compute the offset to $\mathcal{C}$ and observe that every point on the parabola will be generated twice.

This phenomenom motivates the following definition.

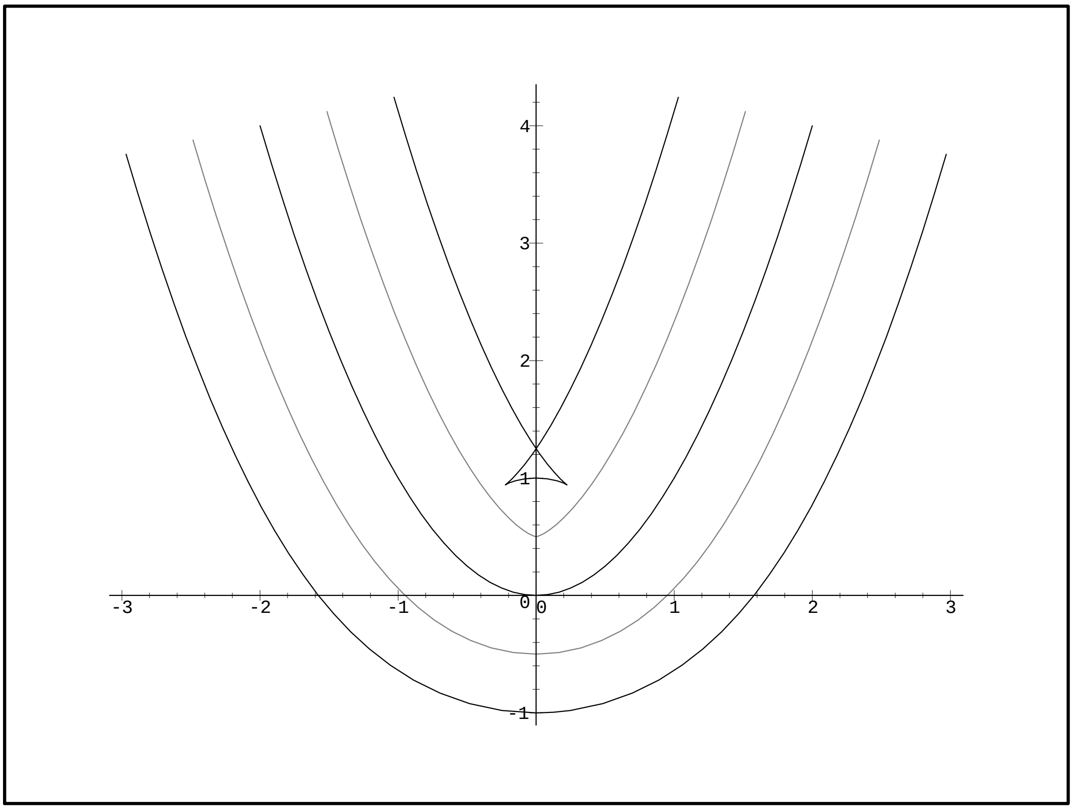

Figure 3.1: special component generated by the offset to the offset

Definition 3.2 *An irreducible component $\mathcal{M}$ of $\mathcal{O}_d$ is called* simple *iff almost all points of $\mathcal{M}$ are generated by exactly one point of $\mathcal{C}$, otherwise $\mathcal{M}$ is called* special.

Remark.

- Special components of offsets are precisely those generated when computing offsets to offsets. Let f be the defining polynomial of a curve $\mathcal{C}$ and g be the defining polynomial of a special component $\mathcal{S}$ of the offset to $\mathcal{C}$ at distance d. If we compute the defining polynomial h of the offset $\mathcal{O}_d$ to $\mathcal{S}$, we observe that f and h must have almost the same roots (i.e. only finitely many exceptions) and therefore $f = h$, i.e. $\mathcal{C} = \mathcal{O}_d$.
- If $\mathcal{O}_d$ is irreducible then it is simple. Assume that $\mathcal{O}_d$ is an irreducible offset curve to $\mathcal{C}$ and $\mathcal{S}$ is a special component of $\mathcal{O}_d$. However if we consider the offset $\mathcal{O}_{2d}^{\mathcal{S}}$ to $\mathcal{S}$ then this has to be another component of $\mathcal{O}_d$ (since a generator

curve and its offset have the same normal vector in the corresponding points). This is a contradiction to the irreducibility of $\mathcal{O}_d$ and therefore an irreducible offset curve has to be simple.

- If one irreducible component of $\mathcal{O}_d$ is special then the other is simple. Thus every offset $\mathcal{O}_d$ to a rational curve has always at least one simple component. Assume that the offset consists of two special components $\mathcal{S}_1$ and $\mathcal{S}_2$. Computing the offset to a special component yields again the generator curve (see above). So $\mathcal{O}_d(\mathcal{S}_1) = \mathcal{C}$ and $\mathcal{O}_d(\mathcal{S}_2) = \mathcal{C}$. But this is impossible because $\mathcal{S}_1$ and $\mathcal{S}_2$ are different curves. So one of the components of the offset curve has to be simple.

Theorem 3.2 *If all the components of the offset curve $\mathcal{O}_d$ are rational then there exists an RPH parametrization $\mathcal{P}(t)$ of $\mathcal{C}$.*

Proof: Let

$$\mathcal{Q}(t) = (q_1(t), q_2(t)),$$

$$\mathcal{P}(t) = (p_1(t), p_2(t))$$

be proper rational parametrizations of a simple component of $\mathcal{O}_d$ and $\mathcal{C}$, respectively, and let

$$\bar{L}(x, y, t) = (x - p_1(t))p_1'(t) + (y - p_2(t))p_2'(t)$$

$$\bar{C}(x, y, t) = (x - p_1(t))^2 + (y - p_2(t))^2 - d^2$$

describe the normal line at $\mathcal{P}(t)$ and the circle of radius d centered on $\mathcal{P}(t)$, respectively.

In order to cross out extraneous factors, take $L(x, y, t)$ and $C(x, y, t)$ as the primitive polynomials w.r.t (x, y) of the numerators of $\bar{L}$ and $\bar{C}$, respectively.

Now we plug in the parametrization of the component of the offset curve and define $M(h, t)$ and $N(h, t)$ as the squarefree factors of the numerators of

$$L(q_1(h), q_2(h), t)$$

$$C(q_1(h), q_2(h), t)$$

respectively. Let $D(h,t) = gcd(M(h,t), N(h,t))$.

Now we want to investigate the degree of D. First we observe that for almost all (i.e. all but a finite number of exceptions) $b \in \bar{K}$, the point $\mathcal{Q}(b)$ has the following property: there exists only one point on $\mathcal{C}$ such that its normal line passes through $\mathcal{Q}(b)$ (since $\mathcal{Q}(b)$ is a simple component there are at most finitely many exceptions corresponding to the singularities of the offset and intersection points of the two components of the offset, if they exist). Since $\mathcal{P}$ is proper, for almost all $b \in \bar{K}$ the polynomials $M(b,t)$ and $N(b,t)$ have only one common root. Thus

$$deg_t(gcd(M(b,t), N(b,t)) = 1$$

for almost all $b \in \bar{K}$, hence $deg_t(D) = 1$.

Similar considerations lead to $deg_h(D) \leq 2$. Here for almost all $a \in \bar{K}$ the point $\mathcal{P}(a)$ generates two points on the offset. Our parametrization $\mathcal{Q}$ describes only one component of the offset. So two cases can appear. If the offset has only one component, for almost all $a \in \bar{K}$ the polynomials $M(h,a)$ and $N(h,a)$ have two common roots, else if the offset has two compenents they have one common root. Hence $deg_h(D) \leq 2$.

Let $t = \varphi(h)$ be the solution of the equation $D(h,t) = 0$ (φ is a rational function of degree at most two). We compute the reparametrization $\tilde{\mathcal{P}} = \mathcal{P}(\varphi)$ and prove that $\tilde{\mathcal{P}}$ is RPH. We observe that

$$(q_1(t) - \tilde{p}_1(t))\tilde{p}_1'(t) + (q_2(t) - \tilde{p}_2(t))\tilde{p}_2'(t) = 0$$

and

$$(q_1(t) - \tilde{p}_1(t))^2 + (q_2(t) - \tilde{p}_2(t))^2 - d^2 = 0.$$

Furthermore, since $\tilde{\mathcal{P}}$ and $\mathcal{Q}$ parametrize different curves, the polynomials $\triangle_1(t) = q_1(t) - \tilde{p}_1(t)$ and $\triangle_2(t) = q_2(t) - \tilde{p}_2(t)$ cannot vanish simultaneously. Let $\triangle_1 \neq 0$. Then

$$\tilde{p}_1'(t)^2 + \tilde{p}_2'(t)^2 = \frac{d^2}{\triangle_1^2}\tilde{p}_2'(t)^2$$

Hence there exists an RPH parametrization of the curve $\mathcal{C}$. $\square$

Corollary 3.3 *If all the components of $\mathcal{O}_d$ are rational, and $\mathcal{P}$ is a proper parametrization of $\mathcal{C}$, then there exists $\varphi \in \bar{K}(t)$ of degree at most two such that $\mathcal{P}(\varphi)$ has normal vector with rational length.*

Proof: Follows from the proof of Theorem 3.2. $\Box$

3.3 Properties of Rational Offsets

Proposition 3.4 *If $\mathcal{O}_d$ is reducible then every simple component is birationally equivalent to $\mathcal{C}$.*

Proof: Let $\mathcal{M}_1$ be a simple component of $\mathcal{O}_d$. We consider the following diagram:

$$\begin{array}{ccc} & \mathcal{B}_d = \Gamma_1 \cup \Gamma_2 \subset \bar{K}^2 \times \bar{K}^2 & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ \mathcal{O}_d = \mathcal{M}_1 \cup \mathcal{M}_2 \subset \bar{K}^2 & & \mathcal{C} \subset \bar{K}^2 \end{array}$$

where $\mathcal{B}_d$ is the set introduced in Definition 2.3., $\Gamma_i = \pi_1^{-1}(\mathcal{M}_i) \quad i = 1, 2$, and π_1, π_2 are the natural projections. Now, one simply has to observe that, although $\pi_2 : \mathcal{B}_d \longrightarrow \mathcal{C}$ is $2:1$, $\pi_2 : \Gamma_1 \longrightarrow \mathcal{C}$ is $1:1$, and that since $\mathcal{M}_1$ is simple then $\pi_1 : \Gamma_1 \longrightarrow \mathcal{M}_1$ is a birational map. $\Box$

Theorem 3.5 *If $\mathcal{C}$ is rational and $\mathcal{O}_d$ is reducible then both components of $\mathcal{O}_d$ are rational.*

Proof: If $\mathcal{M}$ is a simple component of $\mathcal{O}_d$ then Proposition 3.4. implies that $\mathcal{M}$ is rational.
Suppose that $\mathcal{M}$ is a special component of $\mathcal{O}_d$. Let $\mathcal{P}(t)$ be a parametrization of $\mathcal{C}$. Then, since $\pi_2 : \pi_1^{-1}(\mathcal{M}) \longrightarrow \mathcal{C}$ is one-to-one, it follows that $\pi_1(\pi_2^{-1}(\mathcal{P}(t)))$ parametrizes $\mathcal{M}$, and hence $\mathcal{M}$ is rational. $\Box$

Remark. From Theorem 3.5. it follows that offset curves of rational plane curves are either reducible with two rational components, or rational, or irreducible but not rational.

We continue by proving that reducible offset curves with rational components are found by testing whether a proper rational paramatrization of the original curve is RPH.

