A Degree Bound for the Parameterization of a Rational Surface

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September 3, 1997

Abstract

The main result of this paper is an upper bound for the degree of the smallest parameterization of a rational surface of degree d. We also give another such upper bound in terms of the degree and the sectional genus of the surface.

1 Introduction

Rational surfaces have been studied by many authors [14, 25, 19, 5, 15, 17]. In this paper, we study the size of rational parameterizations of a given rational surface.

In the area of computer algebra, rational parameterizations have become especially important, because they are used in NURBS- (non-uniform rational B-spline) -represention, which are used for many applications in Computer Aided Design and Manufacture, such as reliable surface plotting and display, motion display (computing transformations), computing cutter offset surfaces, computing curvatures for shading and colouring, and many others (see also [7, 3, 29, 16, 21]). NURBSs are already a de facto standard in CAD/CAM environments (see [26, 2]), and their popularity is still growing.

For the mentioned applications, it would be rather useful to know how small a parameterization of a given surface can be. If we measure the size in terms of degree, one wants to know the degree of the smallest parameterization. Since the exact number is not easy to calculate, one wants to know at least an upper bound. Such an upper bound has not been known yet; in this paper, we give one, which is polynomial in the degree of the given surface. We give a lower bound, too.

Experiments show that the given upper bound is often much too high. In order to make a better estimate, we give another upper bound in terms of the sectional genus.

As a side result, we give a numerical relation between the degree and the sectional genus, which is interesting for the classification theory of rational surfaces.

The paper is organized as follows. We start by giving the main results. In the following section, we introduce the machinery of repeated adjunction and prove some auxiliary results. In the final section, we prove the main results and establish a theorem which enables us to give a lower bound for particular surfaces. Actually, our method allows to determine the degree of the smallest parameterization up to a factor 2, i.e. the parameterization we construct is at most twice as large as the smallest possible parameterization.

2 Main Results

In this section we introduce the basic concepts and give a survey of the main results. Most proofs are delayed to later sections.

We fix the following conventions. Throughout, let k be an algebraically closed field of characteristic zero, and let $F\in\mathbf{P}^3$ be a surface, given by a homogenous equation $F\in k[x,y,z,w]$ of degree $d\geq 2$. A parameterization of F is a quadruple $\tau=(X,Y,Z,W)\in k[s,t]^4$ of bivariate polynomials, representing a rational map

$$f_\tau : \mathbf{A}^2\to F, (s,t)\mapsto (X(s,t):Y(s,t):Z(s,t):W(s,t))\in\mathbf{P}^3$$

such that the image is Zariski dense in F. By definition, the degree of this parameterization is the maximum of the total degrees of the polynomials X,Y,Z,W.

By Castelnuovo's criterion [12], we know that a parameterization exists iff the arithmetic genus p_a and the second plurigenus P_2 vanish. From now on, we will assume $p_a=P_2=0$, i.e. F is a rational surface.

Here is a lower bound for a parameterization of F.

Theorem 1 *Let F be a surface of degree d. Then every parameterization of F has degree $\geq\sqrt{d}$.*

Proof: Let $\tau=(X,Y,Z,W)$ be an $e:1$-parameterization of F of degree n, i.e. a generic point of F is reached by e points in the parameter space. We may assume that X,Y,Z,W do not have a common divisor (otherwise we may cancel by this divisor and obtain a parameterization of strictly smaller degree). Let E_1,E_2 be two generic planes in $\mathbf{P}^3$. Then the line $E_1\cap E_2$ intersects F in exactly d points. The preimage of the set of these points is a set Σ of de points. On the other hand, Σ is the set of common zeroes of the two linear combinations Q_1, Q_2 of X,Y,Z,W corresponding to E_1 and E_2 minus the set of common zeros of X, Y, Z, W. Since X, Y, Z, W do not have a common factor, and Q_1, Q_2 are generic linear combinations, Q_1 and Q_2 do not have a common factor. By Bezout's theorem, the number of common zeroes is less or equal n^2. Hence $d \leq de = \mathrm{card}(\Sigma) \leq n^2$.

Example 1 Here is an example showing that the lower bound in theorem 1 cannot be improved.

Let $n \geq 2$. The surface of degree n^2 with equation

$$(x^n - yw^{n-1})^n - zw^{n^2-1} = 0$$

has a parameterization of degree n, namely

$$X = t, Y = s + t^n, Z = s^n, W = 1.$$

Obviously, there is no upper bound for parameterizations of F: by suitable parameter substitutions, one can construct parameterizations of arbitrary high degree.

Usually, it is important to find at least one parameterization, which should be as small as possible. Here is a theorem that guarantees the existence of a small parameterization.

