Embedded Desingularization of Hypersurfaces after Villamayor

Josef Schicho

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Abstract

This paper contains an algorithmic description of Villamayor's algorithm for embedded desingularization of hypersurfaces.

1 Introduction

This paper should be seen as a student's reception of the course [3] given by O. Villamayor at the 'working week on mountains and singularities', September 1997, Obergurgl. It is also based on the lecture notes [1] and on the exercises [2].

My motivation for writing this up was to have something on resolution of singularities which is easy enough so that I can still read it in a couple of years.

The content of this paper is a self-contained algorithmic description of Villamayor's algorithm. It does not cover all material presented in [3] (this is quite obvious from the shortness of this paper).

Besides O. Villamayor, I have to thank S. Encinas, who explained to me some details which I could not figure out at the first place.

2 The Problem

Let W be a regular variety. Let $X \subset W$ be a hypersurface, i.e. a variety which is locally given by one equation. A *desingularization* of X is a proper birational map $\tilde{X} \to X$, such that $\tilde{X}$ is regular, which is an isomorphism outside of the singular locus of X. An *embedded desingularization* is a proper birational map $\tilde{W} \to W$, such that $\tilde{W}$ is regular and the inverse image of X in $\tilde{W}$ is a hypersurface with normal crossing (i.e. for any points in $\tilde{W}$, there exist local parameters such that the local equation of the hypersurface is a monomial), the component of the inverse image which birationally maps to X is regular, and the map is an isomorphism outside the singular locus of X. Our problem is to construct an embedded desingularization of a given hypersurface (X, W).

In addition, the embedded desingularization which we will construct has the following nice properties.

1. It will be constructed by blowing up regular subvarieties.
2. At any stage, the regular center of the next blowing up has normal crossing with the exceptional divisor of the map constructed up to this stage.
3. (Equivariance) Any automorphism of (X, W) lifts to an automorphism of all blowing ups and finally to an automorphism of the desingularization.
4. The choice of the center of the next blowing up depends on local data: If $U \subset W$ is open, then the desingularization of $(X \cap U, U)$ coincides with the restriction of the desingularization, except for some identity maps corresponding to blowing ups of centers outside U.

3 Basic Objects

A *basic object* is a quadruple $(W, (J, b), E)$, where

- W is a regular variety,
- J is an ideal which is nowhere zero (i.e. there is no point where the local ideal is the zero ideal),
- b is a positive integer,
- E is a set of regular hypersurfaces with normal crossing.

The singular locus $\mathrm{Sing}(J, b)$ is the set of all points in W where the ideal J has order $\geq b$.

The basic object $(W, (J, b), E)$ is called *simple* iff the order of J is at most b, for all points.

A *weighted basic object* is a basic object $(W, (J, b), E)$ together with a function $a : E \to \mathbf{N}$, such that the following holds: if $\mathrm{mon}(E, a)$ is the ideal locally generated by the monomial which is the product of the coordinates to the power specified by a, then $J = \mathrm{mon}(E, a) \cdot \overline{J}$ for some ideal $\overline{J}$, which is called the *stripped ideal* of the weighted basic object.

If $(W, (J, b), E, a)$ is a weighted basic object, then we define the weighted order at a point $x \in W$ as the order of the stripped ideal J' divided by b, and the weighted order of (J, b) as the maximum value of the weighted orders at all points. The set of all points where the maximum is reached is denoted by w-$\mathrm{Sing}(J, b)$. A weighted basic object is called *monomial* iff its weighted order is zero.

4 Effect of Blowing Up on Basic Objects

Let $(W,(J,b),E)$ be a basic object, and let Y be a regular subvariety having normal crossing with E contained in the singular locus of (J,b) (such an Y is called *permissible* with respect to the basic object). Then the blowing up of W at Y is a proper birational map $W' \to W$. One can show that the inverse image of the hypersurfaces in E has normal crossing, and it contains birational preimages of the hyperplanes and one new regular hypersurface H (the exceptional divisor). The set of these hyperplanes (the old ones and the one new) is denoted by E'.