Theorem 3.6 *The following statements are equivalent:*

1. *$\mathcal{O}_d$ has two rational components*
2. *There exists a proper parametrization of $\mathcal{C}$ which is RPH.*
3. *All the proper parametrizations of $\mathcal{C}$ are RPH.*

Proof: Let $\mathcal{O}_d$ have two rational components. Then, the proof of theorem 3.2. shows that there exists a reparametrizing rational function of degree one. Therefore, (1) implies (2).
Now we observe that any reparametrization of a parametrization, which is RPH, is also RPH. Thus, since two proper parametrizations are always related by means of a linear rational function, it is clear that (2) implies (3).
Finally, let $\mathcal{P}$ be a proper parametrization of the normal vector $\mathcal{N}$ with length $m \in \bar{K}(t)$. Then $\mathcal{O}^+ = \mathcal{P} + \frac{d}{m}\mathcal{N}$ and $\mathcal{O}^- = \mathcal{P} - \frac{d}{m}\mathcal{N}$ parametrize the exterior and interior offset of $\mathcal{O}_d$.$\square$

Theorem 3.7 *The rationality of the components of the offset curve $\mathcal{O}_d$ to a rational curve does not depend on the distance d.*

Proof: Let us assume that there exists a distance $d_0 \in \bar{K}$ such that $\mathcal{O}_{d_0}$ has rational components. Then, applying Theorem 3.1 we see that there exists a parametrization $\mathcal{P}$ of the generator curve $\mathcal{C}$ which is RPH. Therefore, $\mathcal{O}_d = \mathcal{P} \pm \frac{d}{m}\mathcal{N}$ are rational parametrizations of the components of the offset at any distance d unless $m = 0$. $\square$

Theorem 3.8 *If the offset $\mathcal{O}_d$ to a rational curve has rational components then the reducibility of $\mathcal{O}_d$ does not depend on the distance d.*

Proof: If there exists a distance $d_0 \in \bar{K}$ such that $\mathcal{O}_{d_0}$ has two rational components, then applying Theorem 3.6 one has that there exists a proper parametrization which is RPH. Therefore, at any distance d the offset is reducible. □

3.4 A First Algorithm

By the results of the previous section we can formulate an algorithm for computing rational offset parametrizations.

- Given a proper parametrization $\mathcal{P}$ of an affine plane curve $\mathcal{C}$.
- Decide the rationality of the components of the offset $\mathcal{O}_d$ to $\mathcal{C}$ at distance d.
- Determine (if $\mathcal{O}_d$ has rational components) a rational parametrization of each component of $\mathcal{O}_d$.

1. Check whether the normal vector $\mathcal{N}$ of $\mathcal{P}$ has rational length. If so, return that $\mathcal{O}_d$ has two rational components parametrized as:

$$\mathcal{O}^+ = \mathcal{P} + \frac{d}{m}\mathcal{N}$$

$$\mathcal{O}^- = \mathcal{P} - \frac{d}{m}\mathcal{N}$$

   where m is the rational length of $\mathcal{N}$ (the special case $m = 0$ (see Lemma 2.3) is not treated by this algorithm).

2. Determine the implicit equation of the offset $\mathcal{O}_1$ at distance one.
3. Obtain the genus g of $\mathcal{O}_1$.
4. If $g \neq 0$ then return that $\mathcal{O}_d$ is not rational.
5. If $g = 0$ then
   (a) Compute a parametrization of $\mathcal{O}_1$ (see [23] and [24]).

(b) Take $L(x,y,t)$ and $C(x,y,t)$ as the primitive polynomials w.r.t $\{x,y\}$ of the numerators of

$$\bar{L}(x,y,t) = (x - p_1(t))p_1'(t) + (y - p_2(t))p_2'(t)$$

$$\bar{C}(x,y,t) = (x - p_1(t))^2 + (y - p_2(t))^2 - 1$$

(c) Take $M(t,h)$ and $N(t,h)$ as the squarefree representations of the numerators of $L(q_1(h), q_2(h), t)$ and $C(q_1(h), q_2(h), t)$,respectively.

(d) $D = gcd(M,N)$ and $t = \varphi(h)$ the solution of $D = 0$.

(e) $\tilde{\mathcal{P}} = \mathcal{P}(\varphi(t)), \tilde{\mathcal{N}} = \varphi'(t)\mathcal{N}(\varphi(t))$, and m the length of $\tilde{\mathcal{N}}$.

(f) Return that $\mathcal{O}_d$ is a rational curve paramatrized by $\mathcal{O}^+ = \mathcal{P} + \frac{d}{m}\mathcal{N}$. □

Example 3.1 Applying this algorithm to the parabola

$$P := [\,t, t^2\,]$$

by using MAPLE and CASA (see [16] and [26]) we obtain the following computation:

The normal vector is given by

$$N := [\,-2\,t, 1\,].$$

Its length

$$m := \sqrt{4\,t^2 + 1}$$

is not rational. Therefore the offset at any distance is either rational or it consists of one irreducible component of positive genus.

The offset at distance 1 is

$$\mathcal{O}_1 := -25 + 40\,y + 28\,x^2 + 9\,y^2 + 6\,y\,x^2 + 16\,x^4\,y^2 - 40\,x^4\,y - 32\,x^2\,y^3 \\ - 47\,x^4 - 40\,y^3 + 16\,x^6 + 16\,y^4$$

which can be parametrized by

$$q_1(t) = \frac{2}{5} \cdot \frac{26351\,t^4 - 44574\,t^3 + 22416\,t^2 - 4514\,t + 321}{91395\,t^4 - 84364\,t^3 + 29202\,t^2 - 4492\,t + 259},$$
$$q_2(t) = -\frac{1}{25} \cdot \frac{a}{b}$$

where $a = 307089223\,t^6 - 422392458\,t^5 + 240991145\,t^4 - 72992460\,t^3 + 12376745\,t^2 - 1113738\,t + 41543$ and $b = 12338325\,t^6 - 17055630\,t^5 + 9812603\,t^4 - 3007492\,t^3 + 517883\,t^2 - 47502\,t + 1813$.

Thus one may conclude that the offset is rational at any distance. By the algorithm above we obtain the following reparametrization.

$$\varphi := -\frac{-26\,h^2 + 32\,h - 6}{-310\,h + 35 + 675\,h^2}$$

which leads to a normal vector with length

$$m := \frac{4}{25}\,\frac{(-314\,h + 677\,h^2 + 37)^2}{(7 - 62\,h + 135\,h^2)^3}.$$

Now the offset curve can be parametrized by

$$x_{\mathcal{O}_d} := \frac{a}{5\,(135\,t^2 - 62\,t + 7)\,(677\,t^2 - 314\,t + 37)}$$

$$y_{\mathcal{O}_d} := \frac{b}{25\,(135\,t^2 - 62\,t + 7)\,(677\,t^2 - 314\,t + 37)}$$

where $a = (-35100\,d + 17602)\,t^4 + (-29828 + 59320\,d)\,t^3 + (-29760\,d + 15072)\,t^2 + (5960\,d - 3068)\,t - 420\,d + 222$
and $b = (457652 + 307546875\,d)\,t^6 + (-1338792 - 423731250\,d)\,t^5 + (242443125\,d + 1451980)\,t^4 + (-73733500\,d - 741040)\,t^3 + (194380 + 12571125\,d)\,t^2 + (-25512 - 1139250\,d)\,t + 42875\,d + 1332$.

Example 3.2 Let us consider the hyperbola

$$P := \left[t, \frac{1}{t}\right]$$

with the normal vector

$$N := \left[\frac{1}{t^2}, 1\right]$$

whose length

$$m := \frac{\sqrt{1+t^4}}{t^2}$$

is not rational. Therefore the offset curve is irreducible and we have to check if it is rational. Hence we have to compute the offset at distance one:

$$\begin{aligned}\mathcal{O}_1 := & -23\,x^2 - 23\,y^2 + 22\,y\,x^3 + 22\,y^3\,x - 68\,y\,x + y^6\,x^2 + 2\,y^4\,x^4 + y^2\,x^6 \\ & - 2\,y^5\,x + 57\,y^2\,x^2 - 5\,y^4\,x^2 - 5\,y^2\,x^4 + 4\,y^4 + 4\,x^4 - y^6 - x^6 - 2\,y\,x^5 \\ & - 20\,y^3\,x^3 + 9 \qquad .\end{aligned}$$

The genus of this offset curve is one and therefore the offset curve is not rational at any distace.

3.5 The Reparametrizing Curve

The previous example of computing the offset to a rather simple curve is very time consuming and produces rational functions with large coefficients. Since we know that there exists a simpler reparametrization, we should look for improvements. The problem is caused by the paramatrization of the offset curve at distance 1. So we should concentrate on this step. We will try to find a way of avoiding this computation.

Definition 3.3 *Let $\mathcal{P}(t) = (p_1(t), p_2(t))$ be a rational parametrization of $\mathcal{C}$, and $\mathcal{N}(t) = (n_1(t), n_2(t))$ the normal vector of $\mathcal{P}(t)$. We assume w.l.o.g that $n_2(t) \neq 0$. Then, we define the* reparametrizing curve *of $\mathcal{O}_d$* associated with *$\mathcal{P}(t)$ as the curve generated by the primitive part with respect to y of the numerator of*

$$n_2(x)(y^2-1) - 2n_1(x)(y).$$

In the following, we denote by $\mathcal{G}_{\mathcal{P}}$ the reparametrizing curve of $\mathcal{O}_d$ associated with $\mathcal{P}(t)$. $\square$

Theorem 3.9 *All the components of $\mathcal{O}_d$ are rational if and only if for all proper parametrization of $\mathcal{C}$ there exists at least one rational component of $\mathcal{G}_{\mathcal{P}}$.*

Proof: If all components of $\mathcal{O}_d$ are rational then there exists an RPH parametrization $\mathcal{P}(t)$ of $\mathcal{C}$. Let $\mathcal{N}(t) = (n_1(t), n_2(t))$ be the normal vector of $\mathcal{P}(t)$, and $m(t) = n_1^2(t) + n_2^2(t) \in \bar{K}(t)$. Then $(\frac{n_1}{m}, \frac{n_2}{m})$ is a rational parametrization of the circle $x^2 + y^2 = 1$. Hence, since $\mathcal{R}(t) = (\frac{t^2-1}{t^2+1}, \frac{2t}{t^2+1})$ is a proper parametrization of the circle, there exists a $\phi(t) \in \bar{K}(t)$ such that $\mathcal{R}(\phi(t)) = \mathcal{N}(t)$, i.e:

$$\frac{\phi^2 - 1}{\phi^2 + 1} = \frac{n_1}{m},$$

$$\frac{2\phi}{\phi^2 + 1} = \frac{n_2}{m}.$$

This leads to

$$n_2(t)(\phi^2 - 1) - 2n_1(t)(\phi)$$

and therefore $(t, \phi(t))$ parametrizes one component of $\mathcal{G}_{\mathcal{P}}$.

