Parametric Robust Stability
by
Quantifier Elimination

Dissertation

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Eidesstattliche Erklärung

Ich versichere, daß ich diese Dissertation selbständig verfaßt habe, andere als die angegebenen Quellen und Hilfsmittel nicht benutzt habe, und mich auch sonst keiner unerlaubten Hilfe bedient habe. Weiters versichere ich, daß ich diese Dissertation weder im In- noch im Ausland in irgendeiner Form als Prüfungsarbeit vorgelegt habe.

Andreas Neubacher
Linz, 5. Oktober 1997

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My work also depended various freely available software systems; the ones that I used most heavily were Linux, GNU C++, the STURM C++ library for symbolic computation [24, 29], the symbolic computation system MuPAD [18], and the text processing systems TeX [36] and LaTeX [37].

Abstract

Stability analysis is a central aspect of the theory of differential equations, especially for parameterized problems; one application area where parametric stability plays an important rôle is control theory. The main goals of this thesis are to apply a quantifier elimination framework consistently to various stability problems and to develop specialized methods that solve these problems either symbolically or numerically with mathematically guaranteed exactness more efficiently than general quantifier elimination methods.

As a first step, known parametric robust stability problems and known methods for solving parameter-free problems are formulated in a general quantifier elimination framework. This approach extends work done by other authors on applying quantifier elimination theory to robust control and parametric design problems.

A known stability test based on positiveness of the Hurwitz determinant is generalized and new proofs are given for this test as well as for Kharitonov's theorem.

In the case where parameters appear linearly and independently, a new method for exactly computing the stability margin is presented.

In the case where parameters appear nonlinearly and depedently, new methods for computing a lower bound on the stability margin and for exactly computing the stability margin are developed.

Finally, a framework and a general method for approximate quantifier elimination are presented.

Zusammenfassung

Stabilitätsanalyse ist ein zentraler Aspekt der Differentialgleichungstheorie, insbesondere bei parametrisierten Problemen; ein Anwendungsgebiet in welchem parametrische Stabilität eine wichtige Rolle spielt ist die Steuer- und Regelungstechnik. Das Hauptziel dieser Arbeit ist die konsistente Anwendung der Theorie der Quantorenelimination auf verschiedenen Stabilitätsprobleme und die Entwicklung spezieller Methoden, welche diese Probleme symbolisch oder numerisch mit mathematisch garantierter Genauigkeit effizienter lösen als allgemeine Methoden zur Quantorenelimination.

In einem ersten Schritt werden bekannte parametrische Probleme der robusten Stabilität und bekannte Methoden zur Lösung parameterfreier Probleme in einer allgemeinen Form zur Quantorenelimination formuliert. Dieser Ansatz erweitert die Arbeit anderer Autoren in der Anwendung der Theorie der Quantorenelimination auf Probleme der robusten Steuerung und auf parametrische Designprobleme.

Ein bekannter Stabilitätstest basierend auf der Positivheit der Hurwitz Determinante wird verallgemeinert und neue Beweise werden sowohl für diesen Test als auch für den Satz von Kharitonov gegeben.

Für den Fall des Auftretens linearer, unabhängiger Parameter wird eine neue Methode zur exakten Berechnung des Stabilitätslimits vorgestellt.

Für den Fall des Auftretens nichtlinearer, abhängiger Parameter werden neue Methoden zur Berechnung einer unteren Grenze für das Stabilitätslimit und zur exakten Berechnung des Stabilitätslimits entwickelt.

Zuletzt wird eine formale Struktur und eine allgemeine Methode zur näherungsweisen Quantorenelimination vorgestellt.

To B., for bearing with me . . .

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Notation

Standard Constants and Functions

Symbol	Meaning
$\mathbb{N}$, $\mathbb{N}_0$	the set of natural numbers, the set of natural numbers including zero
$\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{R}^-$	the set of real numbers, the set of positive real numbers, the set of negative real numbers
$\mathbb{C}$	the set of complex numbers
j	the imaginary unit $\sqrt{-1}$
$\mathrm{Re}\,(z)\,, \mathrm{Im}\,(z)$	the real and imaginary parts of z
$\lvert z\rvert$	the absolute value of an imaginary number z, i.e. $\sqrt{\mathrm{Re}\,(z)^2 + \mathrm{Im}\,(z)^2}$
$\lVert v\rVert_2$	the l_2 norm of a vector $v = (v_1, \ldots, v_n)$, i.e. $\sqrt{\sum_{i=1}^n v_i^2}$
$[\underline{A}; \overline{A}]$, $]\underline{A}; \overline{A}[$	a real closed and a real open interval A, i.e. the sets $\{a \in \mathbb{R} \mid \underline{A} \leq a \leq \overline{A}\}$ and $\{a \in \mathbb{R} \mid \underline{A} < a < \overline{A}\}$
$\lvert A\rvert$	the width of an interval A, i.e. $\left\lvert \overline{A} - \underline{A} \right\rvert$
$\underline{x}$, $\overline{x}$	the lower and upper bounds of an interval, parameter, or function x
∂F	the boundary of a set F
$\frac{d}{dx}f(x)$	the derivative of f with respect to x
$\left[\frac{d}{dx}f(x)\right]_{x=\tilde{x}}$	the value of $\frac{d}{dx}f(x)$ at $\tilde{x}$
∇f	the gradient of $f(x_1, \ldots, x_n)$, i.e. the vector $(\frac{d}{dx_1}f, \ldots, \frac{d}{dx_n}f)$

Variables

Symbol	Meaning	Defined
f	polynomial whose stability is under consideration, $f(z, p_1, \ldots, p_m, q_1, \ldots, q_n) = a_k(p_1, \ldots, p_m, q_1, \ldots, q_n)z^k + \ldots + a_0(p_1, \ldots, p_m, q_1, \ldots, q_n)$	Section 2.2.1
k, m, n	the degree of f, the number of design parameters, and the number of robustness parameters	Section 2.2.1
g, h	polynomials, $g(z) = \sum b_i z^i$, $h(z) = \sum c_i z^i$	Section 3.3.3
P, Q	sets of parameters, $P \subseteq \mathbb{R}^m$, $Q \subseteq \mathbb{R}^n$	Definitions 2.2 and 2.3, page 9
p, q	elements of P and Q, $p = (p_1, \ldots, p_m)$, $q = (q_1, \ldots, q_n)$	Definitions 2.2 and 2.3, page 9
z	a complex variable	Section 2.2.1
u	a real or complex variable	Section 3.3.3
ω	a real variable, used for parameterizing the imaginary axis by $j\omega$	Figure 3.1, page 16
H, D, D_i	the Hurwitz matrix, Hurwitz determinant, and i-th Hurwitz minor of f	Definition 3.4, page 23
i, j, l	integer variables (used for indices, etc.)	

Special Functions

Symbol	Meaning	Defined
$f(z, Q)$	the set $\{f(z, q) \mid q \in Q\}$	Section 2.2.1, page 11
k^-, k^+	number of roots of f with negative and non-negative real part	Section 3.1.1, page 16
f_R, f_I	real and imaginary parts of $f(j\omega)$	Section 3.1.1, page 17
$\Delta\arg f(\omega)\|_a^b$	argument change of (complex valued) f as ω goes from a to b	Section 3.1.1, page 16
$\mathrm{I}_a^b f(\omega)$	Cauchy index of f as ω goes from a to b	Definition 3.1, page 17
$\mathrm{res}(g, h)$	resultant of g and h	Section 3.3.3
SSM_f, PSM_f	stability margin and positive stability margin of f	Definitions 2.5 and 2.6, page 11
RS_f, $\mathrm{RS}_{f,Q}$	root space stability measure of f (on Q)	Definitions 2.7 and 2.8, page 13
SR_f	stability radius of f	Definition 6.1, page 59
$\hat{B}_f$	absolute positiveness bound of f	Theorem A.6, page 71
$\check{B}_f$	lower bound for positiveness of f	Theorem A.8, page 72

Additional Notation for Appendix B

Symbol	Meaning	Defined
$\mathbb{F}$	a subset of $\mathbb{R}$	Section B.1.1
$\mathbb{IF}$	the set of intervals over $\mathbb{F}$	Section B.1.1
p, q	points, i.e. elements of $\mathbb{F}^m$	Section B.1.2
I, J	intervals, i.e. elements of $\mathbb{IF}$	
$(B_1, \ldots, B_m)$	an m-dimensional box written as a vector of intervals	Definition B.1, page 76
B, C	boxes	Section B.1.1
$\mathcal{B}$, $\mathcal{C}$	box-sets	Section B.1.1
$\bar{\mathcal{B}}$, $\bar{\mathcal{C}}$	point-sets related to the box-sets	Section B.1.1
$\mathbb{B}$	the set of boolean values $\{\text{TRUE}, \text{FALSE}\}$	Section B.1.2
F, G	formulae or constraints	Section B.1.2
x, y	free or quantified variables in constraints	Section B.1.2
f, g	functions (e.g. from $\mathbb{B}^n$ to $\mathbb{B}$, or from $\mathbb{F}^m$ to $\mathbb{F}$)	Section B.1.2
Y, $\mathcal{Y}$	a box / box-set over which the truth value of a constraint is known to be TRUE	Section B.1.3
N, $\mathcal{N}$	a box / box-set over which the truth value of a constraint is known to be FALSE	Section B.1.3
U, $\mathcal{U}$	a box / box-set over which the truth value of a constraint is unknown (undefined)	Section B.1.3
$\mathcal{B}_y$, $\mathcal{B}_n$, $\mathcal{B}_u$	subsets of a box B / box-set $\mathcal{B}$ over which the truth value of a constraint is TRUE, FALSE, and unknown (undefined), resp.	Section B.1.3
ϵ	the relative error of approximation	Section B.1.3

Chapter 1

Introduction

1.1 Goals

Stability analysis is a central aspect of the theory of differential equations, especially in the presence of parameters. One application area where parametric stability plays an important rôle is control theory; there is quite an extensive body of literature on robust control and parametric design problems. Recently researchers started applying quantifier elimination theory to this kind of problems [16, 27, 31]. Due to the general nature of existing quantifier elimination methods, only problems of relatively low computational complexity (small number of parameters, low degrees, etc.) can be solved in acceptable computing times on currently available hardware.

