■ Zero is a Polynomial

Let P be arbitrary but fixed, generated by `PolynomialsBySequenceVariables`.

We prove

(0_P ∈ P) $0_P \in \mathbf{P}$

under the provided assumptions

(⟨ ⟩ ∈ P) ⟨ ⟩ ∈ P ,

(0_P) $0_P :=$ ⟨ ⟩

by simplification.

We simplify

0_P

to ⟨ ⟩ by (0_P)

and prove the equivalent

⟨ ⟩ ∈ P .

This follows from (⟨ ⟩ ∈ P).

■ One is a Polynomial

Let the monomial domain M be arbitrary but fixed .
Let P be `PolynomialsBySequenceVariables[M]` .
We prove
$(1_{P} \in P)$ $1_{P} \in P$
under the provided assumptions
$(\langle 1_{M} \rangle \in P)$ $\langle 1_{M} \rangle \in P$,
(1_{P}) $1_{P} := \langle 1_{M} \rangle$
by simplification.
We simplify
1_{P}
to $\langle 1_{M} \rangle$ by (1_{P})
and prove the equivalent
$\langle 1_{M} \rangle \in P$.
This follows from $(\langle 1_{M} \rangle \in P)$.

■ Negative is Closed

Let the monomial domain $\mathtt{M}$ be arbitrary but fixed .
Let $\mathtt{P}$ be $\mathtt{PolynomialsBySequenceVariables[M]}$.
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{-}$, $\underset{M}{>}$.
Let $\mathtt{p}$, $\langle\bar{\mathtt{p}}\rangle$ range over $\mathtt{P}$.
Let $\mathtt{m}$, $\mathtt{n}$ range over $\mathtt{M}$.
We prove

(– closed) $\underset{p\in P}{\forall}\ -p \in P$

under the provided assumptions

$(\underset{M}{-}\text{ closed})\quad \underset{m\in M}{\forall}\ \underset{M}{-}\, m \in M$,

$(\underset{M}{-} > \underset{M}{-})\quad \underset{m,\,n}{\forall}\ m > n \implies \underset{M}{-}\, m > \underset{M}{-}\, n$

definitions

$(\smile)\quad \underset{e,\,\bar{s}}{\forall}\ e \smile \langle\bar{s}\rangle := \langle e, \bar{s}\rangle$,

$(\langle\,\rangle \in P)\quad \langle\,\rangle \in P$,

$(\langle m\rangle \in P)\quad \underset{m}{\forall}\ \langle m\rangle \in P$,

$(\in P)\quad \underset{m,\,n,\,\bar{p}}{\forall}\ \langle n, \bar{p}\rangle \in P \wedge m > n \implies \langle m, n, \bar{p}\rangle \in P$,

$(-0)\quad -\langle\,\rangle := \langle\,\rangle$,

$(-)\quad \underset{m,\,\bar{p}}{\forall}\ -\langle m, \bar{p}\rangle := \underset{M}{-}m \smile -\langle\bar{p}\rangle$

and lemmata

$(\smile \in P)\quad \underset{m,\,p}{\forall}\ m > \mathrm{ldm}[p] \vee p = \langle\,\rangle \implies m \smile p \in P$,

$(\mathrm{ldm}[\smile])\quad \underset{m,\,p}{\forall}\ \mathrm{ldm}[m \smile p] = m$

by induction.
We represent polynomials in sequence notation and prove

$\underset{\bar{p}}{\forall}\ -\langle\bar{p}\rangle \in P$

by polynomial induction.
Induction base, $\mathtt{p}$ zero polynomial: We prove

$-\langle\,\rangle \in P$.

We simplify

$-\langle\,\rangle$

to $\langle\,\rangle$ by (-0)
and prove the equivalent

$\langle\,\rangle \in P$.

This follows from $(\langle\,\rangle \in P)$.

Induction base, $\mathtt{p}$ single monomial: We take $\dot{p}_0 \in M$ arbitrary but fixed and prove

$-\langle\dot{p}_0\rangle \in P$.

We simplify

$-\langle\dot{p}_0\rangle$

to $\underset{M}{-}\dot{\mathtt{p}}_0 \smile - \langle\,\rangle$ by $(-)$,

to $\underset{M}{-}\dot{\mathtt{p}}_0 \smile \langle\,\rangle$ by (-0),

to $\langle \underset{M}{-}\dot{\mathtt{p}}_0 \rangle$ by $(\smile)$

and prove the equivalent

$\langle \underset{M}{-}\dot{\mathtt{p}}_0 \rangle \in P$.

By $(\langle m \rangle \underset{P}{\in})$, this follows from

$\underset{M}{-}\dot{\mathtt{p}}_0 \in M$.

This follows from ($\underset{M}{-}$ closed).

Induction step for $\mathtt{p}$:

We take $\left\langle \dot{\mathtt{p}}_0, \ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0} \right\rangle \in \mathtt{P}$ arbitrary but fixed, assume the induction hypothesis

(– closed.V.IH) $-\left\langle \ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0} \right\rangle \in \mathtt{P}$

and prove the induction conclusion

$-\left\langle \dot{\mathtt{p}}_0, \ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0} \right\rangle \in \mathtt{P}$.

We simplify

$-\left\langle \dot{\mathtt{p}}_0, \ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0} \right\rangle$

to $\underset{M}{-}\dot{\mathtt{p}}_0 \smile -\left\langle \ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0} \right\rangle$ by $(-)$,

to $\underset{M}{-}\dot{\mathtt{p}}_0 \smile \underset{M}{-}\ddot{\mathtt{p}}_0 \smile - \langle \overline{\mathtt{p}_0} \rangle$ by $(-)$

and prove the equivalent

$\underset{M}{-}\dot{\mathtt{p}}_0 \smile \underset{M}{-}\ddot{\mathtt{p}}_0 \smile - \langle \overline{\mathtt{p}_0} \rangle \in P$.

By $(\smile \in P)$, this follows from

$\underset{M}{-}\dot{\mathtt{p}}_0 > \mathtt{ldm}[\underset{M}{-}\ddot{\mathtt{p}}_0 \smile - \langle \overline{\mathtt{p}_0} \rangle] \vee \underset{M}{-}\ddot{\mathtt{p}}_0 \smile - \langle \overline{\mathtt{p}_0} \rangle = \langle\,\rangle$.

We assume

$\underset{M}{-}\ddot{\mathtt{p}}_0 \smile - \langle \overline{\mathtt{p}_0} \rangle \neq \langle\,\rangle$

and prove

$\underset{M}{-}\dot{\mathtt{p}}_0 > \mathtt{ldm}[\underset{M}{-}\ddot{\mathtt{p}}_0 \smile - \langle \overline{\mathtt{p}_0} \rangle]$.

We simplify

$\mathtt{ldm}[\underset{M}{-}\ddot{\mathtt{p}}_0 \smile - \langle \overline{\mathtt{p}_0} \rangle]$

to $\underset{M}{-}\ddot{\mathtt{p}}_0$ by $(\mathtt{ldm}[\smile])$

and prove the equivalent

$\underset{M}{-}\dot{\mathtt{p}}_0 > \underset{M}{-}\ddot{\mathtt{p}}_0$.

By $(\underset{M}{-} > \underset{M}{-})$, this follows from the definition of $\dot{\mathtt{p}}_0$ and $\ddot{\mathtt{p}}_0$.

■ Lemma: Leading Monomial of a Polynomial constructed by Prepending (polynomial induction)

Let the monomial domain $\mathtt{M}$ be arbitrary but fixed.
Let $\mathtt{P}$ be $\mathtt{PolynomialsBySequenceVariables[M]}$.
Let $\mathtt{p}$, $\langle\bar{\mathtt{p}}\rangle$ range over $\mathtt{P}$.
Let m range over $\mathtt{M}$.
We prove

$$(\mathtt{ldm}[\smile])\quad \underset{\mathtt{m},\,\mathtt{p}}{\forall}\ \mathtt{ldm}[\mathtt{m}\smile\mathtt{p}] = \mathtt{m}$$

under the provided assumptions

$$(\smile)\quad \underset{\mathtt{e},\,\bar{\mathtt{s}}}{\forall}\ \mathtt{e}\smile\langle\bar{\mathtt{s}}\rangle := \langle\mathtt{e},\ \bar{\mathtt{s}}\rangle$$

$$(\mathtt{ldm})\quad \underset{\mathtt{m},\,\bar{\mathtt{p}}}{\forall}\ \mathtt{ldm}[\langle\mathtt{m},\ \bar{\mathtt{p}}\rangle] := \mathtt{m}$$

by induction.
We take $\mathtt{m}_0 \in \mathtt{M}$ arbitrary but fixed and prove

$$\underset{\mathtt{p}}{\forall}\ \mathtt{ldm}[\mathtt{m}_0\smile\mathtt{p}] = \mathtt{m}_0\ .$$

We represent polynomials in sequence notation and prove

$$\underset{\bar{\mathtt{p}}}{\forall}\ \mathtt{ldm}[\mathtt{m}_0\smile\langle\bar{\mathtt{p}}\rangle] = \mathtt{m}_0$$

by polynomial induction.
Induction base, $\mathtt{p}$ zero polynomial: We prove

$$\mathtt{ldm}[\mathtt{m}_0\smile\langle\,\rangle] = \mathtt{m}_0$$

by simplification.
We simplify the left hand side
to $\mathtt{ldm}[\langle\mathtt{m}_0\rangle]$ by ($\smile$),
to $\mathtt{m}_0$ by ($\mathtt{ldm}$)

Induction base, $\mathtt{p}$ single monomial: We take $\dot{\mathtt{p}}_0 \in \mathtt{M}$ arbitrary but fixed and prove

$$\mathtt{ldm}\left[\mathtt{m}_0\smile\left\langle\dot{\mathtt{p}}_0\right\rangle\right] = \mathtt{m}_0$$

by simplification.
We simplify the left hand side
to $\mathtt{ldm}\left[\left\langle\mathtt{m}_0,\ \dot{\mathtt{p}}_0\right\rangle\right]$ by ($\smile$),
to $\mathtt{m}_0$ by ($\mathtt{ldm}$)

Induction step for $\mathtt{p}$:
We take $\left\langle\dot{\mathtt{p}}_0,\ \ddot{\mathtt{p}}_0,\ \overline{\mathtt{p}_0}\right\rangle \in \mathtt{P}$ arbitrary but fixed, assume the induction hypothesis

$$\mathtt{ldm}[\mathtt{m}_0\smile\langle\mathtt{n}_0,\ \overline{\mathtt{p}_0}\rangle] = \mathtt{m}_0$$

and prove the induction conclusion

$$\mathtt{ldm}\left[\mathtt{m}_0\smile\left\langle\dot{\mathtt{p}}_0,\ \ddot{\mathtt{p}}_0,\ \overline{\mathtt{p}_0}\right\rangle\right] = \mathtt{m}_0$$

by simplification.
We simplify the left hand side
to $\mathtt{ldm}\left[\left\langle\mathtt{m}_0,\ \dot{\mathtt{p}}_0,\ \ddot{\mathtt{p}}_0,\ \overline{\mathtt{p}_0}\right\rangle\right]$ by ($\smile$),
to $\mathtt{m}_0$ by ($\mathtt{ldm}$)

■ Lemma: Leading Monomial of a Polynomial constructed by Prepending (sequence induction)

Let the monomial domain $\mathtt{M}$ be arbitrary but fixed.
Let $\mathtt{P}$ be $\mathtt{PolynomialsBySequenceVariables[M]}$.
Let $\mathtt{p}$, $\langle\bar{\mathtt{p}}\rangle$ range over $\mathtt{P}$.
Let $\mathtt{m}$ range over $\mathtt{M}$.
We prove

$$(\mathtt{ldm}[\smile]) \quad \underset{\mathtt{m},\mathtt{p}}{\forall} \; \mathtt{ldm}[\mathtt{m}\smile\mathtt{p}] = \mathtt{m}$$

under the provided assumptions

$$(\smile) \quad \underset{\mathtt{e},\,\bar{\mathtt{s}}}{\forall} \; \mathtt{e}\smile\langle\bar{\mathtt{s}}\rangle := \langle\mathtt{e},\,\bar{\mathtt{s}}\rangle$$

$$(\mathtt{ldm}) \quad \underset{\mathtt{m},\,\bar{\mathtt{p}}}{\forall} \; \mathtt{ldm}[\langle\mathtt{m},\,\bar{\mathtt{p}}\rangle] := \mathtt{m}$$

by induction.
We take $\mathtt{m}_0 \in \mathtt{M}$ arbitrary but fixed and prove

$$\underset{\mathtt{p}}{\forall} \; \mathtt{ldm}[\mathtt{m}_0\smile\mathtt{p}] = \mathtt{m}_0 \,.$$

We represent polynomials in sequence notation and prove

$$\underset{\bar{\mathtt{p}}}{\forall} \; \mathtt{ldm}[\mathtt{m}_0\smile\langle\bar{\mathtt{p}}\rangle] = \mathtt{m}_0$$

by sequence induction.
Induction base, $\bar{\mathtt{p}}$ empty sequence: We prove

$$\mathtt{ldm}[\mathtt{m}_0\smile\langle\,\rangle] = \mathtt{m}_0$$

by simplification.
We simplify the left hand side
to $\mathtt{ldm}[\langle\mathtt{m}_0\rangle]$ by $(\smile)$,
to $\mathtt{m}_0$ by $(\mathtt{ldm})$
Induction step for $\bar{\mathtt{p}}$:
We take $\langle\overset{\bullet}{\mathtt{p}}_0,\,\overline{\mathtt{p}_0}\rangle \in \mathtt{P}$ arbitrary but fixed, assume the induction hypothesis

$$\mathtt{ldm}[\mathtt{m}_0\smile\langle\overline{\mathtt{p}_0}\rangle] = \mathtt{m}_0$$

and prove the induction conclusion

$$\mathtt{ldm}\left[\mathtt{m}_0\smile\left\langle\overset{\bullet}{\mathtt{p}}_0,\,\overline{\mathtt{p}_0}\right\rangle\right] = \mathtt{m}_0$$

by simplification.
We simplify the left hand side
to $\mathtt{ldm}\left[\left\langle\mathtt{m}_0,\,\overset{\bullet}{\mathtt{p}}_0,\,\overline{\mathtt{p}_0}\right\rangle\right]$ by $(\smile)$,
to $\mathtt{m}_0$ by $(\mathtt{ldm})$

▪ Lemma: Conditions for Prepending to give a Polynomial

Let the monomial domain M be arbitrary but fixed.
Let P be PolynomialsBySequenceVariables[M].
In the following, we omit the underscripts of $\underset{M}{>}$.
Let p, $\langle\bar{p}\rangle$ range over P.
Let m, n range over M.
We prove

($\smile\in$P) $\underset{m,p}{\forall}\; m > \mathrm{ldm}[p] \lor p = \langle\rangle \Longrightarrow m \smile p \in P$

under the provided assumptions

($\langle$m$\rangle\in$P) $\underset{m}{\forall}\; \langle m\rangle \in P$,

($\in$P) $\underset{m,n,\bar{p}}{\forall}\; \langle n, \bar{p}\rangle \in P \land m > n \Longrightarrow \langle m, n, \bar{p}\rangle \in P$,

($\smile$) $\underset{e,\bar{s}}{\forall}\; e \smile \langle\bar{s}\rangle := \langle e, \bar{s}\rangle$,

(ldm) $\underset{m,\bar{p}}{\forall}\; \mathrm{ldm}[\langle m, \bar{p}\rangle] = m$

by induction.
We take $m_0 \in M$ arbitrary but fixed and prove

$\underset{p}{\forall}\; m_0 > \mathrm{ldm}[p] \lor p = \langle\rangle \Longrightarrow m_0 \smile p \in P$.