On the other hand let $(\varphi(t), \phi(t))$ be a rational parametrization of one of the components of $\mathcal{G}_{\mathcal{P}}$. Then

$$n_2(\varphi(t))(\phi^2 - 1) - 2n_1(\varphi(t))(\phi) = 0.$$

Note that ϕ is not identically zero; otherwise it would imply that n_2 is identically zero. Thus,

$$n_1^2(\varphi) + n_2^2(\varphi) = \frac{(n_1(\varphi)\phi + n_2(\varphi))^2}{\phi^2}. \tag{3.2}$$

Therefore $\mathcal{P}(\varphi(t))$ is an RPH parametrization of $\mathcal{C}$. $\square$

Corollary 3.10 *Let $\mathcal{P}(t)$ be a proper parametrization of $\mathcal{C}$. $\mathcal{O}_d$ is reducible if and only if $\mathcal{G}_{\mathcal{P}}$ is reducible.*

Proof: Let H be the defining polynomial of $\mathcal{G}_{\mathcal{P}}$. Since H is primitive with respect to y, it follows that any factor of H depends on y. Thus $\mathcal{G}_{\mathcal{P}}$ is reducible if and only if H has two factors being linear in y. Thus the reparametrizing function φ is of degree one and $\mathcal{O}_d$ is reducible. $\square$

Corollary 3.11 *$\mathcal{O}_d$ is rational if and only if for every proper parametrization $\mathcal{P}(t)$ of $\mathcal{C}$ $\mathcal{G}_\mathcal{P}$ is rational.*

Proof: If $\mathcal{O}_d$ is rational, then by Theorem 3.9 there exists a proper parametrization $\mathcal{P}(t)$ of $\mathcal{C}$, such that $\mathcal{G}_\mathcal{P}$ has at least one rational component. Furthermore $\mathcal{P}(t)$ is not RPH (else $\mathcal{O}_d$ would be reducible, Corollary 3.10). Thus, $\mathcal{G}_\mathcal{P}$ is irreducible. Therefore $\mathcal{G}_\mathcal{P}$ is rational. □

These results lead to the following

Algorithm

- Given a proper parametrization $\mathcal{P}$ of a affine plane curve $\mathcal{C}$.
- Decide the rationality of the components of the offset $\mathcal{O}_d$ to $\mathcal{C}$ at distance d.
- Determine (if $\mathcal{O}_d$ has rational components) a rational parametrization of each component of $\mathcal{O}_d$.

1. Check whether the normal vector $\mathcal{N}$ of $\mathcal{P}$ has rational length. If so, return that $\mathcal{O}_d$ has two rational components paramatrized as:
$$\mathcal{O}^+ = \mathcal{P} + \frac{d}{m}\mathcal{N}$$
$$\mathcal{O}^- = \mathcal{P} - \frac{d}{m}\mathcal{N}$$
where m is the rational length of $\mathcal{N}$.
2. Determine $\mathcal{G}_\mathcal{P}$, and decide wether $\mathcal{G}_\mathcal{P}$ is rational. (Note that $\mathcal{G}_\mathcal{P}$ is irreducible).
3. If $\mathcal{G}_\mathcal{P}$ is not rational return that no component of the offset $\mathcal{O}_d$ is rational.
4. Determine a rational parametrization (φ, ϕ) of $\mathcal{G}_\mathcal{P}$.
5. Return that $\mathcal{O}_d$ is a rational curve parametrized by
$$\mathcal{O}_d = \mathcal{P}(\varphi) + \frac{2d\phi}{n_2(\varphi)(\phi^2+1)}\mathcal{N}(\varphi)$$
(Note, that this results by simple term manipulations from 3.2). □

Example 3.3. Again we consider the parabola

$$P := [\,t, t^2\,]$$

Now we compute the reparametrizing curve

$$\mathcal{G}_{\mathcal{P}} := y^2 - 1 + 4\,x\,y$$

This curve obviously is simpler than the offset curve at distance one, which we had to compute in the previous chapter. Since $\mathcal{G}_{\mathcal{P}}$ is linear in x we can parametrize it easily by

$$\mathcal{G}_{\mathcal{P}} := \left[-\frac{1}{4}\,\frac{t^2-1}{t}, t\right]$$

and therefore the reparametrizing function is given by

$$\varphi(t) := -\frac{1}{4}\,\frac{t^2-1}{t}$$

which leads to the following parametrization of the offset curve

$$q1 := -\frac{1}{4}\,\frac{(\,t^2-1\,)\,(\,t^2+1+4\,d\,t\,)}{t\,(\,t^2+1\,)}$$

$$q2 := \frac{1}{16}\,\frac{t^6 - t^4 - t^2 + 1 - 32\,d\,t^3}{t^2\,(\,t^2+1\,)}$$

.

Example 3.4. We consider the cubic curve

$$P := [\,t^2, t^3\,]$$

The primitive part w.r.t. y of the curve described by

$$2\,x\,(\,y^2 - 1 + 3\,x\,y\,)$$

is the reparametrizing curve $\mathcal{G}_{\mathcal{P}}$, i.e.

$$\mathcal{G}_{\mathcal{P}} := y^2 - 1 + 3\,x\,y.$$

$\mathcal{G}_{\mathcal{P}}$ is rational and can be parametrized by

$$\mathcal{G}_{\mathcal{P}} := \left[-\frac{-3+2\,t}{t\,(\,t-3\,)}, \frac{t}{t-3} \right]$$

and we use the x-coordinate of this parametrization as a reparametrizing function for $\mathcal{P}$ and obtain a normal vector with length

$$\frac{4\,t^3 - 18\,t^2 + 36\,t - 27}{t^2\,(\,t-3\,)^2}$$

and therefore we can parametrize the offset curve by

$$x_{\mathcal{O}_d} = \frac{a}{(\,t-3\,)^2\,(\,2\,t^2 - 6\,t + 9\,)\,t^2}$$

$$y_{\mathcal{O}_d} = \frac{b}{(\,t-3\,)^3\,(\,2\,t^2 - 6\,t + 9\,)\,t^3}$$

where $a = 6\,d\,t^5 + (\,-45\,d + 8\,)\,t^4 + (\,108\,d - 48\,)\,t^3 + (\,126 - 81\,d\,)\,t^2 - 162\,t + 81$ and $b = -2\,d\,t^8 + 24\,d\,t^7 - 108\,d\,t^6 + (\,216\,d + 16\,)\,t^5 + (\,-120 - 162\,d\,)\,t^4 + 396\,t^3 - 702\,t^2 + 648\,t - 243$.

Example 3.5. If we take the ellipse parametrized by

$$P := \left[2\,\frac{1-t^2}{1+t^2}, 6\,\frac{t}{1+t^2} \right]$$

as generator curve, we get the normal vector

$$\mathcal{N} := \left[4\,\frac{t}{1+t^2} + 4\,\frac{(\,1-t^2\,)\,t}{(\,1+t^2\,)^2}, 6\,\frac{1}{1+t^2} - 12\,\frac{t^2}{(\,1+t^2\,)^2} \right]$$

which leads to the reparametrizing curve

$$\mathcal{G}_{\mathcal{P}} := 3\,y^2 - 3 - 3\,x^2\,y^2 + 3\,x^2 - 8\,x\,y.$$

Since this curve is irreducible and of genus one the offset curve to the ellipse is not rational.

3.6 Rational Components to Non-Rational Generator Curves

In the previous sections we have seen that the offset curve to a rational curve may be non-rational. But when we consider the offset to the offset it must obviously have a rational component, namely the original curve. The following theorem gives a complete characterization of the rationality of offset curves to non-rational generator curves.

Theorem 3.12 *Let $\mathcal{C}$ be a non-rational generator curve and assume the offset $\mathcal{O}_d$ has at least one rational component. Then $\mathcal{O}_d$ has the following properties.*

- *$\mathcal{O}_d$ is reducible*
- *only one component of $\mathcal{O}_d$ is rational*
- *$\mathcal{C}$ is the offset to the rational component*

Proof: Let $\mathcal{C}$ be a non-rational curve with rational offset components. Assume $\mathcal{O}_d$ is irreducible. But then the offset to $\mathcal{O}_d$ must be reducible (it contains the original curve and the offset at distance $2d$ to $\mathcal{C}$). But a the rational curve $\mathcal{O}_d$ cannot have a reducible offset with a non-rational component (cf. section 3.3). Hence $\mathcal{O}_d$ is reducible. Next consider one rational component $\mathcal{O}_r$ of $\mathcal{O}_d$. Obviously $\mathcal{O}_d$ must be one rational component of $\mathcal{O}_r$, but since it is not rational it must be the only component. The other compenent of $\mathcal{O}_d$ is the offset of $\mathcal{O}_r$ at distance $2d$ and by theorem 3.2.2. it is not rational. $\square$

Now we can summarize. The following cases can occur:

generator curve	offset curve	example
rational curve	2 rational components	circle
	1 rational component	parabola
	1 non-rational component	ellipse
non rational curve	1 rational and 1 non-rational comp.	offset to the ellipse
	2 non-rational components	?
	1 non-rational component	?

Chapter 4

Generalizations of Offsets

In the previous chapters we restricted ourselves to consider only offsets to curves, we took only the normal vector for generating an offset and a constant distance d. So there seem to be some possibilities for generalizations:

- By changing the systems of defining polynomials it is easy to extend the definition of offsets from 2 to any dimension. We can consider hypersurfaces and their offsets. For applications obviously the case of 3 dimensions is of special interest.
- Instead of the normal line in the system of defining polynomials we can use a line in some given direction besides the normal one, which can be determined by a fixed isometry.
- The constant distance d can be replaced by a rational distance function.
- Instead of hypersurfaces we could also consider algebraic varieties of lower dimension. In this case one can describe a pipe around a curve in 3 dimensions.

In this chapter we will concentrate on the first two points. Fortunately many of the concepts can be extended in a very natural way. This means that we will not work out all the details and proofs if they are obvious.

4.1 Basic Notions

In this section the concept of generalized offsets to hypersurfaces are presented and basic properties of these offsets are stated (for more details see Sendra [1]).

Let $\mathcal{V}$ be an irreducible hypersurface in $\mathbb{A}^n$ defined by the polynomial $f(u)$, where $u = (u_1, \ldots, u_n)$, and T an $n \times n$ matrix defining a direct isometry in $\mathbb{A}^n$. For any point $a = (a_1, \ldots, a_n) \in \mathcal{V}$ let $\mathcal{N}(a)$ be a normal vector of $\mathcal{V}$ in a and $\mathcal{M}(a)$ be the direction determined by the vector $\mathcal{N}(a) \cdot T$.

Definition 4.1 *Given an irreducible hypersurface $\mathcal{V}$ defined by $f(u) = 0$ in $\mathbb{A}^n$, a distance d and the system*

$$f(u) = 0 \tag{4.1}$$

$$\sum_{i=1}^{n}(x_i - ui)^2 - d^2 = 0 \tag{4.2}$$

$$(x_1 - u_1)M_i(u) = (x_n - u_n)M_1(u) \quad \mathit{for} \quad i = 2 \ldots n \tag{4.3}$$

where $(x_1 - u_1)M_i(u) = (x_n - u_n)M_1(u) \quad \mathit{for} \quad i = 2 \ldots n$ is the formal expression of the line passing through u in the direction of $M(u)$. We define the algebraic set $\mathcal{A}_d^T$ as

$$\mathcal{A}_d^S = \{(x, u) | (x, y) \ \mathit{satisfy}\ (4.1),\ (4.2)\ \mathit{and}\ (4.3)\} \subseteq \mathbb{A}^{2n}. \quad \square$$

First we have to note that the set of singularities of an n-dimensional hypersurface may have dimension $n-1$. But in analogy to the definitions of Chapter 2 the envelope should be defined as a product of the offset curve and a hypersphere around any singular point. If the generating hypersurface has infinitely many singular points, this will be no algebraic set. Hence a direct extension of Definition 2.1 is not possible. But by applying gcd computations we can avoid the hyperspheres in advance (cf. Chapter 2).

