A Linear Algebra Method for Solving Systems of Algebraic Equations

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We consider a linear algebra method in residue class rings for solving systems of algebraic equations. We construct multiplication tables using a Gröbner basis of a zero-dimensional ideal. Then, we analyze the tables by exactly computing their Frobenius normal forms. The derogatoriness and the diagonalizability are determined by the normal forms, and the problem is divided into four cases:

(1) nonderogatory and diagonalizable case,
(2) nonderogatory and nondiagonalizable case,
(3) derogatory and diagonalizable case,
(4) derogatory and nondiagonalizable case.

Subsequently, we construct common eigenvectors symbolically, and compute all the exact zeros with their multiplicities. In the appendix, we show the minute description of multiple zeros using only the normal set basis, where we do not need explicit computation of differential conditions.

1. Introduction

To solve a system of algebraic equations by computer algebra, the classical method of general elimination has been studied as the computation of a U-resultant since Lazard(1977, 1981). Kobayashi *et al.* (1988a) showed an efficient algorithm for computing a U-resultant using a Gröbner basis of a zero-dimensional ideal. Their method computes the matrices $A_{x_1}, \ldots, A_{x_s}$, which are nowadays called multiplication tables, and obtains the U-resultant as $\det(u_1 A_{x_1} + \cdots + u_s A_{x_s})$ with new indeterminates $u_1, \ldots, u_s$.

On the other hand, Auzinger and Stetter (1988) proposed a method for computing the solutions of a system as the eigenvalues of certain matrices. At first they did not use Gröbner bases, but in these years, multiplication tables $A_{x_1}, \ldots, A_{x_s}$ are usually computed by a Gröbner basis (Stetter, 1993). A brief introduction to this method is given in Corless (1996).

Stetter's method intends to solve matrix eigenproblems using numerical analysis technique. However, we have to assume at present that the multiplication tables should be computed exactly, because we do not yet have a practical algorithm for computing Gröbner bases approximately. Hence, we may apply symbolic computation technique to matrix eigenproblems. Takeshima and Yokoyama (1990) proposes a symbolic method for computing the eigenvalues and the eigenvectors by the Frobenius normal form of a matrix. Their method is very efficient because of avoiding arithmetic operations over an algebraic extension field. In this paper, we extend their idea and give the algorithms according to the properties of multiplication tables.

In the above algorithms, the so-called *normal set basis* for a residue class ring is used to compute the multiplication tables. On the other hand, Möller (1993), Möller and Stetter (1995), and Stetter (1996) show that multiple zeros are characterized by the notion of *a dual basis* (Marinari *et al.*, 1991). They consider a dual space and adopt a basis of differential conditions that describe the local behavior of multiple zeros. To the contrary, we show that the description of multiple zeros can be derived from the multiplication tables by themselves. Computing the Jordan normal form of one nonderogatory matrix, we can transform all the multiplication tables into upper triangular block diagonal forms simultaneously. Since nonderogatoriness is also required in Möller and Stetter's method, our method seems essentially identical with theirs. Nevertheless, we utilize only the multiplication tables themselves, and we do not assume that any other information is given.

We have implemented our method on the computer algebra system Reduce3.6, and we report the result of the computation of several examples.

2. Basic Notions

2.1. Residue class ring and multiplication table

Let $I = (f_1, \dots, f_\ell)$ be an ideal in a polynomial ring with rational number coefficients $R = \mathbf{Q}[x_1, \dots, x_s]$. It is well-known that an ideal I is zero-dimensional if and only if the residue class ring R/I is finite-dimensional as a $\mathbf{Q}$-vector space. The following theorem is fundamental in the Gröbner basis theory (Buchberger, 1965, 1970).

Theorem 2.1. (Normal Set Basis) *Let G be a Gröbner basis of zero-dimensional ideal I with an arbitrary order. Then, the set of power products*

$$B := \{x_1^{e_1} \cdots x_s^{e_s} \mid x_1^{e_1} \cdots x_s^{e_s} \text{ is irreducible with } G\}$$

is a linearly independent basis for R/I as a $\mathbf{Q}$-vector space.

Let $B = \{t_1, \dots, t_n\}$ be the normal set basis for R/I in the above theorem. Then, it is well-known that n coincides with the number of the zeros of I with their multiplicities counted. We define the multiplication tables A_{x_k} $(k = 1, \dots, s)$ in R/I as follows.

Definition 2.1. (Multiplication Table for x_k) Compute the normal form of the product of x_k and each t_i $(t_i \in B)$ with G:

$$x_k \cdot t_i \ \mapsto \ \sum_{j=1}^{n} a_{ij} \cdot t_j .$$

Then, the $n \times n$ matrix $A_{x_k} := [a_{ij}]$ is called the multiplication table for x_k in R/I.

The following fact is important for analyzing the family of multiplication tables $\{A_{x_k}\}$.

LEMMA 2.1. *The matrices A_{x_k}'s are commutative, that is, $A_{x_k}A_{x_\ell} = A_{x_\ell}A_{x_k}$ for every pair $(k, \ell = 1, \ldots, s)$.*

Now we consider the problem of solving a system of algebraic equations

$$f_1(x_1, \ldots, x_s) = 0, \quad \ldots, \quad f_\ell(x_1, \ldots, x_s) = 0, \tag{2.1}$$

that is, the problem of finding all the zeros of an ideal $I = (f_1, \ldots, f_\ell)$. In this paper, we restrict ourselves to the case where I is zero-dimensional, that is, the system (2.1) has finitely many solutions. The relation between the zeros of I and the multiplication tables $\{A_{x_k}\}$ is elucidated by the following theorem (Auzinger and Stetter, 1988) (Yokoyama *et al.*, 1992). Therefore, the problem of solving a system of algebraic equations is interpreted as a matrix eigenproblem.

THEOREM 2.2. (SOLUTIONS BY MATRIX EIGENPROBLEM) *Using one common regular matrix S, the commutative matrices $\{A_{x_k}\}$ are simultaneously transformed into upper triangular forms $U_k = S^{-1}A_{x_k}S$ $(k = 1, \ldots, s)$, where the diagonal elements of U_k are the eigenvalues of A_{x_k}. Let the diagonal elements of U_k be $u_{ii}^{(k)}$ $(i = 1, \ldots, n)$. Then, all the zeros of I, counting multiplicities, coincide with $\left(u_{ii}^{(1)}, u_{ii}^{(2)}, \ldots, u_{ii}^{(s)}\right)$ $(i = 1, \ldots, n)$.*

2.2. FROBENIUS NORMAL FORM

In the following sections, we assume that the elements of a given matrix are rational numbers, that is, $A = [a_{ij}]$, $a_{ij} \in \mathbf{Q}$. The following notions are well-known in the matrix theory.

DEFINITION 2.2. (COMPANION MATRIX) The following $n \times n$ square matrix

$$C = \begin{bmatrix} 0 & 1 & & & \\ 0 & 0 & \ddots & & O \\ \vdots & \vdots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & 1 \\ c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \end{bmatrix} \tag{2.2}$$

is called the companion matrix associated with the polynomial $f(x) = x^n - c_{n-1}x^{n-1} - \cdots - c_1 x - c_0$. In particular, the companion matrix associated with the first degree polynomial $f(x) = x - c_0$ is the 1×1 square matrix $[c_0]$.

LEMMA 2.2. *The characteristic polynomial $\varphi_C(x)$ and the minimal polynomial $\phi_C(x)$ of the companion matrix C are equal to $f(x)$.*

THEOREM 2.3. (FROBENIUS NORMAL FORM) *Using a suitable regular matrix S, every $n \times n$ square matrix A can be transformed into the block diagonal matrix as follows:*

$$F = S^{-1}AS = C_1 \oplus C_2 \oplus \cdots \oplus C_t. \tag{2.3}$$

It is called the Frobenius normal form (or the rational normal form) of A. Each block matrix $C_i\,(i = 1, \cdots, t)$ is an $m_i \times m_i$ companion matrix (2.2), and the associated polynomial $\varphi_{i+1}(x)$ of C_{i+1} divides the associated polynomial $\varphi_i(x)$ of C_i $(i = 1, \ldots, t-1)$. The matrix in the form (2.3) always exists and is uniquely determined for every given matrix. Moreover, the minimal polynomial $\phi_A(x)$ of A is equal to $\varphi_1(x)$, and the characteristic polynomial $\varphi_A(x)$ of A is given by $\varphi_1(x) \cdot \varphi_2(x) \cdots \varphi_t(x)$.

Danilevskii (1937) showed the method for computing a block diagonal matrix like (2.3), but the result of his method did not necessarily satisfy the division condition

$$\varphi_t(x) \mid \varphi_{t-1}(x) \mid \cdots \mid \varphi_1(x),$$

hence his $\varphi_1(x)$ was not necessarily the minimal polynomial of A. However, Gantmacher (1959) shows the existence of strict meaning of the Frobenius normal form in Theorem 2.3. Our implementation is based on the algorithm given by Kan and Iri (1982), which eliminates the matrix elements like Gaussian elimination by computing similar transformation step by step: $\cdots S_3^{-1}\left(S_2^{-1}\left(S_1^{-1} A S_1\right) S_2\right) S_3 \cdots$, and obtains F, S, S^{-1} in (2.3) finally.

Since Danilevskii's method is numerically unstable, it had been abandoned in the field of numerical analysis. However, on the assumption of exact computation, Takeshima and Yokoyama (1990) proposes an efficient algorithm for the symbolic solution of matrix eigenproblems using Frobenius normal forms. The following lemmas are the straightforward extension of their method.

Lemma 2.3. (Eigenvector of Companion Matrix) *Let λ be an eigenvalue of an $n \times n$ companion matrix C (2.2) and we put a vector $\boldsymbol{u} := \left[1, \lambda, \lambda^2, \ldots, \lambda^{n-1}\right]^T$. Then, this vector $\boldsymbol{u}$ is an eigenvector of C with the eigenvalue λ.*

Proof. Since λ is a root of the characteristic polynomial $\varphi_C(x)$, we have

$$\varphi_C(\lambda) = \lambda^n - c_{n-1}\lambda^{n-1} - \cdots - c_1\lambda - c_0 = 0.$$

Then, we obtain

$$\begin{aligned} C\boldsymbol{u} &= \begin{bmatrix} & 1 & & & \\ & & \ddots & & \\ & & & \ddots & \\ & & & & 1 \\ c_0 & c_1 & \cdots & c_{n-2} & c_{n-1} \end{bmatrix} \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix} \\ &= \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ c_0 + c_1\lambda + \cdots + c_{n-1}\lambda^{n-1} \end{bmatrix} = \begin{bmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \\ \lambda^n \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ \lambda \\ \lambda^2 \\ \vdots \\ \lambda^{n-1} \end{bmatrix} = \lambda\boldsymbol{u}, \end{aligned}$$

which proves the assertion. □

Note that, for the companion matrix C, there exists only one linearly independent eigenvector with the eigenvalue λ, even if λ is a multiple root of $\varphi_C(x)$.