We represent polynomials in sequence notation and prove

$\underset{\bar{p}}{\forall}\; m_0 > \mathrm{ldm}[\langle\bar{p}\rangle] \lor \langle\bar{p}\rangle = \langle\rangle \Longrightarrow m_0 \smile \langle\bar{p}\rangle \in P$.

We simplify

$m_0 \smile \langle\bar{p}\rangle$

to $\langle m_0, \bar{p}\rangle$ by ($\smile$)
and prove the equivalent

$\underset{\bar{p}}{\forall}\; m_0 > \mathrm{ldm}[\langle\bar{p}\rangle] \lor \langle\bar{p}\rangle = \langle\rangle \Longrightarrow \langle m_0, \bar{p}\rangle \in P$.

by sequence induction.
Induction base, $\bar{p}$ empty sequence: We prove

$m_0 > \mathrm{ldm}[\langle\rangle] \lor \langle\rangle = \langle\rangle \Longrightarrow \langle m_0\rangle \in P$.

We assume

$m_0 > \mathrm{ldm}[\langle\rangle] \lor \langle\rangle = \langle\rangle$

and show

$\langle m_0\rangle \in P$.

This follows from ($\langle$m$\rangle\in$P).
Induction step for $\bar{p}$:
We take $\left\langle \dot{p}_0, \overline{p_0}\right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis

$m_0 > \mathrm{ldm}[\langle\overline{p_0}\rangle] \lor \langle\overline{p_0}\rangle = \langle\rangle \Longrightarrow \langle m_0, \overline{p_0}\rangle \in P$.

and prove the induction conclusion

$m_0 > \mathrm{ldm}\left[\left\langle \dot{p}_0, \overline{p_0}\right\rangle\right] \lor \langle m_1, \overline{p_0}\rangle = \langle\rangle \Longrightarrow \left\langle m_0, \dot{p}_0, \overline{p_0}\right\rangle \in P$.

We simplify

$\mathrm{ldm}\left[\left\langle \dot{p}_0, \overline{p_0}\right\rangle\right]$

to $\overset{\bullet}{p}_0$ by (ldm)
and prove the equivalent

$m_0 > \overset{\bullet}{p}_0 \vee \langle m_1 , \overline{p_0} \rangle = \langle\,\rangle \Longrightarrow \left\langle m_0 , \overset{\bullet}{p}_0 , \overline{p_0} \right\rangle \in P$.

This is equivalent to

$m_0 > \overset{\bullet}{p}_0 \Longrightarrow \left\langle m_0 , \overset{\bullet}{p}_0 , \overline{p_0} \right\rangle \in P$.

We assume

(assum) $m_0 > \overset{\bullet}{p}_0$.

and prove

$\left\langle m_0 , \overset{\bullet}{p}_0 , \overline{p_0} \right\rangle \in P$.

By ($\in$ P), this follows from (assum) and the definition of $\overset{\bullet}{p}_0$ and $\overline{p_0}$.

■ Plus is Closed

Let the monomial domain $\mathtt{M}$ be arbitrary but fixed .
Let $\mathtt{P}$ be $\mathtt{PolynomialsBySequenceVariables[M]}$.
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{+}$, $\underset{M}{-}$, $\underset{M}{>}$, $\underset{M}{\sim}$.

Let $\mathtt{p}$, $\mathtt{q}$, $\langle \bar{p} \rangle$, $\langle \bar{q} \rangle$ range over $\mathtt{P}$.
Let $\mathtt{l}$, $\mathtt{m}$, $\mathtt{n}$ range over $\mathtt{M}$.
We prove

(+ closed) $\underset{p,q \in P}{\forall} \; p + q \in P$

under the provided assumptions

($\underset{M}{+}$ closed) $\underset{m,n \in M}{\forall} \; m \sim n \wedge m \neq -n \Longrightarrow m \underset{M}{+} n \in M$,

($\underset{M}{+}$ >) $\underset{l,m,n}{\forall} \; m > n \wedge l \sim m \wedge l \neq -m \Longrightarrow l \underset{M}{+} m > n$,

(−−) $\underset{m}{\forall} \; --m = m$,

($\underset{M}{+}$comm) $\underset{m,n}{\forall} \; m \sim n \wedge m \neq -n \Longrightarrow m \underset{M}{+} n = n \underset{M}{+} m$

definitions

($\smile$) $\underset{e,\bar{s}}{\forall} \; e \smile \langle \bar{s} \rangle := \langle e, \bar{s} \rangle$,

(ldm) $\underset{m,\bar{p}}{\forall} \; \mathrm{ldm}[\langle m, \bar{p} \rangle] := m$,

(+0) $\underset{p}{\forall} \; p + \langle \rangle := \langle \rangle$,

(0+) $\underset{p}{\forall} \; \langle \rangle + p := \langle \rangle$,

(+) $$\underset{m,n,\bar{p},\bar{q}}{\forall} \; \langle m, \bar{p} \rangle + \langle n, \bar{q} \rangle := \left\| \begin{array}{rl} m \smile (\langle \bar{p} \rangle + \langle n, \bar{q} \rangle) & \Leftarrow m > n, \\ n \smile (\langle m, \bar{p} \rangle + \langle \bar{q} \rangle) & \Leftarrow n > m, \\ \langle \bar{p} \rangle + \langle \bar{q} \rangle & \Leftarrow m = -n, \\ \left(m \underset{M}{+} n\right) \smile (\langle \bar{p} \rangle + \langle \bar{q} \rangle) & \end{array} \right\|$$

and lemmata

($\smile \in$ P) $\underset{m,p}{\forall} \; m \underset{M}{>} \mathrm{ldm}[p] \vee p = \langle \rangle \Longrightarrow m \smile p \in P$,

(> ldm[+]) $\underset{m,p,q}{\forall} \; m > \mathrm{ldm}[p] \wedge m > \mathrm{ldm}[q] \wedge p + q \neq \langle \rangle \Longrightarrow$
$m > \mathrm{ldm}[p + q]$

by induction.
We represent polynomials in sequence notation and prove

(+ closed.V) $\underset{\bar{p},\bar{q}}{\forall} \; \langle \bar{p} \rangle + \langle \bar{q} \rangle \in P$

by polynomial induction on $\mathtt{p}$.
Base case, $\mathtt{p}$ zero polynomial: We prove

(+ closed.V.IB⟨ ⟩) $\underset{\bar{q}}{\forall} \; \langle \rangle + \langle \bar{q} \rangle \in P$.

Induction base, $\mathtt{p}$ single monomial: We take $\dot{p}_0 \in M$ be arbitrary but fixed and prove

(+ closed.V.IB⟨m⟩) $\underset{\bar{q}}{\forall} \; \left\langle \dot{p}_0 \right\rangle + \langle \bar{q} \rangle \in P$

by polynomial induction on $\mathtt{q}$:

Base case, q zero polynomial: We prove

(+ closed.V.IB⟨m⟩.IB⟨ ⟩) $\langle \dot{p}_0 \rangle + \langle\,\rangle \in P$.

Induction base, q single monomial: We take $\dot{q}_0 \in M$ be arbitrary but fixed and prove

(+ closed.V.IB⟨m⟩.IB⟨m⟩) $\langle \dot{p}_0 \rangle + \langle \dot{q}_0 \rangle \in P$

by case distinction.

Case $\dot{p}_0 > \dot{q}_0$:

We simplify

$\langle \dot{p}_0 \rangle + \langle \dot{q}_0 \rangle$

to $\dot{p}_0 \smile (\langle\,\rangle + \langle \dot{q}_0 \rangle)$ by (+),

to $\dot{p}_0 \smile \langle \dot{q}_0 \rangle$ by (0+),

to $\langle \dot{p}_0, \dot{q}_0 \rangle$ by ($\smile$)

and prove the equivalent

(+ closed.V.IB⟨m⟩.IB⟨m⟩.C1) $\langle \dot{p}_0, \dot{q}_0 \rangle \in P$.

This follows from (ϵ P) and (⟨m⟩ ϵ P).

Case $\dot{q}_0 > \dot{p}_0$:

Case $\dot{p}_0 = -\dot{q}_0$:

We simplify

$\langle \dot{p}_0 \rangle + \langle \dot{q}_0 \rangle$

to $\langle\,\rangle$ by (+)

and prove the equivalent

(+ closed.V.IB⟨m⟩.IB⟨m⟩.C3) $\langle\,\rangle \in P$.

This follows from (⟨ ⟩ ϵ P).

Case otherwise:

We simplify

$\langle \dot{p}_0 \rangle + \langle \dot{q}_0 \rangle$

to $\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \rangle$ by (+)

and prove the equivalent

(+ closed.V.IB⟨m⟩.IB⟨m⟩.C4) $\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \rangle \in P$.

By (⟨m⟩ ϵ P), this follows from

$\dot{p}_0 \underset{M}{+} \dot{q}_0 \in M$.

This follows from ($\underset{M}{+}$ closed).

Induction step for q: We take $\langle \dot{q}_0, \ddot{q}_0, \overline{q_0} \rangle \epsilon$ P arbitrary but fixed, assume the induction hypothesis for q

(+ closed.V.IB⟨m⟩.IH) $\langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle \in P$

and prove the induction conclusion for q

(+ closed.V.IB⟨m⟩.IC) $\langle \dot{p}_0 \rangle + \langle \dot{q}_0, \ddot{q}_0, \overline{q_0} \rangle \in P$

by case distinction.

Case $\dot{p}_0 > \dot{q}_0$:

Case $\dot{q}_0 > \dot{p}_0$:

We simplify

$\langle \dot{p}_0 \rangle + \langle \dot{q}_0, \ddot{q}_0, \overline{q_0} \rangle$

to $\dot{q}_0 \smile (\langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle)$ by (+)

and prove the equivalent

(+ closed.V.IB⟨m⟩.IC.C2) $\dot{q}_0 \smile (\langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle) \in \mathrm{P}$.

By ($\smile \in$ P), this follows from

$\dot{q}_0 > \mathrm{ldm}[\langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle] \vee \langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle = \langle\rangle$.

We assume

(+ closed.V.IB⟨m⟩.IC.C2.or) $\langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle \neq \langle\rangle$

and prove

$\dot{q}_0 > \mathrm{ldm}[\langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle]$.

By (>ldm[+]), this follows from

$\dot{q}_0 > \mathrm{ldm}[\langle \dot{p}_0 \rangle] \wedge \dot{q}_0 > \mathrm{ldm}[\langle \ddot{q}_0, \overline{q_0} \rangle] \wedge$

$\langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle \neq \langle\rangle$

We prove the individual conjunctive parts:

We prove

$\dot{q}_0 > \mathrm{ldm}[\langle \dot{p}_0 \rangle]$

We simplify

$\mathrm{ldm}[\langle \dot{p}_0 \rangle]$

to $\dot{p}_0$ by (ldm)

and prove the equivalent

$\dot{q}_0 > \dot{p}_0$

This follows from the case assumption.

We prove

$\dot{q}_0 > \mathrm{ldm}[\langle \ddot{q}_0, \overline{q_0} \rangle]$

We simplify

$\mathrm{ldm}[\langle \ddot{q}_0, \overline{q_0} \rangle]$

to $\ddot{q}_0$ by (ldm)

and prove the equivalent

$\dot{q}_0 > \ddot{q}_0$

This follows from the definition of $\dot{q}_0$ and $\ddot{q}_0$.

We prove

$\langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle \neq \langle\rangle$.

This follows from (+ closed.V.IB⟨m⟩.IC.C2.or).