Theorem 2 *Let F be a rational surface of degree d. Then there exists a parameterization of degree $\leq 2(d^2 - 4d + 1)^2 + 2d + 2$.*

Proof: Delayed.

Example 2 Here is an example where the degree of the smallest parameterization grows with order d^2.

Let $n \geq 8$. The surface of degree $d = 2n - 1$ with equation

$$x^{2n-1} - y^n w^{n-1} - z^2 w^{2n-3} = 0$$

has the following parameterization of degree n^2:

$$X = (s^n + t^2)^{n-1}, Y = s(s^n + t^2)^{n-2}, Z = t, W = (s^n + t^2)^n.$$

Later (see example 6), we will show that there is no parameterization of degree $< \frac{n^2+2n}{2}$.

I do not know any examples where the degree of the smallest parameterization grows with order faster than d^2. It is possible that the bound in theorem 2 can be improved.

Remark 1 Note that we can - theoretically - solve the parameterization problem with theorem 2, in the following way. Given a surface F of degree d, make an ansatz for a parameterization of degree $\leq 2(d^2 - 4d + 1)^2 + 2d + 2$. The statement that $(X : Y : Z : W)$ is a parameterization of F can be expressed by polynomial equations and inequations:

- $F(X,Y,Z,W) = 0$ in $k[s,t]$ (i.e. the image of the parameterization is contained in F), and
- the rank of the matrix

$$M(\tau) := \begin{pmatrix} X & Y & Z & W \\ \partial_s X & \partial_s Y & \partial_s Z & \partial_s W \\ \partial_t X & \partial_t Y & \partial_t Z & \partial_t W \end{pmatrix}$$

  over the field $k(s,t)$ is 3 (i.e. the image is not only a point or a curve on F).

By deciding if this system has a solution and computing one in the affirmative case, we may decide existence of a parameterization and compute one if one exists. Arbitrary systems of algebraic equations and inequations can be solved if we can perform the field operations and solve univariate equations, using the method of Gröbner bases (see [9, ?, 10, 4]) or resultants (see [13, 8, 11, 24]) or characteristic sets (see [22, 28]).

In practice, this approach is not feasible, because the number of indeterminates is way to large. For more efficient methods, see [1, 32, 27, 31].

The upper bound given in theorem 2 is often much too high. Better estimates can be made with the "sectional genus" p_1 of F, which is the genus of a generic plane section.

Theorem 3 *Let F be a rational surface of degree d with sectional genus $p_1 = 0$. Then there exists a parameterization of degree $\leq d$.*

Proof: Delayed.

Example 3 This example shows that the upper bound in theorem 3 cannot be improved.

Let $d \geq 2$. The surface of degree d with equation

$$x^d - (y+z)w^{d-1} = 0$$

is a cone over a rational curve C of degree d with vertex $(0 : 1 : -1 : 0)$. The generic plane section is a projective image of this rational curve, hence $p_1 = 0$.

We have the parameterization

$$X = s, Y = t, Z = s^d - t, W = 1,$$

which has degree d. There is no parameterization of smaller degree, because otherwise one would be able to find a parameterization of C with degree smaller than d, which is not possible.

Theorem 4 *Let F be a rational surface of degree d with sectional genus $p_1 > 0$. Then there exists a parameterization of degree* $\leq 4 \binom{2p_1 - d - 1}{2} + 4p_1$, *with the convention that a binomial number is zero if the upper argument is smaller than the lower argument.*

Proof: Delayed.

There is some evidence that there are surfaces with sectional genus p_1 and degree $d = p_1 + 1$, such that the degree of the smallest parameterization grows with order p_1^2 (see example 7), but I do not know an explicit example.

It is also possible to give another lower bound for the degree of the parameterization.

Theorem 5 *Let F be a surface of degree d. If F has a parameterization of degree n, then* $\frac{(n-1)(n-2)}{2} gep_1$.

Proof: The parameterization map induces a rational map from the zero set C of a generic linear combination of X, Y, Z, W to a generic plane section. By the Hurwitz genus formula (see [20]), it follows $genus(C) \geq p_1$. But C is a curve of degree n, hence our assertion follows.

We conclude this section by giving a correlation between the numbers d and p_1 by inequalities. One direction is needed for the proof of the main result theorem 2, the other is a side result which is interesting for the classification of rational surfaces. The idea to look for such a relation came from the observation that a weaker inequality follows from theorem 4 and theorem 1.

Theorem 6 *For any surface with degree d and sectional genus p_1, we have* $p_1 \leq \frac{(d-1)(d-2)}{2}$.

Proof: The generic plane section is a plane curve of degree d, hence the inequality follows immediately from the genus formula for plane curves.