LEMMA 2.4. (EIGENVECTOR OF FROBENIUS NORMAL FORM) *Let λ be an eigenvalue of a Frobenius normal form F (2.3), and let $\varphi_i(x)$ be the associated polynomial of C_i $(i = 1, \ldots, t)$. Suppose that λ satisfies $\varphi_1(\lambda) = \cdots = \varphi_k(\lambda) = 0$ and $\varphi_{k+1}(\lambda) \neq 0$ for some k. (When $k = t$, the last condition should be omitted.) For each block C_i with its size m_i, we put vectors $\tilde{\boldsymbol{u}}_i := \left[1, \lambda, \lambda^2, \ldots, \lambda^{m_i - 1}\right]^T$ $(i = 1, \ldots, k)$. Extending them to the vectors of size n, we put*

$$\boldsymbol{u}_1 := \begin{bmatrix} \tilde{\boldsymbol{u}}_1 \\ \boldsymbol{0} \\ \vdots \\ \vdots \\ \boldsymbol{0} \end{bmatrix}, \quad \boldsymbol{u}_2 := \begin{bmatrix} \boldsymbol{0} \\ \tilde{\boldsymbol{u}}_2 \\ \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \end{bmatrix}, \ldots, \quad \boldsymbol{u}_k := \begin{bmatrix} \boldsymbol{0} \\ \vdots \\ \boldsymbol{0} \\ \tilde{\boldsymbol{u}}_k \\ \boldsymbol{0} \end{bmatrix}, \tag{2.4}$$

where the size of each zero vector corresponds to each companion block. Then, these k vectors are eigenvectors of F with the eigenvalue λ.

PROOF. Apparently, the vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_k$ are linearly independent. According to Lemma 2.3, for each block C_i, we have $C_i \tilde{\boldsymbol{u}}_i = \lambda \tilde{\boldsymbol{u}}_i$ $(i = 1, \ldots, k)$. Since the matrix F is block diagonal, it is easy to see that $F\boldsymbol{u}_i = \lambda \boldsymbol{u}_i$ $(i = 1, \ldots, k)$. $\square$

Note that this lemma gives also the dimension k of the eigenspace of F belonging to the eigenvalue λ, that is, k is determined to be such a number that $\varphi_1(\lambda) = \cdots = \varphi_k(\lambda) = 0$ and $\varphi_{k+1}(\lambda) \neq 0$.

LEMMA 2.5. (EIGENVECTOR OF GENERAL MATRIX) *Let F be the Frobenius normal form of a matrix A, and $\boldsymbol{u}$ be one of the eigenvectors (2.4) of F having λ as the eigenvalue. Then, λ is also the eigenvalue of A. The corresponding eigenvector of A is given by $\boldsymbol{v} := S\boldsymbol{u}$, where S is a similar transformation matrix such that $F = S^{-1}AS$ (2.3).*

PROOF. Since A and F are similar, they have the same eigenvalues in common. Moreover, since we have $F\boldsymbol{u} = \lambda \boldsymbol{u}$ and $AS = SF$, we obtain

$$A\boldsymbol{v} = A(S\boldsymbol{u}) = (AS)\boldsymbol{u} = (SF)\boldsymbol{u} = S(F\boldsymbol{u}) = S(\lambda \boldsymbol{u}) = \lambda(S\boldsymbol{u}) = \lambda \boldsymbol{v}.$$

Therefore, $\boldsymbol{v}$ is an eigenvector of A having λ as the eigenvalue. $\square$

EXAMPLE 2.1. (SYMBOLIC COMPUTATION OF EIGENVALUES AND EIGENVECTORS) For a given matrix

$$A = \begin{bmatrix} 3 & -2 & & & 6 \\ 1 & & & & \\ & & & 1 & \\ & & -2 & & \\ & 1 & & & \end{bmatrix},$$

we compute its eigenvalues and eigenvectors in symbolic expressions.

First, we compute its Frobenius normal form F and a transformation matrix S such that $F = S^{-1}AS$. (The inverse S^{-1} of S can be computed simultaneously, but only S is necessary in the following computation.) Then, we obtain

$$F = \left[\begin{array}{ccc|cc} & 1 & & & \\ & & 1 & & \\ 6 & -2 & 3 & & \\ \hline & & & & 1 \\ & & & -2 & 0 \end{array}\right], \quad S = \begin{bmatrix} 1 & & & & \\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6} & & \\ & & & & -\frac{1}{2} \\ & & & 1 & \\ -\frac{7}{18} & & \frac{1}{18} & & \end{bmatrix}.$$

The associated polynomial of each companion block is

$$\varphi_1(x) = x^3 - 3x^2 + 2x - 6 = (x^2+2)(x-3), \quad \varphi_2(x) = x^2 + 2.$$

Let λ_i $(i = 1, 2)$ be the roots of x^2+2. Then, the eigenvalues of the matrix A are expressed as λ_1,λ_2, and 3.

We have $\varphi_1(\lambda_i) = \varphi_2(\lambda_i) = 0$ $(i = 1, 2)$; $\varphi_1(3) = 0$ and $\varphi_2(3) \neq 0$. Hence, for the eigenvalues λ_1, λ_2, and 3, we obtain the following eigenvectors $\boldsymbol{u}_j$ $(j = 1, \ldots, 5)$ of F, where we have reduced $\lambda_i^2 \mapsto -2$:

$$\boldsymbol{u}_j = \begin{bmatrix} 1 \\ \lambda_1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \lambda_1 \end{bmatrix}; \begin{bmatrix} 1 \\ \lambda_2 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \lambda_2 \end{bmatrix}; \begin{bmatrix} 1 \\ 3 \\ 9 \\ 0 \\ 0 \end{bmatrix}.$$

Computing $S\boldsymbol{u}_j$, we obtain the corresponding eigenvectors $\boldsymbol{v}_j$ $(j = 1, \ldots, 5)$ of A as follows:

$$\boldsymbol{v}_j = \begin{bmatrix} 1 \\ -\frac{1}{2}\lambda_1 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\lambda_1 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ -\frac{1}{2}\lambda_2 \\ 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2}\lambda_2 \\ 1 \\ 0 \end{bmatrix}; \begin{bmatrix} 1 \\ \frac{1}{3} \\ 0 \\ 0 \\ \frac{1}{9} \end{bmatrix}.$$

We proceed the exact computation by expressing algebraic numbers over $\mathbf{Q}$ as new symbols. Note that the result has the common expressions for algebraic conjugates. Therefore, in practice, we have only to compute once for a certain symbol, and to substitute other symbols to it.

This method does not need to solve a system of linear equations $(A-\lambda E)\boldsymbol{x} = \mathbf{0}$ over an algebraic extension field $\mathbf{Q}(\lambda)$. We have only to reduce λ^k by the definition polynomial of λ. Thus, symbolic expressions of eigenvalues and eigenvectors of a rational matrix can be efficiently computed.

2.3. MATRIX EIGENPROBLEM

When we consider a multiple eigenvalue of a matrix, we have to distinguish two kinds of multiplicities.

DEFINITION 2.3. (ALGEBRAIC MULTIPLICITY) Let λ be an eigenvalue of a matrix A. If λ is a p-fold root of the characteristic polynomial $\varphi_A(x) = \det(A - xE)$, then it is called that λ has the algebraic multiplicity p.

DEFINITION 2.4. (GEOMETRIC MULTIPLICITY) Let λ be an eigenvalue of a matrix A, and $V(\lambda, A)$ be the eigenspace of A belonging to λ. If the dimension of $V(\lambda, A)$ is m, then it is called that λ has the geometric multiplicity m.

Note that the inequality $m \leq p$ always holds between the both multiplicities. The following property of matrices is crucial in Stetter's method.

DEFINITION 2.5. (NONDEROGATORINESS) If every eigenvalue of a matrix A has the geometric multiplicity 1, then A is called nonderogatory.

LEMMA 2.6. (NONDEROGATORY MATRIX) *An $n \times n$ matrix A is nonderogatory if and only if its Frobenius normal form F consists of only one companion block.*

PROOF. From Definition 2.5, each eigenvalue λ of A has the geometric multiplicity 1. Suppose that F has two companion blocks, that is, $F = S^{-1}AS = C_1 \oplus C_2$. Let $\varphi_i(x)$ be the associated polynomial of C_i respectively. Since $\varphi_2(x)$ divides $\varphi_1(x)$, if an eigenvalue λ satisfies $\varphi_2(\lambda) = 0$, then we have $\varphi_1(\lambda) = 0$. Therefore, from Lemma 2.4, there exist two linearly independent eigenvectors with the eigenvalue λ, and λ has the geometric multiplicity 2. This is a contradiction. Conversely, if F consists of only one companion block, then each eigenvalue λ has only one linearly independent eigenvector from Lemma 2.3. $\square$

Hence, we can determine the nonderogatoriness of a matrix by computing its Frobenius normal form. On the other hand, the diagonalizability of a matrix, which is fundamental in linear algebra, is also determined by its Frobenius normal form.

DEFINITION 2.6. (DIAGONALIZABILITY) If there exists a suitable regular matrix V for a matrix A such that $V^{-1}AV$ is diagonal, then A is called diagonalizable.

LEMMA 2.7. (DIAGONALIZABLE MATRIX) *An $n \times n$ matrix A is diagonalizable if and only if its minimal polynomial $\phi_A(x)$ is square-free.*

PROOF. We show that the following conditions are equivalent.

(1) An $n \times n$ matrix A is diagonalizable.
(2) There exist n linearly independent eigenvectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ for A.
(3) The algebraic multiplicity p_i and the geometric multiplicity m_i coincides for each eigenvalue λ_i of A.
(4) Each eigenvalue λ_i is a simple root of the minimal polynomial $\phi_A(x)$ of A.

(1)$\Leftrightarrow$(2) If we put $V := [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$, then we have

$$\begin{aligned} AV &= A[\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] = [A\boldsymbol{v}_1, \ldots, A\boldsymbol{v}_n] = [\lambda_1 \boldsymbol{v}_1, \ldots, \lambda_n A\boldsymbol{v}_n] \\ &= [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n] \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix}. \end{aligned}$$

Since V is regular, $V^{-1}AV$ is diagonal.