Case $\dot{p}_0 = -\dot{q}_0$:

Case otherwise:
We simplify
and prove the equivalent
(+ closed.V.IB⟨m⟩.IC.C4) $\left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0,\ \ddot{q}_0,\ \overline{q_0} \right\rangle \in P.$
By (ϵ P), this follows from
$\dot{p}_0 \underset{M}{+} \dot{q}_0 \in M \wedge \dot{p}_0 \underset{M}{+} \dot{q}_0 > \ddot{q}_0 \wedge \left\langle \ddot{q}_0,\ \overline{q_0} \right\rangle \in P.$
We prove the individual conjunctive parts:
We prove
$\dot{p}_0 \underset{M}{+} \dot{q}_0 \in M$
This follows from (+ closed).
We prove
$\dot{p}_0 \underset{M}{+} \dot{q}_0 > \ddot{q}_0$
By ($\underset{M}{+}$ >), this follows from the case assumptions and
$\dot{q}_0 > \ddot{q}_0.$
This follows from the definition of $\dot{q}_0$ and $\ddot{q}_0$.
We prove
$\left\langle \ddot{q}_0,\ \overline{q_0} \right\rangle \in P$
This follows from the definition of $\ddot{q}_0$ and $\overline{q_0}$.
Induction step for p:
We take $\left\langle \dot{p}_0,\ \ddot{p}_0,\ \overline{p_0} \right\rangle \epsilon$ P arbitrary but fixed, assume the induction hypothesis for p
(+ closed.V.IH) $\underset{q}{\forall} \left\langle \ddot{p}_0,\ \overline{p_0} \right\rangle + \langle \bar{q} \rangle \in P$
and prove the induction conclusion for p
(+ closed.V.IC) $\underset{q}{\forall} \left\langle \dot{p}_0,\ \ddot{p}_0,\ \overline{p_0} \right\rangle + \langle \bar{q} \rangle \in P$
by polynomial induction on q:
Base case, q zero polynomial: We prove
(+ closed.V.IC.IB⟨ ⟩) $\left\langle \dot{p}_0,\ \ddot{p}_0,\ \overline{p_0} \right\rangle + \langle\,\rangle \in P.$
Induction base, q single monomial: We take $\dot{q}_0 \epsilon$ M be arbitrary but fixed and prove
(+ closed.V.IC.IB⟨m⟩) $\left\langle \dot{p}_0,\ \ddot{p}_0,\ \overline{p_0} \right\rangle + \left\langle \dot{q}_0 \right\rangle \in P$
by case distinction.
Case $\dot{p}_0 > \dot{q}_0$:
Case $\dot{q}_0 > \dot{p}_0$:
Case $\dot{p}_0 = -\dot{q}_0$:
Case otherwise:
We simplify

and prove the equivalent

(+ closed.V.IC.IB⟨m⟩.C4) $\left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0,\ \ddot{p}_0,\ \overline{p_0} \right\rangle \in P.$

By (ϵ P), this follows from

$\dot{p}_0 \underset{M}{+} \dot{q}_0 \in M \wedge \dot{p}_0 \underset{M}{+} \dot{q}_0 > \ddot{p}_0 \wedge \left\langle \ddot{p}_0,\ \overline{p_0} \right\rangle \in P.$

We prove the individual conjunctive parts:

We prove

$\dot{p}_0 \underset{M}{+} \dot{q}_0 \in M$

This follows from (+ closed).

We prove

$\dot{p}_0 \underset{M}{+} \dot{q}_0 > \ddot{p}_0$

We simplify

$\dot{p}_0 \underset{M}{+} \dot{q}_0$

to $\dot{q}_0 \underset{M}{+} \dot{p}_0$ by ($\underset{M}{+}$ comm)

and prove the equivalent

$\dot{q}_0 \underset{M}{+} \dot{p}_0 > \ddot{p}_0$

By ($\underset{M}{+}$ >), this follows from the case assumptions and

$\dot{p}_0 > \ddot{p}_0$.

This follows from the definition of $\dot{p}_0$ and $\ddot{p}_0$.

We prove

$\left\langle \ddot{p}_0,\ \overline{p_0} \right\rangle \in P$

This follows from the definition of $\ddot{p}_0$ and $\overline{p_0}$.

Induction step for q:

We take $\left\langle \dot{q}_0,\ \ddot{q}_0,\ \overline{q_0} \right\rangle \epsilon$ P arbitrary but fixed, assume the induction hypothesis for q

(+ closed.V.IC.IH) $\left\langle \dot{p}_0,\ \ddot{p}_0,\ \overline{p_0} \right\rangle + \left\langle \ddot{q}_0,\ \overline{q_0} \right\rangle \in P$

and prove the induction conclusion for q

(+ closed.V.IC.IC) $\left\langle \dot{p}_0,\ \ddot{p}_0,\ \overline{p_0} \right\rangle + \left\langle \dot{q}_0,\ \ddot{q}_0,\ \overline{q_0} \right\rangle \in P$

by case distinction.

Case $\dot{p}_0 > \dot{q}_0$:

Case $\dot{q}_0 > \dot{p}_0$:

Case $\dot{p}_0 = -\dot{q}_0$:

We simplify

and prove the equivalent

(+ closed.V.IC.IC.C3) $\left\langle \ddot{p}_0,\ \overline{p_0} \right\rangle + \left\langle \ddot{q}_0,\ \overline{q_0} \right\rangle \in P.$

This follows from (+ closed.V.IH) .

Case otherwise:

We simplify

$\left\langle \dot{p}_0, \ddot{p}_0, \overline{p_0} \right\rangle + \left\langle \dot{q}_0, \ddot{q}_0, \overline{q_0} \right\rangle$

to $\left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \right\rangle \smile \left(\left\langle \ddot{p}_0, \overline{p_0} \right\rangle + \left\langle \ddot{q}_0, \overline{q_0} \right\rangle \right)$ by (+)

and prove the equivalent

(+ closed.V.IC.IC.C4) $\left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \right\rangle \smile \left(\left\langle \ddot{p}_0, \overline{p_0} \right\rangle + \left\langle \ddot{q}_0, \overline{q_0} \right\rangle \right) \in \mathtt{P}.$

By ($\smile \in$ P), this follows from

$\dot{p}_0 \underset{M}{+} \dot{q}_0 > \mathtt{ldm}\left[\left\langle \ddot{p}_0, \overline{p_0} \right\rangle + \left\langle \ddot{q}_0, \overline{q_0} \right\rangle \right] \vee$

$\left\langle \ddot{p}_0, \overline{p_0} \right\rangle + \left\langle \ddot{q}_0, \overline{q_0} \right\rangle = \langle\, \rangle$

■ Times is Closed

Let the monomial domain $\mathtt{M}$ be arbitrary but fixed .
Let $\mathtt{P}$ be $\mathtt{PolynomialsBySequenceVariables[M]}$.
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{*}$, $\underset{P}{+}$.

Let p , q , $\langle\bar{p}\rangle$, $\langle\bar{q}\rangle$ range over P.
Let m, n range over M.
We prove
($*$ closed) $\underset{q,p\in P}{\forall}\ p * q \in P$
under the provided assumptions
($\underset{M}{*}$ closed) $\underset{m,n\in M}{\forall}\ m \underset{M}{*} n \in M$
definitions
($\smile$) $\underset{e,\,\bar{s}}{\forall}\ e \smile \langle\bar{s}\rangle := \langle e,\ \bar{s}\rangle$,
($\langle\,\rangle \in$ P) $\langle\,\rangle \in P$,
($\langle m\rangle \in$ P) $\underset{m}{\forall}\ \langle m\rangle \in P$,
($*$ 0) $\underset{p}{\forall}\ p * \langle\,\rangle := \langle\,\rangle$,
(0 $*$) $\underset{p}{\forall}\ \langle\,\rangle * p := \langle\,\rangle$,
($*$) $\underset{m,n,\bar{p},\bar{q}}{\forall}\ \langle m,\ \bar{p}\rangle * \langle n,\ \bar{q}\rangle := \left\langle m \underset{M}{*} n\right\rangle + (\langle m\rangle * \langle\bar{q}\rangle + \langle\bar{p}\rangle * \langle n,\ \bar{q}\rangle)$
and lemmata
(+ closed) $\underset{p,q\in P}{\forall}\ p + q \in P$
by induction.
We represent polynomials in sequence notation and prove
($*$ closed.V) $\underset{\bar{q},\bar{p}}{\forall}\ \langle\bar{p}\rangle * \langle\bar{q}\rangle \in P$
by sequence induction on $\bar{q}$:
Induction base, $\bar{q}$ empty sequence: We prove
($*$ closed.V.IB) $\underset{\bar{p}}{\forall}\ \langle\bar{p}\rangle * \langle\,\rangle \in P$

Induction step for $\bar{q}$:
We take $\left\langle \overset{\bullet}{q}_0\ ,\ \overline{q_0}\right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis
($*$ closed.V.IH) $\underset{\bar{p}}{\forall}\ \langle\bar{p}\rangle * \langle\overline{q_0}\rangle \in P$
and prove the induction conclusion
($*$ closed.V.IC) $\underset{\bar{p}}{\forall}\ \langle\bar{p}\rangle * \left\langle \overset{\bullet}{q}_0\ ,\ \overline{q_0}\right\rangle \in P$
by sequence induction on $\bar{p}$:
Induction base, $\bar{p}$ empty sequence: We prove
($*$ closed.V.IC.IB) $\langle\,\rangle * \left\langle \overset{\bullet}{q}_0\ ,\ \overline{q_0}\right\rangle \in P$.

Induction step for $\bar{p}$:
We take $\left\langle \overset{\bullet}{p}_0\ ,\ \overline{p_0}\right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis
($*$ closed.V.IC.IH) $\langle\overline{p_0}\rangle * \left\langle \overset{\bullet}{q}_0\ ,\ \overline{q_0}\right\rangle \in P$
and prove the induction conclusion

(* closed.V.IC.IC) $\left\langle \overset{\bullet}{p}_0 , \overline{p_0} \right\rangle * \left\langle \overset{\bullet}{q}_0 , \overline{q_0} \right\rangle \in P.$
We simplify
$\left\langle \overset{\bullet}{p}_0 , \overline{p_0} \right\rangle * \left\langle \overset{\bullet}{q}_0 , \overline{q_0} \right\rangle$
to $\left\langle \overset{\bullet}{p}_0 \underset{M}{*} \overset{\bullet}{q}_0 \right\rangle + \left(\left\langle \overset{\bullet}{p}_0 \right\rangle * \langle \bar{q}_0 \rangle + \langle \overline{p_0} \rangle * \left\langle \overset{\bullet}{q}_0 , \overline{q_0} \right\rangle \right)$ by (*)
and prove the equivalent
(* closed.V.IC.IC.S) $\left\langle \overset{\bullet}{p}_0 \underset{M}{*} \overset{\bullet}{q}_0 \right\rangle + \left(\left\langle \overset{\bullet}{p}_0 \right\rangle * \langle \bar{q}_0 \rangle + \langle \overline{p_0} \rangle * \left\langle \overset{\bullet}{q}_0 , \overline{q_0} \right\rangle \right) \in P$
By (+ closed), this follows from
$\left\langle \overset{\bullet}{p}_0 \underset{M}{*} \overset{\bullet}{q}_0 \right\rangle \in P \wedge \left\langle \overset{\bullet}{p}_0 \right\rangle * \langle \bar{q}_0 \rangle + \langle \overline{p_0} \rangle * \left\langle \overset{\bullet}{q}_0 , \overline{q_0} \right\rangle \in P.$
We prove the individual conjunctive parts.
We prove
$\left\langle \overset{\bullet}{p}_0 \underset{M}{*} \overset{\bullet}{q}_0 \right\rangle \in P.$
By ($\langle m \rangle \in P$) this follows from
$\overset{\bullet}{p}_0 \underset{M}{*} \overset{\bullet}{q}_0 \in M.$
This follows from ($\underset{M}{*}$ closed).
We prove
$\left\langle \overset{\bullet}{p}_0 \right\rangle * \langle \bar{q}_0 \rangle + \langle \overline{p_0} \rangle * \left\langle \overset{\bullet}{q}_0 , \overline{q_0} \right\rangle \in P.$
By (+ closed), this follows from
$\left\langle \overset{\bullet}{p}_0 \right\rangle * \langle \bar{q}_0 \rangle \in P \wedge \langle \overline{p_0} \rangle * \left\langle \overset{\bullet}{q}_0 , \overline{q_0} \right\rangle \in P.$
We prove the individual conjunctive parts.
We prove
$\left\langle \overset{\bullet}{p}_0 \right\rangle * \langle \bar{q}_0 \rangle \in P$
This follows from (* closed.V.IH).
We prove
$\langle \overline{p_0} \rangle * \left\langle \overset{\bullet}{q}_0 , \overline{q_0} \right\rangle \in P$
This follows from (* closed.V.IC.IH).

■ Zero is Neutral Element

Let P be arbitrary but fixed, generated by `PolynomialsBySequenceVariables`.
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{+}$.