The main goals of this thesis are to apply the quantifier elimination framework consistently to various stability problems and to develop specialized methods that solve these problems either symbolically or numerically with mathematically guaranteed exactness more efficiently than general quantifier elimination methods.

1.2 Structure

In Chapter 2 we consider the origins of the problem of parametric robust stability and give a formal specification of the problems we will be dealing with in the remaining chapters of this thesis.

In Chapters 3 and 4 we cover well-known methods for solving parameter-free stability problems and show how these methods can be used for simplifying parameterized problems.

In Chapter 5 we treat the case where parameters appear linearly. This case is special because it allows the application of simple methods that cannot be used in the nonlinear case.

In Chapter 6 we present new methods for computing a lower bound on the stability margin and for computing the stability margin exactly.

In Chapter 7 we put the work done in this thesis into context with future work motivated by the results produced so far.

In Appendix A we list known results from various areas that are elementary to the results presented in the main chapters of this thesis.

In Appendix B we present a new iterative branch-and-bound method for solving quantifier elimination problems, most importantly for computing nominal solutions required by stability margin computation problems.

1.3 New Results

New results are scattered throughout the thesis. The following list should serve as a quick reference to them and should allow to clearly separate known and new results.

Section 2.2, Chapter 3: Various known parametric robust stability problems and methods for solving parameter-free problems are formulated in a general quantifier elimination framework. This extends the work done in [27].

Section 3.3.3: A simple new proof of Lemma 3.6, page 26 based on the connection between Hurwitz determinants and resultants is given. The content of this lemma was formulated and proven implicitly in [8].

Section 4.2.3: A new general result on the connection between positiveness of the Hurwitz determinant and the stability margin is presented. Theorem 4.6, page 40 is a generalization to the nonlinear dependent case of an idea from [8].

Section 5.1.1: A new proof of Kharitonov's theorem (first proven in [34, 35]) is given.

Section 5.1.2: A new method for exactly computing the stability margin in the linear independent case is presented. This method is based on an idea from [47].

Chapter 6: New methods for computing a lower bound on the stability margin and for exactly computing the stability margin are developed.

Appendix B: A framework and a general method for approximate quantifier elimination are presented. These results are based on work done in cooperation with Hoon Hong [28].

Chapter 2

The Problem of Parametric Robust Stability

There is an extensive body of literature on solutions to the problems given in Section 2.2, page 8. Two excellent textbooks ([4], [7]) may serve as an up-to-date (at the time of this writing) introduction to and overview of the area of robust stability and state-of-the-art methods for solving stability problems. The overview article [5] and the bibliography [15] may also serve as a starting point.

2.1 Motivation and Background

The problem of parametric robust stability stems from stability theory of differential equations. Given a parameterized family of differential equations resulting in a parameterized solution, we are interested in finding sets of parameter values for which the solution is stable. The exposition in Section 2.1.1 is mostly based on [21] and [19], but any other textbook on stability theory of differential equations should serve as well.

Differential equations are ubiquitous; the main application that we will treat in this thesis are stability problems from control theory. The exposition in Section 2.1.2 of important control theory concepts is mostly based on [40], but any other textbook on control theory covering issues of robust stability should serve as well.

2.1.1 Linear, Time-Invariant Differential Equations

Consider a linear, time-invariant differential equation

$$a_k y^{(k)}(t) + \ldots + a_0 y(t) = b_l u^{(l)}(t) + \ldots + b_0 u(t) \tag{2.1}$$

where a_i, b_i, and $u(t)$ are given, and we are interested in the solution(s) $y(t)$. In a parameter-free problem, the a_i and b_i are numbers, while in a parameterized problem these coefficients are polynomials in the parameters.

This kind of equations are essential in modelling phenomena from a variety of disciplines such as biology, chemistry, economics, engineering, physics, etc.

Equation 2.1 is called **(globally) asymptotically stable** if for all solutions $y(t)$ of the corresponding **homogeneous** equation

$$a_k y^{(k)}(t) + \ldots + a_0 y(t) = 0 \tag{2.2}$$

we have

$$\lim_{t\to\infty} y(t) = 0.$$

A possible solution of equation 2.2 is of the form

$$y(t) = ce^{zt}.$$

Substitution gives the equation

$$(a_k z^k + \ldots + a_0)ce^{zt} = 0$$

and excluding the trivial solution $y \equiv 0$ generated by $c = 0$ we get the condition

$$a_k z^k + \ldots + a_0 = 0. \tag{2.3}$$

From the k solutions z_i of this equation we finally derive the linear combination

$$y(t) = \sum_{i=1}^{k} c_i e^{z_i t},$$

which yields all solutions of equation 2.2. Writing this formula as

$$y(t) = \sum_{i=1}^{k} c_i e^{\mathrm{Re}(z_i)t} \left(\cos(\mathrm{Im}\,(z_i)\,t) + j\sin(\mathrm{Im}\,(z_i)\,t)\right) \tag{2.4}$$

we see that for $y(t)$ to be asymptotically stable, the real parts of all z_i must be negative. This is exactly how stability is defined by Definition 2.1, page 9.

A few remarks for completeness:

- If there are multiple roots of Equation 2.3, e.g. $z_i = \ldots = z_j$ then the corresponding terms in the linear combination are replaced by the single term
$$c_i t^{j-i} e^{z_i t}.$$

- If **starting conditions** fixing $y^{(k)}(0), \ldots, y(0)$ are given, the c_i are uniquely determined.
- It is possible that there are no roots z with $\mathrm{Re}\,(z) > 0$, but there are roots on the imaginary axis (i.e. $\mathrm{Re}\,(z) = 0$ for some z). In general this is treated as a special case in the literature. Since we are only interested in asymptotic stability we will not make such a distinction and only treat the cases $\mathrm{Re}\,(z) < 0$ and $\mathrm{Re}\,(z) \geq 0$.

2.1.2 Control Systems

In control theory an elementary means of studying systems are **transfer functions**. These are rational functions that are derived by applying the Laplace transform to the differential equations describing a system.

Starting from the Laplace transform

$$Y(z) = G(z)U(z)$$

of Equation 2.1 we get a condition for asymptotic stability that is similar to that given in Definition 2.1, page 9. Again we only need to consider the homogeneous case, which implies $U(z) = 0$. If we write $N(z)/D(z)$ for the rational function $G(z)$ we get $D(z)Y(z) = 0$. Excluding the trivial solution $Y \equiv 0$ yields

$$D(z) = 0,$$

which is similar to Equation 2.3. Again asymptotic stability is equivalent to all roots of this equation being in the open left half plane.

In a simplified control systems context, one considers a transfer function $G(z)$ that describes a **plant** (i.e. a system whose behavior should be controlled) and a transfer function $C(z)$ that describes a controller. Figure 2.1 shows a block diagram of a plant with a controller in a (unity) feedback loop.

Such a system is stable if the zeros of $1 + C(z)G(z)$ have negative real parts. With $C(z) = C_N(z)/C_D(z)$ and $G(z) = G_N(z)/G_D(z)$ this condition is equivalent to the roots of $f(z) := C_D(z)G_D(z)+C_N(z)G_N(z)$ having negative real parts, which in turn is how stability is defined by Definition 2.1.

In this context the meaning of robustness and design parameters in general is as follows:

- *Robustness parameters* may appear linearly both in $C(z)$ and $G(z)$ representing uncertainties in the model. If the model incorporates constants that come from measurements, for instance, then errors in the measurements can be robustly handled by replacing the constant by a robustness parameter ranging over an interval that contains the (unknown) exact value of the constant.

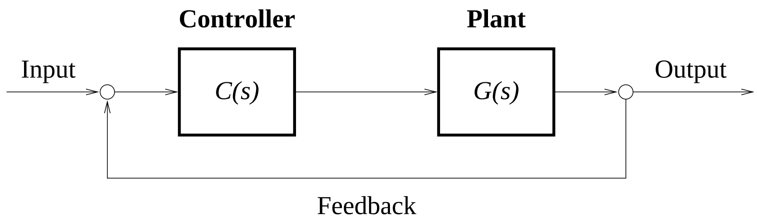

Figure 2.1: Block diagram of plant with controller. *A plant with transfer function $G(z)$ and a controller with transfer function $C(z)$ in a (unity) feedback loop.*

- *Design parameters* appear linearly in $C(z)$ and represent values that the control engineer needs to select for designing a controller that guarantees a certain desired behavior of the system.

Solving the problem of parametric robust stability (Problem 2.4, page 10) means computing a set of values for the design parameters for which the system is guaranteed to be stable. The control engineer may then apply other criteria (such as tracking error, control effort, resonance peak, etc.) to this set to finally select suitable values for the design parameters fulfilling all design goals for the system.

2.2 Specification

The problem of *parametric robust stability* appears in the context of asymptotic stability of linear, time-invariant differential equations. See Section 2.1 for details on the underlying theory, esp. concerning the relationship to control theory, which we chose as the main application area motivating the results presented in this thesis.

The general multivariate, non-linear problems given in Section 2.2.1 are hard quantifier elimination problems. Even if a solution can be computed, the solution set is often too complicated to be usable for practical purposes. For this reason, one may be satisfied with solutions to the related problem of computing *stability margins*. This type of problems is defined in Section 2.2.2.