Let x be a local equation of the exceptional divisor. Then the inverse image of J can be written locally as $x^b \cdot J'$. The basic object $(W',(J',b),E')$ is called the transform of the basic object $(W,(J,b),E)$.

Let $(W,(J,b),E,a)$ be a weighted basic object. The center Y is called *w-permissible* iff it is permissible and contained in w-Sing(J,b). In this case, the transformed weight function has the old values for the old hypersurfaces, and the value on the exceptional divisor is $b' + \sum a(H_i)$, where $b' :=$ w-ord(J,b) and the sum ranges over all hypersurfaces in E which contain Y. Note that the value is chosen in such a way that the blowing up at Y transforms the simple basic object $(W,(\overline{J},b'),E)$ into $(W',(\overline{J}',b'),E')$ (i.e. stripped ideal transforms to stripped ideal).

It is important to note that the weighted order always drops. Moreover, the transform of a simple basic object is again simple.

5 Resolution of Basic Objects

A resolution of a basic object $(W,(J,b),E)$ is a sequence $\langle (W_i,(J_i,b),E_i)\rangle_i$, starting with the given basic object, such that the successor is the transform of the predecessor under a suitable permissible blowing up, and such that the last basic object has empty singular locus. The analogous definition applies in the weighted case.

We are interested in finding resolutions because we can produce desingularizations from them.

Algorithm *Desingularize*(X,W);
  $J :=$ Ideal(X);
  Resolve_weighted$(W,(J,1),\emptyset,$empty function$)$;
  stop one step before w-ord(J,b) becomes zero;

To see that the algorithm is correct, note that the stripped ideal coincides with the ideal of the proper transform of X. Consequently, the weighted order is the highest multiplicity of the proper transform of X (so, a desingularization is already obtained when the weighted order is one). The blowing up at the very next step would be the blowing up at the proper transform, which is necessarily - at this stage - a regular hypersurface having normal crossing with E.

Let $\langle (W_i, (J_i, b), E_i, a_i) \rangle_i$, $i = 1, \ldots, n$ be a weighted resolution. Let $(W, (J, b), E, a)$ be the k-th basic object. Then there is a unique r, $1 \le r \le k$, such that w-ord$(J_i, b) >$ w-ord(J, b) iff $i < r$. We partition E into two disjoint subsets: E^- is the set of all "old" exceptional hypersurfaces coming from E_r, and E^+ is its complement.

The subset E^- is the harmful part, which requires some extra treatment. Below, we give an algorithm that resolves weighted basic object by resolving simple basic objects having only harmless hypersurfaces in their E-set.

Algorithm *Resolve_weighted*$(W, (J, b), E)$;
    $b' :=$ w-ord(J, b);
    $E^- := \emptyset$;
    while $b' > 0$ and Sing(J, b) is not empty do
        $J' :=$ stripped ideal of (J, b);
        if $b' < b$ then
            $J' := J'^{b-b'} + \text{mon}(J, b)^{b'}$;
            $b' := b'(b - b')$;
        $t :=$ the largest number such that t hypersurfaces in E^- intersect within w-Sing(J, b);
        $H_1, \ldots, H_t :=$ the first t hypersurfaces in E^- intersecting within w-Sing(J, b)
            (with respect to lexicographical ordering);
        $J' := J' + \text{Ideal}(H_1)^{b'} + \ldots + \text{Ideal}(H_t)$;
        $E^+ := E - E^-$;
        Resolve_simple$(W, (J', b'), E^+)$;
        if the weighted order has dropped then
            update b' and E^-;
    if Sing(J, b) is not empty then
        Resolve_monomial$(W, (J, b), E)$;

The correctness follows from the following statements.

1. The basic object $(W, (J', b'), E^+)$ constructed in the algorithm is simple.

2. Let $(W, (J, b), E, a)$ be a weighted basic object (at some stage in the middle of the procedure), and let $(W, (J', b'), E^+)$ be the simple basic object constructed as in the algorithm. Then Y is w-permissible with respect to $(W, (J, b), E, a)$ iff it is permissible with respect to $(W, (J', b'), E^+)$.