(2)⇔(3) Let A have r distinct eigenvalues $\lambda_1, \ldots, \lambda_r$. From the definition of algebraic multiplicity, we have $\sum_{i=1}^{r} p_i = n$. On the other hand, the total number of linearly independent eigenvectors is given by $\sum_{i=1}^{r} m_i$. Since $m_i \leq p_i$ holds for each i, we have $\sum_{i=1}^{r} m_i = n$ if and only if $m_i = p_i$ $(i = 1, \ldots, r)$.

(3)⇔(4) We consider the Frobenius normal form F of A such that $F = S^{-1}AS = C_1 \oplus C_2 \oplus \cdots \oplus C_t$ (2.3). Then, the asscoiated polynomial $\varphi_1(x)$ of C_1 is the minimal polynomial $\phi_A(x)$ of A. From Lemmas 2.3 and 2.4, if some λ_i is a multiple root of $\varphi_1(x)$, then $m_i < p_i$.

Now (1) and (4) are equivalent, the lemma follows immediately. □

Commutative matrices have the following property, which is applicable to the multiplication tables.

Lemma 2.8. (Commutative Matrix and Common Eigenvector) *Let matrices A, B be commutative: $AB = BA$. And let $\boldsymbol{v}$ be an eigenvector of A with the eigenvalue λ, whose geometric multiplicity is 1. Then, this vector $\boldsymbol{v}$ is also an eigenvector of B.*

Proof. Since we have $A\boldsymbol{v} = \lambda\boldsymbol{v}$ and $AB = BA$, we obtain

$$A(B\boldsymbol{v}) = (AB)\boldsymbol{v} = (BA)\boldsymbol{v} = B(A\boldsymbol{v}) = B(\lambda\boldsymbol{v}) = \lambda(B\boldsymbol{v}).$$

Hence, $B\boldsymbol{v}$ also belongs to the eigenspace of A with the eigenvalue λ. Since λ has the geometric multiplicity 1, we have $B\boldsymbol{v} \in \mathrm{span}(\boldsymbol{v})$. Therefore, we obtain $B\boldsymbol{v} = \mu\boldsymbol{v}$ for a certain scalar μ. This means that $\boldsymbol{v}$ is an eigenvector of B having μ as the eigenvalue. □

3. Algorithms by Analysis of Multiplication Tables

Let $n \times n$ matrices $A_{x_1}, \ldots, A_{x_s}$ be the multiplication tables in Definition 2.1. We put $A := A_{x_k}$ for an arbitrary variable x_k.

First, we compute the Frobenius normal form F of A and a transformation matrix S such that $F = S^{-1}AS$. According to Lemmas 2.6 and 2.7, we determine the nonderogatoriness and diagonalizability of A. Then, we divide the problem into four cases, which correspond to the following four subsections.

3.1. nonderogatory and diagonalizable case

In this case, F consists of only one companion block C (2.2), that is, $F = C$. Since the minimal polynomial $\phi_A(\lambda) = \varphi_C(\lambda) = \lambda^n - c_{n-1}\lambda^{n-1} - \cdots - c_1\lambda - c_0$ is square-free, the n eigenvalues λ_j $(j = 1, \ldots, n)$ are simple and distinct. Then, Stetter's method (Theorem 2.2) can be directly applied to symbolic computation.

Algorithm 3.1. (Nonderogatory and Diagonalizable) Initialize the list of solutions $Sol := \{\ \}$, and for $j := 1$ to n, do the following steps.
1) $\boldsymbol{u}_j := \left[1, \lambda_j, \lambda_j^2, \ldots, \lambda_j^{n-1}\right]^T$.
2) $\boldsymbol{v}_j := S\boldsymbol{u}_j$.
3) Normalize the first element of $\boldsymbol{v}_j$ to 1; (See Remark 3.1).

4) Let $\boldsymbol{a}_{(i)1}^T$ be the first row of A_{x_i} $(i = 1, \ldots, s)$, and compute a zero

$$\boldsymbol{z}_j = \begin{bmatrix} z_{1j} \\ z_{2j} \\ \vdots \\ z_{sj} \end{bmatrix} := \begin{bmatrix} \boldsymbol{a}_{(1)1}^T \\ \boldsymbol{a}_{(2)1}^T \\ \vdots \\ \boldsymbol{a}_{(s)1}^T \end{bmatrix} \boldsymbol{v}_j.$$

5) $Sol := Sol \cup \{\boldsymbol{z}_j\}$.

PROOF. From Lemma 2.3, $\boldsymbol{u}_j$ is an eigenvector of $F = C$. Hence, $\boldsymbol{v}_j$ is an eigenvector of A from Lemma 2.5. Since λ_j has the geometric multiplicity 1, $\boldsymbol{v}_j$ is a common eigenvector of all the A_{x_i}'s from Lemmas 2.1 and 2.8.

Corresponding eigenvalues are given by $A_{x_i}\boldsymbol{v}_j = \mu_{ij}\boldsymbol{v}_j$. However, since the first element of $\boldsymbol{v}_j$ is normalized to 1, the first element of $\mu_{ij}\boldsymbol{v}_j$ is μ_{ij} itself. Hence, it is sufficient to compute only with the first row of A_{x_i}'s. □

REMARK 3.1. In most cases, the first row of S becomes $[1, 0, \ldots, 0]$ because of the process of computing the Frobenius normal form. Then, the first element of $\boldsymbol{v}_j$ is already 1, and we do not need to apply step 3) in practice.

If it happens to be 0, we have to select another non-zero element and utilize the corresponding rows of A_{x_i}. However, we have not found such an example at present.

EXAMPLE 3.1. (EXAMPLE 1 IN CORLESS (1996)) We consider the following system:

$$\begin{cases} p_1 = 3x^2y + 9x^2 + 2xy + 5x + y - 3 \\ p_2 = 2x^3y + 6x^3 - 2x^2 - xy - 3x - y + 3 \\ p_3 = x^3y + 3x^3 + x^2y + 2x^2. \end{cases}$$

According to the reference, we compute with *gradlex* order $(x > y)$, and we obtain the normal set basis $B = \{1, y, x\}$. Then, the multiplication tables are computed as follows:

$$A_x = \begin{bmatrix} & & 1 \\ -3 & 1 & -1 \\ 3 & -1 & \frac{3}{2} \end{bmatrix}, \quad A_y = \begin{bmatrix} & 1 & \\ \frac{3}{2} & \frac{5}{2} & 4 \\ -3 & 1 & -1 \end{bmatrix}.$$

Computing the Frobenius normal form of A_x, we obtain

$$F = \begin{bmatrix} & 1 & \\ & & 1 \\ 0 & \frac{5}{2} & \frac{5}{2} \end{bmatrix}, \quad S = \begin{bmatrix} 1 & & \\ 3 & \frac{3}{2} & -1 \\ & 1 & \end{bmatrix},$$

hence, A_x is nonderogatory. The minimal polynomial of A_x is $\phi(\lambda) = \lambda^3 - (5/2)\lambda^2 - (5/2)\lambda$, which is square-free, then A_x is diagonalizable.

Let λ_1, λ_2 be the roots of $\lambda^2 - (5/2)\lambda - 5/2$, and we get the eigenvectors of F with the eigenvalues $\lambda_1, \lambda_2, 0$ respectively:

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ \lambda_1 \\ \frac{5}{2}\lambda_1 + \frac{5}{2} \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ \lambda_2 \\ \frac{5}{2}\lambda_2 + \frac{5}{2} \end{bmatrix}, \quad \boldsymbol{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Then, the eigenvectors of A_x are given by $\boldsymbol{v}_j = S\boldsymbol{u}_j$:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ -\lambda_1 + \frac{1}{2} \\ \lambda_1 \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -\lambda_2 + \frac{1}{2} \\ \lambda_2 \end{bmatrix}, \quad \boldsymbol{v}_3 = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

Consequently, we obtain the 3 simple zeros:

$$\begin{bmatrix} \boldsymbol{a}_{(x)1}^T \\ \boldsymbol{a}_{(y)1}^T \end{bmatrix} \boldsymbol{v}_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -\lambda_1 + \frac{1}{2} \\ \lambda_1 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ -\lambda_1 + \frac{1}{2} \end{bmatrix},$$

$$\begin{bmatrix} \boldsymbol{a}_{(x)1}^T \\ \boldsymbol{a}_{(y)1}^T \end{bmatrix} \boldsymbol{v}_2 = \begin{bmatrix} \lambda_2 \\ -\lambda_2 + \frac{1}{2} \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{a}_{(x)1}^T \\ \boldsymbol{a}_{(y)1}^T \end{bmatrix} \boldsymbol{v}_3 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

3.2. NONDEROGATORY AND NONDIAGONALIZABLE CASE

In this case, F consists of only one companion block C (2.2), that is, $F = C$. Let the characteristic polynomial $\varphi_C(\lambda) = \lambda^n - c_{n-1}\lambda^{n-1} - \cdots - c_1\lambda - c_0$ have r $(r < n)$ roots. We let the algebraic multiplicity of each eigenvalue λ_j be p_j $(j = 1, \ldots, r)$.

ALGORITHM 3.2. (NONDEROGATORY AND NONDIAGONALIZABLE) Initialize the list of solutions $Sol := \{\ \}$, and the list of multiplicities $Mul := \{p_1, p_2, \ldots, p_r\}$. For $j := 1$ to r, do the following steps.
1) $\boldsymbol{u}_j := \left[1, \lambda_j, \lambda_j^2, \ldots, \lambda_j^{n-1}\right]^T$.
2) $\boldsymbol{v}_j := S\boldsymbol{u}_j$.
3) Normalize the first element of $\boldsymbol{v}_j$ to 1; (See Remark 3.1 too).
4) Let $\boldsymbol{a}_{(i)1}^T$ be the first row of A_{x_i} $(i = 1, \ldots, s)$, and compute a zero

$$\boldsymbol{z}_j = \begin{bmatrix} z_{1j} \\ z_{2j} \\ \vdots \\ z_{sj} \end{bmatrix} := \begin{bmatrix} \boldsymbol{a}_{(1)1}^T \\ \boldsymbol{a}_{(2)1}^T \\ \vdots \\ \boldsymbol{a}_{(s)1}^T \end{bmatrix} \boldsymbol{v}_j.$$

5) $Sol := Sol \cup \{\boldsymbol{z}_j\}$.