Let p range over P.
We prove
(neutral 0**)** $\underset{p}{\forall}\; p + 0 = p$
under the provided assumptions
(0) $0 := \langle\,\rangle$,
(+0) $\underset{p}{\forall}\; p + \langle\,\rangle := p$
by simplification.
We simplify the left hand side
to $p + \langle\,\rangle$ by (0),
to p by (+0)

■ Negative is an Inverse of Plus

Let the monomial domain M be arbitrary but fixed .
Let P be PolynomialsBySequenceVariables[M] .
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{+}$, $\underset{P}{-}$, $\underset{M}{>}$.
Let p , q , r, $\langle\bar{p}\rangle$, $\langle\bar{q}\rangle$ range over P.
Let m, n range over M.
We prove

(+−) $\underset{p}{\forall}\ p + -p = 0$

under the provided assumptions

$(\underset{M}{-}\underset{M}{-})\quad \underset{m}{\forall}\ \underset{M}{-}\underset{M}{-}m = m$

definitions

$(\langle\rangle \in P)\quad \langle\rangle \in P,$

$(0)\quad 0 := \langle\rangle,$

$(+0)\quad \underset{p}{\forall}\ p + \langle\rangle := \langle\rangle,$

$(0+)\quad \underset{p}{\forall}\ \langle\rangle + p := \langle\rangle,$

$$(+)\quad \underset{m,n,\bar{p},\bar{q}}{\forall}\ \langle m, \bar{p}\rangle + \langle n, \bar{q}\rangle := \left\| \begin{array}{ll} m \smile (\langle\bar{p}\rangle + \langle n, \bar{q}\rangle) & \Leftarrow m > n, \\ n \smile (\langle m, \bar{p}\rangle + \langle\bar{q}\rangle) & \Leftarrow n > m, \\ \langle\bar{p}\rangle + \langle\bar{q}\rangle & \Leftarrow m = \underset{M}{-}n, \\ (m \underset{M}{+} n) \smile (\langle\bar{p}\rangle + \langle\bar{q}\rangle) & \end{array} \right\|$$

$(-0)\quad -\langle\rangle := \langle\rangle,$

$(-)\quad \underset{m,\bar{p}}{\forall}\ -\langle m, \bar{p}\rangle := \underset{M}{-}m \smile -\langle\bar{p}\rangle$

and lemmata

$(-\ \text{closed})\quad \underset{p\in P}{\forall}\ -p \in P$

$(+\ \text{assoc})\quad \underset{p,q,r}{\forall}\ (p + q) + r = p + (q + r)$

$(\smile\ =\ +)\quad \underset{m,p}{\forall}\ m \smile p \in P \Longrightarrow m \smile p = \langle m\rangle + p$

by induction.
We represent polynomials in sequence notation and prove

$(+-.V)\quad \underset{\bar{p}}{\forall}\ \langle\bar{p}\rangle + -\langle\bar{p}\rangle = 0.$

We simplify

and prove the equivalent

$(+-.V.S)\quad \underset{\bar{p}}{\forall}\ \langle\bar{p}\rangle + -\langle\bar{p}\rangle = \langle\rangle$

by sequence induction on $\bar{p}$.
Base case, $\bar{p}$ empty sequence: We prove

$(+-.V.S.IB)\quad \langle\rangle + -\langle\rangle = \langle\rangle .$

We simplify

$\langle\rangle + -\langle\rangle$

to $-\langle\rangle$ by (0+)

to $\langle\,\rangle$ by (-0)
and prove the equivalent
(+–.V.S.IB.S) $\langle\,\rangle \in \mathtt{P}$
This follows from $(\langle\,\rangle \in \mathtt{P})$.
Induction step for $\bar{\mathtt{p}}$:
We take $\left\langle \dot{\mathtt{p}}_0\,,\ \overline{\mathtt{p}_0} \right\rangle \in \mathtt{P}$ arbitrary but fixed, assume the induction hypothesis for $\bar{\mathtt{p}}$
(+–.V.S.IH) $\langle \overline{\mathtt{p}_0} \rangle + -\langle \overline{\mathtt{p}_0} \rangle = \langle\,\rangle$
and prove the induction conclusion for $\bar{\mathtt{p}}$
(+–.V.S.IC) $\left\langle \dot{\mathtt{p}}_0\,,\ \overline{\mathtt{p}_0} \right\rangle + -\left\langle \dot{\mathtt{p}}_0\,,\ \overline{\mathtt{p}_0} \right\rangle = \langle\,\rangle$
by simplification.
We simplify the left hand side
to $\left\langle \dot{\mathtt{p}}_0\,,\ \overline{\mathtt{p}_0} \right\rangle + \bar{_{\mathtt{M}}}\dot{\mathtt{p}}_0 \smile -\langle \overline{\mathtt{p}_0} \rangle$ by $(-)$,
to $\left\langle \dot{\mathtt{p}}_0\,,\ \overline{\mathtt{p}_0} \right\rangle + \left(\left\langle \bar{_{\mathtt{M}}}\dot{\mathtt{p}}_0 \right\rangle + -\langle \overline{\mathtt{p}_0} \rangle\right)$ by $(\smile\ =\ +)$,
which we can use because
$\bar{_{\mathtt{M}}}\dot{\mathtt{p}}_0 \smile -\langle \overline{\mathtt{p}_0} \rangle \in \mathtt{P}$.
We simplify
$\bar{_{\mathtt{M}}}\dot{\mathtt{p}}_0 \smile -\langle \overline{\mathtt{p}_0} \rangle$
to $-\left\langle \dot{\mathtt{p}}_0\,,\ \overline{\mathtt{p}_0} \right\rangle$ by $(-)$
and prove the equivalent
$-\left\langle \dot{\mathtt{p}}_0\,,\ \overline{\mathtt{p}_0} \right\rangle \in \mathtt{P}$.
This follows from (– closed).
to $\left(\left\langle \dot{\mathtt{p}}_0\,,\ \overline{\mathtt{p}_0} \right\rangle + \left\langle \bar{_{\mathtt{M}}}\dot{\mathtt{p}}_0 \right\rangle\right) + -\langle \overline{\mathtt{p}_0} \rangle$ by (+ assoc),
to $(\langle \overline{\mathtt{p}_0} \rangle + \langle\,\rangle) + -\langle \overline{\mathtt{p}_0} \rangle$ by (+),
which we can use because
$\dot{\mathtt{p}}_0 = \bar{_{\mathtt{M}}}\bar{_{\mathtt{M}}}\dot{\mathtt{p}}_0$.
This follows from $(\bar{_{\mathtt{M}}}\ \bar{_{\mathtt{M}}})$.
to $\langle \overline{\mathtt{p}_0} \rangle + -\langle \overline{\mathtt{p}_0} \rangle$ by (+0),
to $\langle\,\rangle$ by (+–.V.IH)

■ Lemma: Prepending is Summation

Let the monomial domain M be arbitrary but fixed .
Let P be PolynomialsBySequenceVariables[M] .
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{+}$, $\underset{P}{-}$, $\underset{M}{>}$.
Let p , q , $\langle\bar{p}\rangle$, $\langle\bar{q}\rangle$ range over P.
Let m, n range over M.
We prove

(⌣ = +) $\underset{m,p}{\forall}\ m \smile p \in P \Longrightarrow m \smile p = \langle m\rangle + p$

under the provided assumptions

(+0) $\underset{p}{\forall}\ p + \langle\rangle := \langle\rangle,$

(0+) $\underset{p}{\forall}\ \langle\rangle + p := \langle\rangle,$

(+) $$\underset{m,n,\bar{p},\bar{q}}{\forall}\ \langle m, \bar{p}\rangle + \langle n, \bar{q}\rangle := \begin{Vmatrix} m \smile (\langle\bar{p}\rangle + \langle n, \bar{q}\rangle) \Longleftarrow m > n, \\ n \smile (\langle m, \bar{p}\rangle + \langle\bar{q}\rangle) \Longleftarrow n > m, \\ \langle\bar{p}\rangle + \langle\bar{q}\rangle \Longleftarrow m \underset{M}{=} n, \\ \left(m \underset{M}{+} n\right) \smile (\langle\bar{p}\rangle + \langle\bar{q}\rangle) \end{Vmatrix}$$

(⌣) $\underset{e,\bar{s}}{\forall}\ e \smile \langle\bar{s}\rangle := \langle e, \bar{s}\rangle$

by induction.
We take $m_0 \in M$ arbitrary but fixed and prove

(⌣ = +.F) $\underset{p}{\forall}\ m_0 \smile p \in P \Longrightarrow m_0 \smile p = \langle m_0\rangle + p$

We represent polynomials in sequence notation and prove

(⌣ = +.F.V) $\underset{\bar{p}}{\forall}\ m_0 \smile \langle\bar{p}\rangle \in P \Longrightarrow m_0 \smile \langle\bar{p}\rangle = \langle m_0\rangle + \langle\bar{p}\rangle.$

We simplify
$m_0 \smile \langle\bar{p}\rangle$
to $\langle m_0, \bar{p}\rangle$ by (⌣)
and prove the equivalent

(⌣ = +.F.V.S) $\underset{\bar{p}}{\forall}\ \langle m_0, \bar{p}\rangle \in P \Longrightarrow \langle m_0, \bar{p}\rangle = \langle m_0\rangle + \langle\bar{p}\rangle.$

by sequence induction on $\bar{p}$.
Base case, $\bar{p}$ empty sequence: We prove

(⌣ = +.F.V.S.IB) $\langle m_0\rangle \in P \Longrightarrow \langle m_0\rangle = \langle m_0\rangle + \langle\rangle.$

We assume
$\langle m_0\rangle \in P$
and prove
$\langle m_0\rangle = \langle m_0\rangle + \langle\rangle$
by simplification.
We simplify the right hand side
to $\langle m_0\rangle$ by (+0) .
Induction step for $\bar{p}$:
We take $\left\langle \dot{p}_0, \overline{p_0}\right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis for $\bar{p}$

(⌣ = +.F.V.S.IH) $\langle m_0, \overline{p_0}\rangle \in P \Longrightarrow \langle m_0, \overline{p_0}\rangle = \langle m_0\rangle + \langle\overline{p_0}\rangle.$

and prove the induction conclusion for $\bar{\mathtt{p}}$

$(\smile\ =\ +$.F.V.S.IC) $\left\langle \mathtt{m}_0 ,\ \dot{\mathtt{p}}_0 ,\ \overline{\mathtt{p}_0} \right\rangle \in \mathtt{P} \implies$

$\left\langle \mathtt{m}_0 ,\ \dot{\mathtt{p}}_0 ,\ \overline{\mathtt{p}_0} \right\rangle = \langle \mathtt{m}_0 \rangle + \left\langle \dot{\mathtt{p}}_0 ,\ \overline{\mathtt{p}_0} \right\rangle .$

We assume

$(\smile\ =\ +$.F.V.S.IC.assum) $\left\langle \mathtt{m}_0 ,\ \dot{\mathtt{p}}_0 ,\ \overline{\mathtt{p}_0} \right\rangle \in \mathtt{P}$

and prove

$\left\langle \mathtt{m}_0 ,\ \dot{\mathtt{p}}_0 ,\ \overline{\mathtt{p}_0} \right\rangle = \langle \mathtt{m}_0 \rangle + \left\langle \dot{\mathtt{p}}_0 ,\ \overline{\mathtt{p}_0} \right\rangle$

by simplification.

We simplify the right hand side

to $\mathtt{m}_0 \smile \left(\langle\,\rangle + \left\langle \dot{\mathtt{p}}_0 ,\ \overline{\mathtt{p}_0} \right\rangle \right)$ by (+),

which we can use because

$\mathtt{m}_0 > \dot{\mathtt{p}}_0 .$

This follows from $(\smile\ =\ +$.F.V.S.IC.assum).

to $\mathtt{m}_0 \smile \left\langle \dot{\mathtt{p}}_0 ,\ \overline{\mathtt{p}_0} \right\rangle$ by (0+),

to $\left\langle \mathtt{m}_0 ,\ \dot{\mathtt{p}}_0 ,\ \overline{\mathtt{p}_0} \right\rangle$ by $(\smile)$.

■ Plus is Commutative

Let the monomial domain $\mathtt{M}$ be arbitrary but fixed .
Let $\mathtt{P}$ be $\mathtt{PolynomialsBySequenceVariables[M]}$.
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{+}$, $\underset{M}{-}$, $\underset{M}{>}$.

Let $\mathtt{p}$, $\mathtt{q}$, $\langle \bar{p} \rangle$, $\langle \bar{q} \rangle$ range over $\mathtt{P}$.
Let $\mathtt{m},\ \mathtt{n}$ range over $\mathtt{M}$.
We prove

(+comm) $\underset{p,q}{\forall}\ p + q = q + p$

under the provided assumptions

$(--)\ \underset{m}{\forall}\ --m = m$

$(\underset{M}{+}\mathrm{comm})\ \underset{m,n}{\forall}\ m \underset{M}{\sim} n \Longrightarrow m \underset{M}{+} n = n \underset{M}{+} m$

and definitions

$(+0)\ \underset{p}{\forall}\ p + \langle\rangle := \langle\rangle,$

$(0+)\ \underset{p}{\forall}\ \langle\rangle + p := \langle\rangle,$

$$(+)\ \underset{m,n,\bar{p},\bar{q}}{\forall}\ \langle m, \bar{p}\rangle + \langle n, \bar{q}\rangle := \left\| \begin{array}{ll} m \smile (\langle \bar{p}\rangle + \langle n, \bar{q}\rangle) & \Longleftarrow m > n, \\ n \smile (\langle m, \bar{p}\rangle + \langle \bar{q}\rangle) & \Longleftarrow n > m, \\ \langle \bar{p}\rangle + \langle \bar{q}\rangle & \Longleftarrow m = -n, \\ \left(m \underset{M}{+} n\right) \smile (\langle \bar{p}\rangle + \langle \bar{q}\rangle) & \end{array} \right\|$$

by induction.
We represent polynomials in sequence notation and prove

$(+\mathrm{comm.V})\ \underset{\bar{p},\bar{q}}{\forall}\ \langle \bar{p}\rangle + \langle \bar{q}\rangle = \langle \bar{q}\rangle + \langle \bar{p}\rangle$

by sequence induction on $\bar{p}$:
Induction base, $\bar{p}$ empty sequence: We prove

$(+\mathrm{comm.V.IB})\ \underset{\bar{q}}{\forall}\ \langle\rangle + \langle \bar{q}\rangle = \langle \bar{q}\rangle + \langle\rangle$

by simplification.
We simplify the left hand side
to $\langle \bar{q}\rangle$ by (0+)
and the right hand side

Induction step for $\bar{p}$:
We take $\langle \dot{p}_0,\ \overline{p_0}\rangle \in$ P arbitrary but fixed, assume the induction hypothesis for $\bar{p}$:

$(+\mathrm{comm.V.IH})\ \underset{\bar{q}}{\forall}\ \langle \overline{p_0}\rangle + \langle \bar{q}\rangle = \langle \bar{q}\rangle + \langle \overline{p_0}\rangle$

and prove the induction conclusion for $\bar{p}$

$(+\mathrm{comm.V.IC})\ \underset{\bar{q}}{\forall}\ \langle \dot{p}_0,\ \overline{p_0}\rangle + \langle \bar{q}\rangle = \langle \bar{q}\rangle + \langle \dot{p}_0,\ \overline{p_0}\rangle$

by sequence induction on $\bar{q}$:
Induction base, $\bar{q}$ empty sequence: We prove

$(+\mathrm{comm.V.IC.IB})\ \langle \dot{p}_0,\ \overline{p_0}\rangle + \langle\rangle = \langle\rangle + \langle \dot{p}_0,\ \overline{p_0}\rangle$

Induction step for $\bar{q}$:
We take $\left\langle \dot{q}_0 , \overline{q_0} \right\rangle \in$ P arbitrary but fixed, assume the induction hypothesis for $\bar{q}$

(+comm.V.IC.IH) $\left\langle \dot{p}_0 , \overline{p_0} \right\rangle + \langle \overline{q_0} \rangle = \langle \overline{q_0} \rangle + \left\langle \dot{p}_0 , \overline{p_0} \right\rangle$

and prove the induction conclusion for $\bar{q}$

(+comm.V.IC.IC) $\left\langle \dot{p}_0 , \overline{p_0} \right\rangle + \left\langle \dot{q}_0 , \overline{q_0} \right\rangle = \left\langle \dot{q}_0 , \overline{q_0} \right\rangle + \left\langle \dot{p}_0 , \overline{p_0} \right\rangle$

by case distinction.