Example 4 Here is an example that shows that equality can be reached for rational surfaces. Let F be the surface of degree d with equation

$$x^d + y^d + z^{d-1} w = 0.$$

Since F has no curve singularities, the generic plane section is a smooth curve of degree n, with genus $p_1 = \frac{(d-1)(d-2)}{2}$. The surface is rational because it has the parameterization

$$X = s, Y = t, Z = 1, W = -s^d - t^d.$$

Theorem 7 *For any rational surface with degree d and sectional genus $p_1 > 0$, we have $d \leq 9$ for $p_1 = 1$ and $d \leq 4p_1 + 4$ for $p_1 \geq 2$.*

Proof: Delayed. Note that the weaker inequality $d \leq 16p_1^2$ follows immediately from theorem 4 and theorem 1.

Example 5 Let $n \geq 2$. The surface of degree $4n$ with equation

$$(yw^{2e-1} - x^{2e})^2 - zw^{4e-1} = 0$$

is rational, because it has the parameterization

$$X = s, Y = s^{2e} + t, Z = t^2, W = 1.$$

The intersection with a generic plane $z = ax + by + cw$ is a plane curve C with inhomogenous equation

$$(y - x^{2e})^2 - ax - by - c = 0.$$

The projection onto the x-axis is a $2:1$-map with $2n$ ramification points, therefore C has genus $p_1 = n - 1$ by the Hurwitz genus formula (see [20]).

This shows that the inequality in theorem 7 is sharp.

3 Preliminaries

In this section, we introduce the classical method of repeated adjunction (see also [5, 6, 31]) and prove some bound on this sequence. We also introduce rational ruled surfaces (see also [18, 23]).

We study pairs (S, D), where S is a smooth projective surface and D is a divisor class of S.

A curve C is called a -1-curve iff $C^2 = C \cdot K = -1$ (see [23]). A pair (S, D) is called *minimal* iff there exists no -1-curve E such that $D \cdot E = 0$. If (S, D) is not minimal, then we construct a minimalization of (S, D) recursively by blowing down some -1-curve E with $D \cdot E = 0$. Eventually, we arrive at a minimal pair (S_0, D_0), and a birational regular minimalization map $\pi : S \to S_0$, such that $\pi^* D_0 = D$ (where π^* is the pullback of divisor classes).

Let (S, D) be a minimal pair. Let K be the canonical class of S. A minimalization (S', D') of $(S, D + K)$ is called an *adjoint pair* of (S, D).

From now on, we fix the following notation. Starting from the input surface $F \in \mathbf{P}^3$, we construct a chain of minimal pairs in the following way.

Let $\pi_0 : S_0 \to F$ be the minimal desingularization of F. Let D_0 be the class of the pullback of a plane section. The pair (S_0, D_0) is minimal, because there is no -1-curve which is contracted by π_0.

For $i \geq 0$, define (S_{i+1}, D_{i+1}) as an adjoint pair of (S_i, D_i). Denote by $\tau_i : S \to S_i$ the composite of the minimalization maps. Define $\pi_i : S_i \to F$ as $\pi \circ \tau_i^{-1}$. Moreover, K_i is the canonical class of S_i, and $r_i := \frac{D_i(D_i - K_i)}{2}$. Note that the numbers r_i are integers by the Riemann-Roch formula.

Lemma 1 *We have $r_0 + r_1 = d$ and $r_1 + 1 = p_2$.*

Proof: Since a 3-dimensional subsystem of $|D_0|$ without base points defines the map $\pi_0 : S_0 \to F$, we have $D_0^2 = d$. Since $D_0 + K_0$ is the pullback of D_1 and $D_1 - K_1$ is the direct image of D_0, we get

$$r_0 + r_1 = \frac{D_0(D_0 - K_0)}{2} + \frac{D_1(D_1 - K_1)}{2} =$$

$$= \frac{D_0(D_0 - K_0)}{2} + \frac{(D_0 + K_0)D_0}{2} = D_0^2 = d.$$

Let $\Gamma \in |D_0|$ be the 3-dimensional subsystem that defines π_0. This map induces a birational equivalence from a generic divisor in Γ to a generic plane section of F. Hence both curves have the same genus. The generic divisor of Γ is smooth by Bertini's theorem (see [20]), hence we have

$$p_1 = \frac{D_0(D_0 + K_0)}{2} + 1 = \frac{(D_1 - K_1)D_1}{2} + 1 = r_1 + 1.$$

We define μ as the smallest integer such that the linear system $|D_{\mu+1}|$ is empty, or ∞ if there is no such integer (however, we will see that μ is always finite). Note that $\mu \geq 0$, because $|D_0|$ has a 3-dimensional subsystem.