2.2.1 Parametric Robust Stability

Definition 2.1 (Stability)

Let $f \in \mathbb{R}[z]$. f is called ***(Hurwitz) stable***[1] *iff*

$$\forall z \in \mathbb{C} : [f(z) = 0 \Rightarrow \mathrm{Re}\,(z) < 0]\,.$$

This definition motivates the following decision problem.

Problem 2.1 (Stability)
Let $f \in \mathbb{R}[z]$. Decide whether f is stable.

Here we assume that the polynomial coefficients a_i are real numbers. For real-life problems these coefficients may be polynomials in **robustness parameters** that either represent quantities known only to a finite precision (e.g. values measured in experiments) or represent possible changes in the system (e.g. the changing mass of a vehicle).

Definition 2.2 (Robust stability)
Let $Q \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[z, q_1, \ldots, q_n]$. f is called ***robustly stable with respect to** Q iff*

$$\forall q \in Q : \forall z \in \mathbb{C} : [f(z,q) = 0 \Rightarrow \mathrm{Re}\,(z) < 0]\,,$$

or in other words if and only if $f(z,q)$ is stable for all $q \in Q$.

This definition also leads to an associated decision problem.

Problem 2.2 (Robust stability)
Let $Q \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[z, q_1, \ldots, q_n]$. Decide whether f is robustly stable with respect to Q.

It is also possible that the a_i depend on **design parameters** that cannot be specified a priory. In this case the question is whether there exist values for these parameters such that the resulting polynomial is stable.

Definition 2.3 (Parametric stability)
Let $f \in \mathbb{R}[z, p_1, \ldots, p_m]$. f is called ***stable under parameters*** *iff*

$$\exists P \subseteq \mathbb{R}^m : \forall p \in P : \forall z \in \mathbb{C} : [f(z,p) = 0 \Rightarrow \mathrm{Re}\,(z) < 0]\,,$$

or in other words if and only if there exists a $P \subseteq \mathbb{R}^m$ such that $f(z,p)$ is stable for all $p \in P$.

This definition motivates the following computational problem.

[1] The literature distinguishes various types of stability. In this thesis, we exclusively deal with *Hurwitz stability*. Therefore, we will use the term *stability* synomymously for Hurwitz stability.

Problem 2.3 (Parametric stability)
Let $f \in \mathbb{R}[p_1, \ldots, p_m]$. Decide whether f is stable under parameters and if it is, find the maximal set of parameters $P \subseteq \mathbb{R}^m$ for which f is stable.

By combining Definitions 2.2 and 2.3 we derive the following notion.

Definition 2.4 (Parametric robust stability)
*Let $Q \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[z, p_1, \ldots, p_m, q_1, \ldots, q_n]$. f is called **robustly stable with respect to Q under parameters** iff*

$$\exists P \subseteq \mathbb{R}^m : \forall p \in P : \forall q \in Q : \forall z \in \mathbb{C} : [f(z, p, q) = 0 \Rightarrow \mathrm{Re}(z) < 0],$$

or in other words if and only if there exists a $P \subseteq \mathbb{R}^m$ such that $f(z, p, q)$ is stable for all $p \in P$ and all $q \in Q$.

The associated computational problem is as follows.

Problem 2.4 (Parametric robust stability)
Let $Q \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[z, p_1, \ldots, p_m, q_1, \ldots, q_n]$. Decide whether f is robustly stable with respect to Q under parameters and if it is, find the maximal set of parameters $P \subseteq \mathbb{R}^m$ for which f is robustly stable with respect to Q.

Remarks

- In most cases we are not interested in parameter sets P and Q that are of a "complicated" structure. For applications it is often necessary to compute elements of these sets or to test whether a given parameter is an element of these sets. To simplify these tasks we will restrict most results concerning computations to **boxes**, i.e. sets $P = \{p \in \mathbb{R}^m \,|\, p_i \in [\underline{p_i}; \overline{p_i}]$ with $\underline{p_i} \leq \overline{p_i}$ for $i = 1, \ldots, m\}$ and $Q = \{q \in \mathbb{R}^n \,|\, q_i \in [\underline{q_i}; \overline{q_i}]$ with $\underline{q_i} \leq \overline{q_i}$ for $i = 1, \ldots, n\}$.

- To simplify results for parameterized functions, we will assume that $f(z, p, q)$ is always **of constant degree** with respect to P and Q, i.e. that there always exists a $k \in \mathbb{N}$ such that $\deg f(z, p, q) = k$ for all $p \in P, q \in Q$.

Notation:

- We will call $p \in P$ *design parameter(s)*, "$\forall p \in P$" *design quantifier(s)*, $q \in Q$ *robustness parameter(s)*, and "$\forall q \in Q$" *robustness quantifier(s)*. If we do not want to distinguish between design and robustness parameters, we will only use q and Q instead of p, P, q, and Q.

- Given some function f we will use the phrases "*stabilizing set* Q" and "*stabilizing parameter* p" when it is clear from the context that we mean a set Q such that $f(z,q)$ is stable for all $q \in Q$, and that we mean a parameter p for which $f(z,p)$ is stable. We will use the phrases "*nonstabilizing set* Q" and "*nonstabilizing parameter* p" when it is clear from the context that we mean a set Q for which there exists a $q \in Q$ such that $f(z,q)$ is instable, and that we mean a parameter p for which $f(z,p)$ is instable.

- We will denote the set $\{f(z,q) | \, q \in Q\}$ by $f(z,Q)$ and say "$f(z,Q)$ is stable" meaning that $f(z,q)$ is stable for all $q \in Q$. We will say "$f(z,Q)$ is instable" meaning that there exists a $q \in Q$ such that $f(z,q)$ is instable.

□

2.2.2 Stability Margins

In the literature there are various definitions for the notion of stability margin. The basic idea is to start from a so-called **nominal point** $\tilde{q}$ in parameter space for which it is known that a polynomial $f(z,\tilde{q})$ is stable. Given such a point, we want to find a value ϵ with

$$0 < \epsilon \leq \sup\{e \in \mathbb{R} | \, \forall q : [\|q - \tilde{q}\| \leq e \Rightarrow f(z,q) \text{ stable}]\}.$$

If equality holds, then ϵ is called a **stability margin** for f with respect to $\tilde{q}$, otherwise ϵ is called a **lower bound on the stability margin**.

The main variations to be found in the literature concern the norm $\|q - \tilde{q}\|$, which may be weighted, l_2, l_∞, etc. The following definition utilizes the maximum norm, assumes that the nominal point is at the origin, and does not explicitly mention weights. The latter two restrictions do not cause any loss of generality because a nominal point $\tilde{q}$ and weights $w_1, \ldots, w_n$ can always be introduced by replacing $f(z, q_1, \ldots, q_n)$ by $f(z, \tilde{q}_1 + w_1 q_1, \ldots, \tilde{q}_n + w_n q_n)$.

Definition 2.5 (Stability margin)
Let $f \in \mathbb{R}[z, q_1, \ldots, q_n]$ such that $f(z,0)$ is stable. Then

$$SSM_f := \sup\{e \in \mathbb{R}^+ | \, \forall q \in [-e; e]^n : \, f(z,q) \textit{ is stable}\}$$

is called ***the (symmetric) stability margin of*** f.

The set of stable polynomials is open because by definition the roots of a stable polynomial must be contained in an open set and the coefficients of a polynomial depend continuously on the roots. Therefore there always exists a non-zero stability margin.

Sometimes we will use the following notion of **positive stability margins**. It is weaker than the usual notion but will also prove to be more flexible because it applies only to one quadrant in parameter space.

Definition 2.6 (Positive stability margin)
Let $f \in \mathbb{R}[z, q_1, \ldots, q_n]$ such that $f(z, 0)$ is stable. Then

$$PSM_f := \sup\{e \in \mathbb{R}^+ \mid \forall q \in [0; e]^n : f(z, q) \text{ is stable}\}$$

is called ***the positive (symmetric) stability margin of*** *f.*

Obviously we have $\mathrm{SSM}_f \leq \mathrm{PSM}_f$.

These stability margins provide a partial solution to Problem 2.3 (parametric stability), because f is stable for all $q \in]-\mathrm{SSM}_f; \mathrm{SSM}_f[^n$ and for all $q \in]0; \mathrm{PSM}_f[^n$. It also provides a partial solution to Problem 2.2 (robust stability), because if Q is contained in $]-\mathrm{SSM}_f; \mathrm{SSM}_f[^n$ or in $]0; \mathrm{PSM}_f[^n$, then f is robustly stable with respect to Q; if Q is not contained in this set and Q is a box, then f is not robustly stable with respect to Q. Otherwise the stability margins do not give sufficient information on robust stability. This observation leads to the following recipes:

Recipe 2.1 ((Partial) test for robust stability on a box by SSM_f)
Given *$f \in \mathbb{R}[z, q_1, \ldots, q_n]$ and $Q =]\underline{q_1}; \overline{q_1}[\times \ldots \times]\underline{q_n}; \overline{q_n}[\subset \mathbb{R}^n$;*
decide *whether f is robustly stable with respect to Q.*

1. *Let $\tilde{q} := (\frac{\overline{q_1}+\underline{q_1}}{2}, \ldots, \frac{\overline{q_n}+\underline{q_n}}{2})$. Test whether $f(z, \tilde{q})$ is stable, e.g. by applying the Routh-Hurwitz criterion (Section 3.3). If $f(z, \tilde{q})$ is not stable, f is not robustly stable with respect to Q and we are done; otherwise continue.*

2. *Let $g(z, r_1, \ldots, r_n) := f(z, \tilde{q}_1 + r_1 \frac{\overline{q_1}-\underline{q_1}}{2}, \ldots, \tilde{q}_n + r_n \frac{\overline{q_1}-\underline{q_1}}{2})$.*

3. *Compute SSM_g. If $1 \leq SSM_g$ then g is robustly stable with respect to $]-1; 1[^n$ and therefore f is robustly stable with respect to Q; otherwise f is not robustly stable with respect to Q.*

4. *If SSM_g cannot be computed exactly, compute a bound $B \leq SSM_g$ (see Section 6.2). If $1 \leq B$ then g is robustly stable with respect to $]-1; 1[^n$ and therefore f is robustly stable with respect to Q; otherwise we only*

know that g is robustly stable with respect to $]-B;B[^n$ and therefore that f is robustly stable with respect to $]\tilde{q}_1 - B\frac{\overline{q_1}-\underline{q_1}}{2}; \tilde{q}_1 + B\frac{\overline{q_1}-\underline{q_1}}{2}[\times \ldots \times]\tilde{q}_n - B\frac{\overline{q_n}-\underline{q_n}}{2}; \tilde{q}_n + B\frac{\overline{q_n}-\underline{q_n}}{2}[$.