Definition 4.2 *Let $G = \gcd(M_1(u), \ldots, M_n(u)$ and replace (4.3) by*

$$\frac{(x_1 - u_1)M_i(u)}{G(u)} = \frac{(x_n - u_n)M_1(u)}{G(u)} \quad \mathit{for} \quad i = 2 \ldots n \tag{4.4}$$

Then we define the generalized offset to an irreducible hypersurface $\mathcal{V}$ at distance d in the direction determined by a direct isomtry S *as the result of eliminating the variables* $u = (u_1, \ldots, u_n)$ *from the algebraic set defined by (4.1), (4.2) and (4.4) and we denote it by* $\mathcal{O}_d^T$. $\Box$

Remark. Note that when the direct isometry is the identity and the dimension $n = 2$ the concept of generalized offset particularizes to the notion of offset curves used in previous chapters.

The notions of open offsets and two-sided parallel curve can be generalized immediatly.

Definition 4.3 *We define the algebraic set* $\mathcal{S}_d \subseteq \mathbb{A}^{2n}$ *by the following polynomials:*

$$f(u) = 0 \tag{4.5}$$

$$\sum_{i=1}^{n} (x_i - ui)^2 - d^2 = 0 \tag{4.6}$$

$$M_i(u) = 0 \quad for \quad i = 2, \ldots, n \tag{4.7}$$

and define $\mathcal{B}_d$ *by* $\mathcal{A}_d \setminus \mathcal{S}_d$ *which is a constructible set.* $\Box$

Definition 4.4 *Let* π_1 *be the projection*

$$\begin{array}{cccc} \pi_1 : & \mathbb{A}^n \times \mathbb{A}^n & \longrightarrow & \mathbb{A}^n \\ & (x, u) & \longrightarrow & (x) \end{array}$$

We call the constructible set $\mathcal{Q}_d^T$ *defined by* $\pi_1(\mathcal{B}_d)$ *the* open offset. $\Box$

By similar arguments as in Lemma 2.1 we obtain that there exists a one-to-one correspondence between the points of $\mathcal{O}_d^T$ and the Zariski-closure of $\mathcal{Q}_d^T$.

Definition 4.5 *Given a parametrized hypersurface* $\mathcal{V}$ *defined by the rational functions* $f(t) = (f_1(t), \ldots, f_n(t))$ *we define the* two-sided parallel hypersurface $\mathcal{P}_d^T$ *as subset of* $\mathbb{A}^n$ *generated by:*

$$\mathcal{P}_d^T(t) := f(t) \pm \frac{d}{m(t)} \mathcal{N}(t) \cdot T \tag{4.8}$$

where $\mathcal{N}(t)$ *is the normal vector to* $\mathcal{V}$ *at* $f(t)$ *and* $m(t)$ *is its legth for all except the singular points of* $\mathcal{V}$. $\Box$

Remark.

- From the definition of generalized offset it follows that $\mathcal{O}_d^T$ has at most two components.
- Whenever the set $\{f(t)|m(t)=0\}$ has dimension less than $n-1$ the algebraic closure of $\mathcal{P}_d^T(t)$ is equal to the generalized offset $\mathcal{O}_d^T$. The proof is similar to the proof of Lemma 2.3.
- In order to compute the implicit equation of $\mathcal{O}_d^T$ one can apply any method of elimination theory (e.g. Gröbner bases, Characteristic Sets, ...). Furthermore, if $\mathcal{V}$ is a rational curve, given in parametrized form, the techniques described by Farouki and Neff [6] can be generalized.
- Similarly to classical offsets, the normal vectors to the generalized offset are parallel to the normal vectors to the variety in the corresponding point.

Example 4.1 Let $\mathcal{V}$ be the parabola given by the parametrization $f(t)=(t,t^2)$, and take the direct isometry given by

$$T=\begin{pmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

that corresponds to a rotation of angle $\frac{\pi}{4}$. Then the implicit equation of the generalized offset curve at distance d in direction T is given by

$$\begin{aligned}\mathcal{O}_d^T(x,y) :=\ & 4\,y^2-2\,d^2-44\,x^2\,d^2-8\,x^2\,y+128\,x^2\,y^2+20\,d^2\,y-32\,d^2\,y^2+4\,x^4\\ &-11\,d^4-32\,y^3-128\,x^4\,d^2-160\,x^4\,y+64\,x^4\,y^2+80\,x^2\,d^4\\ &-128\,x^2\,y^3+56\,d^4\,y+16\,d^4\,y^2-16\,d^4\,x-64\,d^2\,y^3+64\,x^2\,d^2\,y\\ &-64\,x^2\,d^2\,y^2+64\,x^6-16\,d^6+64\,y^4\end{aligned}$$

which is also irreducible. Moreover, it is also rational and can be parametrized by

$$x := -\frac{1}{4}\,\frac{t^2-1}{t}-\frac{d\sqrt{2}\,(\,t^2-1-2\,t\,)}{2\,t^2+2}$$

$$y := \frac{1}{16}\,\frac{(\,t^2-1\,)^2}{t^2}-\frac{d\sqrt{2}\,(\,t^2-1+2\,t\,)}{2\,t^2+2}$$

The affine singularities of $\mathcal{O}_d^T$ are the double points

$$(\alpha, -7\alpha^3 + \frac{21}{88} + \frac{969}{176}\alpha + \frac{987}{88}\alpha^2 + \frac{31}{11}\alpha^4 - \frac{9}{11}\alpha^5 - \frac{38}{11}\alpha^6)$$

where $64\alpha^7 - 48\alpha^5 + 144\alpha^4 + 180\alpha^3 - 140\alpha^2 + 29\alpha - 2 = 0$. Hence there is a real isolated root that can be approximated as $(0.2967198899, 0.7195404261)$.

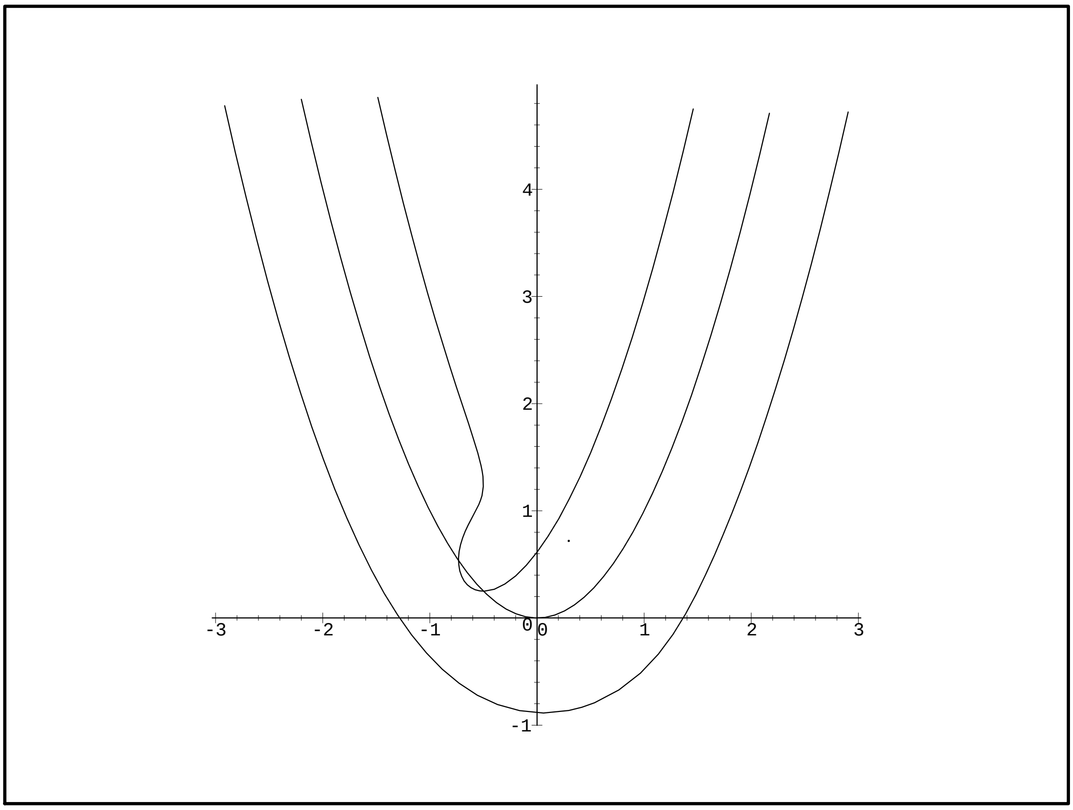

Figure 4.1: Generalized offset to a parabola

4.2 Rationality

When speaking about rationality one has to be more careful than in dimension 2. Let $\mathcal{V}$ be a variety of dimension r over K. Then $\mathcal{V}$ is said to be *unirational*, or *parametric*, if there exists a rational map $\mathcal{P} : K^r \longrightarrow \mathcal{V}$ such that $\mathcal{P}(K^r)$ is dense in $\mathcal{V}$. We call $\mathcal{P}(t)$ a rational parametrization of $\mathcal{V}$. If $\mathcal{P}$ defines a birational map then we say that $\mathcal{P}(t)$ is a proper parametrization. A variety $\mathcal{V}$ is rational if and only if it has a proper parametrization (for curves the notions of rationality and unirationality are equivalent for any field, for surfaces over an algebraically closed field the notions are also the same (Castelnuovo's theorem), however, in general the equivalence is not true).

We give a new definition of simple and special components

Definition 4.6 *An irreducible component $\mathcal{M}$ of $\mathcal{O}_d^T$ is called* simple *if there exists a dense set M of points on $\mathcal{M}$ such that every $q \in M$ is generated by exactly one point $p \in \mathcal{V}$, otherwise $\mathcal{M}$ is called special.* □

Again special components of generalized offsets are precisely those generated when computing offsets to offsets. Also it holds that every generalized offset has at least one simple component.

In the following we show how any parametrization of a rational variety is related to a proper rational parametrization of the same variety, which is shown in the following lemma.

Lemma 4.1 *Let $\mathcal{V}$ be a rational variety of dimension r over K, and $\mathcal{P}(t)$ a proper rational parametrization of $\mathcal{V}$. Then for every parametrization $\mathcal{Q}(t)$ of $\mathcal{V}$ there exists $\varphi \in K(t)^r$ such that $\mathcal{P}(\varphi(t)) = \mathcal{Q}(t)$.*

Proof: Consider the following diagram:

$$\begin{array}{ccc} Q : K^r & \longrightarrow & \mathcal{V} \\ & \searrow & \uparrow \mathcal{P} \\ & & K^r \end{array}$$

Then, since $\mathcal{P}$ is birational, we can take $\varphi(t) = \mathcal{P}^{-1}(\mathcal{Q}(t)) \in K(t)^r$ and $\mathcal{P}(\varphi(t)) = \mathcal{Q}(t)$. □

First we consider reducible generalized offsets and investigate the unirationality of its components.

Proposition 4.2 *Let $\mathcal{V}$ be a variety and $\mathcal{O}_d^T$ the generalized offset in the direction determined by T.*

1. *If $\mathcal{O}_d^T$ is reducible then every simple component of $\mathcal{O}_d^T$ is birationally equivalent to $\mathcal{V}$.*
2. *If $\mathcal{V}$ is rational and $\mathcal{O}_d^T$ is reducible then all the simple components of $\mathcal{O}_d^T$ are rational.*
3. *If $\mathcal{V}$ is parametric and $\mathcal{O}_d^T$ is reducible then all the components of $\mathcal{O}_d^T$ are parametric.*

Proof: See Sendra [1]. □

Remark Let the notion RPH (Rational Pythagorean Hodograph) be the natural extension from the 2-dimensional case. Note the if $\mathcal{P}(t)$ is an RPH parametrization and $\mathcal{Q}(t) = \mathcal{P}(\varphi(t))$ is a reparametrization of $\mathcal{P}(t)$, then the normal vector of $\mathcal{Q}(t)$ is $\mathcal{N}_{\mathcal{Q}}(t) = det(J_\varphi \cdot \mathcal{N}_{\mathcal{P}}(\varphi(t))$, where $\mathcal{N}_{\mathcal{P}}(t)$ is the normal vector to $\mathcal{P}(t)$, and J_φ is the Jacobian matrix of $\varphi(t) \in K(t)^{n-1}$. Therefore $\mathcal{Q}(t)$ is also RPH.