Lemma 2 *We have $\mu = 0$ iff $p_1 = 0$.*

Proof: $\to$: If $|D_0 + K_0|$ is empty, then $D_0(D_0 + K_0) \leq -2$ by the Riemann-Roch formula, and the assertion follows from the genus formula.

$\leftarrow$: In this case, we have $D_0(D_0 + K_0) = -2$. Because D_0 is nef, $|D_0 + K_0|$ must be empty.

A curve C is called *rigid* iff $\dim|C| = 0$ (the only divisor in $|C|$ is C itself).

Lemma 3 *Let S be a surface such that S is a rational surface. Let C be a rigid curve. Then $C \cdot K \geq -1$, and equality holds iff C is a -1-curve.*

Proof: The genus formula gives $C \cdot K + C^2 \geq -2$. The Riemann-Roch formula gives $C \cdot K - C^2 \geq 0$. This shows the inequality. In case of equality, we have $C^2 = C \cdot K = -1$, i.e. C is a -1-curve.

Lemma 4 *Let (S, D) a minimal pair such that S is a rational surface. Let (S', D') be an adjoint pair. Suppose that D is nef (numerically eventually free), and the linear systems $|D|$, $|D'|$ are not empty. Then D' is also nef.*

Proof: Minimalization does not change the nef property and the dimension. Hence, we may equivalently show that $D + K$ is nef under the assumption that $|D + K|$ is not empty.

Assume, indirectly, that there is a curve C with $(D + K)\cdot C < 0$. Then C must be a fixed component of the non-empty linear system $|D+K|$. By lemma 3 and nefness of D, C is a -1-curve and $C\cdot D = 0$. But this contradicts minimality of (S, D).

Lemma 5 *Let (S, D) a minimal pair such that S is a rational surface. Let (S', D') be an adjoint pair. Suppose that D is nef, $D^2 > 0$, the linear systems $|D|$, $|D'|$, $|D' + K'|$ are not empty (where K' is the canonical class of S'). Then $D'^2 > 0$.*

Proof: The direct image of D is $D' - K'$. By factoring the map $S \to S'$ into blowing downs, one shows easily that $D' - K'$ is nef and $(D' - K')^2 \geq D^2 > 0$.

By lemma 4, D' is nef. Hence,

$$0 \leq D'\cdot(D' + K') + D'\cdot(D' - K') = 2D'^2.$$

Assume, indirectly, that equality holds. Then $D'\cdot(D' - K') = 0$, hence D' is numerically zero by the Hodge index theorem. But $|D'|$ is not empty, hence $D' = 0$. But then, $|K'|$ is not empty, contradicting $p_g = 0$.

Lemma 6 *For $i \leq \mu$, D_i is nef and $D_i^2 \geq 0$. For $i < \mu$, $D_i^2 > 0$.*

Proof: This follows easily by induction on i, using lemma 4 and lemma 5.

Lemma 7 *For $i \leq \mu - 1$, we have $r_{i+1} < r_i$ or $r_{i+2} - r_{i+1} < r_{i+1} - r_i$.*

Proof: Since $D_i + K_i$ is the pullback of D_{i+1} and $D_{i+1} - K_{i+1}$ is the direct image of D_i, we get

$$r_{i+1} - r_i = \frac{D_{i+1}(D_{i+1} - K_{i+1})}{2} - \frac{D_i(D_i - K_i)}{2} =$$

$$= \frac{(D_i + K_i)D_i}{2} - \frac{D_i(D_i - K_i)}{2} = D_i\cdot K_i.$$

Since K_{i+1} is the direct image of K_i, we have

$$(r_{i+2} - r_{i+1}) - (r_{i+1} - r_i) = D_{i+1}\cdot K_{i+1} - D_i\cdot K_i = (D_i + K_i)K_i - D_i\cdot K_i = K_i^2.$$

Suppose $K_i^2 \geq 0$. Then $|-K_i|$ has effective divisors by Riemann-Roch. By lemma 6, D_i is nef, hence $D_i \cdot K_i \leq 0$. Suppose, indirectly, that $D_i \cdot K_i = 0$. By lemma 6 again, we have $D_i^2 > 0$. By the Hodge index theorem, K_i is numerically trivial. Then $-K_i$ is also numerically trivial. But $|-K_i|$ is not empty, therefore $K_i = 0$. This contradicts $p_g = 0$.

Remark 2 Lemma 7 is actually Castelnuovo's original argument to prove his criterion [12]. The assumption $P_2 = 0$ is used in the implication $K_i^2 \geq 0 \to |-K_i|$ has effective divisors.