Recipe 2.2 ((Partial) test for robust stability on a box by PSM$_f$)
Given *$f \in \mathbb{R}[z, q_1, \ldots, q_n]$ and $Q =]\underline{q_1}; \overline{q_1}[\times \ldots \times]\underline{q_n}; \overline{q_n}[\subset \mathbb{R}^n$;*
decide *whether f is robustly stable with respect to Q.*

1. *Let $\underline{q} := (\underline{q_1}, \ldots, \underline{q_n})$. Test whether $f(z, \underline{q})$ is stable, e.g. by applying the Routh-Hurwitz criterion (Section 3.3). If $f(z, \underline{q})$ is not stable, f is not robustly stable with respect to Q and we are done; otherwise continue.*

2. *Let $g(z, r_1, \ldots, r_n) := f(z, \underline{q_1} + r_1(\overline{q_1} - \underline{q_1}), \ldots, \underline{q_n} + r_n(\overline{q_1} - \underline{q_1}))$.*

3. *Compute PSM_g. If $1 \leq PSM_g$ then g is robustly stable with respect to $]0;1[^n$ and therefore f is robustly stable with respect to Q; otherwise f is not robustly stable with respect to Q.*

4. *If PSM_g cannot be computed exactly, compute a bound $B \leq PSM_g$ (see Section 6.2). If $1 \leq B$ then g is robustly stable with respect to $]0;1[^n$ and therefore f is robustly stable with respect to Q; otherwise we only know that g is robustly stable with respect to $]0;B[^n$ and therefore that f is robustly stable with respect to $]\underline{q_1} - B(\overline{q_1} - \underline{q_1}); \underline{q_1} + B(\overline{q_1} - \underline{q_1})[\times \ldots \times [\underline{q_n} - B(\overline{q_n} - \underline{q_n}); \underline{q_n} + B(\overline{q_n} - \underline{q_n})]$.*

2.2.3 A Stability Measure in Root Space

The "quality" of a polynomial's stability depends on the distance of the roots from the imaginary axis: A stable polynomial that has roots close to the imaginary axis can be regarded as being "less stable" than a polynomial all of whose roots have negative real parts "far away" from the imaginary axis. This qualitative measure is motivated by the discussion in Section 2.1.1 (esp. Equation 2.4, page 6): The solution $y(t)$ of the differential equation under consideration approaches zero faster if the real parts of the associated polynomial's roots are more negative. The following notion is motivated by this observation.

Definition 2.7 (Root space stability measure)
Let $f \in \mathbb{R}[z]$ be a polynomial of degree k with roots $z_1, \ldots, z_k$. Then we call

$$RS_f := \max_{1 \leq i \leq k} \mathrm{Re}\,(z_i)$$

*the **root space stability measure of** f.*

Thus by definition stability of f is equivalent to $\mathrm{RS}_f < 0$ and if $\mathrm{RS}_f < \mathrm{RS}_g$ then f can be regarded as being "more stable" than g.

In control theory a measure for the quality of a stable polynomial's stability is the **damping ratio**. It essentially depends on the real and imaginary parts of the roots closest to the imaginary axis (the "dominant poles"). Assuming we can compute RS_f (i.e. the real part of the roots defining the damping ratio), we can compute the imaginary parts as the real roots with respect to ω of $f(j\omega + \mathrm{RS}_f)$. This relationship immediately follows from Lemma 3.7, page 27, which formalizes the observation that $f(z + \mathrm{RS}_f)$ has zeros on the imaginary axis.

We can straightforwardly extend the definition of the root space stability measure to the parameterized case.

Definition 2.8 (Parameterized root space stability measure) *Let $f \in \mathbb{R}[z, q_1, \ldots, q_n]$, $Q \subseteq \mathbb{R}^n$. Then we call*

$$RS_{f,Q} := \sup_{q \in Q} RS_{f(z,q)}$$

the ***root space stability measure of*** *f* ***on*** *Q.*

Thus by definition robust stability of f with respect to Q is equivalent to $\mathrm{RS}_{f,Q} < 0$. In other words (recalling Definition 2.2, page 9 of robust stability)

$$\begin{array}{c} \forall q \in Q : \forall z \in \mathbb{C} : [f(z,q) = 0 \Rightarrow \mathrm{Re}\,(z) < 0] \\ \Leftrightarrow \\ \mathrm{RS}_{f,Q} < 0 \end{array} .$$

This means that we can solve the problem of robust stability (Problem 2.2, page 9) by computing $\mathrm{RS}_{f,Q}$.

Chapter 3

Eliminating the Non-Parametric Universal Quantifier

The methods presented in this chapter were originally invented for dealing with the parameter-free stability problem (Problem 2.1). We will see that they can be also be used for eliminating the quantifier "$\forall z \in \mathbb{C}:$" in parametric robust stability problems (Problems 2.2, 2.3, and 2.4).

This theory introduces several important concepts used in other parts of this thesis, therefore we will cover it in some detail. Unless noted otherwise, the exposition given in this chapter follows that in [19] and [21].

3.1 From the Principle of the Argument to Cauchy Indices

In this section we will see how applying the *principle of the argument* and *Cauchy indices* to stability problems transforms them into equivalent problems that form the basis for Routh's solution described in Section 3.2 and Hurwitz's solution described in Section 3.3.

3.1.1 The Principle of the Argument

Recalling the stability problem

$$\forall z \in \mathbb{C} : [f(z) = 0 \Rightarrow \mathrm{Re}\,(z) < 0]$$

we see that the exact locations of the roots of f are not relevant. What we need to check is whether all roots are in the left open complex halfplane (in

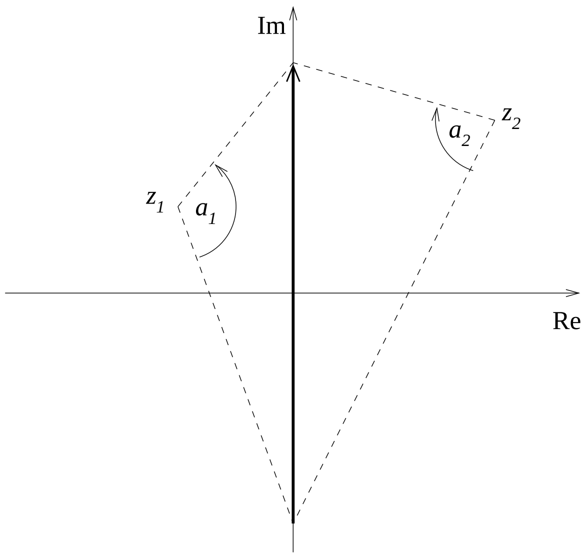

Figure 3.1: Argument change for values in the left and in the right halfplane. *For a value z_1 to the left of the imaginary axis the argument change of $j\omega - z_1$ is $a_1 = \pi$; for a value z_2 to the right of the imaginary axis the argument change of $j\omega - z_2$ is $a_2 = -\pi$.*

which case f is asymptotically stable), or whether there are roots on the imaginary axis or in the right halfplane (in which case f is not stable).

For this purpose one may apply the "principle of the argument": Consider a point $j\omega$ traversing the imaginary axis as shown in Figure 3.1. For a point z_1 to the left of the imaginary axis, the argument of $j\omega - z_1$ changes by $a_1 = \pi$ as ω goes from $-\infty$ to ∞. For a point z_2 to the right of the imaginary axis, the argument changes by $a_2 = -\pi$.

Now let $z_1, \dots, z_k$ be the complex roots of f so that we have $f(z) = a_k \prod_{i=1}^{k}(z - z_i)$ and $\arg(f(z)) = \sum_{i=1}^{k} \arg(z - z_i)$. We denote the number of roots with negative real part by k^- and the number of roots with non-negative real part by k^+. For now we will assume that no roots of f are located on the imaginary axis, i.e. there is no i with $\mathrm{Re}\,(z_i) = 0$; we will treat this case in Section 3.1.2. By $\Delta\arg f(j\omega)|_{-R}^{R}$ we denote the change in the argument of $f(j\omega)$ along the imaginary axis as ω goes from $-R$ to R.

From $\arg(f(z)) = \sum_{i=1}^{k} \arg(z - z_i)$ follows

$$\Delta\arg f(j\omega)|_{-R}^{R} = \sum_{i=1}^{k} \Delta\arg(j\omega - z_i)|_{-R}^{R}.$$

The argument change of a single term is

$$\lim_{R\to\infty} \Delta\arg(j\omega - z_i)|_{-R}^{R} = \begin{cases} \pi & \text{for } \mathrm{Re}\,(z_i) < 0 \\ -\pi & \text{for } \mathrm{Re}\,(z_i) > 0 \end{cases}$$

and thus the total argument change along the imaginary axis is given by

$$\lim_{R\to\infty} \Delta\arg f(j\omega)|_{-R}^{R} = \pi(k^- - k^+) = \pi(k - 2k^+). \tag{3.1}$$

From this we see that if we can compute the argument change we can also compute k^- and k^+.