PROOF. Since each eigenvalue has the geometric multiplicity 1, $\boldsymbol{v}_j$ is a common eigenvector similarly to Algorithm 3.1. The multiplicities of each zero $\boldsymbol{z}_j$ coincides with p_j from Theorem 2.2. □

REMARK 3.2. We obtain the locations and the multiplicities of all the zeros by this algorithm. For more minute description of multiple zeros, see the appendix, which shows the formula for the simultaneous transformation of the A_{x_i}'s into upper triangular block diagonal forms.

EXAMPLE 3.2. (EXAMPLE 2 IN CORLESS (1996)) We consider the following system:

$$\begin{cases} p_1 = x^2 + y^2 - 1 \\ p_2 = (3x+4y)^2 + \frac{(-4x+3y)^2}{2} - 25. \end{cases}$$

According to the reference, we compute with *gradlex* order ($x > y$), and we obtain the normal set basis $B = \{1, y, x, y^2\}$. Then, the multiplication tables are computed as follows:

$$A_x = \begin{bmatrix} & & 1 & \\ \frac{2}{3} & & & -\frac{7}{24} \\ 1 & & & -1 \\ & \frac{216}{625} & \frac{112}{625} & \end{bmatrix}, \quad A_y = \begin{bmatrix} & 1 & & \\ & & & 1 \\ \frac{2}{3} & & & -\frac{7}{24} \\ & \frac{688}{625} & -\frac{384}{625} & \end{bmatrix}.$$

Computing the Frobenius normal form of A_x, we obtain

$$F = \begin{bmatrix} & 1 & & \\ & & 1 & \\ & & & 1 \\ -\frac{81}{625} & 0 & \frac{18}{25} & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & & & \\ & \frac{19}{8} & & -\frac{625}{216} \\ & 1 & & \\ 1 & & -1 & \end{bmatrix},$$

hence, A_x is nonderogatory. The minimal polynomial of A_x is

$$\phi(\lambda) = \lambda^4 - \frac{18}{25}\lambda^2 + \frac{81}{625} = \left(\lambda - \frac{3}{5}\right)^2 \left(\lambda + \frac{3}{5}\right)^2,$$

which is not square-free, then A_x is nondiagonalizable.

The eigenvalues $\lambda_1 = 3/5$ and $\lambda_2 = -3/5$ respectively have the algebraic multiplicity 2 and the geometric multiplicity 1. Hence, both $x = 3/5$ and $x = -3/5$ constitute double zeros. Corresponding eigenvectors of F are

$$\boldsymbol{u}_1 = \begin{bmatrix} 1 \\ \frac{3}{5} \\ \frac{9}{25} \\ \frac{27}{125} \end{bmatrix}, \quad \boldsymbol{u}_2 = \begin{bmatrix} 1 \\ -\frac{3}{5} \\ \frac{9}{25} \\ -\frac{27}{125} \end{bmatrix}.$$

Then, the eigenvectors of A_x are given by $\boldsymbol{v}_j = S\boldsymbol{u}_j$:

$$\boldsymbol{v}_1 = \begin{bmatrix} 1 \\ \frac{4}{5} \\ \frac{3}{5} \\ \frac{16}{25} \end{bmatrix}, \quad \boldsymbol{v}_2 = \begin{bmatrix} 1 \\ -\frac{4}{5} \\ -\frac{3}{5} \\ \frac{16}{25} \end{bmatrix}.$$

Consequently, we obtain the 2 double zeros:

$$\begin{bmatrix} \boldsymbol{a}_{(x)1}^T \\ \boldsymbol{a}_{(y)1}^T \end{bmatrix} \boldsymbol{v}_1 = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \frac{4}{5} \\ \frac{3}{5} \\ \frac{16}{25} \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{4}{5} \end{bmatrix}, \quad \begin{bmatrix} \boldsymbol{a}_{(x)1}^T \\ \boldsymbol{a}_{(y)1}^T \end{bmatrix} \boldsymbol{v}_2 = \begin{bmatrix} -\frac{3}{5} \\ -\frac{4}{5} \end{bmatrix}.$$

3.3. DEROGATORY AND DIAGONALIZABLE CASE

We assume that the multiplication table A_{x_k} is derogatory and diagonalizable. In this case, we compute the Frobenius normal forms of other multiplication tables A_{x_i} $(i = 1, \ldots, s;\ i \neq k)$. If there exists even one nonderogatory A_{x_ℓ}, then we can let $A := A_{x_\ell}$ and apply Algorithm 3.1 or Algorithm 3.2 according to its diagonalizability.

Now we consider the case where all the multiplication tables A_{x_i} $(i = 1, \ldots, s)$ are derogatory. If there exists even one nondiagonalizable A_{x_ℓ} among them, then we have to apply the case §3.4.

Otherwise, if all the A_{x_i}'s are diagonalizable, the following theorem in linear algebra is applicable to the family $\{A_{x_i}\}$.

THEOREM 3.1. (COMMUTATIVE AND DIAGONALIZABLE MATRICES) *Let A, B be commutative matrices: $AB = BA$. Moreover, we assume that both A and B are diagonalizable. Then, there exists a set of common eigenvectors $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ of A and B such that both $V^{-1}AV$ and $V^{-1}BV$ are diagonal, where $V = [\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n]$.*

To find such common eigenvectors for the family $\{A_{x_i}\}$, we put $A_w := \sum_{i=1}^{s} c_i A_i$ with randomly selected integers c_i. Then, A_w becomes nonderogatory for almost every s-tuple $(c_1, \ldots, c_s)$ (Yokoyama *et al.*, 1992). Since $V^{-1}A_wV = V^{-1}\left(\sum c_iA_i\right)V = \sum c_i\left(V^{-1}A_iV\right)$, A_w is also diagonalizable, then we can apply Algorithm 3.1 to A_w.

From the viewpoint of a system of algebraic equations, this situation corresponds to the case where all the zeros are simple but some of them have the same x_i-coordinate in common. In such a case, we can make the system into general position by almost every linear coordinate transformation $w := c_1x_1 + \cdots + c_sx_s$.

Now we summerize the above flow.

ALGORITHM 3.3. (DEROGATORY AND DIAGONALIZABLE)
% Assumption: The A_{x_k} is derogatory and diagonalizable.
Compute all the Frobenius normal forms of the A_{x_i}'s and apply one of the following.
% Case (A): Some A_{x_ℓ} is nonderogatory.
Apply Algorithm 3.1 or 3.2 according to its diagonalizability.
% Case (B): All the A_{x_i}'s are derogatory and some A_{x_ℓ} is nondiagonalizable.
Apply the case §3.4.
% Case (C): All the A_{x_i}'s are derogatory and diagonalizable.
Put $A_w := \sum_{i=1}^{s} c_iA_i$ with random $(c_1, \ldots, c_s)$ so that A_w is nonderogatory. Then, apply Algorithm 3.1 to A_w.

REMARK 3.3. In case (A), it is sufficient to find at least one nonderogatory matrix A_{x_ℓ}. On the other hand, in case (C), it is necessary to confirm that all the A_{x_i}'s are diagonalizable.

EXAMPLE 3.3. (CYCLIC 3RD ROOT) We consider the following system:

$$\begin{cases} p_1 = x + y + z \\ p_2 = xy + yz + zx \\ p_3 = xyz - 1. \end{cases}$$

We compute with *gradlex* order $(x > y > z)$, and we obtain the normal set basis $B =$

$\{1, z, y, z^2, yz, yz^2\}$. Then, the multiplication tables are computed as follows:

$$A_x = \begin{bmatrix} & -1 & -1 & & & \\ & & & -1 & -1 & \\ & & & 1 & & \\ -1 & & & & & -1 \\ 1 & & & & & \\ & 1 & & & & \end{bmatrix}, \quad A_y = \begin{bmatrix} & & 1 & & & \\ & & & & 1 & \\ & & & -1 & -1 & \\ & & & & & 1 \\ -1 & & & & & -1 \\ & -1 & -1 & & & \end{bmatrix},$$

$$A_z = \begin{bmatrix} & 1 & & & & \\ & & & 1 & & \\ & & & & 1 & \\ 1 & & & & & \\ & & & & & 1 \\ & & 1 & & & \end{bmatrix}.$$

The Frobenius normal form of each matrix consists of 2 companion blocks, with the associated polynomials $\varphi_1(\lambda) = \varphi_2(\lambda) = \lambda^3 - 1$. This means that all of A_x, A_y, A_z are derogatory and diagonalizable. (They have naturally the same structure because of the symmetry of variables x, y, z.)

Then, we try putting $A_w := A_x + 2A_y - A_z$ and computing its Frobenius normal form F and a transformation matrix S such that $A_w S = SF$. As a result, we obtain

$$F = \begin{bmatrix} & 1 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ -343 & 0 & 0 & -20 & 0 & 0 \end{bmatrix},$$

thus, A_w is nonderogatory as expected. The minimal polynomial of A_w is

$$\phi(\lambda) = \lambda^6 + 20\lambda^3 + 343 = \left(\lambda^2 + 5\lambda + 7\right)\left(\lambda^2 - \lambda + 7\right)\left(\lambda^2 - 4\lambda + 7\right),$$

which is square-free, then A_w is also diagonalizable.

Hence, we apply Algorithm 3.1 to A_w. We let the eigenvalues as follows:

$\lambda_{11}, \lambda_{12}$ are the roots of $\lambda^2 + 5\lambda + 7$,
$\lambda_{21}, \lambda_{22}$ are the roots of $\lambda^2 - \lambda + 7$,
$\lambda_{31}, \lambda_{32}$ are the roots of $\lambda^2 - 4\lambda + 7$.