Lemma 8 *If $\mu > 0$, then*

$$\mu + r_\mu \leq p_1 + \binom{2p_1 - d - 1}{2}.$$

Proof: By lemma 7, $r_{i+1} < r_i$ implies $r_{i+2} < r_{i+1}$ for $i < \mu$. Hence there exists an integer $\nu \leq \mu$, such that $r_{i+1} \geq r_i$ for $i < \nu$ and $r_{i+1} < r_i$ for $\nu \leq i < \mu$. If $r_1 < r_0$, then $\nu = 0$. If $r_1 \geq 0$, then the sequence of nonnegative integers $r_1 - r_0, \ldots, r_\nu - r_{\nu-1}$ is decreasing, hence $\nu \leq r_1 - r_0 + 1$ and

$$r_\nu = r_0 + (r_1 - r_0) + \ldots + (r_\nu - r_{\nu-1}) \leq r_0 + \binom{r_1 - r_0 + 1}{2}.$$

The sequence $r_\nu + \nu, \ldots, r_\mu + \mu$ is non-increasing, hence

$$\mu + r_\mu \leq 1 + r_1 = p_1$$

in the case $r_1 - r_0 = 2p_1 - d - 2 < 0$ and

$$\mu + r_\mu \leq \nu + r_\nu \leq r_1 + 1 + \binom{r_1 - r_0 + 1}{2} = p_1 + \binom{2p_1 - d - 1}{2}$$

in the case $r_1 - r_0 = 2p_1 - d - 2 \geq 0$.

Remark 3 There are many interesting facts about the numbers r_i. For instance, $r_i = \dim|D_i|$ for $1 \leq i \leq \mu$. This is a consequence of the Riemann-Roch theorem and the Grauert-Riemenschneider vanishing theorem. One can also show that the integers $\mu, r_1, \ldots, r_\mu$ depend only on F, and not on the choice of the minimalization maps. See [31] for details.

Another ingredience for our bound is a dehomogenization of rational ruled surfaces, described below.

By R_e, we denote the rational ruled surface with characteristic number $e \geq 0$. The class P is the class of the ruling (or one of the two rulings in case $e = 0$), and B is the class of the fundamental cross section (or the class of the traverse ruling if $e = 0$), so that $B^2 = -e$.

We introduce a bi-graded coordinate ring for R_e, as follows. The ring is $k[p,q,u,v]$, and the bi-degrees of the generators are given by

$$\deg p = \deg q = (1,0), \deg u = (0,1), \deg v = (e,1).$$

The surface R_e is the set of all 4-tuples $(p:q:u:v)$ such that $(p,q)\neq(0,0)$ and $(u,v)\neq(0,0)$, modulo the equivalence relation

$$(p:q:u:v)\sim(\lambda p:\lambda q:\mu u:\lambda^e\mu v).$$

All divisors have a global bi-homogenous equation. Two divisors are equivalent iff their bi-degree coincides, so that we can identify classes and bi-degrees.

The ruling has bi-degree $(1,0)$. The fundamental cross section B has bi-degree $(0,1)$ (its equation is $u=0$). The canonical class corresponds to bi-degree $(-2-e,-2)$.

Our standard-dehomogenization of R_e is given by the the affine coordinates $s := \frac{p}{q}$, $t := \frac{v}{q^e u}$. If $a\geq be\geq 0$, then the map associated to the linear system of divisors of bi-degree (a,b) is given by

$$(s^a : \ldots : 1 : s^{a-e}t : \ldots : t : \ldots : s^{a-be}t^b : \ldots t^b),$$

therefore the map has degree a if $e\geq 1$ and degree $a+b$ if $e=0$. If $a<be$, then the class (a,b) has negative intersection number with B.

4 Proofs of the Main Results

In this section, we are going to characterize the minimal pair (S_μ, D_μ). Then we give an upper bound of the degree of a parameterization in terms of the numbers μ and r_μ. We also give a lower bound, which is needed in order to construct examples close to the upper bound.

Theorem 8 *Suppose that $D_\mu^2 = 0$. Then one of the following holds.*

a) $S_\mu = \mathbf{P}^2$, $D_\mu = 0$, $r_\mu = 0$.

b) *S_μ is a rational ruled surface, $D_\mu = r_\mu P$ where P is a ruling, $r_\mu \geq 0$.*

Proof: If $D_\mu = 0$, then $r_\mu = 0$ and S_μ is a minimal surface, which is either $\mathbf{P}^2$ or a rational ruled surface (see [18, 23]). It remains to prove the case $D_\mu\neq 0$.