The function $f(j\omega)$ can be written as

$$f(j\omega) = f_R(\omega) + jf_I(\omega)$$

so that for the argument of f we get

$$\arg(f(j\omega)) = \arctan\frac{f_I(\omega)}{f_R(\omega)}.$$

From the fact that $\Delta\arg f(j\omega))|_{-\infty}^{\infty}$ is a multiple of π and from the form of arctan we see that $\frac{f_I(\omega)}{f_R(\omega)}$ must pass $k - 2k^+$ times through the open interval $]-\infty,\infty[$ as ω goes from $-\infty$ to ∞. So if we know the behavior of $\frac{f_I(\omega)}{f_R(\omega)}$ we also know the number of roots with non-negative real part.

For $f(z) = a_k z^k + \ldots + a_0$ we have

$$\frac{f_I(\omega)}{f_R(\omega)} = \begin{cases} -\frac{a_{k-1}\omega^{k-1} - a_{k-3}\omega^{k-3} + \ldots}{a_k\omega^k - a_{k-2}\omega^{k-2} + \ldots} & \text{for } k \text{ even} \\ \frac{a_k\omega^k - a_{k-2}\omega^{k-2} + \ldots}{a_{k-1}\omega^{k-1} - a_{k-3}\omega^{k-3} + \ldots} & \text{for } k \text{ odd} \end{cases}.$$

We can unify the cases for even and odd k to $\frac{a_{k-1}\omega^{k-1} - a_{k-3}\omega^{k-3} + \ldots}{a_k\omega^k - a_{k-2}\omega^{k-2} + \ldots}$ by applying $\tan(-x) = -\tan(x) = \cot(x + \frac{\pi}{2})$ and the fact that both tan and cot range over $]-\infty,\infty[$ for every interval of length π.

3.1.2 The Cauchy Index

As we have seen by the end of Section 3.1.1, information on the poles of a certain rational function gives us information on the argument change (and therefore on the locations of the roots) of a related polynomial. In this section we will show how these poles can be analyzed.

Definition 3.1 (Cauchy index)
*The **Cauchy index** $\mathrm{I}_a^b F(z)$ of a real-valued rational function $F(z)$ in the range from a to b is defined to be the number of poles in the interval $[a;b]$ where $F(z)$ goes from $-\infty$ to ∞ minus the number of poles where $F(z)$ goes from ∞ to $-\infty$.*

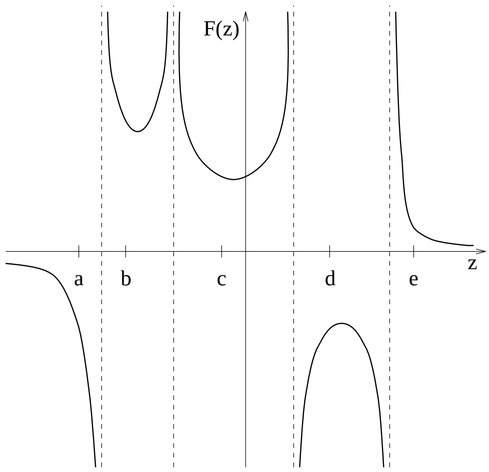

Figure 3.2: Cauchy index example. $\mathrm{I}_a^b F(z) = \mathrm{I}_a^c F(z) = 1$, $\mathrm{I}_c^d F(z) = -1$, $\mathrm{I}_a^d F(z) = 0$, $\mathrm{I}_a^e F(z) = 1$. *Poles where the function does not change its sign are not counted.*

This definition is illustrated in Figure 3.2.

We see that the Cauchy index is defined such that it can be applied to Equation 3.1 by

$$\begin{aligned}\pi(k - 2k^+) &= \Delta\arg f(j\omega))|_{-\infty}^{\infty} \\ &= \pi \mathrm{I}_{-\infty}^{\infty} \frac{a_{k-1}\omega^{k-1} - a_{k-3}\omega^{k-3} + \dots}{a_k\omega^k - a_{k-2}\omega^{k-2} + \dots}.\end{aligned}$$

This means that if we can compute the Cauchy index of such a rational function we can also compute the number of roots with non-negative real parts of the corresponding function f. We will show how the Cauchy index can be computed in Sections 3.2 and 3.3.

3.2 Cauchy Indices by Routh Tables

As we have seen by the end of Section 3.1.2, we can get information on the argument change (and therefore on the locations of the roots) of a polynomial from the Cauchy index of a certain rational function that is derived from the polynomial. In this section we will present Routh's solution ([19] gives [44] as the original reference, whereas [21] refers to [43]) for computing the Cauchy index.

3.2.1 Generalized Sturm Sequences

Definition 3.2 (Generalized Sturm sequence)
Let $f_1, f_2 \in \mathbb{R}[x]$ with $\deg(f_1) > \deg(f_2)$. The ***generalized Sturm sequence*** *of f_1 and f_2 is the sequence of polynomials $f_1, f_2, f_3, \ldots, f_l$ defined by*

$$f_i(x) = f_{i-1}(x)q_i(x) - f_{i-2}(x),$$

where q_i is such that $\deg(f_i) < \deg(f_{i-1})$ and $\deg(f_l) = 0$.

Definition 3.3 (Sign variations)
Let $f_1, f_2, \ldots, f_l \in \mathbb{R}[x]$. By $V(\tilde{x})$ we denote the number of ***sign variations*** *in the sequence of real numbers $f_1(\tilde{x}), f_2(\tilde{x}), \ldots, f_l(\tilde{x})$ ignoring 0.*

For example the number of sign variations in the sequence 3, 8, -1, 2 is 2, the number of sign variations in the sequence -5, -2, 0, 1, 0, 7 is 1.

The following theorem relates the Cauchy index of a rational function to sign variations in the Sturm sequence generated by the denominator and numerator.

Theorem 3.1 (Sturm)
Let $f_1, f_2 \in \mathbb{R}[x]$ with $\deg(f_1) > \deg(f_2)$. If f_1 and f_2 are relatively prime, then

$$V(a) - V(b) = \mathrm{I}_a^b \frac{f_2(x)}{f_1(x)}.$$

By now we have reduced the computation of the number of roots with non-negative real parts of a function f to counting sign variations in Sturm sequences.

For proving the theorem we need a lemma.

Lemma 3.2
Let $f_1, f_2 \in \mathbb{R}[x]$ with $\deg(f_1) > \deg(f_2)$. If f_1 and f_2 are relatively prime then the following holds:

i. $f_l \neq 0$

ii. If $f_i(\tilde{x}) = 0$ *then* $f_{i-1}(\tilde{x}) = -f_{i+1}(\tilde{x}) \neq 0$.

If f_1 and f_2 are not relatively prime then $f_l = 0$.

Proof: Assume that f_1 and f_2 are relatively prime.

i. If $f_l = 0$ then $f_{l-1}(x)q_l(x) = f_{l-2}(x)$, from which follows that f_{l-1} is a common factor of all $f_i(x)$, a contradiction to $f_1(x)$ and $f_2(x)$ being relatively prime.

ii. If $f_i(\tilde{x}) = f_{i-1}(\tilde{x}) = f_{i+1}(\tilde{x}) = 0$ then $f_j(\tilde{x}) = 0$ for $j \geq i-1$, a contradiction to $f_l \neq 0$. $f_{i-1}(\tilde{x}) = -f_{i+1}(\tilde{x})$ follows from the definition of the Sturm sequence.

Now assume that f_1 and f_2 are not relatively prime. Then their common factor is a factor of all f_i with $i \leq l-1$ and f_l must be 0.

□

Proof of Theorem 3.1: $V(x)$ can change only when a polynomial in the Sturm sequence passes through zero. Therefore for some $i \geq 2$ let $\tilde{x} \in [a;b]$ be such that $f_i(\tilde{x}) = 0$ and the signs of $f_i(\tilde{x}-\epsilon)$ and $f_i(\tilde{x}+\epsilon)$ are different. Now we consider the behavior of $f_{i-1}(x)$, $f_i(x)$, and $f_{i+1}(x)$ sufficiently close to $\tilde{x}$. Due to Lemma 3.2 only the following situations are possible:

x	$f_{i-1}(x)$	$f_i(x)$	$f_{i+1}(x)$	$f_{i-1}(x)$	$f_i(x)$	$f_{i+1}(x)$
$\tilde{x}-\epsilon$	-	-	+	+	-	-
$\tilde{x}$	-	0	+	+	0	-
$\tilde{x}+\epsilon$	-	+	+	+	+	-
$\tilde{x}-\epsilon$	-	+	+	+	+	-
$\tilde{x}$	-	0	+	+	0	-
$\tilde{x}+\epsilon$	-	-	+	+	-	-

This shows that $V(x)$ does not change when x passes through a zero of a polynomial with $i \geq 2$. Therefore, $V(x)$ can only change when $f_1(x)$ passes through zero. This gives the following situations:

	$f_1(x)$	$f_2(x)$	$f_1(x)$	$f_2(x)$	$f_1(x)$	$f_2(x)$	$f_1(x)$	$f_2(x)$
$x = \tilde{x}-\epsilon$	-	+	+	+	-	-	+	-
$x = \tilde{x}$	0	+	0	+	0	-	0	-
$x = \tilde{x}+\epsilon$	+	+	-	+	+	-	-	-
$V(\tilde{x}-\epsilon)-$ $V(\tilde{x}+\epsilon)$	1		-1		-1		1	
$\mathrm{I}_{\tilde{x}-\epsilon}^{\tilde{x}+\epsilon}\frac{f_2(x)}{f_1(x)}$	1		-1		-1		1	

Here we see that the difference of the sign changes is the same as the Cauchy index on the interval from $\tilde{x}-\epsilon$ to $\tilde{x}+\epsilon$. This holds for all roots of f_1 which proves Sturm's theorem.