Case $\dot{p}_0 > \dot{q}_0$:
We prove (+comm.V.IC.IC) by simplification.
We simplify the left hand side
to $\dot{p}_0 - \left(\langle \overline{p_0} \rangle + \left\langle \dot{q}_0 , \overline{q_0} \right\rangle \right)$ by (+)
and the right hand side
to $\dot{p}_0 - \left(\left\langle \dot{q}_0 , \overline{q_0} \right\rangle + \langle \overline{p_0} \rangle \right)$ by (+),
to $\dot{p}_0 - \left(\langle \overline{p_0} \rangle + \left\langle \dot{q}_0 , \overline{q_0} \right\rangle \right)$ by (+comm.V.IH)

Case $\dot{q}_0 > \dot{p}_0$:

Case $\dot{p}_0 = -\dot{q}_0$:
We prove (+comm.V.IC.IC) by simplification.
We simplify the left hand side
to $\langle \overline{p_0} \rangle + \langle \overline{q_0} \rangle$ by (+)
and the right hand side
to $\langle \overline{q_0} \rangle + \langle \overline{p_0} \rangle$ by (+),
which we can use because
$\dot{q}_0 = -\dot{p}_0$
by simplification.
We simplify the right hand side
to $--\dot{q}_0$ by the case assumption,
to $\dot{q}_0$ by (−−)
to $\langle \overline{p_0} \rangle + \langle \overline{q_0} \rangle$ by (+comm.V.IH)

Case otherwise:
We prove (+comm.V.IC.IC) by simplification.
We simplify the left hand side
to $\left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \right\rangle - \left(\langle \overline{p_0} \rangle + \langle \overline{q_0} \rangle \right)$ by (+)
and the right hand side
to $\left\langle \dot{q}_0 \underset{M}{+} \dot{p}_0 \right\rangle - \left(\langle \overline{q_0} \rangle \underset{P}{+} \langle \overline{p_0} \rangle \right)$ by (+),
to $\left\langle \dot{q}_0 \underset{M}{+} \dot{p}_0 \right\rangle - \left(\langle \overline{p_0} \rangle + \langle \overline{q_0} \rangle \right)$ by (+comm.V.IH)
to $\left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \right\rangle - \left(\langle \overline{p_0} \rangle + \langle \overline{q_0} \rangle \right)$ by $(\underset{M}{+}\text{comm})$

■ Plus is Associative (case subproofs closed)

Let the monomial domain M be arbitrary but fixed .
Let P be PolynomialsBySequenceVariables[M] .
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{+}$, $\underset{M}{-}$, $\underset{M}{>}$, $\underset{M}{\sim}$.
Let p , q , r, $\langle\bar{p}\rangle$, $\langle\bar{q}\rangle$, $\langle\bar{r}\rangle$ range over P.
Let l, m, n range over M.
We prove

(+ assoc) $\underset{p,q,r}{\forall} \ (p + q) + r = p + (q + r)$

under the provided assumptions

by induction.
We represent polynomials in sequence notation and prove

(+ assoc.V) $\underset{\bar{p},\bar{q},\bar{r}}{\forall} \ (\langle\bar{p}\rangle + \langle\bar{q}\rangle) + \langle\bar{r}\rangle = \langle\bar{p}\rangle + (\langle\bar{q}\rangle + \langle\bar{r}\rangle)$

by polynomial induction on p.
Base case, p zero polynomial: We prove

(+ assoc.V.IB⟨ ⟩) $\underset{\bar{q},\bar{r}}{\forall} \ (\langle\rangle + \langle\bar{q}\rangle) + \langle\bar{r}\rangle = \langle\rangle + (\langle\bar{q}\rangle + \langle\bar{r}\rangle)$

Induction base, p single monomial: We take $\dot{p}_0 \in M$ be arbitrary but fixed and prove

(+ assoc.V.IB⟨m⟩) $\underset{\bar{q},\bar{r}}{\forall} \ \left(\left\langle\dot{p}_0\right\rangle + \langle\bar{q}\rangle\right) + \langle\bar{r}\rangle = \left\langle\dot{p}_0\right\rangle + (\langle\bar{q}\rangle + \langle\bar{r}\rangle)$

by polynomial induction on q:
Base case, q zero polynomial: We prove

(+ assoc.V.IB⟨m⟩.IB⟨ ⟩) $\underset{\bar{r}}{\forall} \ \left(\left\langle\dot{p}_0\right\rangle + \langle\rangle\right) + \langle\bar{r}\rangle = \left\langle\dot{p}_0\right\rangle + (\langle\rangle + \langle\bar{r}\rangle)$

Induction base, q single monomial: We take $\dot{q}_0 \in M$ be arbitrary but fixed and prove

(+ assoc.V.IB⟨m⟩.IB⟨m⟩)

$\underset{\bar{r}}{\forall} \ \left(\left\langle\dot{p}_0\right\rangle + \left\langle\dot{q}_0\right\rangle\right) + \langle\bar{r}\rangle = \left\langle\dot{p}_0\right\rangle + \left(\left\langle\dot{q}_0\right\rangle + \langle\bar{r}\rangle\right)$

by polynomial induction on r:
Base case, r zero polynomial: We prove

(+ assoc.V.IB⟨m⟩.IB⟨m⟩.IB⟨ ⟩)

$\left(\left\langle\dot{p}_0\right\rangle + \left\langle\dot{q}_0\right\rangle\right) + \langle\rangle = \left\langle\dot{p}_0\right\rangle + \left(\left\langle\dot{q}_0\right\rangle + \langle\rangle\right)$

Induction base, r single monomial: We take $\dot{r}_0 \in M$ be arbitrary but fixed and prove

(+ assoc.V.IB⟨m⟩.IB⟨m⟩.IB⟨m⟩)

$\left(\left\langle\dot{p}_0\right\rangle + \left\langle\dot{q}_0\right\rangle\right) + \left\langle\dot{r}_0\right\rangle = \left\langle\dot{p}_0\right\rangle + \left(\left\langle\dot{q}_0\right\rangle + \left\langle\dot{r}_0\right\rangle\right)$

by case distinction.

Case $\dot{p}_0 > \dot{q}_0$:
We simplify

and prove the equivalent

$\langle \dot{p}_0, \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle = \langle \dot{p}_0 \rangle + (\langle \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle)$
by case distinction.

Case $\dot{p}_0 > \dot{r}_0$:
We simplify

and prove the equivalent
$\dot{p}_0 - (\langle \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle) = \langle \dot{p}_0 \rangle + (\langle \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle)$
by case distinction.

Case $\dot{q}_0 > \dot{r}_0$:

Case $\dot{r}_0 > \dot{q}_0$:

Case $\dot{q}_0 = -\dot{r}_0$:

Case otherwise:

Case $\dot{r}_0 > \dot{p}_0$:

Case $\dot{p}_0 = -\dot{r}_0$:

Case otherwise:

Case $\dot{q}_0 > \dot{p}_0$:
We simplify

and prove the equivalent
$\langle \dot{q}_0, \dot{p}_0 \rangle + \langle \dot{r}_0 \rangle = \langle \dot{p}_0 \rangle + (\langle \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle)$
by case distinction.

Case $\dot{q}_0 > \dot{r}_0$:

Case $\dot{r}_0 > \dot{q}_0$:

Case $\dot{q}_0 = -\dot{r}_0$:

Case otherwise:

Case $\dot{p}_0 = -\dot{q}_0$:
We simplify

and prove the equivalent
$\langle \dot{r}_0 \rangle = \langle \dot{p}_0 \rangle + (\langle \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle)$
by case distinction.

Case $\dot{q}_0 > \dot{r}_0$:

Case $\dot{r}_0 > \dot{q}_0$:

Case $\dot{q}_0 = -\dot{r}_0$:

Case otherwise:

Case otherwise:
We simplify

and prove the equivalent

$\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle = \langle \dot{p}_0 \rangle + (\langle \langle \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle)$

by case distinction.

Case $\dot{q}_0 > \dot{r}_0$:

Case $\dot{r}_0 > \dot{q}_0$:

Case $\dot{q}_0 = -\dot{r}_0$:

Case otherwise:

We simplify

and prove the equivalent

$\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle = \langle \dot{p}_0 \rangle + \langle \dot{q}_0 \underset{M}{+} \dot{r}_0 \rangle$

by case distinction.

Case $\dot{p}_0 \underset{M}{+} \dot{q}_0 = -\dot{r}_0$:

Case otherwise:

Induction step for r:

We take $\langle \dot{r}_0, \ddot{r}_0, \overline{r_0} \rangle \in P$ arbitrary but fixed, assume the induction hypothesis for r

(+ assoc.V.IB⟨m⟩.IB⟨m⟩.IH)

$(\langle \dot{p}_0 \rangle + \langle \dot{q}_0 \rangle) + \langle \ddot{r}_0, \overline{r_0} \rangle = \langle \dot{p}_0 \rangle + (\langle \dot{q}_0 \rangle + \langle \ddot{r}_0, \overline{r_0} \rangle)$

and prove the induction conclusion for r

(+ assoc.V.IB⟨m⟩.IB⟨m⟩.IC)

$(\langle \dot{p}_0 \rangle + \langle \dot{q}_0 \rangle) + \langle \dot{r}_0, \ddot{r}_0, \overline{r_0} \rangle = \langle \dot{p}_0 \rangle + (\langle \dot{q}_0 \rangle + \langle \dot{r}_0, \ddot{r}_0, \overline{r_0} \rangle)$

Induction step for q:

We take $\langle \dot{q}_0, \ddot{q}_0, \overline{q_0} \rangle \in P$ arbitrary but fixed, assume the induction hypothesis for q

(+ assoc.V.IB⟨m⟩.IH)

$\underset{\bar{r}}{\forall} \; (\langle \dot{p}_0 \rangle + \langle \ddot{q}_0, \overline{q_0} \rangle) + \langle \bar{r} \rangle = \langle \dot{p}_0 \rangle + (\langle \ddot{q}_0, \overline{q_0} \rangle + \langle \bar{r} \rangle)$

and prove the induction conclusion for q

(+ assoc.V.IB⟨m⟩.IC)

$\underset{\bar{r}}{\forall} \; (\langle \dot{p}_0 \rangle + \langle \dot{q}_0, \ddot{q}_0, \overline{q_0} \rangle) + \langle \bar{r} \rangle = \langle \dot{p}_0 \rangle + (\langle \dot{q}_0, \ddot{q}_0, \overline{q_0} \rangle + \langle \bar{r} \rangle)$

Induction step for p:

We take $\langle \dot{p}_0, \ddot{p}_0, \overline{p_0} \rangle \in P$ arbitrary but fixed, assume the induction hypothesis for p

(+ assoc.V.IH) $\underset{\bar{q},\bar{r}}{\forall} \; (\langle \ddot{p}_0, \overline{p_0} \rangle + \langle \bar{q} \rangle) + \langle \bar{r} \rangle = \langle \ddot{p}_0, \overline{p_0} \rangle + (\langle \bar{q} \rangle + \langle \bar{r} \rangle)$

and prove the induction conclusion for p

(+ assoc.V.IC)

$\underset{\bar{q},\bar{r}}{\forall} \; (\langle \dot{p}_0, \ddot{p}_0, \overline{p_0} \rangle + \langle \bar{q} \rangle) + \langle \bar{r} \rangle = \langle \dot{p}_0, \ddot{p}_0, \overline{p_0} \rangle + (\langle \bar{q} \rangle + \langle \bar{r} \rangle)$

■ Plus is Associative (more details)

Let the monomial domain M be arbitrary but fixed .
Let P be PolynomialsBySequenceVariables[M] .
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{+}$, $\underset{M}{-}$, $\underset{M}{>}$, $\underset{M}{\sim}$.