Let C be any prime component of a divisor in $|D_\mu|$. Then $D_\mu\cdot C = 0$ since D_μ is nef. We show that C is not rigid. Assume the contrary. By lemma 3 and minimality of (S_μ, D_μ), we have $C\cdot K_\mu \geq 0$. The class $D_\mu - K_\mu$ is nef (as the direct image of $D_{\mu-1}$), hence $C\cdot K_\mu = 0$. Moreover, $(D_\mu - K_\mu)^2 \geq D_{\mu-1}^2 > 0$. Hence, C is numerically zero by the Hodge index theorem. But this is absurd, because C is positive.

Because C is not rigid, we have $C^2 \geq 0$. This holds for any component of a divisor in D_μ. But $D_\mu^2 = 0$, hence $C_1 \cdot C_2 = 0$ for any two components of two divisors in $|D_\mu|$. This implies that all possible components lie in one pencil $|P|$, every divisor in $|P|$ is irreducible and $P^2 = 0$.

We show $P \cdot K_\mu = -2$. Suppose, indirectly, that this were not the case. Since the divisors in $|P|$ are irreducible, $P \cdot K_\mu \geq 0$ by the genus formula. Then $D_\mu \cdot K_\mu \geq 0$ and $|D_\mu + K_\mu|$ is not empty by Riemann-Roch. This contradicts the definition of μ.

Therefore, $|P|$ is a pencil of smooth rational curves with $P^2 = 0$, i.e. a ruled surface. The class D_μ is a multiple of the class of the ruling P, and because $D_\mu(D_\mu - K_\mu) = 2r_\mu$, we have $D = r_\mu P$.

Theorem 9 *Suppose that $D_\mu^2 = 0$. Then there exists a parameterization of degree $\leq 4\mu + 2r_\mu$.*

Proof: We will construct such a parameterization by a suitable dehomogenization of the rational map $\pi_\mu : S_\mu \to F$.

If $S_\mu = \mathbf{P}^2$, then $|-\mu K_\mu| = |3L|$, where L is the class of lines. The map π_μ is associated to a subsystem of $|D_\mu - \mu K_\mu| = |3\mu L|$, because this class is the direct image of D_0. By dehomogenizing, we get a parameterization of degree $\leq 3\mu$.

Now, suppose that $S_\mu = R_e$, $D_\mu = r_\mu P$ for a ruled surface R_e and ruling P. The map π_μ is associated to a subsystem of $|D_\mu - \mu K_\mu|$, which has bidegree $(r_\mu + \mu(e+2), 2\mu)$. Using the standard-dehomogenization, we get a parameterization of degree $n \leq r_\mu + \mu(e+2)$ for $e > 0$, and a parameterization of degree $\leq r_\mu + 4\mu$ if $e = 0$. In the first case, we use the fact that $D_\mu - \mu K_\mu$ is nef, hence

$$0 \leq B(D_\mu - \mu K_\mu) = r_\mu + \mu(2-e),$$

which implies that

$$n \leq r_\mu + \mu(e+2) \leq 2r_\mu + 4\mu.$$

Lemma 9 *Suppose that $D_\mu^2 = 0$. Then any parameterization has degree $\geq 3\mu + r_\mu$.*

Proof: Assume that there is a parameterization of degree n. The parameterization induces a rational map $f : \mathbf{P}^2 \to F$ associated to a subsystem of $|nL|$. We resolve the base points of the map $\pi_0^{-1} \circ f$ and obtain a birational regular map $\pi : \mathbf{Y} \to \mathbf{P}^2$, such that $f' := \pi_0^{-1} \circ f \circ \pi : \mathbf{Y} \to S_0$ is regular. The divisor class $L' := \pi^* L$ is nef, and we have $(f')^* D_0 \cdot L' = n$ and $K_{\mathbf{Y}} \cdot L' = -3$.

By the ramification formula (see [20]), we have $K_{\mathbf{Y}} = (f')^* K_0 + R$ for some effective divisor class R.

For $i = 1, \ldots, \mu$, let $\tau_i : S_0 \to S_i$ be the minimalization map. Then $K_0 = \tau_i^* K_i + R_i$ for some effective divisor class R_i, and

$$D_0 = \tau_\mu^* D_\mu - \tau_\mu^* K_\mu - \ldots - \tau_1^* K_1 = \tau_\mu^* D_\mu - \mu K_0 + R_1 + \ldots + R_\mu.$$

If $S_\mu = \mathbf{P}^2$, then $D_\mu = 0$ and

$$n = (f')^* D_0 \cdot L' \geq -\mu f'^* K_0 \cdot L' \geq -\mu K_{\mathbf{Y}} \cdot L' = 3\mu = 3\mu + r_\mu.$$

In the other case, S_μ has a ruling P and $D_\mu = r_\mu P$. Let $P' = (\tau_\mu \circ f')^* P$. Then $(P')^2 = 0$ and P' is effective. Since L' is nef and $(L')^2 = 1 > 0$, we have $P' \cdot L' \geq 1$ by the Hodge index theorem. Thus,

$$n = (f')^* D_0 \cdot L' \geq (r_\mu P' - \mu f'^* K_0) \cdot L' \geq r_\mu - \mu K_{\mathbf{Y}} \cdot L' = 3\mu + r_\mu.$$

Example 6 Here is an example where the degree of the smallest parameterization grows with order d^2.