□

Now we are ready to formulate a stability criterion. If $f(z)$ has no root on the imaginary axis we have shown that

$$\begin{aligned} \pi(k-2k^+) &= \Delta \arg f(j\omega))|_{-\infty}^{\infty} \\ &= \pi \mathrm{I}_{-\infty}^{\infty} \frac{a_{k-1}\omega^{k-1} - a_{k-3}\omega^{k-3} + \ldots}{a_k\omega^k - a_{k-2}\omega^{k-2} + \ldots} \\ &= \pi(V(-\infty) - V(\infty)) \end{aligned}$$

and thus we can compute k^+ from the sign variations in the leading coefficients of the Sturm sequence of $a_k\omega^k - a_{k-2}\omega^{k-2} + \ldots$ and $a_{k-1}\omega^{k-1} - a_{k-3}\omega^{k-3} + \ldots$. If $f(z)$ does have a root on the imaginary axis, f_R and f_I have a common zero and therefore the last element of the Sturm sequence must be 0 according to Lemma 3.2. So we have proven the criterion

> f is stable if and only if the last element of the Sturm sequence is non-zero and $V(-\infty) - V(\infty) = k$.

3.2.2 The Routh Criterion

We have seen that we can check the stability of $f(z)$ by computing the Sturm sequence of $f_1(\omega) = a_k\omega^k - a_{k-2}\omega^{k-2} + \ldots$ and $f_2(\omega) = a_{k-1}\omega^{k-1} - a_{k-3}\omega^{k-3} + \ldots$. This sequence has a special form, therefore we can formulate a scheme for computing the coefficients of the polynomials. For example we get

$$f_3(\omega) = \frac{a_{k-1}a_{k-2} - a_k a_{k-3}}{a_{k-1}}\omega^{k-2} - \frac{a_{k-1}a_{k-4} - a_k a_{k-5}}{a_{k-1}}\omega^{k-4} \ldots$$

Organizing these computations in tabular form yields the following **Routh table**:

r_{ij}				
$f_1(\omega)$	a_k	a_{k-2}	a_{k-4}	$\ldots$
$f_2(\omega)$	a_{k-1}	a_{k-3}	a_{k-5}	$\ldots$
$f_3(\omega)$	$\frac{a_{k-1}a_{k-2}-a_k a_{k-3}}{a_{k-1}}$	$\frac{a_{k-1}a_{k-4}-a_k a_{k-5}}{a_{k-1}}$	$\frac{a_{k-1}a_{k-6}-a_k a_{k-7}}{a_{k-1}}$	$\ldots$
$f_4(\omega)$	$\frac{r_{31}a_{k-3}-a_{k-1}r_{32}}{r_{31}}$	$\frac{r_{31}a_{k-5}-a_{k-1}r_{33}}{r_{31}}$	$\ldots$	
$\ldots$	$\ldots$			

The elements r_{i1} in the table are equal to the leading coefficients of $f_i(\omega)$, therefore the sign variations in the sequence $r_{11}, r_{21}, r_{31}, \ldots$ give $V(\infty)$ and the sign variations in the sequence $r_{11}, -r_{21}, r_{31}, \ldots$ or $-r_{11}, r_{21}, -r_{31}, \ldots$ give $V(-\infty)$ for even or odd k, respectively.

There are $k+1$ rows in the Routh table, therefore we must have $V(-\infty)+V(\infty)=k$. This condition together with $V(-\infty)-V(\infty)=k-2k^+$ yields

$$V(\infty)=k^+,$$

which means that k^+ is equal to the number of sign variations in the first column of the Routh table. Again, this only holds if there are no zeros of f on the imaginary axis. If there are, according to Lemma 3.2 there must be an element $r_{i1}=0$ with $i \leq k+1$. This leads to the Routh criterion

> f is stable if and only if the elements of the first column of the Routh table are all positive or all negative.

In the notation used for formulating stability problem 2.1 this reads

$$\begin{gathered}\forall z \in \mathbb{C} : [f(z)=0 \Rightarrow \mathrm{Re}\,(z)<0] \\ \Leftrightarrow \\ (r_{11}>0 \,\wedge\, r_{21}>0 \,\wedge\, \ldots \,\wedge\, r_{k+1,1}>0)\vee \\ (r_{11}<0 \,\wedge\, r_{21}<0 \,\wedge\, \ldots \,\wedge\, r_{k+1,1}<0).\end{gathered}$$

If there are design and/or robustness parameters, the r_{ij} are rational functions and we arrive at the following formulation for the criterion:

Theorem 3.3 (Parametric robust Routh stability criterion)
Let $P \subseteq \mathbb{R}^m$, $Q \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[z, p_1, \ldots, p_m, q_1, \ldots, q_n]$. Then

$$\begin{array}{rl} & \forall p \in P : \forall q \in Q : \forall z \in \mathbb{C} : [f(z,p,q)=0 \Rightarrow \mathrm{Re}\,(z)<0] \\ & \Leftrightarrow \\ \forall p \in P : \forall q \in Q : & (r_{11}(p,q)>0 \,\wedge\, r_{21}(p,q)>0 \,\wedge\, \ldots \,\wedge\, r_{k+1,1}(p,q)>0)\vee \\ & (r_{11}(p,q)<0 \,\wedge\, r_{21}(p,q)<0 \,\wedge\, \ldots \,\wedge\, r_{k+1,1}(p,q)<0).\end{array}$$

where the r_{i1} are the entries of the first column of the Routh table.

3.3 Cauchy Indices by Hurwitz Determinants

As we have seen by the end of Section 3.1.2, we can get information on the argument change (and therefore on the locations of the roots) of a polynomial from the Cauchy index of a certain rational function that is derived from the polynomial. In this section we will present Hurwitz's solution for computing the Cauchy index. Instead of repeating the proofs given in [19] or [30] we will follow [21] and derive Hurwitz's stability criterion from Routh's criterion given in Section 3.2.

3.3.1 The Hurwitz Criterion

By arranging the coefficients of f in a matrix it is possible to formulate a stability criterion based on determinants that — contrary to the Routh criterion — does not require fractions.

Definition 3.4 (Hurwitz matrix, minors, and determinant)
Let $f(z) = a_k z^k + \ldots + a_0 \in \mathbb{R}[z]$. The $k \times k$ matrix

$$H := \begin{pmatrix} a_{k-1} & a_{k-3} & a_{k-5} & a_{k-7} & \ldots & 0 \\ a_k & a_{k-2} & a_{k-4} & a_{k-6} & \ldots & 0 \\ 0 & a_{k-1} & a_{k-3} & a_{k-5} & \ldots & 0 \\ 0 & a_k & a_{k-2} & a_{k-4} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & 0 & \ldots & a_2 & a_0 \end{pmatrix}$$

*is called **Hurwitz matrix of** f. The i-th leading principal minor of H*

$$D_i := \begin{vmatrix} a_{k-1} & a_{k-3} & a_{k-5} & a_{k-7} & \ldots & 0 \\ a_k & a_{k-2} & a_{k-4} & a_{k-6} & \ldots & 0 \\ 0 & a_{k-1} & a_{k-3} & a_{k-5} & \ldots & 0 \\ 0 & a_k & a_{k-2} & a_{k-4} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \ddots & a_{k-i-1} \\ 0 & 0 & 0 & \ldots & a_{k-i+2} & a_{k-i} \end{vmatrix}$$

*is called i**-th Hurwitz minors of** f. $D := D_k = \det H$ is also called **the Hurwitz determinant of** f.*

We easily verify that the entries of the Routh table are related to the D_i in the following manner:

$$\begin{array}{rcl} r_{11} & = & a_k \\ r_{21} & = & a_{k-1} & = & D_1 \\ r_{31} & = & \frac{a_{k-1}a_{k-2}-a_k a_{k-3}}{a_{k-1}} & = & \frac{D_2}{D_1} \\ \vdots & & & & \vdots \\ r_{k+1,1} & = & & = & \frac{D_k}{D_{k-1}} \end{array}$$

Thus we can derive the Routh-Hurwitz[1] criterion from the Routh criterion, getting

[1] In the literature the criterion is called "Routh-Hurwitz", although according to [19] Hurwitz developed it independently of Routh's result.

f is stable if and only if a_k and all D_i are positive or if a_k is negative and all D_i are negative for odd i and positive for even i.