Let p , q , r, ⟨p̄⟩ , ⟨q̄⟩, ⟨r̄⟩ range over P.
Let l, m, n range over M.
We prove

(+ assoc) $\underset{p,q,r}{\forall}\ (p + q) + r = p + (q + r)$

under the provided assumptions

$(>\underset{M}{+})\quad \underset{l,m,n}{\forall}\ l > m \wedge l > n \wedge m \sim n \wedge m \neq -n \Longrightarrow l > m \underset{M}{+} n$

$(> \text{trans})\quad \underset{l,m,n}{\forall}\ l > m \wedge m > n \Longrightarrow l > n$

$(\sim\ >)\quad \underset{l,m,n}{\forall}\ l > m \wedge l \sim n \Longrightarrow n > m$

$(>\ \sim)\quad \underset{l,m,n}{\forall}\ l > n \wedge m \sim n \Longrightarrow l > m$

$(\underset{M}{+}\ >)\quad \underset{l,m,n}{\forall}\ m > n \wedge l \sim m \wedge l \neq -m \Longrightarrow l \underset{M}{+} m > n$

$(\sim \underset{M}{+})\quad \underset{l,m,n}{\forall}\ l \sim m \wedge m \sim n \wedge m \neq -n \Longrightarrow l \sim m \underset{M}{+} n$

$(--)\quad \underset{m}{\forall}\ --m = m$

$(-\underset{M}{+})\quad \underset{m,n}{\forall}\ m \sim n \wedge m \neq -n \Longrightarrow -\left(m \underset{M}{+} n\right) = -m \underset{M}{+} -n$

$(\text{no} \underset{M}{+} 0)\quad \underset{m,n}{\forall}\ m \sim n \wedge m \neq -n \Longrightarrow m \neq m \underset{M}{+} n$

$(-m \underset{M}{+} \left(m \underset{M}{+}\right))\quad \underset{m,n}{\forall}\ m \sim n\ m \neq -n \Longrightarrow -m \underset{M}{+} \left(m \underset{M}{+} n\right) = n$

$(\underset{M}{+} \text{comm})\quad \underset{m,n}{\forall}\ m \sim n \wedge m \neq -n \Longrightarrow m \underset{M}{+} n = n \underset{M}{+} m$

$(\underset{M}{+} \text{assoc})\quad \underset{l,m,n}{\forall}\ l \underset{M}{\sim} m \wedge m \underset{M}{\sim} n \wedge l \neq -m \wedge m \neq -n \Longrightarrow$
$\left(l \underset{M}{+} m\right) \underset{M}{+} n = l \underset{M}{+} \left(m \underset{M}{+} n\right)$

definitions

$(\smile)\quad \underset{e,\bar{s}}{\forall}\ e \smile \langle \bar{s} \rangle := \langle e, \bar{s} \rangle,$

$(\text{ldm})\quad \underset{m,\bar{p}}{\forall}\ \text{ldm}[\langle m, \bar{p} \rangle] := m,$

$(+0)\quad \underset{p}{\forall}\ p + \langle\rangle := \langle\rangle,$

$(0+)\quad \underset{p}{\forall}\ \langle\rangle + p := \langle\rangle,$

$$(+)\quad \underset{m,n,\bar{p},\bar{q}}{\forall}\ \langle m, \bar{p}\rangle + \langle n, \bar{q}\rangle := \left\| \begin{array}{ll} m \smile (\langle \bar{p}\rangle + \langle n, \bar{q}\rangle) & \Leftarrow m > n, \\ n \smile (\langle m, \bar{p}\rangle + \langle \bar{q}\rangle) & \Leftarrow n > m, \\ \langle \bar{p}\rangle + \langle \bar{q}\rangle & \Leftarrow m = -n, \\ \left(m \underset{M}{+} n\right) \smile (\langle \bar{p}\rangle + \langle \bar{q}\rangle) & \end{array} \right\|$$

and lemmata

$(\smile \in P)\quad \underset{m,p}{\forall}\ m > \text{ldm}[p] \vee p = \langle\rangle \Longrightarrow m \smile p \in P,$

$(>\mathtt{ldm}[+])\ \underset{m,p,q}{\forall}\ m > \mathtt{ldm}[p] \wedge m > \mathtt{ldm}[q] \wedge p + q \neq \langle\rangle \Longrightarrow$
$m > \mathtt{ldm}[p+q]$
by induction.
We represent polynomials in sequence notation and prove
(+ assoc.V) $\underset{\bar{p},\bar{q},\bar{r}}{\forall}\ (\langle\bar{p}\rangle + \langle\bar{q}\rangle) + \langle\bar{r}\rangle = \langle\bar{p}\rangle + (\langle\bar{q}\rangle + \langle\bar{r}\rangle)$
by polynomial induction on p.
Base case, p zero polynomial: We prove
(+ assoc.V.IB⟨ ⟩) $\underset{\bar{q},\bar{r}}{\forall}\ (\langle\rangle + \langle\bar{q}\rangle) + \langle\bar{r}\rangle = \langle\rangle + (\langle\bar{q}\rangle + \langle\bar{r}\rangle)$

Induction base, p single monomial: We take $\dot{p}_0 \in M$ be arbitrary but fixed and prove
(+ assoc.V.IB⟨m⟩) $\underset{\bar{q},\bar{r}}{\forall}\ \left(\left\langle\dot{p}_0\right\rangle + \langle\bar{q}\rangle\right) + \langle\bar{r}\rangle = \left\langle\dot{p}_0\right\rangle + (\langle\bar{q}\rangle + \langle\bar{r}\rangle)$
by polynomial induction on q:
Base case, q zero polynomial: We prove
(+ assoc.V.IB⟨m⟩.IB⟨ ⟩) $\underset{\bar{r}}{\forall}\ \left(\left\langle\dot{p}_0\right\rangle + \langle\rangle\right) + \langle\bar{r}\rangle = \left\langle\dot{p}_0\right\rangle + (\langle\rangle + \langle\bar{r}\rangle)$

Induction base, q single monomial: We take $\dot{q}_0 \in$ M be arbitrary but fixed and prove
(+ assoc.V.IB⟨m⟩.IB⟨m⟩)
$\underset{\bar{r}}{\forall}\ \left(\left\langle\dot{p}_0\right\rangle + \left\langle\dot{q}_0\right\rangle\right) + \langle\bar{r}\rangle = \left\langle\dot{p}_0\right\rangle + \left(\left\langle\dot{q}_0\right\rangle + \langle\bar{r}\rangle\right)$
by polynomial induction on r:
Base case, r zero polynomial: We prove
(+ assoc.V.IB⟨m⟩.IB⟨m⟩.IB⟨ ⟩)
$\left(\left\langle\dot{p}_0\right\rangle + \left\langle\dot{q}_0\right\rangle\right) + \langle\rangle = \left\langle\dot{p}_0\right\rangle + \left(\left\langle\dot{q}_0\right\rangle + \langle\rangle\right)$

Induction base, r single monomial: We take $\dot{r}_0 \in$ M be arbitrary but fixed and prove
(+ assoc.V.IB⟨m⟩.IB⟨m⟩.IB⟨m⟩)
$\left(\left\langle\dot{p}_0\right\rangle + \left\langle\dot{q}_0\right\rangle\right) + \left\langle\dot{r}_0\right\rangle = \left\langle\dot{p}_0\right\rangle + \left(\left\langle\dot{q}_0\right\rangle + \left\langle\dot{r}_0\right\rangle\right)$
by case distinction.
Case $\dot{p}_0 > \dot{q}_0$:
We simplify
$\left\langle\dot{p}_0\right\rangle + \left\langle\dot{q}_0\right\rangle$
to $\dot{p}_0 \smile \left(\langle\rangle + \left\langle\dot{q}_0\right\rangle\right)$ by (+)
to $\dot{p}_0 \smile \left\langle\dot{q}_0\right\rangle$ by (0+)
to $\left\langle\dot{p}_0, \dot{q}_0\right\rangle$ by ($\smile$)
and prove the equivalent
$\left\langle\dot{p}_0, \dot{q}_0\right\rangle + \left\langle\dot{r}_0\right\rangle = \left\langle\dot{p}_0\right\rangle + \left(\left\langle\dot{q}_0\right\rangle + \left\langle\dot{r}_0\right\rangle\right)$
by case distinction.
Case $\dot{p}_0 > \dot{r}_0$:
We simplify

and prove the equivalent

$\dot{p}_0 \smile (\langle \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle) = \langle \dot{p}_0 \rangle + (\langle \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle)$

by case distinction.

Case $\dot{q}_0 > \dot{r}_0$:

We prove by simplification.

We simplify the left hand side

to $\dot{p}_0 \smile (\dot{q}_0 \smile (\langle\rangle + \langle \dot{r}_0 \rangle))$ by (+)

to $\dot{p}_0 \smile (\dot{q}_0 \smile \langle \dot{r}_0 \rangle)$ by (0+)

to $\dot{p}_0 \smile \langle \dot{q}_0, \dot{r}_0 \rangle$ by ($\smile$)

to $\langle \dot{p}_0, \dot{q}_0, \dot{r}_0 \rangle$ by ($\smile$)

and the right hand side

to $\langle \dot{p}_0 \rangle + (\dot{q}_0 \smile (\langle\rangle + \langle \dot{r}_0 \rangle))$ by (+)

to $\langle \dot{p}_0 \rangle + (\dot{q}_0 \smile \langle \dot{r}_0 \rangle)$ by (0+)

to $\langle \dot{p}_0 \rangle + \langle \dot{q}_0, \dot{r}_0 \rangle$ by ($\smile$)

to $\dot{p}_0 \smile (\langle\rangle + \langle \dot{q}_0, \dot{r}_0 \rangle)$ by (+)

to $\dot{p}_0 \smile \langle \dot{q}_0, \dot{r}_0 \rangle$ by (0+)

to $\langle \dot{p}_0, \dot{q}_0, \dot{r}_0 \rangle$ by ($\smile$)

Case $\dot{r}_0 > \dot{q}_0$:

Case $\dot{q}_0 = -\dot{r}_0$:

Case otherwise:

We prove by simplification.

We simplify the left hand side

and the right hand side

to $\langle \dot{p}_0 \rangle + ((\dot{q}_0 \underset{M}{+} \dot{r}_0) \smile (\langle\rangle + \langle\rangle))$ by (+),

to $\langle \dot{p}_0 \rangle + ((\dot{q}_0 \underset{M}{+} \dot{r}_0) \smile \langle\rangle)$ by (0+),

to $\dot{p}_0 + \langle \dot{q}_0 \underset{M}{+} \dot{r}_0 \rangle$ by ($\smile$),

to $\dot{p}_0 \smile (\langle\rangle + \langle \dot{q}_0 \underset{M}{+} \dot{r}_0 \rangle)$ by (+),

which we can use because

$\dot{p}_0 > \dot{q}_0 \underset{M}{+} \dot{r}_0$.

By ($> \underset{M}{+}$), this follows from the case assumptions.

to $\dot{p}_0 \smile \langle \dot{q}_0 \underset{M}{+} \dot{r}_0 \rangle$ by (0+),

to $\langle \dot{p}_0, \dot{q}_0 \underset{M}{+} \dot{r}_0 \rangle$ by ($\smile$)

Case $\dot{r}_0 > \dot{p}_0$:

We prove by simplification.

We simplify the left hand side

and the right hand side

to $\langle \dot{p}_0 \rangle + (\dot{r}_0 \smile (\langle \dot{q}_0 \rangle + \langle\rangle))$ by (+),

which we can use because

$\dot{r}_0 > \dot{q}_0$.

By (> trans), this follows from the case assumptions.

to $\langle \dot{p}_0 \rangle + (\dot{r}_0 \smile \langle \dot{q}_0 \rangle)$ by (+0)

to $\langle \dot{p}_0 \rangle + \langle \dot{r}_0 , \dot{q}_0 \rangle$ by ($\smile$)

to $\dot{r}_0 \smile (\langle \dot{p}_0 \rangle + \langle \dot{q}_0 \rangle)$ by (+)

to $\dot{r}_0 \smile (\dot{p}_0 \smile (\langle\rangle + \langle \dot{q}_0 \rangle))$ by (+)

to $\dot{r}_0 \smile (\dot{p}_0 \smile \langle \dot{q}_0 \rangle)$ by (0+)

to $\dot{r}_0 \smile \langle \dot{p}_0 , \dot{q}_0 \rangle$ by ($\smile$)

to $\langle \dot{r}_0 , \dot{p}_0 , \dot{q}_0 \rangle$ by ($\smile$)

Case $\dot{p}_0 = - \dot{r}_0$:

We prove by simplification.

We simplify the left hand side

and the right hand side

to $\langle \dot{p}_0 \rangle + (\dot{r}_0 \smile (\langle \dot{q}_0 \rangle + \langle\rangle))$ by (+),

which we can use because

$\dot{r}_0 > \dot{q}_0$.

By ($\sim$ >), this follows from the case assumptions.

to $\langle \dot{p}_0 \rangle + (\dot{r}_0 \smile \langle \dot{q}_0 \rangle)$ by (+0)

to $\langle \dot{p}_0 \rangle + \langle \dot{r}_0 , \dot{q}_0 \rangle$ by ($\smile$)

to $\langle\rangle + \langle \dot{q}_0 \rangle$ by (+)

to $\langle \dot{q}_0 \rangle$ by (0+)

Case otherwise:

Case $\dot{q}_0 > \dot{p}_0$:

We simplify

and prove the equivalent

$\langle \dot{q}_0 , \dot{p}_0 \rangle + \langle \dot{r}_0 \rangle = \langle \dot{p}_0 \rangle + (\langle \dot{q}_0 \rangle + \langle \dot{r}_0 \rangle)$

by case distinction.

Case $\dot{q}_0 > \dot{r}_0$:

Case $\dot{r}_0 > \dot{q}_0$:

Case $\dot{q}_0 = - \dot{r}_0$:

Case otherwise:

We prove by simplification.