Definition 4.7 *Let $\mathcal{P}(t) = (f_1(t), \ldots, f_n(t))$ be a rational parametrization of $\mathcal{V}$, and $\mathcal{N}(t) = (n_1(t), \ldots, n_n(t))$ the normal vector to $\mathcal{P}(t)$. We assume w.l.o.g. the $n_2(t)$ is not identically zero. Then we define the* reparametrizing hypersurface of $\mathcal{O}_d^T$ associated with $\mathcal{P}(t)$ *as the hypersurface generated by the primitive part with respect to x_n of the numerator of*

$$x_n^2(n_2^2(\bar{x}) + \cdots + n_n^2(\bar{x})) - n_2^2(\bar{x}) - 2x_n n_1(\bar{x}) n_2(\bar{x})$$

where $\bar{x} = (x_1, \ldots, x_{n-1})$. In the following, we denote by $\mathcal{G}_{\mathcal{P}}$ the reparametrizing hypersurface of $\mathcal{O}_d^T$ associated with $\mathcal{P}$. □

Remark. If $\mathcal{G}_{\mathcal{P}}$ is reducible then each factor is linear in x_n, and therefore $\mathcal{G}_{\mathcal{P}}$ has two parametric components.

Theorem 4.3 *Let $\mathcal{V}$ be parametric. Then following statements are equivalent:*

1. *There exists an RPH parametrization of $\mathcal{V}$.*

2. *$\mathcal{O}_d^T$ has at least one simple parametric component.*

3. *There exists a parametrization $\mathcal{P}(t)$ of $\mathcal{V}$ such that $\mathcal{G}_\mathcal{P}$ has at least one parametric component.*

4. *All the components of $\mathcal{O}_d^T$ are parametric.*

Proof: See Sendra [1]. □

Remark. Theorem 4.3 implies that generalized offsets with at least one simple component can only be generated by parametric hypersurfaces. Therefore a generalized offset to a non-rational hypersurface with a parametric component $\mathcal{M}$ can only occur if $\mathcal{M}$ is special. This is only possible when computing the offset to the offset to a rational hypersurface. By Theorem 4.3 and Proposition 4.2 the offset to a parametric hypersurface can be either reducible with two parametric components, or parametric, or irreducible and not unirational.

The following two corollaries describe some useful properties for the computation of offsets.

Corollary 4.4 *If $\mathcal{V}$ is parametric, and $\mathcal{P}(t)$ a rational parametrization of $\mathcal{V}$ such that $\mathcal{G}_\mathcal{P}$ has at least one parametric component $\mathcal{M}$ and $(\phi_1, \ldots, \phi_n)$ is a rational parametrization of $\mathcal{M}$, then $\mathcal{P}((\phi_1, \ldots, \phi_{n-1})$ is RPH. This leads immediately to parametrization of the offset.*

Proof: Follows from Theorem 4.3. □

Corollary 4.5 *If $\mathcal{V}$ is parametric, the unirationality of the components of the generalized offset does neither depend on the distance d nor on the isometry T.*

Proof: Follows directly from Theorem 4.3. □

In addition to Theorem 4.3 for rational hypersurfaces we get the following theorem.

Theorem 4.6 *Let $\mathcal{V}$ be rational. Then the following statements are equivalent:*

1. *There exists an RPH parametrization of $\mathcal{V}$.*

2. *There exists a proper parametrization $\mathcal{P}(t)$ of $\mathcal{V}$ such that $\mathcal{G}_{\mathcal{P}}$ has at least one parametric component.*

3. *For every proper parametrization $\mathcal{P}(t)$ of $\mathcal{V}$, $\mathcal{G}_{\mathcal{P}}$ has at least one parametric component.*

Proof: See Sendra [1]. □

The next two corollaries, which are also proved in [1], characterize the reducibility of the offsets to a rational hypersurface $\mathcal{V}$ by analyzing the RPH property and the reparametrizing hypersurface $\mathcal{G}_{\mathcal{P}}$ of $\mathcal{V}$.

Corollary 4.7 *Let $\mathcal{V}$ be rational. Then the following statements are equivalent.*

1. *$\mathcal{O}_d^T$ is reducible.*

2. *$\mathcal{O}_d^T$ has exactly two parametric components.*

3. *There exists a proper RPH parametrization of $\mathcal{V}$.*

4. *Every proper parametrization of $\mathcal{V}$ is RPH.*

5. *There exists a proper parametrization $\mathcal{P}(t)$ of $\mathcal{V}$ such that $\mathcal{G}_{\mathcal{P}}$ is reducible.*

6. *For every proper parametrization $\mathcal{P}(t)$ of $\mathcal{V}$, $\mathcal{G}_{\mathcal{P}}$ is reducible.* □

Corollary 4.8 *Let $\mathcal{V}$ be rational. Then the following statements are equivalent.*

1. *$\mathcal{O}_d^T$ is parametric (and therefore irreducible).*

2. *There exists a proper parametrization $\mathcal{P}(t)$ of $\mathcal{V}$ such that $\mathcal{G}_{\mathcal{P}}$ is parametric.*

3. *For every proper parametrization $\mathcal{P}(t)$ of $\mathcal{V}$, $\mathcal{G}_{\mathcal{P}}$ is parametric.* □

By using Theorem 4.6 and these two corollaries one can outline an algorithm for computing a parametrization of the rational components of the offset. To decide whether the generalized offset to a rational hypersurface $\mathcal{V}$ is reducible, and has therefore two parametric components, we have to check the reducibility of $\mathcal{G}_{\mathcal{P}}$ of $\mathcal{V}$. The offset consists of one rational component if and only if the reparametrizing hypersurface $\mathcal{G}_{\mathcal{P}}$ is irreducible and paramatric. By parametrizing $\mathcal{G}_{\mathcal{P}}$ we can compute a reparametrizing function which leads to an RPH parametrization of $\mathcal{V}$ and hence to a parametrization of the offset.

Appendix A

Implementation

A.1 Introduction

We implemented several algorithms to compute offsets in MAPLE V and by using the software package CASA ([16] and [26]). CASA offers data structures to represent algebraic sets and to compute with them (parametrization, implicitation, decomposition, ...).

The package for computing offset curves consists of the following procedures:

- mkImplOffset
- mkParaOffset
- RPHcurve

Using mkImplOffset we can compute the implicit equation of an offset curve to a given generator curve (in implicit or parametric form) at a specified or an unspecified distance. Since we use Gröbner basis computations when computing the implicit offset to an implicit generator curve this method has a rather bad computational performance and therefore can only be applied to examples of low degree.

By application of mkParaOffset we can compute a parametric representation of the rational components of the offset curve to a given generator curve (in implicit or rational form) at a specified or an unspecified distance if there exist rational components of the offset curve.

RPHcurve checks if there exists an RPH parametrization of a rational algebraic curve and, if there exits one, it computes the necessary reparametrizing function.

A.2 Description of the Offset Procedures

Procedure: mkImplOffset

FUNCTION: mkImplOffset - compute the offset curve in implicit representation to a given curve

CALLING SEQUENCE:

mkImplOffset(C)
mkImplOffset(C, d)

PARAMETERS

C - algebraic curve in implicit or parametric representation
d - distance, a variable or a rational number

SYNOPSIS

- The command mkImplOffset(C, d) computes the offset curve in implicit representation at distance d to a given plane algebraic curve (in implicit or parametric representation). The input curve has to be be given in the data structure "algebraic set" of CASA.
- If the parameter d is not provided it is assumed that the distance is unspecified.
- The function is in the CASA package, and so can be used in the form mkImplOffset(...) only after performing the command with(casa). The function can always be accessed in the long form casa[mkImplOffset].

EXAMPLES:

```
>a:=mkImplAlgSet([x^2-y],[x,y]);
```

$$a := \text{algebraic\_set}(\,[\,x^2 - y\,], [\,x, y\,]\,)$$

```
>mkImplOffset(a,2);
```

$$\text{algebraic\_set}([-16\,x^6 - 16\,x^4\,y^2 + 40\,x^4\,y + 32\,y^3\,x^2 + 191\,x^4 + 96\,x^2\,y^2 - 16\,y^4 - 30\,x^2\,y - 688\,x^2 - 544\,y - 225\,y^2 + 136\,y^3 + 1156], [\,x, y\,])$$

```
>a:=mkParaAlgSet([2*t/(t^2+1),(t^2-1)/(t^2+1)],[t]);
```

$$a := \text{algebraic\_set}\left(\left[2\,\frac{t}{t^2+1}, \frac{t^2-1}{t^2+1}\right], [\,t\,]\right)$$

```
>mkImplOffset(a);
```

$$\text{algebraic\_set}(\,[\,x^2 - 1 + y^2 + 2\,d - d^2\,], [\,x, y\,]\,),$$
$$\text{algebraic\_set}(\,[\,x^2 - 1 + y^2 - 2\,d - d^2\,], [\,x, y\,]\,)$$

SEE ALSO: casa, mkImplAlgSet, mkParaAlgSet

Procedure: mkParaOffset

FUNCTION: mkParaOffset - compute the offset curve in parametric representation to a given curve

CALLING SEQUENCE:

mkParaOffset(C)
mkParaOffset(C, d)

PARAMETERS

C - algebraic curve in implicit or parametric representation
d - distance, a variable or a rational number

SYNOPSIS

- The command mkParaOffset(C, d) computes the offset curve in parametric representation at distance d to a given plane algebraic curve (in implicit or parametric representation). The input curve has to be be given in the data structure "algebraic set" of CASA.
- If the parameter d is not provided it is assumed that the distance is unspecified.
- The function is in the CASA package, and so can be used in the form mkParaOffset(...) only after performing the command with(casa). The function can always be accessed in the long form casa[mkParaOffset].

EXAMPLES:

```
>a:=mkImplAlgSet([x^2-y],[x,y]);
```

$$a := \mathrm{algebraic\_set}(\,[\,x^2 - y\,], [\,x, y\,]\,)$$

```
>mkParaOffset(a);
```

$$\mathrm{algebraic\_set}\left(\left[-\frac{1}{4}\,\frac{(\,t^2+1+4\,d\,t\,)\,(\,t^2-1\,)}{t\,(\,t^2+1\,)}, \frac{1}{16}\,\frac{t^6-t^4-t^2+1-32\,d\,t^3}{t^2\,(\,t^2+1\,)}\right],\right.$$
$$\left.[\,t\,]\right)$$

```
>a:=mkParaAlgSet([2*t/(t^2+1),(t^2-1)/(t^2+1)],[t]);
```

$$a := \text{algebraic\_set}\left(\left[2\,\frac{t}{t^2+1}, \frac{t^2-1}{t^2+1}\right], [\,t\,]\right)$$

```
>mkParaOffset(a,1/2);
```

$$\text{algebraic\_set}\left(\left[\frac{t}{t^2+1}, \frac{1}{2}\,\frac{t^2-1}{t^2+1}\right], [\,t\,]\right),$$
$$\text{algebraic\_set}\left(\left[3\,\frac{t}{t^2+1}, \frac{3}{2}\,\frac{t^2-1}{t^2+1}\right], [\,t\,]\right)$$

SEE ALSO: casa, mkImplAlgSet, mkParaAlgSet

Procedure: RPHcurve

FUNCTION: RPHcurve checks if there exists an RPH parametrization of the given curve.