In the notation used for formulating stability problem 2.1 this reads

$$\begin{gathered}\forall z \in \mathbb{C} : [f(z) = 0 \Rightarrow \mathrm{Re}\,(z) < 0] \\ \Leftrightarrow \\ (a_k > 0 \;\wedge\; D_1 > 0 \;\wedge\; \ldots \;\wedge\; D_k > 0) \vee \\ (a_k < 0 \;\wedge\; D_1 < 0 \;\wedge\; D_2 > 0 \;\wedge\; \ldots).\end{gathered}$$

In the presence of design and/or robustness parameters the D_i are polynomials in the parameters and we get the following criterion:

Theorem 3.4 (Parametric robust Routh-Hurwitz stability criterion)

Let $P \subseteq \mathbb{R}^m$, $Q \subseteq \mathbb{R}^n$, $f = \sum_{i=0}^{k} a_i(p,q) z^i \in \mathbb{R}[z, p_1, \ldots, p_m, q_1, \ldots, q_n]$. *Then*

$$\begin{gathered}\forall p \in P : \forall q \in Q : \forall z \in \mathbb{C} : [f(z,p,q) = 0 \Rightarrow \mathrm{Re}\,(z) < 0] \\ \Leftrightarrow \\ \forall p \in P : \forall q \in Q : \begin{array}{l}(a_k(p,q) > 0 \;\wedge\; D_1(p,q) > 0 \;\wedge\; \ldots \;\wedge\; D_k(p,q) > 0) \vee \\ (a_k(p,q) < 0 \;\wedge\; D_1(p,q) < 0 \;\wedge\; D_2(p,q) > 0 \;\wedge\; \ldots).\end{array}\end{gathered}$$

3.3.2 The Liénard-Chipart Criterion

A necessary (but not sufficient) condition for f to be stable is that all non-zero coefficients of f are of the same sign. This observation indicates that the D_i are not independent of each other. That this is indeed the case is shown by the stability criterion of Liénard and Chipart (see [13] and [19]), which states (assuming without loss of generality $a_k > 0$) that f is stable if and only if a_0 and its odd or its even coefficients and its odd or its even Hurwitz minors are positive. In the notation used for formulating stability problem 2.1 this reads (for $a_k > 0$)

$$\begin{gathered}\forall z \in \mathbb{C} : [f(z) = 0 \Rightarrow \mathrm{Re}\,(z) < 0] \\ \Leftrightarrow \\ ((a_0 > 0 \;\wedge\; a_2 > 0 \;\wedge\; a_4 > 0 \ldots) \vee (a_0 > 0 \;\wedge\; a_1 > 0 \;\wedge\; a_3 > 0 \ldots)) \wedge \\ ((D_1 > 0 \;\wedge\; D_3 > 0 \;\wedge\; D_5 > 0 \ldots) \vee (D_2 > 0 \;\wedge\; D_4 > 0 \;\wedge\; D_6 > 0 \ldots)).\end{gathered}$$

In the presence of design and/or robustness parameters the D_i are polynomials in the parameters and we get the following criterion:

Theorem 3.5 (Parametric robust Liénard-Chipart stability criterion)

Let $P \subseteq \mathbb{R}^m$, $Q \subseteq \mathbb{R}^n$, $f = \sum_{i=0}^{k} a_i(p,q) z^i \in \mathbb{R}[z, p_1, \ldots, p_m, q_1, \ldots, q_n]$.

If $a_k(p,q) > 0$ *then*

$$\begin{array}{rl} & \forall p \in P : \forall q \in Q : \forall z \in \mathbb{C} : [f(z,p,q) = 0 \Rightarrow \mathrm{Re}\,(z) < 0] \\ & \Leftrightarrow \\ \forall p \in P : \forall q \in Q : & ((a_0(p,q) > 0 \wedge a_2(p,q) > 0 \wedge a_4(p,q) > 0 \ldots)\vee \\ & (a_0(p,q) > 0 \wedge a_1(p,q) > 0 \wedge a_3(p,q) > 0 \ldots))\wedge \\ & ((D_1(p,q) > 0 \wedge D_3(p,q) > 0 \ldots)\vee \\ & (D_2(p,q) > 0 \wedge D_4(p,q) > 0 \ldots)). \end{array}$$

3.3.3 The Hurwitz Determinant as a Resultant

The Hurwitz minor D_{k-1} can be interpreted as a resultant of two polynomials derived from f: If we write $f(z) = a_k z^k + a_{k-1} z^k + \ldots + a_0$ in the form $f(z) = h(z^2) + zg(z^2)$ then for $k = 2l+1$ we have

$$\begin{aligned} g(u) &= a_k u^l + a_{k-2} u^{l-1} + \ldots + a_1 \\ h(u) &= a_{k-1} u^l + a_{k-3} u^{l-1} + \ldots + a_0 \end{aligned}$$

and for $k = 2l$ we have

$$\begin{aligned} g(u) &= a_{k-1} u^{l-1} + a_{k-3} u^{l-2} + \ldots + a_1 \\ h(u) &= a_k u^l + a_{k-2} u^{l-1} + \ldots + a_0. \end{aligned}$$

Consider the case where $k = 2l+1$. By exchanging rows $k-2$ times we can transform the $(k-1)\times(k-1)$ left upper submatrix of the Hurwitz matrix of f into the Sylvester matrix (Definition A.1, page 65)

$$\begin{pmatrix} a_k & a_{k-2} & \cdots & a_1 & & & \\ & a_k & a_{k-2} & \cdots & a_1 & \ddots & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_k & a_{k-2} & \cdots & a_1 \\ a_{k-1} & a_{k-3} & \cdots & a_0 & & & \\ & a_{k-1} & a_{k-3} & \cdots & a_0 & & \\ & & \ddots & \ddots & & \ddots & \\ & & & a_{k-1} & a_{k-3} & \cdots & a_0 \end{pmatrix}.$$

From the definition of the resultant (Definition A.2, page 65) immediately follows $D_{k-1} = (-1)^{k-2}\mathrm{res}(g,h)$. A similar argument shows that this equality also holds for $k = 2l$. This leads to the following lemma:

Lemma 3.6 (Hurwitz test for purely imaginary[2] roots)
Let $f \in \mathbb{R}[z]$. f has a root at the origin or a pair of roots $\tilde{z}$ and $-\tilde{z}$ if and only if the Hurwitz determinant of f is zero.

Proof: Let $g, h \in \mathbb{R}[u]$ with $f(z) = h(z^2) + zg(z^2)$.

As we have seen earlier, $D_{k-1} = (-1)^{k-2}\mathrm{res}(g,h)$. Furthermore, $D_k = a_0 D_{k-1} = a_0(-1)^{k-2}\mathrm{res}(g,h)$. By the fundamental theorem of resultants (Theorem A.1, page 66) $\mathrm{res}(g,h)$ is zero if and only if $g(u)$ and $h(u)$ have a common zero.

If f has a root at the origin, then a_0 and consequently D_k are zero. If $f(z)$ has a pair of purely imaginary roots $\tilde{z}$ and $-\tilde{z}$ then $\tilde{z}^2$ is a common root of g and h and therefore D_k is zero.

If D_k is zero and D_{k-1} is non-zero, then a_0 must be zero and f has a root at the origin. If D_{k-1} is zero, then g and h have a common zero $\tilde{u}$ and f has a pair of roots $\sqrt{\tilde{u}}$ and $-\sqrt{\tilde{u}}$.

□

3.3.4 Orlando's Formula

The Hurwitz minor D_{k-1} can be expressed in terms of the roots $z_1, \ldots, z_k$ of f. This relationship gives rise to Orlando's formula [41] (see e.g. [19] for a proof):

$$D_{k-1} = (-1)^{\frac{k(k-1)}{2}} a_k^{k-1} \prod_{\substack{i=2 \\ j<i}}^{k} (z_i + z_k)$$

We see that D_{k-1} is zero if and only if the sum of any two roots of f is zero. This observation provides an alternative proof of Lemma 3.6.

3.3.5 Stability Measure Computation by Hurwitz Determinants

The following lemma gives a formula that relates the root space stability measure RS_f (Definition 2.7, page 13) to roots on the imaginary axis of a certain shifted polynomial. The underlying intuition is illustrated in Figure 3.3. We will use this lemma in combination with the Hurwitz test for purely imaginary roots (Lemma 3.6, page 26) to prove a theorem relating RS_f to the Hurwitz determinant of f.

[2]The lemma is more general, but we will apply it only for testing whether a polynomial has roots on the imaginary axis. We give a more specialized test for this purpose in Lemma 4.2, page 34.

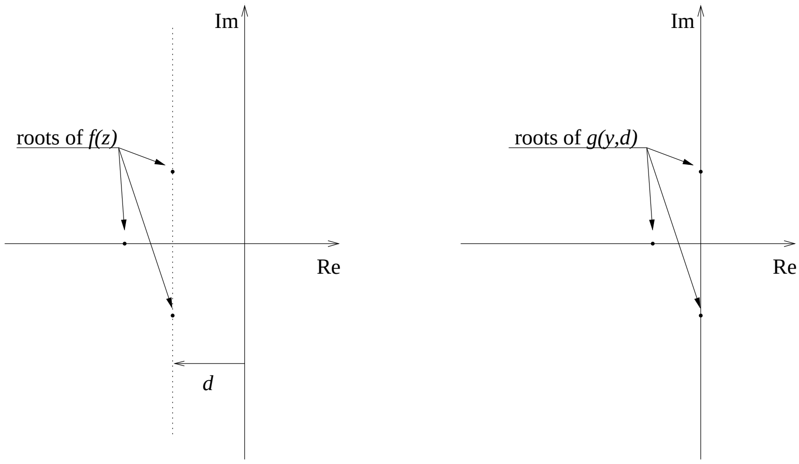

Figure 3.3: Root space stability measure by symbolic shifting. *The figure on the left shows the roots of a polynomial $f(z)$ with $RS_f = d$. The figure on the right shows the roots of the polynomial $g(y,d) = f(y+d)$. This shifted polynomial has roots on the imaginary axis.*

Lemma 3.7 ($\mathbf{RS}_f$ by shifting)
Given $f \in \mathbb{R}[z]$ let $g \in \mathbb{R}[y,d]$ such that $g(y,d) = f(y+d)$. Then

$$RS_f = \max\{d \in \mathbb{R} | \, \exists y \in \mathbb{C} : \, \mathrm{Re}\,(y) = 0 \, \wedge \, g(y,d) = 0\},$$

or in other words RS_f is the largest d such that $g(y,d)$ has a root with respect to y on the imaginary axis.