We simplify the left hand side

and the right hand side

to $\left\langle \dot{\mathtt{p}}_0 \right\rangle + \left(\left(\dot{\mathtt{q}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{r}}_0 \right) \smile \left(\langle\rangle + \langle\rangle \right) \right)$ by (+),

to $\left\langle \dot{\mathtt{p}}_0 \right\rangle + \left(\left(\dot{\mathtt{q}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{r}}_0 \right) \smile \langle\rangle \right)$ by (+0),

to $\left\langle \dot{\mathtt{p}}_0 \right\rangle + \left\langle \dot{\mathtt{q}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{r}}_0 \right\rangle$ by ($\smile$),

to $\left(\dot{\mathtt{q}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{r}}_0 \right) \smile \left(\left\langle \dot{\mathtt{p}}_0 \right\rangle + \langle\rangle \right)$ by (+),

which we can use because

$\dot{\mathtt{q}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{r}}_0 > \dot{\mathtt{p}}_0$.

We simplify

$\dot{\mathtt{q}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{r}}_0$

to $\dot{\mathtt{r}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{q}}_0$ by ($\underset{\mathtt{M}}{+}$ comm)

and prove the equivalent

$\dot{\mathtt{r}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{q}}_0 > \dot{\mathtt{p}}_0$.

By ($\underset{\mathtt{M}}{+}$ >), this follows from the case assumptions.

to $\left(\dot{\mathtt{q}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{r}}_0 \right) \smile \left\langle \dot{\mathtt{p}}_0 \right\rangle$ by (+0)

to $\left\langle \dot{\mathtt{q}}_0 \underset{\mathtt{M}}{+} \dot{\mathtt{r}}_0 , \dot{\mathtt{p}}_0 \right\rangle$ by ($\smile$)

Case $\dot{\mathtt{p}}_0 = - \dot{\mathtt{q}}_0$:

We simplify

and prove the equivalent

$\left\langle \dot{\mathtt{r}}_0 \right\rangle = \left\langle \dot{\mathtt{p}}_0 \right\rangle + \left(\left\langle \dot{\mathtt{q}}_0 \right\rangle + \left\langle \dot{\mathtt{r}}_0 \right\rangle \right)$

by case distinction.

Case $\dot{\mathtt{q}}_0 > \dot{\mathtt{r}}_0$:

Case $\dot{\mathtt{r}}_0 > \dot{\mathtt{q}}_0$:

Case $\dot{\mathtt{q}}_0 = - \dot{\mathtt{r}}_0$:

We prove by simplification.

We simplify the right hand side

to $\left\langle \dot{\mathtt{p}}_0 \right\rangle + \left(\langle\rangle + \langle\rangle \right)$ by (+),

to $\left\langle \dot{\mathtt{p}}_0 \right\rangle + \langle\rangle$ by (0+)

to $\left\langle \dot{\mathtt{p}}_0 \right\rangle$ by (+0),

to $\left\langle - \dot{\mathtt{q}}_0 \right\rangle$ by the case assumption,

to $\left\langle - - \dot{\mathtt{r}}_0 \right\rangle$ by the case assumption,

to $\left\langle \dot{\mathtt{r}}_0 \right\rangle$ by ($--$)

Case otherwise:

We prove by simplification.

We simplify the right hand side

to $\left\langle \dot{p}_0 \right\rangle + \left(\left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right) \smile \left(\langle\rangle + \langle\rangle \right) \right)$ by (+),

to $\left\langle \dot{p}_0 \right\rangle + \left(\left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right) \smile \langle\rangle \right)$ by (0+),

to $\left\langle \dot{p}_0 \right\rangle + \left\langle \dot{q}_0 \underset{M}{+} \dot{r}_0 \right\rangle$ by ($\smile$),

to $\left(\dot{p}_0 \underset{M}{+} \left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right) \right) \smile \left(\langle\rangle + \langle\rangle \right)$ by (+),

which we can use because

$\dot{p}_0 \neq -\left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right)$.

We prove by contradiction.

We assume

$\dot{p}_0 = -\left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right)$

and prove a contradiction.

We simplify

$\dot{p}_0$

to $-\dot{q}_0$ by the case assumption, and

$-\left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right)$

to $-\dot{q}_0 \underset{M}{+} -\dot{r}_0$ by $(-\underset{M}{+})$

and assume the equivalent

$-\dot{q}_0 = -\dot{q}_0 \underset{M}{+} -\dot{r}_0$.

This contradicts (no $\underset{M}{+}0$) .

to $\left(\dot{p}_0 \underset{M}{+} \left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right) \right) \smile \langle\rangle$ by (0+),

to $\left\langle \dot{p}_0 \underset{M}{+} \left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right) \right\rangle$ by ($\smile$),

to $\left\langle -\dot{q}_0 \underset{M}{+} \left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right) \right\rangle$ by the case assumption,

to $\left\langle \dot{r}_0 \right\rangle$ by $\left(-m \underset{M}{+} \left(m \underset{M}{+}\right)\right)$

Case otherwise:

We simplify

and prove the equivalent

$\left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \right\rangle + \left\langle \dot{r}_0 \right\rangle = \left\langle \dot{p}_0 \right\rangle + \left(\left\langle \dot{q}_0 \right\rangle + \left\langle \dot{r}_0 \right\rangle \right)$

by case distinction.

Case $\dot{q}_0 > \dot{r}_0$:

Case $\dot{r}_0 > \dot{q}_0$:

We prove by simplification.

We simplify the left hand side

to $\dot{r}_0 \smile \left(\left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \right\rangle + \langle\rangle \right)$ by (+),

which we can use because

$\dot{r}_0 > \dot{p}_0 \underset{M}{+} \dot{q}_0$.
By ($>$ $\underset{M}{+}$), this follows from the case assumption and
$\dot{r}_0 > \dot{p}_0$.
By ($>$ $\sim$), this follows from the case assumptions.
to $\dot{r}_0 \smile \left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \right\rangle$ by (+0),
to $\left\langle \dot{r}_0 , \dot{p}_0 \underset{M}{+} \dot{q}_0 \right\rangle$ by ($\smile$)
and the right hand side
Case $\dot{q}_0 = - \dot{r}_0$:
Case otherwise:
We simplify
and prove the equivalent
$\left\langle \dot{p}_0 \underset{M}{+} \dot{q}_0 \right\rangle + \left\langle \dot{r}_0 \right\rangle = \left\langle \dot{p}_0 \right\rangle + \left\langle \dot{q}_0 \underset{M}{+} \dot{r}_0 \right\rangle$
by case distinction.
Case $\dot{p}_0 \underset{M}{+} \dot{q}_0 = - \dot{r}_0$:
We prove by simplification.
We simplify the left hand side
and the right hand side
to $\langle\rangle + \langle\rangle$ by (+),
which we can use because
$\dot{p}_0 = - \left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right)$.
We prove by simplification.
We simplify the right hand side
to $- \dot{q}_0 \underset{M}{+} - \dot{r}_0$ by ($-\underset{M}{+}$)
to $- \dot{q}_0 \underset{M}{+} \left(\dot{p}_0 \underset{M}{+} \dot{q}_0 \right)$ by the case assumption
to $- \dot{q}_0 \underset{M}{+} \left(\dot{q}_0 \underset{M}{+} \dot{p}_0 \right)$ by ($\underset{M}{+}$ comm)
to $\dot{p}_0$ by ($-\mathrm{m} \underset{M}{+} \left(\mathrm{m} \underset{M}{+} \right)$)
to $\langle\rangle$ by (0+)
Case otherwise:
We prove by simplification.
We simplify the left hand side
and the right hand side
to $\left(\dot{p}_0 \underset{M}{+} \left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right) \right) \smile (\langle\rangle + \langle\rangle)$ by (+),
which we can use because
$\dot{p}_0 \neq - \left(\dot{q}_0 \underset{M}{+} \dot{r}_0 \right)$.
We prove by contradiction.
We assume

$\dot{\mathtt{p}}_0 = -\left(\dot{\mathtt{q}}_0 \underset{M}{+} \dot{\mathtt{r}}_0\right)$

and prove a contradiction

We prove

$\dot{\mathtt{p}}_0 \underset{M}{+} \dot{\mathtt{q}}_0 = -\dot{\mathtt{r}}_0$

This contradicts the case assumption.

to $\left(\dot{\mathtt{p}}_0 \underset{M}{+} \left(\dot{\mathtt{q}}_0 \underset{M}{+} \dot{\mathtt{r}}_0\right)\right) \smile \langle\rangle$ by (0+),

to $\left\langle \dot{\mathtt{p}}_0 \underset{M}{+} \left(\dot{\mathtt{q}}_0 \underset{M}{+} \dot{\mathtt{r}}_0\right)\right\rangle$ by ($\smile$),

to $\left\langle \left(\dot{\mathtt{p}}_0 \underset{M}{+} \dot{\mathtt{q}}_0\right) \underset{M}{+} \dot{\mathtt{r}}_0\right\rangle$ by ($\underset{M}{+}$ assoc)

Induction step for $\mathtt{r}$:

We take $\left\langle \dot{\mathtt{r}}_0, \ddot{\mathtt{r}}_0, \overline{\mathtt{r}_0}\right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis for $\mathtt{r}$

(+ assoc.V.IB⟨m⟩.IB⟨m⟩.IH)

$\left(\left\langle\dot{\mathtt{p}}_0\right\rangle + \left\langle\dot{\mathtt{q}}_0\right\rangle\right) + \left\langle\ddot{\mathtt{r}}_0, \overline{\mathtt{r}_0}\right\rangle = \left\langle\dot{\mathtt{p}}_0\right\rangle + \left(\left\langle\dot{\mathtt{q}}_0\right\rangle + \left\langle\ddot{\mathtt{r}}_0, \overline{\mathtt{r}_0}\right\rangle\right)$

and prove the induction conclusion for $\mathtt{r}$

(+ assoc.V.IB⟨m⟩.IB⟨m⟩.IC)

$\left(\left\langle\dot{\mathtt{p}}_0\right\rangle + \left\langle\dot{\mathtt{q}}_0\right\rangle\right) + \left\langle\dot{\mathtt{r}}_0, \ddot{\mathtt{r}}_0, \overline{\mathtt{r}_0}\right\rangle = \left\langle\dot{\mathtt{p}}_0\right\rangle + \left(\left\langle\dot{\mathtt{q}}_0\right\rangle + \left\langle\dot{\mathtt{r}}_0, \ddot{\mathtt{r}}_0, \overline{\mathtt{r}_0}\right\rangle\right)$

Induction step for $\mathtt{q}$:

We take $\left\langle \dot{\mathtt{q}}_0, \ddot{\mathtt{q}}_0, \overline{\mathtt{q}_0}\right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis for $\mathtt{q}$

(+ assoc.V.IB⟨m⟩.IH)

$\underset{\bar{\mathtt{r}}}{\forall} \left(\left\langle\dot{\mathtt{p}}_0\right\rangle + \left\langle\ddot{\mathtt{q}}_0, \overline{\mathtt{q}_0}\right\rangle\right) + \langle\bar{\mathtt{r}}\rangle = \left\langle\dot{\mathtt{p}}_0\right\rangle + \left(\left\langle\ddot{\mathtt{q}}_0, \overline{\mathtt{q}_0}\right\rangle + \langle\bar{\mathtt{r}}\rangle\right)$

and prove the induction conclusion for $\mathtt{q}$

(+ assoc.V.IB⟨m⟩.IC)

$\underset{\bar{\mathtt{r}}}{\forall} \left(\left\langle\dot{\mathtt{p}}_0\right\rangle + \left\langle\dot{\mathtt{q}}_0, \ddot{\mathtt{q}}_0, \overline{\mathtt{q}_0}\right\rangle\right) + \langle\bar{\mathtt{r}}\rangle = \left\langle\dot{\mathtt{p}}_0\right\rangle + \left(\left\langle\dot{\mathtt{q}}_0, \ddot{\mathtt{q}}_0, \overline{\mathtt{q}_0}\right\rangle + \langle\bar{\mathtt{r}}\rangle\right)$

Induction step for $\mathtt{p}$:

We take $\left\langle \dot{\mathtt{p}}_0, \ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0}\right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis for $\mathtt{p}$

(+ assoc.V.IH) $\underset{\bar{\mathtt{q}},\bar{\mathtt{r}}}{\forall} \left(\left\langle\ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0}\right\rangle + \langle\bar{\mathtt{q}}\rangle\right) + \langle\bar{\mathtt{r}}\rangle = \left\langle\ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0}\right\rangle + \left(\langle\bar{\mathtt{q}}\rangle + \langle\bar{\mathtt{r}}\rangle\right)$

and prove the induction conclusion for $\mathtt{p}$

(+ assoc.V.IC)

$\underset{\bar{\mathtt{q}},\bar{\mathtt{r}}}{\forall} \left(\left\langle\dot{\mathtt{p}}_0, \ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0}\right\rangle + \langle\bar{\mathtt{q}}\rangle\right) + \langle\bar{\mathtt{r}}\rangle = \left\langle\dot{\mathtt{p}}_0, \ddot{\mathtt{p}}_0, \overline{\mathtt{p}_0}\right\rangle + \left(\langle\bar{\mathtt{q}}\rangle + \langle\bar{\mathtt{r}}\rangle\right)$