3. The transform of $(W, (J', b'), E^+)$ coincides with the simple basic object constructed from the transform of $(W, (J, b), E, a)$.

4. When the simple basic object $(W, (J', b'), E^+)$ is resolved, then either the weighted order has dropped or the exceptional hyperplanes $H^1, \ldots, H^t$ do not intersect in the weighted singular locus of $(W, (J, b), E, a)$.

5. Eventually, the weighted order drops.

Resolution of monomial basic objects is not so difficult, it is purely combinatorial.

6 The Inductive Step

Suppose that $(W,(J,b),E)$ is a simple basic object. Then the ideal Δ generated by all derivatives of order $\leq b-1$ of elements in J has order 1. Locally, we can choose an element of order 1, which defines a regular hypersurface $V\subset W$. Suppose that V has normal crossing with E and does not coincide with any of the hypersurfaces in E, for all possible choices of V. Then we call $(W,(J,b),E)$ a *normal simple basic object*. For a normal simple basic object, the set of E_V of all intersections of hyperplanes in E with V has normal crossing.

We construct an ideal $I\subset k[V]$, such that a resolution of the basic object $(V,(I,b!),E_V)$ yields a resolution of $(W,(J,b),E)$, in the following sense.

- A center $Y\subset V\subset W$ is permissible with respect to $(V,(I,b!),E_V)$ iff it is permissible with respect to $(W,(J,b),E)$.
- The transform of a normal simple basic object is again a normal simple basic object.
- The transform of $(V,(I,b!),E_V)$ coincides with the basic object constructed from the transform of $(W,(J,b),E)$ (which is again a normal simple object).
- The singular locus of $(V,(I,b!),E_V)$ coincides with the singular locus of $(W,(J,b),E)$ (so that the second becomes resolved when the first becomes resolved).

The construction of I goes as follows. Suppose that W has local coordinates $x_1,\ldots,x_n,z$, and V has equation $z=0$. Let

$$f_{0,1}+\ldots+f_{d_1,1}z^{d_1},\ldots,f_{0,r}+\ldots+f_{d_r,r}z^{d_r}$$

be generators of J, where $f_{i,j}$ are functions in $x_1,\ldots,x_n$. Then we we define J is the ideal generated by

$$f_{0,1}^{\frac{b!}{b}},\ldots,f_{b-1,1}^{\frac{b!}{1}},\ldots,f_{0,r}^{\frac{b!}{b}},\ldots,f_{b-1,r}^{\frac{b!}{1}}.$$

We denote this ideal by $\mathrm{coeff}(J,b,z)$ (the "coefficient ideal"). It can be proved that this construction satisfies the conditions above.

Note the $\mathrm{coeff}(J,b,z)$ is the zero ideal iff $J=z^bJ'$ for some ideal J'. In this case, we cannot construct a basic object of one dimension less. But this case is very easy: since $J=z^bJ'$ has no points with multiplicity $>b$, we have $J'=(1)$. We just have to blowup the hyperplane $V:z=0$. The blowing up map is the identity, and the transform is $(W,(J',b),E\cup\{V\})$, which has empty singular locus.

If V is reducible, then it might be the case that $\mathrm{coeff}(J,b,z)$ is the zero ideal at points in one component and different from the the zero ideal in some other component. But we may shrink the open sets, so that (locally) the coefficient ideal is either zero or nowhere zero.

We have the following local algorithm for resolution of normal simple basic objects.

Algorithm *Resolve_simple*$(W,(J,b),E)$;
  $\Delta :=$ the ideal generated by the partial derivatives of order $\leq b-1$ of J;
  $z :=$ choose an element of order 1 in Δ;
  $V :=$ zero set of z;
  if $J = z^b J'$ (for some ideal J') then
    BlowUp(V);
  else $I := \mathrm{coeff}(J,b,z)$;
    Resolve_weighted$(V,(I,b!),E_V$,zero function);

The correctness of this algorithm follows from the above properties of the construction coeff.