CALLING SEQUENCE:

RPHcurve(C, phi)

PARAMETERS

C	- rational curve in parametric representation
phi	- unevaluated name for the reparametrizing function

SYNOPSIS

- RPHcurve checks if there exists a parametrization R such that an algebraic curve C is RPH (a Rational Phytagorean odograph) by analyzing reducibility and rationality of the reparametrizing curve.
- If the reparametrizing curve is irreducible and rational then a quadratic reparametrizing function is assigned to the parameter phi.
- The function is in the CASA package, and so can be used in the form mkParaOffset(...) only after performing the command with(casa). The function can always be accessed in the long form casa[mkParaOffset].

EXAMPLE

```
>a:=mkParaAlgSet([t^7,t^6],[t]);
```

$$a := \text{algebraic\_set}(\,[\,t^7, t^6\,], [\,t\,]\,)$$

```
>RPHcurve(a,'phi');
```

$$true$$

```
>phi;
```

$$-12\,\frac{t}{7\,t^2-7}$$

SEE ALSO: casa, mkParaAlgSet

A.3 Examples

Here we give some more examples computed by the procedures of the offset package. Furthermore for each generator cure we give a timing table of all possibilities of computing an offset curve. In this table we use the following abbreviations:

**	solution does not exist
***	computation interrupted after 5000 sec.
ud	undetermined

Parabola

```
>pparabola:=mkParaAlgSet([t,t^2],[t]);
```

$$pparabola := \text{algebraic\_set}(\,[\,t, t^2\,], [\,t\,]\,)$$

```
>mkImplOffset(pparabola);
```

$$\text{algebraic\_set}([-d^2 - 32\,x^2\,d^2\,y^2 + 8\,x^2\,y\,d^2 - 20\,x^2\,d^2 + 32\,x^2\,y^2 + y^2 + x^4 - 8\,d^4 - 8\,y^3 - 8\,d^2\,y^2 - 2\,y\,x^2 + 8\,y\,d^2 - 48\,x^4\,d^2 + 16\,x^4\,y^2 + 48\,x^2\,d^4 - 40\,x^4\,y - 32\,x^2\,y^3 + 16\,d^4\,y^2 + 32\,d^4\,y - 32\,d^2\,y^3 + 16\,x^6 - 16\,d^6 + 16\,y^4], [\,x, y\,])$$

```
>mkParaOffset(pparabola);
```

$$\text{algebraic\_set}\left(\left[-\frac{1}{4}\,\frac{(\,t^2+1-4\,d\,t\,)\,(\,t^2-1\,)}{t\,(\,t^2+1\,)}, \frac{1}{16}\,\frac{t^6-t^4-t^2+1+32\,d\,t^3}{t^2\,(\,t^2+1\,)}\right], [\,t\,]\right)$$

```
>RPHcurve(pparabola,'phi');
```

$$true$$

```
>phi;
```

$$-\frac{1}{4}\,\frac{t^2-1}{t}$$

function	generator	distance	time
mkImplOffset	para	ud	0.650
	para	2	0.266
	impl	ud	85.050
	impl	2	5.434
mkParaOffset	para	ud	0.684
	para	2	0.133
	impl	ud	2.216
	impl	2	0.433
RPHcurve	para	-	0.033

Table A.1: Example: Parabola

Cusp

```
>pcusp:=mkParaAlgSet([t^2,t^3],[t]);
```

$$pcusp := \text{algebraic\_set}(\,[t^2, t^3\,], [\,t\,]\,)$$

```
>mkImplOffset(pcusp);
```

$$\begin{aligned}
&\text{algebraic_set}([-504\,x^4\,d^2 + 288\,d^4\,x + 4104\,x^3\,d^4 - 1944\,d^6\,x\\
&\quad - 2916\,d^2\,x^6 - 32\,d^2\,x^3 + 1728\,x^2\,d^4 - 2916\,d^6\,x^2 - 432\,x^4\,y^2 + 16\,x^6\\
&\quad + 16\,y^4 + 4374\,d^4\,x^4 - 32\,x^3\,y^2 - 2376\,x^5\,d^2 - 2484\,x^2\,y^2\,d^2\\
&\quad - 4860\,x^3\,y^2\,d^2 - 504\,y^2\,d^2\,x - 4374\,y^4\,d^2\,x + 6318\,y^2\,d^4\,x\\
&\quad - 2187\,x^4\,y^2\,d^2 + 2187\,y^2\,d^4\,x^2 - 32\,y^2\,d^2 + 729\,y^6 + 16\,d^4\\
&\quad - 1458\,x^3\,y^4 - 1458\,x^5\,y^2 + 729\,x^2\,y^4 + 216\,y^4\,x - 1701\,y^4\,d^2\\
&\quad + 1188\,y^2\,d^4 + 729\,x^6\,y^2 - 729\,d^6\,y^2 - 216\,d^6 + 216\,x^7 + 729\,d^8\\
&\quad + 729\,x^8], [\,x, y\,])
\end{aligned}$$

```
>icusp := para2impl(pcusp,[x,y]);
```

$$icusp := \text{algebraic\_set}(\,[\,y^2 - x^3\,], [\,x, y\,]\,)$$

```
>mkParaOffset(icusp);
```

$$\text{algebraic\_set}\left(\left[\frac{1}{9}\,\frac{(t^2-1)(t^4-1+9\,d\,t^2)}{t^2(t^2+1)}, -\frac{1}{27}\,\frac{t^8-2\,t^6+2\,t^2-1-54\,d\,t^4}{t^3(t^2+1)}\right],[t]\right)$$

```
>mkParaOffset(icusp,2);
```

$$\text{algebraic\_set}\left(\left[\frac{1}{9}\,\frac{(t^2-1)(t^4-1+18\,t^2)}{t^2(t^2+1)}, -\frac{1}{27}\,\frac{t^8-2\,t^6+2\,t^2-1-108\,t^4}{t^3(t^2+1)}\right],[t]\right)$$

```
>RPHcurve(pcusp,'phi');
```

$$true$$

```
>phi;
```

$$-\frac{1}{3}\,\frac{t^2-1}{t}$$

function	generator	distance	time
mkImplOffset	para	ud	0.616
	para	2	1.033
	impl	ud	***
	impl	2	***
mkParaOffset	para	ud	0.700
	para	2	0.150
	impl	ud	3.350
	impl	2	0.566
RPHcurve	para	-	0.300

Table A.2: Example: Cusp

Circle

```
>pcircle := mkParaAlgSet([(t^2-1)/(t^2+1),2*t/(t^2+1)],[t]);
```

$$pcircle := \text{algebraic\_set}\left(\left[\frac{t^2-1}{t^2+1}, 2\,\frac{t}{t^2+1}\right], [\,t\,]\right)$$

```
>mkParaOffset(pcircle);
```

$$\text{algebraic\_set}\left(\left[\frac{(\,t^2-1\,)\,(\,1+d\,)}{t^2+1}, 2\,\frac{t\,(\,1+d\,)}{t^2+1}\right], [\,t\,]\right),$$
$$\text{algebraic\_set}\left(\left[-\frac{(\,t^2-1\,)\,(\,-1+d\,)}{t^2+1}, -2\,\frac{t\,(\,-1+d\,)}{t^2+1}\right], [\,t\,]\right)$$

```
>icircle := para2impl(pcircle,[x,y]);
```

$$icircle := \text{algebraic\_set}(\,[\,-1+y^2+x^2\,], [\,x, y\,]\,)$$

```
>mkParaOffset(icircle);
```

$$\text{algebraic\_set}\left(\left[2\,\frac{t\,(\,1+d\,)}{t^2+1}, -\frac{(\,t^2-1\,)\,(\,1+d\,)}{t^2+1}\right], [\,t\,]\right),$$
$$\text{algebraic\_set}\left(\left[-2\,\frac{t\,(\,-1+d\,)}{t^2+1}, \frac{(\,t^2-1\,)\,(\,-1+d\,)}{t^2+1}\right], [\,t\,]\right)$$

```
>RPHcurve(pcircle,'phi');
```

$$true$$

```
>phi;
```

$$t$$

Ellipse

```
>iellipse := mkImplAlgSet([-225+9*y^2+25*x^2],[x,y]);
```

$$iellipse := \text{algebraic\_set}(\,[\,-225+9\,y^2+25\,x^2\,], [\,x, y\,]\,)$$

```
>mkImplOffset(iellipse);
```

function	generator	distance	time
mkImplOffset	para	ud	2.873
	para	2	1.167
	impl	ud	3.267
	impl	2	4.283
mkParaOffset	para	ud	0.450
	para	2	0.067
	impl	ud	2.167
	impl	2	0.467
RPHcurve	para	-	0.134

Table A.3: Example: Circle

$$\begin{aligned}
&\mathrm{algebraic_set}([256\,d^8 - 17408\,d^6 + 411136\,d^4 - 3916800\,d^2 + 12960000\\
&\quad + (\,-224\,d^6 - 20576\,d^4 + 498528\,d^2 - 2656800\,)\,y^2\\
&\quad + (\,1986\,d^4 + 19516\,d^2 + 446850\,)\,y^2\,x^2 + 625\,x^8 + 1606\,y^4\,x^4\\
&\quad + 1700\,x^6\,y^2 + (\,2481\,d^4 - 49650\,d^2 - 149375\,)\,x^4\\
&\quad + (\,126\,d^2 - 6642\,)\,y^6 + 81\,y^8\\
&\quad + (\,-1312\,d^6 + 55392\,d^4 - 514400\,d^2 - 1260000\,)\,x^2\\
&\quad + (\,-3462\,d^2 - 32150\,)\,x^4\,y^2 + (\,-2050\,d^2 + 8750\,)\,x^6\\
&\quad + (\,-239\,d^4 - 17874\,d^2 + 200961\,)\,y^4 + 612\,y^6\,x^2\\
&\quad + (\,-1286\,d^2 - 31158\,)\,x^2\,y^4], [\,x, y\,])
\end{aligned}$$

```
>mkParaOffset(iellipse);
no component of the offset is rational
```

```
>pellipse:= mkParaAlgSet([3*(t^2-1)/(t^2+1),10*t/(t^2+1)],[t]);
```

$$pellipse := \mathrm{algebraic\_set}\left(\left[3\,\frac{t^2-1}{t^2+1}, 10\,\frac{t}{t^2+1}\right], [\,t\,]\right)$$

```
>RPHcurve(pellipse,'phi');
```

false

function	generator	distance	time
mkImplOffset	para	ud	11.150
	para	2	5.584
	impl	ud	308.433
	impl	2	15.067
mkParaOffset	para	ud	**
	para	2	**
	impl	ud	**
	impl	2	**
RPHcurve	para	-	0.550