■ One is Neutral Element

Let the monomial domain M be arbitrary but fixed .
Let P be PolynomialsBySequenceVariables[M] .
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{1}$, $\underset{P}{*}$, $\underset{P}{+}$, $\underset{M}{\text{zero-divisor}}$.
Let p, q, $\langle\bar{p}\rangle$, $\langle\bar{q}\rangle$ range over P.
Let m, n range over M .
We prove
(neutral 1) $\underset{p}{\forall}\ p * 1 = p$
under the provided assumptions
($\underset{M}{1}$ not zerodiv) $\underset{m}{\forall}\ \neg\,\text{zero-divisor}\left[m, \underset{M}{1}\right]$
(neutral $\underset{M}{1}$) $\underset{m}{\forall}\ m \underset{M}{*} \underset{M}{1} = m$
and definitions
(1) $1 := \left\langle \underset{M}{1} \right\rangle$,
(0 +) $\underset{p}{\forall}\ \langle\rangle + p := p$,
(0 $*$) $\underset{p}{\forall}\ \langle\rangle * p := \langle\rangle$,
($*$) $\underset{m,n,\bar{p},\bar{q}}{\forall}\ \langle m, \bar{p}\rangle * \langle n, \bar{q}\rangle :=$
$\left\| r \Leftarrow \text{zero-divisor}[m, n] , \left(m \underset{M}{*} n\right) \smile r \right\|$,
where $r = \langle m\rangle * \langle\bar{q}\rangle + \langle\bar{p}\rangle * \langle n, \bar{q}\rangle$
by induction.
We represent polynomials in sequence notation and prove
(neutral 1.V) $\underset{\bar{p}}{\forall}\ \langle\bar{p}\rangle * 1 = \langle\bar{p}\rangle$
by sequence induction on p:
Base case, $\bar{p}$ empty sequence: We prove
(neutral 1.V.IB) $\langle\rangle * 1 = \langle\rangle$
by simplification.
We simplify the left hand side
to $\langle\rangle$ by (0 $*$)
Induction step:
We take $\left\langle \dot{p}_0 , \overline{p_0} \right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis for $\bar{p}$
(neutral 1.V.IH) $\langle\overline{p_0}\rangle * 1 = \langle\overline{p_0}\rangle$
and prove the induction conclusion for $\bar{p}$
(neutral 1.V.IC) $\left\langle \dot{p}_0 , \overline{p_0} \right\rangle * 1 = \left\langle \dot{p}_0 , \overline{p_0} \right\rangle$
by simplification.
We simplify the left hand side
to $\left\langle \dot{p}_0 , \overline{p_0} \right\rangle * \left\langle \underset{M}{1} \right\rangle$ by (1),
to $\left(\dot{p}_0 \underset{M}{*} \underset{M}{1}\right) \smile \left(\left\langle \dot{p}_0 \right\rangle * \langle\rangle + \langle\overline{p_0}\rangle * \left\langle \underset{M}{1} \right\rangle\right)$ by ($*$),
which we can use because
$\neg\,\text{zero-divisor}\left[\dot{p}_0 , \underset{M}{1}\right]$.

This follows from ($\underset{\mathtt{M}}{\mathtt{1}}$ not zerodiv).

to $\left(\overset{\bullet}{\mathtt{p}}_0 \underset{\mathtt{M}}{\star} \underset{\mathtt{M}}{\mathtt{1}}\right) \smile \left(\langle\,\rangle + \langle\overline{\mathtt{p}_0}\rangle \star \left\langle \underset{\mathtt{M}}{\mathtt{1}} \right\rangle\right)$ by ($\star$ 0),

to $\left(\overset{\bullet}{\mathtt{p}}_0 \underset{\mathtt{M}}{\star} \underset{\mathtt{M}}{\mathtt{1}}\right) \smile \left(\langle\overline{\mathtt{p}_0}\rangle \star \left\langle \underset{\mathtt{M}}{\mathtt{1}} \right\rangle\right)$ by (0 +),

to $\left(\overset{\bullet}{\mathtt{p}}_0 \underset{\mathtt{M}}{\star} \underset{\mathtt{M}}{\mathtt{1}}\right) \smile (\langle\overline{\mathtt{p}_0}\rangle \star \mathtt{1})$ by (1),

to $\left(\overset{\bullet}{\mathtt{p}}_0 \underset{\mathtt{M}}{\star} \underset{\mathtt{M}}{\mathtt{1}}\right) \smile \langle\overline{\mathtt{p}_0}\rangle$ by (neutral 1.V.IH),

to $\left\langle\overset{\bullet}{\mathtt{p}}_0 \underset{\mathtt{M}}{\star} \underset{\mathtt{M}}{\mathtt{1}},\ \overline{\mathtt{p}_0}\right\rangle$ by ($\smile$),

to $\left\langle\overset{\bullet}{\mathtt{p}}_0\,,\ \overline{\mathtt{p}_0}\right\rangle$ by (neutral $\underset{\mathtt{M}}{\mathtt{1}}$)

■ Times is Commutative

Let the monomial domain $\mathtt{M}$ be arbitrary but fixed .
Let $\mathtt{P}$ be $\mathtt{PolynomialsBySequenceVariables[M]}$.
In the following, we omit the underscripts of $\underset{P}{0}$, $\underset{P}{*}$, $\underset{P}{+}$, $\underset{M}{\mathtt{zero\text{-}divisor}}$.
Let p , q , r, $\langle \bar{p} \rangle$, $\langle \bar{q} \rangle$ range over $\mathtt{P}$.
Let $m,\ n$ range over $\mathtt{M}$.
We prove

($*$ comm) $\underset{p,q}{\forall}\ p * q = q * p$

under the provided assumptions

($\underset{M}{*}$ comm) $\underset{m,n}{\forall}\ m \underset{M}{*} n = n \underset{M}{*} m$,

(zerodiv symm) $\underset{m,n}{\forall}\ \mathtt{zero\text{-}divisor}[m, n] \Longleftrightarrow \mathtt{zero\text{-}divisor}[n, m]$,

definitions

($0\ *$) $\langle\rangle * p := \langle\rangle$,

($*\ 0$) $p * \langle\rangle := \langle\rangle$,

($*$) $\langle m, \bar{p}\rangle \underset{P}{*} \langle n, \bar{q}\rangle :=$

$$\left\| r \Leftarrow \mathtt{zero\text{-}divisor}[m, n],\ \left(m \underset{M}{*} n\right) \smile r \right\|,$$

$$\text{where } r = \langle m\rangle \underset{P}{*} \langle \bar{q}\rangle \underset{P}{+} \langle \bar{p}\rangle \underset{P}{*} \langle n, \bar{q}\rangle$$

and lemmata

($*\ \langle\rangle$ distrib) $\underset{m,p,\bar{q}}{\forall}\ \langle m, \bar{q}\rangle \in P \Longrightarrow p * \langle m, \bar{q}\rangle = p * \langle m\rangle + p * \langle \bar{q}\rangle$,

($\langle\rangle\ *$ distrib) $\underset{m,p,\bar{q}}{\forall}\ \langle m, \bar{q}\rangle \in P \Longrightarrow \langle m, \bar{q}\rangle * p = \langle m\rangle * p + \langle \bar{q}\rangle * p$,

($+$ assoc) $\underset{p,q,r}{\forall}\ (p + q) + r = p + (q + r)$,

($+$ comm) $\underset{p,q}{\forall}\ p + q = q + p$

by induction.
We represent polynomials in sequence notation and prove

($*$ comm.V) $\underset{\bar{p},\bar{q}}{\forall}\ \langle \bar{p}\rangle * \langle \bar{q}\rangle = \langle \bar{q}\rangle * \langle \bar{p}\rangle$

by sequence induction on p:
Base case, $\bar{p}$ empty sequence: We prove

($*$ comm.V.IB) $\underset{\bar{q}}{\forall}\ \langle\rangle * \langle \bar{q}\rangle = \langle \bar{q}\rangle * \langle\rangle$

Induction step:
We take $\left\langle \dot{p}_0 ,\ \overline{p_0} \right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis for $\bar{p}$

($*$ comm.V.IH) $\underset{\bar{q}}{\forall}\ \langle \overline{p_0}\rangle * \langle \bar{q}\rangle = \langle \bar{q}\rangle * \langle \overline{p_0}\rangle$

and prove the induction conclusion for $\bar{p}$

($*$ comm.V.IC) $\underset{\bar{q}}{\forall}\ \left\langle \dot{p}_0 ,\ \overline{p_0} \right\rangle * \langle \bar{q}\rangle = \langle \bar{q}\rangle * \left\langle \dot{p}_0 ,\ \overline{p_0} \right\rangle$

by sequence induction on $\bar{q}$:
Base case, $\bar{q}$ empty sequence: We prove

(⋆ comm.V.IC.IB) $\left\langle \dot{p}_0, \overline{p_0} \right\rangle \star \langle\rangle = \langle\rangle \star \left\langle \dot{p}_0, \overline{p_0} \right\rangle$

Induction step:

We take $\left\langle \dot{q}_0, \overline{q_0} \right\rangle \in P$ arbitrary but fixed, assume the induction hypothesis for $\bar{q}$

(⋆ comm.V.IC.IH) $\left\langle \dot{p}_0, \overline{p_0} \right\rangle \star \langle \overline{q_0} \rangle = \langle \overline{q_0} \rangle \star \left\langle \dot{p}_0, \overline{p_0} \right\rangle$

and prove the induction conclusion for $\bar{q}$

(⋆ comm.V.IC.IC) $\left\langle \dot{p}_0, \overline{p_0} \right\rangle \star \left\langle \dot{q}_0, \overline{q_0} \right\rangle = \left\langle \dot{q}_0, \overline{q_0} \right\rangle \star \left\langle \dot{p}_0, \overline{p_0} \right\rangle$

by case distinction

Case $\mathtt{zero\text{-}divisor}\left[\dot{p}_0, \dot{q}_0\right]$:

Case otherwise :

We prove by simplification.

We simplify the left hand side

to $\left\langle \dot{p}_0 \underset{M}{\star} \dot{q}_0 \right\rangle \smile \left(\left\langle \dot{p}_0 \right\rangle \star \langle \overline{q_0} \rangle + \langle \overline{p_0} \rangle \star \left\langle \dot{q}_0, \overline{q_0} \right\rangle \right)$ by (⋆),

to $\left\langle \dot{p}_0 \underset{M}{\star} \dot{q}_0 \right\rangle \smile \left(\left\langle \dot{p}_0 \right\rangle \star \langle \overline{q_0} \rangle + \left(\langle \overline{p_0} \rangle \star \left\langle \dot{q}_0 \right\rangle + \langle \overline{p_0} \rangle \star \langle \overline{q_0} \rangle \right) \right)$

by (⋆ ⟨⟩ distrib),

to $\left\langle \dot{p}_0 \underset{M}{\star} \dot{q}_0 \right\rangle \smile \left(\left(\left\langle \dot{p}_0 \right\rangle \star \langle \overline{q_0} \rangle + \langle \overline{p_0} \rangle \star \left\langle \dot{q}_0 \right\rangle \right) + \langle \overline{p_0} \rangle \star \langle \overline{q_0} \rangle \right)$

by (+ assoc)

and the right hand side

to $\left\langle \dot{q}_0 \underset{M}{\star} \dot{p}_0 \right\rangle \smile \left(\left\langle \dot{q}_0 \right\rangle \star \langle \bar{p}_0 \rangle + \langle \overline{q_0} \rangle \star \left\langle \dot{p}_0, \overline{p_0} \right\rangle \right)$ by (⋆),

which we can use because

$\neg\, \mathtt{zero\text{-}divisor}\left[\dot{q}_0, \dot{p}_0\right]$.

By (zerodiv symm), this follows from the case assumption.

to $\left\langle \dot{q}_0 \underset{M}{\star} \dot{p}_0 \right\rangle \smile \left(\langle \bar{p}_0 \rangle \star \left\langle \dot{q}_0 \right\rangle + \langle \overline{q_0} \rangle \star \left\langle \dot{p}_0, \overline{p_0} \right\rangle \right)$

by (⋆ comm.V.IH),

to $\left\langle \dot{q}_0 \underset{M}{\star} \dot{p}_0 \right\rangle \smile \left(\langle \bar{p}_0 \rangle \star \left\langle \dot{q}_0 \right\rangle + \left\langle \dot{p}_0, \overline{p_0} \right\rangle \star \langle \overline{q_0} \rangle \right)$

by (⋆ comm.V.IC.IH),

to $\left\langle \dot{q}_0 \underset{M}{\star} \dot{p}_0 \right\rangle \smile \left(\langle \bar{p}_0 \rangle \star \left\langle \dot{q}_0 \right\rangle + \left(\left\langle \dot{p}_0 \right\rangle \star \langle \overline{q_0} \rangle + \langle \overline{p_0} \rangle \star \langle \overline{q_0} \rangle \right) \right)$

by (⟨⟩ ⋆ distrib),

to $\left\langle \dot{q}_0 \underset{M}{\star} \dot{p}_0 \right\rangle \smile \left(\left(\langle \bar{p}_0 \rangle \star \left\langle \dot{q}_0 \right\rangle + \left\langle \dot{p}_0 \right\rangle \star \langle \overline{q_0} \rangle \right) + \langle \overline{p_0} \rangle \star \langle \overline{q_0} \rangle \right)$

by (+ assoc),

to $\left\langle \dot{q}_0 \underset{M}{\star} \dot{p}_0 \right\rangle \smile \left(\left(\left\langle \dot{p}_0 \right\rangle \star \langle \overline{q_0} \rangle + \langle \bar{p}_0 \rangle \star \left\langle \dot{q}_0 \right\rangle \right) + \langle \overline{p_0} \rangle \star \langle \overline{q_0} \rangle \right)$

by (+ comm),

to $\left\langle \dot{p}_0 \underset{M}{\star} \dot{q}_0 \right\rangle \smile \left(\left(\left\langle \dot{p}_0 \right\rangle \star \langle \overline{q_0} \rangle + \langle \bar{p}_0 \rangle \star \left\langle \dot{q}_0 \right\rangle \right) + \langle \overline{p_0} \rangle \star \langle \overline{q_0} \rangle \right)$

by ($\underset{M}{\star}$ comm)