Because resolution of simple basic object requires only resolution of basic objects of one dimension less, the resolution algorithm terminates (by induction on the dimension). Unfortunately, it is impossible to give a complexity bound, because we do not know any bound for the weighted order of the basic object constructed in *Resolve_simple*.

It remains to prove that the simple basic objects constructed in the algorithm *Resolve_weighted* are normal simple. This can be done.

7 Patching Local Data Together

In the inductive step, we had an ideal Δ of order 1, and we had to choose an element of order 1. The zero set is a regular subspace of one dimension less, which allows us to do induction on the dimension.

The problem is that the choice is not unique. Therefore, the desingularization might - a priori - depend on the choice of the regular subspace. Since we have chosen the elements of order 1 locally, our algorithm computes resolutions of local patches. It might be the case that the local patches would not fit together for a global resolution.

To overcome this difficulty, we will prove that the choice of the next center of the blowing up is independant of the choice of the element of order 1, in the local case. This proven, the local centers will automatically patch together to give a global center of the next blowing up step.

Let W be a regular variety, and let E be a set of regular hypersurfaces having normal crossing. Let V_1, V_2 be two regular subvarieties of the same dimension, which have normal crossing with E but are not contained in any of the hypersurfaces in E. Then the set E_j of intersections with V_j has normal crossing, for $j = 1, 2$. A *simultanous sequence of blowing ups* is a sequence of triples $(V_1^{(r)}, (V_2^{(r)}, W^r)$, starting with $((V_1, V_2, W)$, such that

- $W^{(r+1)}$ is the blowing up of $W^{(r)}$ at some regular center $Y^{(r)} \subset V_1^{(r)} \cap V_2^{(r)}$,
- $V_j^{(r+1)} \subset W^{(r+1)}$ is the proper transform of $V_i^{(r+1)}$ (which is also the blowing up $V_i^{(r+1)}$ at $Y^{(r+1)}$), for $i = 1, 2$.

It can be shown that Y has normal crossing with respect to E_1 iff Y has normal crossing with respect E_2.

Lemma 1 *Let $V_1, V_2 \subset W$ and E, E_1, E_2 be as above. Let a be a function from E to $\mathbf{N}$, inducing functions $a_i : E_i \to \mathbf{N}$, for Let $(V_i, (J_i, b), E_i)$ be basic objects. Suppose that every simultanous resolution is permissible with respect to $(V_1, (J_1, b), E_1)$ iff it is permissible with respect to $(V_2, (J_2, b), E_2)$. Then the following hold.*

The two sets Sing(J_1, b) and Sing(J_2, b) are equal. Let x be a point in this set. Then the weighted order of $(V_1, (J_1, b), E_1, a_1)$ at x is equal to the weighted order of $(V_2, (J_2, b), E_2, a_2)$ at x.

Using this lemma, one can show that all (weighted) singular loci of (weighted) basic objects constructed during the algorithm do not depend on the choices of the element of order 1 in algorithm *Resolve_simple*. Therefore, the centers of the blowing up do not depend on the choices of the element of order 1.

In the case where we had to shrink the open sets in order to separate the part where the coefficient ideal is zero from the ideal where it is not, we first blow up the part where it is zero and then proceed with the rest. This algorithm yields an embedded desingularization as described in section 2 above: obtained by blowing up of regular subvarieties with normal crossing with respect to the exceptional hyperplanes, the center is chosen equivariantly and determined by local data.

References

[1] S. Encinas and O. Villamayor. Constructive desingularization and equivariance. In *Working Week on Mountains and Singularities*. Obergurgl, 1997. lecture notes.

[2] S. Encinas and O. Villamayor. Constructive desingularization and equivariance: Problems revised with general comments and definitions. In *Working Week on Mountains and Singularities*. Obergurgl, 1997.

[3] O. Villamayor. Constructive resolution. In *Working Week on Mountains and Singularities*. Obergurgl, 1997. lecture.