For $n \geq 8$, let F be the surface of degree $d = 2n - 1$ with equation

$$x^{2n-1} - y^n w^{n-1} - z^2 w^{2n-3} = 0,$$

as in example 2. The numbers μ and r_μ can be computed either from the equation itself or from some birational parameterization - which is the easier way in this case, since we have a birational parameterization in example 2. See [30, 31] for details. One finds

$$\mu = n, r_\mu = n^2 - 3n - nr, \text{ where } r = \left[\frac{n-2}{2}\right].$$

By lemma 9, any parameterization has degree $\geq \frac{n^2+2n}{2}$.

Example 7 Here is an idea to find an example where the degree of the smallest parameterization grows with order p_1^2 (which is the largest possible order, according to theorem 4).

Let $a_1, \ldots, a_{10}$ be generic points in the plane. Let S be the blowup at these points. Let L be the pullback of the class of lines, and let $A_1, \ldots, A_{10}$ be the exceptional classes. Let $m \geq 0$, and let $n := \frac{m(m+1)}{2}$. Then we construct the surface F as the image of the rational map f associated to

$$D := 3nL - nA_1 - \ldots - nA_8 - (n-1)A_9 - mA_{10}.$$

Then $\dim|D| \geq 3$ by Riemann-Roch. By genericness, one would expect $\dim|D| = 3$, f is birational. The linear system has a base point, because it intersects the one elliptic curve in $|3L - A_1 - \ldots - A_9|$ in one point b. By genericness, one would expect that b is the only base point. In this case, the surface has the following parameters:

$$d = 9n^2 - 8n^2 - (n-1)^2 - m^2 - 1 = m + 4,$$

$$p_1 = \frac{(3n-1)(3n-2)}{2} - 8\frac{n(n-1)}{2} - \frac{(n-1)(n-2)}{2} - \frac{m(m-1)}{2} = m + 3,$$

$$\mu = n, r_\mu = 0.$$

By lemma 9, the smallest parameterization has degree $3n$.

Lemma 10 *Suppose that $D_\mu^2 > 0$. Then one of the following holds.*

a) $S_\mu = \mathbf{P}^2$, $D_\mu = L$ (the class of lines), $r_\mu = 2$.

b) $S_\mu = \mathbf{P}^2$, $D_\mu = 2L$, $r_\mu = 5$.

c) S_μ is a rational ruled surface, $2D_\mu + K_\mu = (r_\mu - 3)P$ for a ruling P, $r_\mu \geq 3$.

Proof: Let us write $D := D_\mu$ and $K := K_\mu$. By Riemann-Roch and the assumption that $|D+K|$ is empty, we have $D^2 + D\cdot K \leq -2$, and also $D\cdot K \leq -3$.

Let $m > 1$ be the integer uniquely defined by the inequality

$$0 \leq mD^2 + D\cdot K \leq D^2 - 1.$$

By Riemann-Roch, $|mD + K|$ is not empty. mD is nef, hence also $mD + K$ is nef by lemma 4. Then

$$\begin{aligned} 0 \leq (mD+K)^2 = m^2D^2 + mD\cdot K + mD\cdot K + K^2 \leq \\ \leq mD^2 - m + D\cdot K + (m-1)D\cdot K + K^2 \leq \\ \leq D^2 - 1 - m + D\cdot K + (m-2)D\cdot K + K^2 \leq \\ \leq -2 - 1 - m + (m-2)(-3) + K^2 = 3 - 4m + K^2 \leq 12 - 4m, \end{aligned}$$

hence $m \leq 3$.

If $m = 3$, then we must have equality everywhere: $K^2 = 9$, hence $S_\mu = \mathbf{P}^2$ (see [23]), and $D^2 = 1$, hence $D = L$ and $r_\mu = 2$.

If $m = 2$, then

$$0 \leq (2D+K)^2 = 4(D^2 + D\cdot K) + K^2 \leq -8 + K^2.$$

The right inequality must be an equation: $D^2 + D\cdot K = -2$ (because $K^2 < 12$).

If $K^2 = 9$ (i.e. $S_\mu = \mathbf{P}^2$), then $D^2 = 4$, hence $D = 2L$ and $r_\mu = 5$.