We put the eigenvectors of F as $\boldsymbol{u}_{ij} := \left[1, \lambda_{ij}, \lambda_{ij}^2, \ldots, \lambda_{ij}^5\right]^T$, and reduce each element using the definition polynomials of λ_{ij} $(i = 1, 2, 3;\ j = 1, 2)$. As a result, we obtain

$$\boldsymbol{u}_{1j} = \begin{bmatrix} 1 \\ \lambda_{1j} \\ -5\lambda_{1j} - 7 \\ 18\lambda_{1j} + 35 \\ -55\lambda_{1j} - 126 \\ 149\lambda_{1j} + 385 \end{bmatrix}, \quad \boldsymbol{u}_{2j} = \begin{bmatrix} 1 \\ \lambda_{2j} \\ \lambda_{2j} - 7 \\ -6\lambda_{2j} - 7 \\ -13\lambda_{2j} + 42 \\ 29\lambda_{2j} + 91 \end{bmatrix}, \quad \boldsymbol{u}_{3j} = \begin{bmatrix} 1 \\ \lambda_{3j} \\ 4\lambda_{3j} - 7 \\ 9\lambda_{3j} - 28 \\ 8\lambda_{3j} - 63 \\ -31\lambda_{3j} - 56 \end{bmatrix}.$$

Then, the eigenvectors of A_w are given by $\boldsymbol{v}_{ij} = S\boldsymbol{u}_{ij}$:

$$\boldsymbol{v}_{1j} = \begin{bmatrix} 1 \\ 1 \\ \lambda_{1j}+2 \\ 1 \\ \lambda_{1j}+2 \\ \lambda_{1j}+2 \end{bmatrix}, \quad \boldsymbol{v}_{2j} = \begin{bmatrix} 1 \\ -\frac{1}{3}\lambda_{2j}-\frac{1}{3} \\ \frac{1}{3}\lambda_{2j}-\frac{2}{3} \\ \frac{1}{3}\lambda_{2j}-\frac{2}{3} \\ 1 \\ -\frac{1}{3}\lambda_{2j}-\frac{1}{3} \end{bmatrix}, \quad \boldsymbol{v}_{3j} = \begin{bmatrix} 1 \\ -\frac{1}{2}\lambda_{3j}+\frac{1}{2} \\ 1 \\ \frac{1}{2}\lambda_{3j}-\frac{3}{2} \\ -\frac{1}{2}\lambda_{3j}+\frac{1}{2} \\ \frac{1}{2}\lambda_{3j}-\frac{3}{2} \end{bmatrix}.$$

Computing with each i and j

$$\begin{bmatrix} \boldsymbol{a}_{(x)1}^T \\ \boldsymbol{a}_{(y)1}^T \\ \boldsymbol{a}_{(z)1}^T \end{bmatrix} \boldsymbol{v}_{ij} = \begin{bmatrix} 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \boldsymbol{v}_{ij},$$

we obtain the 6 simple zeros:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -\lambda_{1j}-3 \\ \lambda_{1j}+2 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ \frac{1}{3}\lambda_{2j}-\frac{2}{3} \\ -\frac{1}{3}\lambda_{2j}-\frac{1}{3} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{2}\lambda_{3j}-\frac{3}{2} \\ 1 \\ -\frac{1}{2}\lambda_{3j}+\frac{1}{2} \end{bmatrix} \quad (j=1,2).$$

When numerical values are needed, we solve the definition polynomials, substitute each eigenvalue to the above zeros, and obtain

$$\begin{bmatrix} \omega^2 \\ \omega \\ 1 \end{bmatrix}, \begin{bmatrix} \omega \\ \omega^2 \\ 1 \end{bmatrix}; \quad \begin{bmatrix} 1 \\ \omega \\ \omega^2 \end{bmatrix}, \begin{bmatrix} 1 \\ \omega^2 \\ \omega \end{bmatrix}; \quad \begin{bmatrix} \omega \\ 1 \\ \omega^2 \end{bmatrix}, \begin{bmatrix} \omega^2 \\ 1 \\ \omega \end{bmatrix}; \quad \omega = \frac{-1+\sqrt{-3}}{2}.$$

3.4. DEROGATORY AND NONDIAGONALIZABLE CASE

In this case, all of the A_{x_i}'s are derogatory, and some of them are nondiagonalizable. We cannot immediately distinguish a multiple zero from simples zeros with the same x_i-coordinate. Therefore, we have to handle the system depending on the case.

In some cases, it is still effective to make a linear combination with random $(c_1, \ldots, c_s)$. If we obtain a nonderogatory $A_w = \sum_{i=1}^{s} c_i A_i$, then we can apply Algorithm 3.1 or 3.2 to A_w. The following example belongs to such a case. (See also Example A.1 in the appendix.)

EXAMPLE 3.4. (EXAMPLE 3 IN CORLESS (1996)) We consider the following system:

$$\begin{cases} p_1 = x^2 + y^2 - 1 \\ p_2 = x^3 + (2+z)xy + y^3 - 1 \\ p_3 = z^2 - 2. \end{cases}$$

According to the reference, we compute with *gradlex* order $(x > y > z)$, and we obtain the normal set basis with 12 elements. Then, this system has 12 solutions and the multiplication tables A_x, A_y, A_z become 12×12 matrices.

The Frobenius normal form of A_x contains 2 companion blocks, and its minimal polynomial is

$$\phi_{A_x}(\lambda) = (\lambda^4 - 2\lambda^3 + \lambda^2 + 8\lambda + 8)\lambda(\lambda - 1)\left(\lambda^2 - \frac{1}{2}\right)^2,$$

hence, A_x is derogatory and nondiagonalizable. The matrix A_y has the same structure as A_x. The Frobenius normal form of A_z contains 6 companion blocks, and its minimal polynomial is $\phi_{A_z}(\lambda) = \lambda^2 - 2$, hence, A_z is derogatory but diagonalizable. However, the family $\{A_x, A_y, A_z\}$ are not diagonalizable.

Then, we try putting $A_w := A_x - A_y + A_z$ and computing its Frobenius normal form. Consequently, we find it nonderogatory and its minimal polynomial to be

$$\phi_{A_w}(\lambda) = \left(\lambda^4 + 10\lambda^2 + 32\lambda + 49\right)\left(\lambda^2 + 2\lambda - 1\right)\left(\lambda^2 - 2\lambda - 1\right)\left(\lambda^2 - 2\right)^2.$$

We let the eigenvalues as follows:

$\lambda_{11}, \ldots, \lambda_{14}$ are the roots of $\lambda^4 + 10\lambda^2 + 32\lambda + 49$,
$\lambda_{21}, \lambda_{22}$ are the roots of $\lambda^2 + 2\lambda - 1$,
$\lambda_{31}, \lambda_{32}$ are the roots of $\lambda^2 - 2\lambda - 1$,
$\lambda_{41}, \lambda_{42}$ are the roots of $\lambda^2 - 2$.

The eigenvalues $\lambda_{41}, \lambda_{42}$ have the algebraic multiplicity 2 and the geometric multiplicity 1. Hence, the matrix A_w is nondiagonalizable, and we apply Algorithm 3.2 to it. Constructing 10 common eigenvectors, we obtain the following zeros:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underset{(j=1,\ldots,4)}{\begin{bmatrix} -\frac{1}{52}\lambda_{1j}^3 - \frac{1}{26}\lambda_{1j}^2 + \frac{25}{52}\lambda_{1j} - \frac{2}{13} \\ -\frac{3}{52}\lambda_{1j}^3 - \frac{3}{26}\lambda_{1j}^2 - \frac{29}{52}\lambda_{1j} - \frac{19}{13} \\ -\frac{1}{26}\lambda_{1j}^3 - \frac{1}{13}\lambda_{1j}^2 - \frac{1}{26}\lambda_{1j} - \frac{17}{13} \end{bmatrix}}, \underset{(j=1,2)}{\begin{bmatrix} 0 \\ 1 \\ \lambda_{2j} + 1 \end{bmatrix}}, \underset{(j=1,2)}{\begin{bmatrix} 1 \\ 0 \\ \lambda_{3j} - 1 \end{bmatrix}}, \underset{(j=1,2)}{\begin{bmatrix} -\frac{1}{2}\lambda_{4j} \\ -\frac{1}{2}\lambda_{4j} \\ \lambda_{4j} \end{bmatrix}}.$$

The first 8 zeros are simple, and the last 2 are double. In Corless (1996), the derogatoriness of A_x has caused confusion and misled the 4 simple zeros with $x = 0, 1$ into 2 double zeros.

In other cases, we may not find a nonderogatory A_w by a linear combination $\sum_{i=1}^{s} c_i A_i$. Such situation indicates the existence of multiple zeros, but it is difficult to find common eigenvectors without heuristics. Therefore, we try to compute the radical $\sqrt{I}$ of the ideal I, because the locations of their zeros are coincident. The multiplicities can be counted by the algebraic multiplicities of the eigenvalues in the original system.

Note that all the Frobenius normal forms of the A_{x_i} $(i = 1, \ldots, s)$ have been already computed, that is, we have obtained all the minimal polynomials of the A_{x_i}'s at this point. Then, the following facts are applicable.

Lemma 3.1. (Minimal Univariate Polynomial in the Ideal) *Let A_{x_i} be the multiplication table in $\mathbf{Q}[x_1, \ldots, x_s]/I$, where I is zero-dimensional. And let $\phi_i(\lambda)$ be the minimal polynomial of the matrix A_{x_i}. Then, $\phi_i(x_i)$ is the unique monic univariate polynomial of minimal degree in $I \cap \mathbf{Q}[x_i]$.*

Proof. As a matrix polynomial, we have $\phi_i(A_{x_i}) = O$, which is equivalent to $\phi_i(x_i) \equiv 0$ in $\mathbf{Q}[x_1, \ldots, x_s]/I$. $\square$

THEOREM 3.2. (THEOREM 8.22 OF BECKER AND WEISPFENNING (1993)) *Let $h_i(x_i)$ be the square-free part of $\phi_i(x_i)$ in the above lemma. Then, we have*

$$\sqrt{I} = (I, h_1(x_1), \ldots, h_s(x_s)) .$$

We have already computed the Gröbner basis G of ideal I. Hence, we compute again with a new (smaller) system $\{G, h_1, \ldots, h_s\}$. Since all the zeros of $\sqrt{I}$ are simple, the family of new multiplication tables is always diagonalizable. Therefore, we can find a nonderogatory matrix (by a linear combination, if necessarily).

EXAMPLE 3.5. (EXAMPLE 2 IN KOBAYASHI *et al.* (1988B)) For the polynomials $f_1 = x^2 - xy - y^2 + 1$ and $f_2 = xy + 3y^2 + 1$, we define $p_1 := f_1^2 - f_2^2$ and $p_2 := 2f_1^2 + 3f_2^2$. Here, We consider the zeros of the ideal $I = (p_1, p_2)$.

We compute with *gradlex* order $(x > y)$, and we obtain the normal set basis with 16 elements. Then, this system has 16 solutions and the multiplication tables A_x, A_y become 16×16 matrices.

The Frobenius normal form of A_x contains 2 companion blocks (12×12 and 4×4), and its minimal polynomial is $\phi_{A_x}(\lambda) = \left(\lambda^4 + (28/11)\lambda^2 + 16/11\right)^3$, hence, A_x is derogatory and nondiagonalizable. Similarly, the Frobenius normal form of A_y contains 2 companion blocks (12×12 and 4×4), and its minimal polynomial is $\phi_{A_y}(\lambda) = \left(\lambda^4 + (8/11)\lambda^2 + 1/11\right)^3$. In this case, we cannot find a nonderogatory matrix by linear combination $c_1A_x + c_2A_y$. For example, $A_w := 3A_x + 7A_y$ has the same structure as A_x and A_y.