Table A.4: Example: Ellipse

Cardioid

```
>pcardioid := mkParaAlgSet([(1-t^2)/(1+t^2)^2,2*t/(1+t^2)^2],[t]);
```

$$pcardioid := \text{algebraic\_set}\left(\left[\frac{-t^2+1}{(t^2+1)^2}, 2\,\frac{t}{(t^2+1)^2}\right], [\,t\,]\right)$$

```
>mkImplOffset(pcardioid,2);
```

$$\begin{aligned}&\text{algebraic\_set}([64\,x^8 - 2784\,y^2 + 1408\,x + 1152\,x^5 - 800\,y^6 - 1984\,x^2\\&\quad + 2884\,y^4 - 128\,x^7 - 3040\,x\,y^2 + 1184\,x\,y^4 - 2944\,x^3 + 5184\,y^2\,x^2\\&\quad - 192 + 2304\,x^4 - 2304\,x^2\,y^4 - 2208\,x^4\,y^2 - 704\,x^6 + 256\,y^6\,x^2\\&\quad + 384\,y^4\,x^4 + 256\,x^6\,y^2 + 64\,y^8 + 2336\,y^2\,x^3 - 128\,x\,y^6 - 384\,x^3\,y^4\\&\quad - 384\,x^5\,y^2], [\,x, y\,])\end{aligned}$$

```
>mkParaOffset(pcardioid,2);
```

$$\begin{aligned}
&\mathrm{algebraic_set}\Bigg(\bigg[(6725\,t^2 - 13295562\,t^{11} + 603750\,t - 7500\,t^{16}\\
&\quad - 3361370\,\mathrm{RootOf}(_Z^2-2)\,t^4 + 6725\,t^{14} - 603750\,t^{15} + 138902\,t^{12}\\
&\quad + 4351550\,t^{13} + 13295562\,t^5 + 212345\,\mathrm{RootOf}(_Z^2-2)\,t^5\\
&\quad - 110985\,\mathrm{RootOf}(_Z^2-2)\,t^3 + 1411500\,\mathrm{RootOf}(_Z^2-2)\,t^2\\
&\quad + 20125\,\mathrm{RootOf}(_Z^2-2)\,t - 22354850\,t^7 + 716804\,t^8\\
&\quad + 22354850\,t^9 - 496529\,t^{10} + 3400620\,\mathrm{RootOf}(_Z^2-2)\,t^6\\
&\quad - 121485\,\mathrm{RootOf}(_Z^2-2)\,t^9 - 121485\,\mathrm{RootOf}(_Z^2-2)\,t^7\\
&\quad - 496529\,t^6 - 3400620\,\mathrm{RootOf}(_Z^2-2)\,t^{10} + 138902\,t^4\\
&\quad - 4351550\,t^3 - 228125\,\mathrm{RootOf}(_Z^2-2)\\
&\quad + 228125\,t^{16}\,\mathrm{RootOf}(_Z^2-2) - 7500\\
&\quad - 1411500\,t^{14}\,\mathrm{RootOf}(_Z^2-2) + 212345\,t^{11}\,\mathrm{RootOf}(_Z^2-2)\\
&\quad + 3361370\,t^{12}\,\mathrm{RootOf}(_Z^2-2) + 20125\,t^{15}\,\mathrm{RootOf}(_Z^2-2)\\
&\quad - 110985\,t^{13}\,\mathrm{RootOf}(_Z^2-2))\Big/\\
&\quad (625\,t^8 - 2400\,t^6 + 3554\,t^4 - 2400\,t^2 + 625)^2, -\frac{1}{2}(-41650\,t^2\\
&\quad + 4882932\,t^{11} + 547500\,t + 4375\,t^{16}\\
&\quad + 30738680\,\mathrm{RootOf}(_Z^2-2)\,t^4 + 41650\,t^{14} + 547500\,t^{15}\\
&\quad - 215922\,t^{12} - 2723300\,t^{13} + 4882932\,t^5\\
&\quad - 99830\,\mathrm{RootOf}(_Z^2-2)\,t^5 - 32710\,\mathrm{RootOf}(_Z^2-2)\,t^3\\
&\quad - 8473500\,\mathrm{RootOf}(_Z^2-2)\,t^2 + 18250\,\mathrm{RootOf}(_Z^2-2)\,t\\
&\quad - 2707100\,t^7 - 2707100\,t^9 + 289394\,t^{10}\\
&\quad - 62788180\,\mathrm{RootOf}(_Z^2-2)\,t^6 - 335290\,\mathrm{RootOf}(_Z^2-2)\,t^9\\
&\quad + 335290\,\mathrm{RootOf}(_Z^2-2)\,t^7 - 289394\,t^6\\
&\quad + 79033500\,\mathrm{RootOf}(_Z^2-2)\,t^8 - 4375\\
&\quad - 62788180\,\mathrm{RootOf}(_Z^2-2)\,t^{10} + 215922\,t^4 - 2723300\,t^3\\
&\quad + 1006250\,\mathrm{RootOf}(_Z^2-2) + 1006250\,t^{16}\,\mathrm{RootOf}(_Z^2-2)\\
&\quad - 8473500\,t^{14}\,\mathrm{RootOf}(_Z^2-2) + 99830\,t^{11}\,\mathrm{RootOf}(_Z^2-2)\\
&\quad + 30738680\,t^{12}\,\mathrm{RootOf}(_Z^2-2) - 18250\,t^{15}\,\mathrm{RootOf}(_Z^2-2)\\
&\quad + 32710\,t^{13}\,\mathrm{RootOf}(_Z^2-2))\Big/\\
&\quad (625\,t^8 - 2400\,t^6 + 3554\,t^4 - 2400\,t^2 + 625)^2\bigg], [t]\Bigg)
\end{aligned}$$

function	generator	distance	time
mkImplOffset	para	ud	5.350
	para	2	1.634
	impl	ud	***
	impl	2	***
mkParaOffset	para	ud	384.850
	para	2	299.734
	impl	ud	163.317
	impl	2	61.200
RPHcurve	para	-	278.650