If $K^2 = 8$, then S_μ is ruled (see [23]). Moreover, $(2D+K)^2 = 0$, hence $2D + K = nP$ for some $n \geq 0$. By

$$8r_\mu = 2D(2D - 2K) = (nP - K)(nP - 3K) = 8n + 24,$$

we get $n = r_\mu - 3$.

Theorem 10 *Suppose that $D_\mu^2 > 0$. Then there exists a parameterization of degree $\leq 4\mu + r_\mu - 1$.*

Proof: The proof is analogous to the proof of theorem 9.

In subcase (a), we obtain a parameterization of degree $\leq 3\mu + 1$. In subcase (b), we obtain a parameterization of degree $\leq 3\mu + 2$. In subcase (c), we obtain a parameterization of degree $\leq 4\mu + r_\mu - 1$.

The fact below is not really needed anywhere in this paper, we include it because it is the counterpart of lemma 9, which might be used to construct more examples which do not have a small parameterization.

Proposition 1 *Suppose that $D_\mu^2 > 0$. Then any parameterization has degree $\geq 3\mu + \frac{1}{2}(r_\mu - 1)$.*

Proof: The proof is analogous to the proof of lemma 9.

In subcase (a), we can show $n \geq 3\mu + 1$. In subcase (b), we can show $n \geq 3\mu + 2$. In subcase (c), we can show $n \geq 3\mu + \frac{1}{2} r_\mu$.

Remark 4 It seems worthwile to note that the parameterization constructed in the proof of theorem 9 or theorem 10 is at most twice as large as the smallest possible parameterization. In the cases where $S_\mu = \mathbf{P}^2$, the parameterization is already the smallest.

Proof of theorem 3: If $p_1 = 0$, then $\mu = 0$ and $r_\mu = d + 1$ by lemma 2 and lemma 1. Moreover $D_\mu^2 = d > 0$, hence there is a parameterization of degree $\leq d$ by lemma 10.

Proof of theorem 4: Since D_μ is nef and $|D_\mu|$ is not empty, we have $D_\mu^2 \geq 0$. By lemma 8 and lemma 10, we have always $r_\mu \geq 0$. By lemma 9 and lemma 10, there is always a parameterization of degree $\leq 4(\mu + r_\mu)$, which is less than or equal to $4 \begin{pmatrix} 2p_1 - d - 1 \\ 2 \end{pmatrix} + 4p_1$ by lemma 8.

Proof of theorem 2: This is a purely numerical consequence of theorem 3, theorem 4, and theorem 6.

Proof of theorem 7: Suppose that F is a rational surface with sectional genus $p_1 > 1$ and degree $d \geq 4p_1 + 5$. Let us write $D := D_0$, $K = K_0$. Let C be a generic divisor of $|D|$. Then C is a curve of genus p_1 (see the proof of lemma 1).

Let $B \in \mathrm{Div} C$ be the divisor class cut out by $-K$. We have

$$\deg B = (-K) \cdot D = D^2 - D(D + K) = d - (2p_1 - 2) \geq 2p_1 + 7,$$

hence $\deg |B| \geq p_1 + 7$ by the Riemann-Roch theorem for curves. By the exact sequence

$$0 \to \mathcal{L}(-D - K) \to \mathcal{L}(-K) \to \mathcal{L}(B) \to 0,$$

we get the inequality

$$p_1 + 8 \leq \dim |-B| + 1 = h^0(\mathcal{L}(B)) \leq h^0(\mathcal{L}(-K)) + h^1(\mathcal{L}(-D - K)).$$

We distinguish two cases.

If $h^1(\mathcal{L}(-D - K)) = 0$, then $h^0(\mathcal{L}(-K)) \geq p_1 + 8 \geq 9$. This is only possible if $X_0 = \mathbf{P}^2$, and in this case we have equality everywhere: $p_1 = 1$ and $d = 4p_1 + 5 = 9$.

If $h^1(\mathcal{L}(-D - K)) > 0$, then either $(D + K)^2 = 0$ or $D + K$ is not nef, by the Grauert-Riemenschneider vanishing theorem [?]. If $D + K$ is not nef, then

D_1 is not nef and $\mu = 0$ by lemma 6. This contradicts our assumption $p_1 > 0$ (by lemma 2). Hence $(D+K)^2 = 0$. Then, it follows

$$0 = -D^2 + 2(D^2 + D{\cdot}K) + K^2 = -d + 2(2p_1 - 2) + K^2 \leq K^2 - 9,$$

hence $d = 4p_1 + 5$ and $K^2 = 9$, which implies $X_0 = \mathbf{P}^2$. The only class with zero self-intersection on $\mathbf{P}^2$ is the zero class, hence $D + K = 0$, which contradicts to $h^1(\mathcal{L}(-D-K)) > 0$.

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