Now we compute the radical of I. Let $q_x := x^4 + (28/11)x^2 + 16/11$, $q_y := y^4 + (8/11)y^2 + 1/11$, and we consider the system $\{G, q_x, q_y\}$, where G is the Gröbner basis of $I = (p_1, p_2)$. Eventually, the new matrix $\bar{A}_x$ is nonderogatory and its minimal polynomial is $\phi_{\bar{A}_x}(\lambda) = \lambda^4 + (28/11)\lambda^2 + 16/11$. Consequently, we obtain the 4 simple zeros of $\sqrt{I}$:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda_i \\ -\frac{11}{8}\lambda_i^3 - 2\lambda_i \end{bmatrix}, \quad \text{where } \phi_{\bar{A}_x}(\lambda_i) = 0 \quad (i = 1, \ldots, 4).$$

In this example, the ideal I is primary, therefore, each solution has the same multiplicity $16/4 = 4$ by Theorem 3.3 in Kobayashi *et al.* (1988b).

4. Implementation and Timing Data

We have implemented the above-mentioned algorithms on the computer algebra system Reduce3.6. Gröbner bases are computed using the library package in Reduce, and Frobenius normal forms are computed using the program written by the authors (Moritsugu and Kuriyama, 1992). Using IBM ThinkPad 535 (Pentium 120MHz processor) with 32MB memory, we solved several problems.

In addition to the examples shown in the previous section, the following examples are computed. The timing data is shown in the Table 1. The result shows that the efficient computation of Frobenius normal forms is crucial, especially for a large-sized dense system like Example 4.2.

Table 1. CPU-Time(msec) for the computation of some examples

Problem	(#sol)	Multip.Table	Frobenius N.Form		Solutions	Total
Example 3.1	(3)	539	391	(A_x)	109	1,039
Example 3.2	(4)	720	550	(A_x)	49	1,319
Example A.1	(5)	829	539	(A_x)		
			610	(A_y)		
			720	(A_w)	171	2,869
Example 3.3	(6)	1,429	1,100	(A_x)		
			1,100	(A_y)		
			1,040	(A_z)		
			1,480	(A_w)	341	6,490
Example 3.4	(12)	5,331	7,970	(A_x)		
			7,860	(A_y)		
			3,790	(A_z)		
			8,230	(A_w)	1,260	34,441
Example 3.5	(16)	7,640	15,600	(A_x)		
			15,110	(A_y)		
			16,530	(A_w)	(unable)	(54,880)
$\sqrt{I}$	(4)	769	550	(A_x)	110	1,429
Example 4.1	(32)	744,539	487,920	(A_{u_0})	3,850	1,236,309
Example 4.2	(64)	12,342,440	87,320,730	(A_{u_0})	172,520	99,835,690
Example 4.3	(70)	519,999	2,255,561	(A_{x_1})		
			1,888,410	(A_w)	56,300	4,720,270

Example 4.1. (Katsura 5)

$$\begin{cases} p_1 = 2u_5^2 + 2u_4^2 + 2u_3^2 + 2u_2^2 + 2u_1^2 + u_0^2 - u0 \\ p_2 = 2u_5u_4 + 2u_4u_3 + 2u_3u_2 + 2u_2u_1 + 2u_1u_0 - u_1 \\ p_3 = 2u_5u_3 + 2u_4u_2 + 2u_3u_1 + 2u_2u_0 - u_2 + u_1^2 \\ p_4 = 2u_5u_2 + 2u_4u_1 + 2u_3u_0 - u_3 + 2u_2u_1 \\ p_5 = 2u_5u_1 + 2u_4u_0 - u_4 + 2u_3u_1 + u_2^2 \\ p_6 = 2u_5 + 2u_4 + 2u_3 + 2u_2 + 2u_1 + u_0 - 1 \end{cases}$$

We compute its Gröbner basis with *revgradlex* order ($u_0 > u_1 > \cdots > u_5$). This system has 32 simple zeros.

Example 4.2. (Katsura 6)

$$\begin{cases} p_1 = 2u_6^2 + 2u_5^2 + 2u_4^2 + 2u_3^2 + 2u_2^2 + 2u_1^2 + u_0^2 - u0 \\ p_2 = 2u_6u_5 + 2u_5u_4 + 2u_4u_3 + 2u_3u_2 + 2u_2u_1 + 2u_1u_0 - u_1 \\ p_3 = 2u_6u_4 + 2u_5u_3 + 2u_4u_2 + 2u_3u_1 + 2u_2u_0 - u_2 + u_1^2 \\ p_4 = 2u_6u_3 + 2u_5u_2 + 2u_4u_1 + 2u_3u_0 - u_3 + 2u_2u_1 \\ p_5 = 2u_6u_2 + 2u_5u_1 + 2u_4u_0 - u_4 + 2u_3u_1 + u_2^2 \\ p_6 = 2u_6u_1 + 2u_5u_0 - u_5 + 2u_4u_1 + 2u_3u_2 \\ p_7 = 2u_6 + 2u_5 + 2u_4 + 2u_3 + 2u_2 + 2u_1 + u_0 - 1 \end{cases}$$

We compute its Gröbner basis with *revgradlex* order ($u_0 > u_1 > \cdots > u_6$). This system has 64 simple zeros.

EXAMPLE 4.3. (CYCLIC 5TH ROOTS)

$$\begin{cases} p_1 = x_1 + x_2 + x_3 + x_4 + x_5 \\ p_2 = x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_1 \\ p_3 = x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 \\ p_4 = x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 \\ p_5 = x_1x_2x_3x_4x_5 - 1 \end{cases}$$

We compute its Gröbner basis with *revgradlex* order ($x_1 > x_2 > \cdots > x_5$). This system has 70 simple zeros. The matrix for each variable is expected to have the same structure analogously to Example 3.1. Therefore, after confirming that matrix A_{x_1} is derogatory and diagonalizable, we omit the computation of the Frobenius normal forms of $A_{x_2}, \ldots, A_{x_5}$. Then, we put $A_w := A_{x_1} - 3A_{x_2} + 5A_{x_3} - 7A_{x_4} + 11A_{x_5}$ and compute its Frobenius normal form, to find it nonderogatory.

5. Conclusion

We have shown the symbolic computation algorithm for solving a system of algebraic equations by matrix eigenproblems, which is similar to Stetter's numerical method. Computing the Frobenius normal forms of multiplication tables gives a good insight into multiple eigenvalues. Resting upon the analysis of the normal forms, we divide the problem into four cases:

(1) nonderogatory and diagonalizable case,
(2) nonderogatory and nondiagonalizable case,
(3) derogatory and diagonalizable case,
(4) derogatory and nondiagonalizable case.

For nonderogatory cases (1)(2), the solutions can be symbolically determined including multiple zeros. In the appendix, we show the minute description of multiple zeros using only the normal set basis, which is in contrast to Möller and Stetter's method.

If the family of matrices is diagonalizable in case (3), there always exist nonderogatory matrices among linear combinations $A_w := \sum_{i=1}^{s} c_iA_i$ with randomly chosen integers c_i. Then, A_w becomes nonderogatory for almost every s-tuple $(c_1, \ldots, c_s)$.

Only for the case (4), we need minute treatment depending on each problem. However, on the assumption of exact computation, the locations and the multiplicities of zeros can be algorithmically determined through the radical of the ideal.

Our program is still naive implementation, but it seems practical except for very large-sized and dense systems. If a system has a multiple zero or distinct zeros with the same x_i-coordinate, they are exactly recognized because of algebraic algorithms.

Since the numerical computation of Frobenius normal forms is unstable, our method cannot be directly applied to numerical computation. Nevertheless, we believe that our method is useful when exact computation is needed especially for purely mathematical models, even though symbolic computation usually takes more CPU-time than numerical computation.

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A. Appendix

In this appendix, we assume that a matrix A is nonderogatory and not necessarily diagonalizable, that is, its Frobenius normal form $C = S^{-1}AS$ consists of only one companion block and some of its eigenvalues may be multiple. We show a method to construct a transformation matrix V for A into its Jordan normal form J such that $J = V^{-1}AV$. Each column of V corresponds to a generalized eigenvector of A. When A is diagonalizable, each column of V naturally becomes an eigenvector of A. In the following, according to Kan and Iri (1982), the construction of V for a diagonalizable A in Lemma 2.7 is generalized to a nondiagonalizable A.

Moreover, we show that if matrices A and B are commutative: $AB = BA$, then $V^{-1}BV$ becomes upper triangular block diagonal with its diagonal elements being the eigenvalues of B. As a result, both A and B are simultaneously transformed into upper triangular block diagonal forms using the same V in common, which is constructively determined only by A.

When such construction of V is applied to the multiplication tables $A_{x_1}, \ldots, A_{x_s}$, multiple zeros are minutely described. We consider our result to be identical with the method shown in Möller and Stetter (1995), which does not necessarily seem constructive. Nevertheless, our elementary formulation is somewhat simpler and more algorithmic, because the notion of dual basis is not required explicitly.

First, we consider one eigenvalue of a companion matrix C.

Lemma A.1. (From Companion Matrix into Jordan Block) *Let C be an $n \times n$ companion matrix* (2.2) *and λ be its p-fold eigenvalue. We define an $n \times p$ matrix U with each element as follows:*

$$u_{ij} := \begin{cases} 0 & (i < j) \\ 1 & (i = j) \\ \binom{i-1}{j-1} \lambda^{i-j} & (i > j) \end{cases} .$$

Then, we have $CU = UJ(\lambda, p)$, where $J(\lambda, p)$ is the $p \times p$ Jordan block with the eigenvalue λ

Proof. From the definition, the elements of U are given as follows:

$$\begin{aligned} U &= [\boldsymbol{u}_1, \boldsymbol{u}_2, \boldsymbol{u}_3, \ldots, \boldsymbol{u}_p] \\ &= \begin{bmatrix} 1 & & & & \\ \lambda & 1 & & & O \\ \lambda^2 & \binom{2}{1}\lambda & 1 & & \\ \lambda^3 & \binom{3}{1}\lambda^2 & \binom{3}{2}\lambda & \ddots & \\ \vdots & \vdots & \vdots & \ddots & 1 \\ \vdots & \vdots & \vdots & & \binom{p}{p-1}\lambda \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ \lambda^{n-1} & \binom{n-1}{1}\lambda^{n-2} & \binom{n-1}{2}\lambda^{n-3} & \cdots & \binom{n-1}{p-1}\lambda^{n-p} \end{bmatrix} . \end{aligned}$$

We show the relation $CU = UJ(\lambda, p)$ by computing with each column vector $\boldsymbol{u}_j$ in U. Note that the characteristic polynomial of C is $\varphi(x) = x^n - c_{n-1}x^{n-1} - \cdots - c_1 x - c_0$, and, since λ is a p-fold root of $\varphi(x)$, it satisfies $\varphi(\lambda) = \varphi'(\lambda) = \cdots = \varphi^{(p-1)}(\lambda) = 0$. (Differential conditions are implicitly used here.)