Table A.5: Example: Cardioid

A.4 Source Code

```
# Offset Curves
#
# mkImplOffset computes an offset curve in implicit
# representation to a given curve at given distance d
#
# param.: curve ... an algebraic curve in implicit or
#                   parametric represent.
#         d ... optional arg. for a distance (rational
#               number or unspecified)
#
# output.: the offset curve to the given curve at distance
#          d in implicit representation
#--------------------------------------------------------------

`offset/mkImplOffset` := proc(curve)

    local Offset, curve1, curve2,d;

    global offset;

#--------------------------------------------------------------
# check input parameters
#--------------------------------------------------------------

    if nargs = 0 then
         ERROR(`wrong number of arguments`)
    elif nargs = 2 then
        d := args[2];
    elif nargs > 2 then
        ERROR(`wrong number of arguments`)
    fi;

    if not type(curve,algset) then
        ERROR (`first argument must be an algebraic curve`)
    fi;

#--------------------------------------------------------------
# read the necessary procedures
#--------------------------------------------------------------

    offset[impl2implOffset] :=
          'readlib(''`offset/impl2implOffset`'')';
    offset[impl2implOffsetFD] :=
          'readlib(''`offset/impl2implOffsetFD`'')';
    offset[polyPara2implOffset] :=
          'readlib(''`offset/polyPara2implOffset`'')';
    offset[ratPara2implOffset] :=
```

```
          'readlib(''offset/ratPara2implOffset'')';

#-------------------------------------------------------------
# if the input curve is an irreducible algebraic curve in
# implicit represent; we call the corresp.  procedure to
# compute its offset
#-------------------------------------------------------------

    if type(curve,algset(impl)) then
        if nops([decompose(curve)]) > 1 then
            ERROR (`input curve must be irreducible`)
        fi;
        if not type(curve,planecurve) then
            ERROR (`first argument must be an algebraic curve`)
        fi;
        if type(d,rational) then
            Offset := offset[impl2implOffsetFD](curve,d);
        else
            Offset := offset[impl2implOffset](curve,d);
        fi;

#-------------------------------------------------------------
# if the input curve is an irreducible algebraic curve in
# parametric represent; we distinguish between polynomial and
# rational parametrizations when calling the  procedure to
# compute its offset
#-------------------------------------------------------------

    elif type(curve,algset(para)) then
        curve := mkParaAlgSet(properParametrization(curve),
                              varlist(curve));
        curve2:= represent(curve);
        if nops(curve2) <> 2 then
            ERROR (`first argument must be an algebraic curve`)
        fi;
        if type(curve2[1],polynom) and type(curve2[2],polynom)
        then
            Offset:=offset[polyPara2implOffset](curve,d);
        else
            Offset := offset[ratPara2implOffset](curve,d);
        fi;
    fi

end:
```

```
# Offset Curves
#
# mkParaOffset computes an offset curve in parametric
# representation to a given curve at given distance d,
# if this is possible
#
# param.: curve ... an algebraic curve in implicit or
#                   parametric represent.
#         d ... optional arg. for a distance (rational number
#               or unspecified)
#
# output.: the offset curve to the given curve at distance d
#          in parametric  representation
#--------------------------------------------------------------

`offset/mkParaOffset` := proc(curve)

    local Offset, curve1, d, g;
    global offset;

#--------------------------------------------------------------
# check input parameters
#--------------------------------------------------------------

    if nargs = 0 then
        ERROR(`wrong number of arguments`)
    elif nargs = 2 then
        d := args[2];
    elif nargs > 2 then
        ERROR(`wrong number of arguments`)
    fi;

    if not type(curve,algset) then
        ERROR (`first argument must be an algebraic curve`)
    fi;

#--------------------------------------------------------------
# read the necessary procedures
#--------------------------------------------------------------

    offset[para2paraOffset]:=
          'readlib(''offset/para2paraOffset'')';
    offset[impl2implOffset]:=
          'readlib(''offset/impl2implOffset'')';

#--------------------------------------------------------------
# if the input curve is given in implicit represent. we first
# check if it is an irreducible algebraic plane curve
#--------------------------------------------------------------
```

```
    if type(curve,algset(impl)) then
        if nops([decompose(curve)])>1 then
            ERROR(`input curve must be irreducible`)
        fi;
        if not type(curve,planecurve) then
            ERROR (`first argument must be an algebraic curve`)
        fi;

#--------------------------------------------------------------
# we try to parametrize the input curve in order to compute
# its offset
#--------------------------------------------------------------

        g := genus(curve);
        if g = 0 then
            curve1 := impl2para(curve);
            Offset := offset[para2paraOffset](curve1,d);

#--------------------------------------------------------------
# if the offset curve is not rational the the offset curve is
# either not rational of it consists of 1 rational and 1 non
# rational component
#--------------------------------------------------------------

        else
            curve1 := offset[impl2implOffset](curve,d);
            if nops([curve1])=1 then
                lprint(`There does not exist a rational
                        component of the Offset`)
            elif nops([curve1])=2 then
                g := genus(op(1,[curve1]));
                if g = 0 then
                    Offset := impl2para(op(1,[curve1]))
                else
                    Offset := impl2para(op(2,[curve1]))
                fi;
                lprint(`There exist one rational and one non
                        rational component`);
                lprint(`Here  is the rational component in
                        parametric representation`);
                Offset;
            fi;
        fi;
#--------------------------------------------------------------
# if the input curve is in parametric representation we first
# compute a proper parametrization of the curve, then we
# compute its Offset
#--------------------------------------------------------------
```

```
    elif type(curve,algset(para)) then
        curve := mkParaAlgSet(properParametrization(curve),
                              varlist(curve));
        curve1 := represent(curve);
        if nops(curve1) <> 2 then
            ERROR (`first argument must be an algebraic curve`)
        fi;
        Offset := offset[para2paraOffset](curve,d);
    fi;

end:
```

```
# Offset Curves
#
# impl2implOffset computes an offset curve in implicit
# representation to a given curve at given distance d
#
# param.: curve ... an algebraic curve in implicit represent.
#         d ... optional arg. for a distance (rational number
#               or unspecified)
#
# output.: the offset curve to the given curve at distance d
#          in implicit representation
#--------------------------------------------------------------

`offset/impl2implOffset` := proc(curve,d)

    local C,L,P,R,Offset,var, circle, R1, r;

#--------------------------------------------------------------
# represent the curve by the polynomial P and use the
# variables u and v
#--------------------------------------------------------------

    P := represent(curve)[1];
    var :=   varlist(curve);
    P := subs(var[1]=u,var[2]=v,P);

#--------------------------------------------------------------
#  compute the corresponding circles and normal lines and use
#  Groebner basis computation to obtain the offset curve
#--------------------------------------------------------------

    C := expand((x-u)^2+(y-v)^2-d^2);
    L := expand(primpart(-diff(P,v)*(x-u)+diff(P,u)*(y-v),
                          [x,y]));
    R := grobner[gbasis]([expand(P),C,L],[u,v,x,y], plex);
    R1 := [];
    for r in R do
        if not(member(u,indets(r)) or member(v,indets(r)))
        then
            R1 := [op(R1),r]
        fi
    od;
    Offset := decompose(mkImplAlgSet(R1,[x,y]));

end:
```

```
# Offset Curves
#
# impl2implOffset computes an offset curve in implicit
# representation to a given curve at given distance d
#
# param.: curve ... an algebraic curve in implicit represent.
#         d ... optional arg. for a distance (rational
#               number or unspecified)
#
# output.: the offset curve to the given curve at distance d
#          in implicit representation
#----------------------------------------------------------------

`offset/impl2implOffsetFD` := proc(curve,d)

    local C,L,P,R,Offset,var, circle, R1, r;

#----------------------------------------------------------------
# represent the curve by the polynomial P and use the
# variables u and v
#----------------------------------------------------------------

    P := represent(curve)[1];
    var :=   varlist(curve);
    P := subs(var[1]=u,var[2]=v,P);

#----------------------------------------------------------------
#  compute the corresponding circles and normal lines and use
#  Groebner basis computation to obtain the offset curve
#----------------------------------------------------------------

    C := expand((x-u)^2+(y-v)^2-d^2);
    L := expand(primpart(-diff(P,v)*(x-u)+diff(P,u)*(y-v),
                         [x,y]));
    R := GWalk([expand(P),C,L],[u,v,x,y]);
    R1 := [];
    for r in R do
        if not(member(u,indets(r)) or member(v,indets(r)))
        then
            R1 := [op(R1),r]
        fi
    od;
    Offset := decompose(mkImplAlgSet(R1,[x,y]));

end:
```

```
# Offset Curves
#
# polyPara2implOffset computes an offset curve in implicit
# representation to a given curve at given distance d
#
# param.: curve ... an algebraic curve in polynomial
#                   parametric represent.
#         d ... optional arg. for a distance (rational number
#               or unspecified)
#
# output.: the offset curve to the given curve at distance d
#          in implicit representation
#-------------------------------------------------------------

`offset/polyPara2implOffset` := proc(curve,d)

    local Offset,P,C,L,R,var;

#-------------------------------------------------------------
# represent the curve by the polynomial function P and use
# the variable t as parameter
#-------------------------------------------------------------

    P := represent(curve);
    var := varlist(curve);
    P := subs(var[1]=t,P);

#-------------------------------------------------------------
# compute the defining polynomial system for the offset curve
# and compute the offset curve by taking the resultant
# w.r.t. t
#-------------------------------------------------------------

    C := (x-P[1])^2+(y-P[2])^2-d^2;
    L := primpart(diff(P[1],t)*(x-P[1])+diff(P[2],t)*(y-P[2]),
                  [x,y]);
    R := resultant(C,L,t);
    Offset := decompose(mkImplAlgSet([R],[x,y]));

end:
```

```
# Offset Curves
#
# ratPara2implOffset computes an offset curve in implicit
# representation to a given curve at given distance d
#
# param.: curve ... an algebraic curve in rational parametric
#                   represent.
#         d ... optional arg. for a distance (rational number
#               or unspecified)
#
# output.: the offset curve to the given curve at distance d
#          in implicit representation
#--------------------------------------------------------------

`offset/ratPara2implOffset` := proc(curve,d)

    local Offset,P,g,f1,f2,R,S,G1,R1,U,V,U1,V1,Q,M,N,m,P1,L,
          k,n,Offset1,K,var, Offset2;

#--------------------------------------------------------------
# represent the curve by the polynomial function P and use
# the variable t as parameter
#--------------------------------------------------------------

    P := represent(curve);
    var := varlist(curve);
    P := subs(var[1]=t,P);

#--------------------------------------------------------------
# compute the polynomials corresponding to Theorem 2.6.
#--------------------------------------------------------------

    g := lcm(denom(P[1]),denom(P[2]));
    f1 := simplify(numer(P[1])*g/denom(P[1]),symbolic);
    f2 := simplify(numer(P[2])*g/denom(P[2]),symbolic);
    R := f1^2+f2^2;
    S := gcd(g,R);
    G1 := simplify(g/S,symbolic);
    R1 := simplify(R/S,symbolic);
    U := g*diff(f1,t)-diff(g,t)*f1;
    V := g*diff(f2,t)-diff(g,t)*f2;
    U1 := simplify(U/gcd(U,V),symbolic);
    V1 := simplify(V/gcd(U,V),symbolic);
    P1 := g*G1*(x^2+y^2)-2*G1*(f1*x+f2*y)*w+(R1-d^2*g*G1)*w^2;
    Q := simplify(G1*(U1*x+V1*y)-(U1*f1+V1*f2)/S*w,symbolic);

#--------------------------------------------------------------
# compute the extranous factors
#--------------------------------------------------------------
```

```
    M := [solve(g=0,t)];
    N := {};
    for m in M do
        if subs(t=m,G1) <> 0 then
            N:={subs(t=m,f1*x+f2*y-w*R1/(2*G1))} union N;
        fi;
    od;
    L := expand(product(N[i],i=1..nops(N)));

#--------------------------------------------------------------
# compute the offset curve by taking the resultant of P1 and Q
# and dividing by the extranous factors
#--------------------------------------------------------------

    K := resultant(P1,Q,t);
    n := max(deg(f1),deg(f2),deg(g));
    k := deg(S);
    Offset1 := subs(w=1,simplify(K/(L*w^(n-k)),symbolic));
    if divide(Offset1,x^2+y^2,'q') then
        Offset2:= [x*q,y*q]
    else
        Offset2:=[Offset1]
    fi;
    Offset := decompose(mkImplAlgSet(Offset2,[x,y]));

end:
```

```
# Offset Curves
#
# para2paraOffset computes an offset curve in parametric
# representation to a given curve at given distance d,
# if this is possible
#
# param.: curve ... an algebraic curve in parametric represent.
#         d ... distance (rational number or unspecified)
#
# output.: the offset curve to the given curve at distance d
#          in parametric representation
#--------------------------------------------------------------

`offset/para2paraOffset` := proc(curve,d)

    local p,N,Pp,m,r,m1,Oplus,Ominus,Gp,g,phi,Offset,gp,var,
          Phi;

#--------------------------------------------------------------
# represent the curve by the rational function p, choose the
# variable t as parameter, compute the normal vector N and its
# length m.
#--------------------------------------------------------------

    p := represent(curve);
    var := varlist(curve);
    p := subs(var[1]=t,p);
    Pp := diff(p,t);
    N := [-Pp[2],Pp[1]];
    m := simplify(sqrt(N[1]^2+N[2]^2),sqrt,symbolic);

#--------------------------------------------------------------
# check if p is a proper RPH parametrization and if it is then
# we can immidiately compute the two rational components of
# the offset curve
#--------------------------------------------------------------

    r := not type(m,sqrt);
    for m1 in op(m) do
       if type(m1,sqrt) then
           r := false
       fi;
    od;

    if r then
        Oplus := [normal(p[1]+d/m*N[1]),
                  normal(p[2]+d/m*N[2])];
        Ominus := [normal(p[1]-d/m*N[1]),
                   normal(p[2]-d/m*N[2])];
```

```
        RETURN(mkParaAlgSet(Oplus,[t]),
               mkParaAlgSet(Ominus,[t]));

#--------------------------------------------------------------
# else we compute the reparametrizing curve Gp
#--------------------------------------------------------------

    else
        gp := subs(t=x,primpart(numer(normal(N[2]*
                   (y^2-1)-2*N[1]*y)),y));
        Gp := mkImplAlgSet([gp],[x,y]);

#--------------------------------------------------------------
# if the genus of Gp <> 0 then the offset curve has no
# rational component
#--------------------------------------------------------------

        g := genus(Gp);
        if g <> 0 then
            lprint(`no component of the offset is rational`);
            RETURN();

#--------------------------------------------------------------
# if the genus of Gp = 0 then the offset curve has rational
# components and we have to compute a parametrization phi for
# the x-coordinate of the curve Gp, which is just solving a
# linear equation if Gp is linear in t
#--------------------------------------------------------------

        else
            if degree(gp,x) = 1 then
                phi := subs(y=t,solve(gp=0,x));
                Phi := t;
            else
                Gp := impl2para(Gp);
                Gp := represent(Gp);
                phi := Gp[1];
                Phi := Gp[2];
            fi;

#--------------------------------------------------------------
# phi now can be used as a reparametrizing function for the
# input curve which leads to a RPH-parametrization and
# therefore to a parametrization of the irreducibel offset
# curve
#--------------------------------------------------------------

            p := subs(t=phi,p);
            N := [-diff(p[2],t),diff(p[1],t)];
```

```
                Offset := simplify(normal([p[1]+2*d*Phi*N[1]/
                          (N[2]*(Phi^2+1)), p[2]+2*d*N[2]*Phi/
                          (N[2]*(Phi^2+1))]),symbolic);
                Offset := mkParaAlgSet(Offset,[t]);
             fi;
         fi
    end:
```

```
# Offset Curves
#
# RPHcurve checks if there exists a parametrization R such
# that an algebraic curve C is RPH (a Rational
# Phytagorean Hodograph) and computes a reparametrization
# if necessary.
#
# param.: curve ... an algebraic curve in parametric
#                   represent.
#         phi ... reparametrizing function (output)
#
# output.: true if there exists an RPH-parametrization and
#               the reparametrizing function (as parameter).
#--------------------------------------------------------------

`offset/RPHcurve` := proc(curve, phi)

    local curve1, curve2, Gp, gp, var, N;

#--------------------------------------------------------------
# check input parameters
#--------------------------------------------------------------

    if nargs = 0 then
         ERROR(`wrong number of arguments`)
    elif nargs > 2 then
        ERROR(`wrong number of arguments`)
    fi;

    curve2:= represent(curve);
    var := varlist(curve);
    curve2:= subs(var[1]=t, curve2);
    if nops(curve2) <> 2 then
         ERROR (`first argument must be an algebraic curve`)
    fi;

#--------------------------------------------------------------
# we compute the reparametrizing curve Gp
#--------------------------------------------------------------

    N := [-diff(curve2[2],t), diff(curve2[1],t)];
    gp := primpart(numer(normal(N[2]*(y^2-1)-2*N[1]*y)),y);
    Gp := mkImplAlgSet([gp],[t,y]);

#--------------------------------------------------------------
# if Gp is reducible then the given param. is already RPH
#--------------------------------------------------------------

    if nops([decompose(Gp)]) > 1 then
```

```
        phi:=t;
        RETURN(true)

#--------------------------------------------------------------
# if GP is irreducible and its genus g<>0 then there does
# not exist an RPH-parametrization
#--------------------------------------------------------------

    else
        if genus(Gp)<>0 then
            phi := 'phi';
            RETURN(false);

#--------------------------------------------------------------
# if GP is irreducible and its genus g=0 then we compute
# a suitable reparametrizing function
#--------------------------------------------------------------

        else
            if degree(gp,t) = 1 then
                phi := subs(y=t,solve(gp=0,t))
            else
                Gp := impl2para(Gp);
                Gp := represent(Gp);
                phi := Gp[1]
            fi;
            RETURN(true)
        fi;
    fi;

end:
```

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