(Step 1) We have $C\boldsymbol{u}_1 = \lambda \boldsymbol{u}_1$ from Lemma 2.3.

(Step 2) For $\boldsymbol{u}_{j+1}$ $(j = 1, \ldots, p-1)$, we show $C\boldsymbol{u}_{j+1} = \lambda \boldsymbol{u}_{j+1} + \boldsymbol{u}_j$.

The left hand side is

$$C\boldsymbol{u}_{j+1} = C\begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 1 \\ \binom{j+1}{j}\lambda \\ \binom{j+2}{j}\lambda^2 \\ \vdots \\ \binom{n-1}{j}\lambda^{n-j-1} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \binom{j+1}{j}\lambda \\ \binom{j+2}{j}\lambda^2 \\ \vdots \\ \binom{n-1}{j}\lambda^{n-j-1} \\ \triangle \end{bmatrix}\begin{matrix} \uparrow \\ {\scriptstyle j-1} \\ \downarrow \\ \\ \\ \\ \\ \\ \\ \end{matrix},$$

where the nth element is

$$\triangle = c_j + \binom{j+1}{j} c_{j+1}\lambda + \binom{j+2}{j} c_{j+2}\lambda^2 + \cdots + \binom{n-1}{j} c_{n-1}\lambda^{n-j-1}.$$

Since the jth differential of the polynomial $\varphi(x)$ is given by

$$\varphi^{(j)}(x) = \frac{n!}{(n-j)!}x^{n-j} - \frac{(n-1)!}{(n-1-j)!}c_{n-1}x^{n-j-1} - \cdots - (j+1)!c_{j+1}x - j!c_j,$$

we have $\varphi^{(j)}(x)/j! = \binom{n}{j} x^{n-j} - \binom{n-1}{j} c_{n-1}x^{n-j-1} - \cdots - \binom{j+1}{j} c_{j+1}x - c_j$. Hence, we obtain $\triangle = \binom{n}{j}\lambda^{n-j}$, because λ satisfies $\varphi^{(j)}(\lambda) = 0$.

On the other hand, we have

$$\lambda\boldsymbol{u}_{j+1} + \boldsymbol{u}j = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ \binom{j}{j}\lambda \\ \binom{j+1}{j}\lambda^2 \\ \vdots \\ \binom{n-1}{j}\lambda^{n-j} \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \binom{j}{j-1}\lambda \\ \binom{j+1}{j-1}\lambda^2 \\ \vdots \\ \binom{n-1}{j-1}\lambda^{n-j} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ \binom{j+1}{j}\lambda \\ \binom{j+2}{j}\lambda^2 \\ \vdots \\ \binom{n}{j}\lambda^{n-j} \end{bmatrix},$$

because $\binom{k-1}{l} + \binom{k-1}{l-1} = \binom{k}{l}$ holds.

Therefore, we obtain $C\boldsymbol{u}_{j+1} = \lambda\boldsymbol{u}_{j+1} + \boldsymbol{u}_j$ $(j = 1, \ldots, p-1)$.

(Step 3) For the whole U, we have

$$\begin{aligned} CU &= [\lambda\boldsymbol{u}_1, \lambda\boldsymbol{u}_2 + \boldsymbol{u}_1, \lambda\boldsymbol{u}_3 + \boldsymbol{u}_2, \ldots, \lambda\boldsymbol{u}_p + \boldsymbol{u}_{p-1}] \\ &= [\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_p]\begin{bmatrix} \lambda & 1 & & \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} = UJ(\lambda, p), \end{aligned}$$

which completes the proof. $\square$

Next, we consider a nonderogatory matrix A in general.

Lemma A.2. (From Nonderogatory Matrix into Jordan Block) *Let A be an $n \times n$ nonderogatory matrix, and C be its Frobenius normal form. We define an $n \times p$ matrix U for C as Lemma A.1. If we put an $n \times p$ matrix $V := SU$, where S is a transformation matrix such that $C = S^{-1}AS$, then we have $AV = VJ(\lambda, p)$.*

Proof. Since we have $AS = SC$ and $CU = UJ(\lambda, p)$, we obtain

$AV = A(SU) = (AS)U = (SC)U = S(CU) = S(UJ(\lambda, p)) = (SU)J(\lambda, p) = VJ(\lambda, p)$. □

From the process of the proof of Lemma A.1, we have

$$C\boldsymbol{u}_1 = \lambda \boldsymbol{u}_1, \quad C\boldsymbol{u}_j = \lambda \boldsymbol{u}_j + \boldsymbol{u}_{j-1} \ (j = 2, \ldots, p).$$

The definition of V means that $\boldsymbol{v}_j = S\boldsymbol{u}_j$ $(j = 1, \ldots, p)$, where $V = [\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_p]$. Then, we get

$$\begin{aligned} A\boldsymbol{v}_1 &= AS\boldsymbol{u}_1 = SC\boldsymbol{u}_1 = S(\lambda \boldsymbol{u}_1) = \lambda S\boldsymbol{u}_1 = \lambda \boldsymbol{v}_1 \\ A\boldsymbol{v}_j &= AS\boldsymbol{u}_j = SC\boldsymbol{u}_j = S(\lambda \boldsymbol{u}_j + \boldsymbol{u}_{j-1}) = \lambda S\boldsymbol{u}_j + S\boldsymbol{u}_{j-1} \\ &= \lambda \boldsymbol{v}_j + \boldsymbol{v}_{j-1} \quad (j = 2, \ldots, p). \end{aligned}$$

Thus, we have

$$(A - \lambda E)\boldsymbol{v}_1 = \mathbf{0}, \quad (A - \lambda E)\boldsymbol{v}_j = \boldsymbol{v}_{j-1} \ (j = 2, \ldots, p), \tag{A.1}$$

which means that $\boldsymbol{v}_j$'s are generalized eigenvectors of A with the eigenvalue λ.

Next, we consider a matrix B that is commutative with A.

Lemma A.3. (Transformation of Commutative Matrices) *Let V be the $n \times p$ matrix that is determined for A in Lemma A.2. Moreover, we assume that $AB = BA$. Then, we have a $p \times p$ upper triangular matrix $M(\mu, p)$ such that*

$$BV = VM(\mu, p), \quad M(\mu, p) = \begin{bmatrix} \mu & \star & \cdots & \star \\ & \mu & \ddots & \vdots \\ & & \ddots & \star \\ & & & \mu \end{bmatrix},$$

where μ is a p-fold eigenvalue of B and $\star$'s are not necessarily zero.

Proof. We note that the eigenspace of A belonging to λ has dimension 1 from the nonderogatoriness of A. Therefore, the eigenspace is given by $\mathrm{span}(\boldsymbol{v}_1)$.

(Step 1) For the first column vector $\boldsymbol{v}_1$, we have $A(B\boldsymbol{v}_1) = BA\boldsymbol{v}_1 = B\lambda\boldsymbol{v}_1 = \lambda(B\boldsymbol{v}_1)$. This means $B\boldsymbol{v}_1 \in \mathrm{span}(\boldsymbol{v}_1)$, hence, $B\boldsymbol{v}_1 = \mu\boldsymbol{v}_1$ for a certain scalar μ. Therefore, μ is an eigenvalue of B similarly to Lemma 2.8.

(Step 2) For the second column vector $\boldsymbol{v}_2$, we have

$$A(B\boldsymbol{v}_2) = B(A\boldsymbol{v}_2) = B(\lambda\boldsymbol{v}_2 + \boldsymbol{v}_1) = \lambda(B\boldsymbol{v}_2) + B\boldsymbol{v}_1 = \lambda(B\boldsymbol{v}_2) + \mu\boldsymbol{v}_1.$$

Hence, we have $(A - \lambda E)(B\boldsymbol{v}_2) = \mu\boldsymbol{v}_1$. On the other hand, from the relations (A.1), we have $(A-\lambda E)(\mu \boldsymbol{v}_2) = \mu \boldsymbol{v}_1$. Eliminating the right hand sides by subtraction, we obtain

$$(A-\lambda E)(B\boldsymbol{v}_2 - \mu \boldsymbol{v}_2) = \mathbf{0}.$$

This means $B\boldsymbol{v}_2 - \mu\boldsymbol{v}_2 \in \mathrm{span}(\boldsymbol{v}_1)$, hence, $B\boldsymbol{v}_2 - \mu\boldsymbol{v}_2 = s_2\boldsymbol{v}_1$ for a certain scalar s_2. Therefore, we obtain $B\boldsymbol{v}_2 = \mu\boldsymbol{v}_2 + s_2\boldsymbol{v}_1$.

(Step 3) To show the jth case, we assume the $(j-1)$th case for an induction hypothesis:

$$B\boldsymbol{v}_{j-1} = \mu\boldsymbol{v}_{j-1} + s_2\boldsymbol{v}_{j-2} + s_3\boldsymbol{v}_{j-3} + \cdots + s_{j-2}\boldsymbol{v}_2 + s_{j-1}\boldsymbol{v}_1.$$

For the jth column vector $\boldsymbol{v}_j$, we have

$$A(B\boldsymbol{v}_j) = B(A\boldsymbol{v}_j) = B(\lambda\boldsymbol{v}_j + \boldsymbol{v}_{j-1}) = \lambda(B\boldsymbol{v}_j) + B\boldsymbol{v}_{j-1}.$$

Hence, we have $(A-\lambda E)(B\boldsymbol{v}_j) = \mu\boldsymbol{v}_{j-1} + s_2\boldsymbol{v}_{j-2} + \cdots + s_{j-1}\boldsymbol{v}_1$. On the other hand, from the relations (A.1) we have

$$\begin{aligned} (A-\lambda E)(\mu\boldsymbol{v}_j) &= \mu\boldsymbol{v}_{j-1}, \qquad & (A-\lambda E)(s_2\boldsymbol{v}_{j-1}) &= s_2\boldsymbol{v}_{j-2}, \\ \ldots\ldots, & & (A-\lambda E)(s_{j-1}\boldsymbol{v}_2) &= s_{j-1}\boldsymbol{v}_1. \end{aligned}$$

Eliminating the right hand sides by subtraction, we obtain

$$(A-\lambda E)\left\{B\boldsymbol{v}_j - \mu\boldsymbol{v}_j - s_2\boldsymbol{v}_{j-1} - \cdots - s_{j-1}\boldsymbol{v}_2\right\} = \mathbf{0}.$$

This means the vector $\{\cdots\} \in \mathrm{span}(\boldsymbol{v}_1)$, hence,

$$B\boldsymbol{v}_j - \mu\boldsymbol{v}_j - s_2\boldsymbol{v}_{j-1} - \cdots - s_{j-1}\boldsymbol{v}_2 = s_j\boldsymbol{v}_1$$

for a certain scalar s_j. Therefore, we obtain

$$B\boldsymbol{v}_j = \mu\boldsymbol{v}_j + s_2\boldsymbol{v}_{j-1} + s_3\boldsymbol{v}_{j-2} + \cdots + s_{j-1}\boldsymbol{v}_2 + s_j\boldsymbol{v}_1,$$

which holds for every j $(j = 2, \ldots, p)$ by mathematical induction.

(Step 4) For the whole V, we have

$$\begin{aligned} BV &= [\mu\boldsymbol{v}_1, \mu\boldsymbol{v}_2 + s_2\boldsymbol{v}_1, \mu\boldsymbol{v}_3 + s_2\boldsymbol{v}_2 + s_3\boldsymbol{v}_3, \ldots, \mu\boldsymbol{v}_p + s_2\boldsymbol{v}_{p-1} + \cdots + s_{p-1}\boldsymbol{v}_2 + s_p\boldsymbol{v}_1] \\ &= [\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3, \ldots, \boldsymbol{v}_p] \begin{bmatrix} \mu & s_2 & s_3 & \cdots & s_p \\ & \mu & s_2 & \ddots & \vdots \\ & & \mu & \ddots & s_3 \\ & & & \ddots & s_2 \\ & & & & \mu \end{bmatrix}, \end{aligned}$$

which completes the proof. □

Now we consider an $n \times n$ companion matrix C and its every eigenvalue λ_i with the algebraic multiplicity p_i $(i = 1, \ldots, r)$. We put $n \times p_i$ matrices $U_1, U_2, \ldots, U_r$ according to Lemma A.1, and put $U := [U_1, U_2, \ldots, U_r]$. Since $\sum_{i=1}^{r} p_i = n$, the matrix U is $n \times n$ square.

Lemma A.4. (From Companion Matrix into Jordan Normal Form) *If we construct the above-mentioned matrix U for a given companion matrix C, then we have*

$$CU = UJ, \quad J = J(\lambda_1, p_1) \oplus \cdots \oplus J(\lambda_r, p_r),$$

where J is an $n \times n$ block diagonal Jordan normal form.

PROOF. Since $CU_i = U_i J(\lambda_i, p_i)$ $(i = 1, \ldots, r)$ from Lemma A.1, we have

$$C[U_1, \ldots, U_r] = [U_1, \ldots, U_r] \begin{bmatrix} J(\lambda_1, p_1) & & \\ & \ddots & \\ & & J(\lambda_r, p_r) \end{bmatrix}. \square$$

Next, we consider a nonderogatory matrix A whose Frobenius normal form is C. The Jordan normal form J of C in the above lemma is also the Jordan normal form of A.

LEMMA A.5. (FROM NONDEROGATORY MATRIX INTO JORDAN NORMAL FORM) *Let A be an $n \times n$ nonderogatory matrix, and C be its Frobenius normal form. We define an $n \times n$ matrix U for C as Lemma A.4. If we put an $n \times n$ matrix $V := SU$, where S is a transformation matrix such that $C = S^{-1}AS$, then we have $AV = VJ$.*

PROOF. Since we have $AS = SC$ and $CU = UJ$, we obtain

$$AV = A(SU) = (AS)U = (SC)U = S(CU) = S(UJ) = (SU)J = VJ. \square$$

Finally, we consider a matrix B that is commutative with A.

THEOREM A.1. (TRANSFORMATION OF COMMUTATIVE MATRICES) *Let V be the $n \times n$ matrix that is determined for A in Lemma A.5. Moreover, we assume that $AB = BA$. Then, we have an $n \times n$ upper triangular matrix M such that*

$$BV = VM, \quad M = M(\mu_1, p_1) \oplus \cdots \oplus M(\mu_r, p_r),$$

where μ_i's are the eigenvalues of B, although they are not necessarily distinct.

PROOF. If we let $V = [V_1, \ldots, V_r] = [SU_1, \ldots, SU_r]$, then we have $BV_i = V_i M(\mu_i, p_i)$ $(i = 1, \ldots, r)$ from Lemma A.3. The conclusion follows immediately. $\square$

The regularity of V is confirmed by the construction rule of U_i (Kan and Iri, 1982). Therefore, we have shown that both $V^{-1}AV$ and $V^{-1}BV$ are upper triangular block diagonal, where the diagonal elements are the eigenvalues of A and B respectively.

REMARK A.1. The proof of Lemma A.3 is based on the assumption that the geometric multiplicity of λ is 1. Hence, we cannot omit the nonderogatoriness of A. The construction of V can be generalized to a derogatory A, but $V^{-1}BV$ will not be always upper triangular block diagonal in such a case.

EXAMPLE A.1. (EXAMPLE 1 IN MÖLLER AND STETTER (1995)) We consider the following system:

$$\begin{cases} p_1 = x^2 + 2y^2 - 2y \\ p_2 = xy^2 - xy \\ p_3 = y^3 - 2y^2 + y. \end{cases}$$

We compute with *gradlex* order $(x > y)$, and we obtain the normal set basis $B =$

$\{1, y, x, y^2, xy\}$. Then, the multiplication tables are computed as follows:

$$A_x = \begin{bmatrix} 0 & & 1 & & \\ & & & & 1 \\ \vdots & 2 & & -2 & \\ & & & & 1 \\ 0 & 2 & & -2 & \end{bmatrix}, \quad A_y = \begin{bmatrix} 0 & 1 & 0 & & \\ & & & 1 & \\ \vdots & & \vdots & & 1 \\ & -1 & & 2 & \\ 0 & & 0 & & 1 \end{bmatrix}.$$

The Frobenius normal form of each matrix consists of two companion blocks:

$$F_x = \left[\begin{array}{ccc|cc} & 1 & & & \\ & & 1 & & \\ 0 & 0 & 0 & & \\ \hline & & & & 1 \\ & & & 0 & 0 \end{array}\right], \quad F_y = \left[\begin{array}{ccc|cc} & 1 & & & \\ & & 1 & & \\ 0 & -1 & 2 & & \\ \hline & & & & 1 \\ & & & 0 & 1 \end{array}\right],$$

whose minimal polynomial is respectively

$$\phi_x(\lambda) = \lambda^3, \quad \phi_y(\lambda) = \lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda - 1)^2.$$

Hence, both A_x and A_y are derogatory and nondiagonalizable. Then, we try putting $A_w := A_x - A_y$ and computing its Frobenius normal form. As a result, we obtain

$$F_w = \begin{bmatrix} & 1 & & & \\ & & 1 & & \\ & & & 1 & \\ & & & & 1 \\ 0 & 0 & -1 & -3 & -3 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & & & & \\ & & \frac{11}{2} & 7 & \frac{5}{2} \\ & 1 & \frac{11}{2} & 7 & \frac{5}{2} \\ & & 5 & 6 & 2 \\ & & \frac{5}{2} & 4 & \frac{3}{2} \end{bmatrix},$$

where $A_w S = S F_w$. We find A_w nonderogatory and its minimal polynomial to be

$$\phi_w(\lambda) = \lambda^5 + 3\lambda^4 + 3\lambda^3 + \lambda^2 = \lambda^2(\lambda + 1)^3,$$

and we have the eigenvalues $\lambda_1 = 0$ (2-fold) and $\lambda_2 = -1$ (3-fold). Applying Algorithm 3.2, we obtain the following zeros:

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{(2-fold)}, \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix} \text{(3-fold)}.$$

To get the locations and the multiplicities of zeros, it is sufficient to finish the computation here. However, we proceed to give the detailed description of the zeros. According to Lemmas A.1 and A.4, we construct the matrix $U = [U_1, U_2]$ corresponding to $\lambda_1 = 0$, $\lambda_2 = -1$:

$$U = \left[\begin{array}{cc|ccc} 1 & & 1 & & \\ & 1 & -1 & 1 & \\ & & 1 & -2 & 1 \\ & & -1 & 3 & -3 \\ & & 1 & -4 & 6 \end{array}\right].$$

Then, we have the following relation between the Frobenius normal form F_w and the Jordan normal form J_w of A_w:

$$F_w U = U J_w, \quad J_w = \left[\begin{array}{cc|ccc} 0 & 1 & & & \\ & 0 & & & \\ \hline & & -1 & 1 & \\ & & & -1 & 1 \\ & & & & -1 \end{array}\right].$$

According to Lemma A.5, if we put

$$V := SU = \left[\begin{array}{cc|ccc} 1 & & 1 & & \\ & & 1 & & -\frac{1}{2} \\ & 1 & & 1 & -\frac{1}{2} \\ & & 1 & & -1 \\ & & & 1 & -\frac{1}{2} \end{array}\right],$$

then we have $A_w V = V J_w$, which shows the transformation of A_w into its Jordan normal form J_w. As a conclusion, we obtain the following relations using the same V in common:

$$A_x V = V \left[\begin{array}{cc|ccc} 0 & 1 & & & \\ & 0 & & & \\ \hline & & 0 & 1 & -\frac{1}{2} \\ & & & 0 & 1 \\ & & & & 0 \end{array}\right], \quad A_y V = V \left[\begin{array}{cc|ccc} 0 & 0 & & & \\ & 0 & & & \\ \hline & & 1 & 0 & -\frac{1}{2} \\ & & & 1 & 0 \\ & & & & 1 \end{array}\right].$$

Thus, both A_x and A_y are simultaneously transformed into upper triangular block diagonal forms as described in Theorem A.1.

Remark A.2. This result is similar to Möller and Stetter (1995), where differential conditions $\partial_x p(0,0)$, $\partial_x p(0,1)$, $\partial_{x^2} p(0,1) - (1/2)\partial_y p(0,1)$ are used for constructing a dual basis. But we could not understand whether they got them, especially for the last one, algorithmically or heuristically.

To the contrary, we assumed that only the multiplication tables are given. In our formulation, differential conditions are deduced from the Frobenius normal form of a nonderogatory matrix, and they are implicitly used for constructing U_i in Lemma A.1.