Proof: Let $\tilde{d} = \max\{d \in \mathbb{R} | \, \exists y \in \mathbb{C} : \, \mathrm{Re}\,(y) = 0 \, \wedge \, g(y,d) = 0\}$. We will show that $\tilde{d}$ is equal to RS_f.

For $\tilde{y} \in \mathbb{C}$ with $\mathrm{Re}\,(y) = 0$ and $g(\tilde{y}, \tilde{d}) = 0$ we have $f(\tilde{y}+\tilde{d}) = 0$. Therefore $\tilde{d}$ is equal to the real part of a root of f and thus $\tilde{d} \leq \mathrm{RS}_f$.

On the other hand, for any $\tilde{z} \in \mathbb{C}$ with $\mathrm{Re}\,(\tilde{z}) = \mathrm{RS}_f$ and $f(\tilde{z}) = 0$ there exists a $\tilde{y} = \tilde{z} - \mathrm{RS}_f$ such that $\mathrm{Re}\,(\tilde{y}) = 0$ and $g(\tilde{y}, \mathrm{RS}_f) = f(\tilde{z}) = 0$. Therefore RS_f is in the set of which $\tilde{d}$ is a maximum and thus $\tilde{d} \geq \mathrm{RS}_f$.

□

Theorem 3.8 ($\mathbf{RS}_f$ as univariate root)
Given $f \in \mathbb{R}[z]$ let $g \in \mathbb{R}[y,d]$ such that $g(y,d) = f(y+d)$ and let $D(d) \in \mathbb{R}[d]$ be the determinant of the Hurwitz matrix of g with respect to y. Then RS_f is equal to the largest real root of $D(d)$.

Proof: According to Lemma 3.6 the real roots of $D(d)$ are exactly those values for which $g(y,d)$ has a root with respect to y on the imaginary axis. Applying Lemma 3.7 then implies that the largest real root of $D(d)$ is equal to RS_f.

□

Similar to the parameter-free case we can also compute $\mathrm{RS}_{f,Q}$ (Definition 2.8, page 14) from the roots of the Hurwitz determinant of f.

Theorem 3.9 (RS$_{f,Q}$ as multivariate root)
Given $f \in \mathbb{R}[z, q_1, \ldots, q_n]$ let $g \in \mathbb{R}[y, d, q_1, \ldots, q_n]$ such that $g(y, d, q) = f(y+d, q)$ and let $D(d, q) \in \mathbb{R}[d, q_1, \ldots, q_n]$ be the determinant of the Hurwitz matrix of g with respect to y. Then

$$RS_{f,Q} = \sup\{d \in \mathbb{R} \mid \exists q \in Q : \; D(d, q) = 0\}.$$

Proof: The proof follows immediately from Definition 2.8 and Theorem 3.8.

□

Theorem 3.8 reduces the computation of RS_f to a symbolic evaluation of the Hurwitz matrix and the task of finding the largest real root of a univariate polynomial of degree $\frac{k(k+1)}{2}$. Alternatively, if we only need to *test* whether f is stable we have seen that we can simply apply the Liénard-Chipart method, which consists of $\frac{k}{2}$ determinant evaluations.

If we are satisfied with computing an *upper bound* on RS_f (because the root computation is too expensive and testing for stability is not sufficient), then we can apply Recipe 3.1 given below. It is based on the following observation: If f is stable, its largest real root $\tilde{z}$ is negative and thus $-\tilde{z}$ is equal to the smallest (positive) real root of $f(-z)$. Therefore a lower positive real root bound (which can be computed by a simple formula) for $f(-z)$ leads to an upper bound on RS_f.

Recipe 3.1 (Bound on RS$_f$ for stable f)
Given *stable $f \in \mathbb{R}[z]$;*
compute *$B \in \mathbb{R}$ such that $RS_f \leq B < 0$.*

1. *Do a symbolic shift of f, i.e. let $g(y, d) := f(y + d)$.*

2. *Regard g as a polynomial in y whose coefficients (except for the leading coefficient) are polynomials in d, i.e. $g(y, d) = a_k y^k + b_{k-1}(d) y^{k-1} +$*

$\ldots + b_0(d)$. *Compute the Hurwitz determinant of* g *wrt.* y, *i.e. compute*

$$D(d) := \begin{vmatrix} b_{k-1}(d) & b_{k-3}(d) & b_{k-5}(d) & \ldots & 0 \\ a_k & b_{k-2}(d) & b_{k-4}(d) & \ldots & 0 \\ 0 & b_{k-1}(d) & b_{k-3}(d) & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \ldots & b_2(d) & b_0(d) \end{vmatrix}.$$

3. *Now to apply the lower bound on positiveness given in Section A.3.2 let* $h = \operatorname{sign}(a_k)^k D(-d)$. *Finally let* $B = -\check{B}_h$.

3.4 Cauchy Indices by Markov Parameters

As we have seen by the end of Section 3.1.2, we can get information on the argument change (and therefore on the locations of the roots) of a polynomial from the Cauchy index of a certain rational function that is derived from the polynomial. In this section we will present a solution for computing the Cauchy index that seems[3] to have been developed by Gantmacher [19].

For our purposes (robust parameterized stability), the resulting stability criterion is equivalent to the Routh-Hurwitz criterion. Therefore, we mention it only for completeness, giving only an outline of the criterion and omitting proofs. See [22] for an application of Markov parameters to the computation of a stability margin for interval polynomials and [32] for an analysis of the relationship between Markov parameters and Hurwitz minors.

3.4.1 Markov Parameters

In Section 3.1.2 we have seen how the Cauchy index of the rational function $\frac{a_{k-1}z^{k-1} - a_{k-3}z^{k-3} + \ldots}{a_k z^k - a_{k-2}z^{k-2} + \ldots}$ derived from the polynomial $f(z) = a_k z^k + \ldots + a_0$ is related to the stability of f. Representing this rational function as a Laurent series leads to a new criterion for stability.

Definition 3.5 (Markov parameters)
Given $f \in \mathbb{R}[z]$ *let* $k = \deg f$ *and let* $g, h \in \mathbb{R}[u]$ *be such that* $f(z) =$

[3]Chapter XV of Gantmacher's monograph [19] collects many results on stability and presents them as a consistent whole. While the author gives references for known results, he rarely points out which results are new. The credit for discovering the connection between stability, Markov parameters, and positive definite Hankel forms seems to belong to Gantmacher.

$h(z^2)+zg(z^2)$. The first $k+1$ coefficients $s_{-1}, s_0, \ldots, s_{k-1}$ of the expansion

$$\frac{g(u)}{h(u)} = s_{-1} + \frac{s_0}{z} - \frac{s_1}{u^2} + \frac{s_2}{u^3} - \frac{s_3}{u^4} + \ldots$$

are called ***Markov parameters of*** *f.*

Note that any s_i with $i \geq k$ is uniquely defined by the preceding $\left\lfloor \frac{k}{2} \right\rfloor$ coefficients $s_{i-1}, \ldots, s_{i-\lfloor k/2 \rfloor}$. More specifically

$$s_i = \frac{(-1)^i}{a_k} a_{k-2i-1} + \sum_{j=1}^{i} (-1)^{j-1} a_{k-2j} s_{i-j}.$$

Thus, f is uniquely defined by its Markov parameters.

3.4.2 The Stability Criterion

The stability of a polynomial can be determined from its Markov parameters as follows:

Theorem 3.10 (Theorem 17 in Chapter XV of [19])
Let $f \in \mathbb{R}$, $k = \deg f$, $l = \lfloor k/2 \rfloor$, $s_{-1}, s_0, \ldots, s_{k-1}$ the Markov parameters of f. Then f is stable if and only if the finite Hankel matrices

$$S^{(0)} := \begin{pmatrix} s_0 & s_1 & \ldots & s_{l-1} \\ s_1 & s_2 & \ldots & s_l \\ \vdots & \vdots & \ddots & \vdots \\ s_{l-1} & s_l & \ldots & s_{k-2} \end{pmatrix}$$

and

$$S^{(1)} := \begin{pmatrix} s_1 & s_2 & \ldots & s_l \\ s_2 & s_3 & \ldots & s_{l+1} \\ \vdots & \vdots & \ddots & \vdots \\ s_l & s_{l+1} & \ldots & s_{k-1} \end{pmatrix}$$

are positive definite and if k is odd then also $s_{-1} > 0$.

See [19] for a proof. To get some idea about the relationship between the leading principal minors of $S^{(0)}$ and $S^{(1)}$ and those of the Hurwitz matrix H of f note that

$$H = \begin{pmatrix} s_0 & -s_1 & s_2 & -s_3 & \ldots & 0 \\ 1 & 0 & 0 & 0 & \ldots & 0 \\ 0 & s_0 & -s_1 & s_2 & \ldots & 0 \\ 0 & 1 & 0 & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 1 & 0 & \ldots & 0 \end{pmatrix} \begin{pmatrix} a_k & a_{k-2} & a_{k-4} & \ldots & \\ 0 & a_k & a_{k-2} & \ddots & \vdots \\ 0 & 0 & a_k & \ddots & a_{k-4} \\ \vdots & & \ddots & \ddots & a_{k-2} \\ 0 & \ldots & \ldots & 0 & a_k \end{pmatrix},$$

where the two matrices are square $k \times k$ matrices. The leading principal minors of the left matrix are the leading principal minors of $S^{(0)}$ and $S^{(1)}$. Thus we see that we can write the latter in the form $a_k^{-i} D_i$, where D_i is some i-th Hurwitz minor.

More results in this spirit can be found in [32].

Chapter 4

Rewriting the Non-Parametric Universal Quantifier

In Chapter 3 we present methods for eliminating the quantifier "$\forall z \in \mathbb{C}$" in the stability problems (Problems 2.1, 2.2, 2.3, and 2.4). As is often the case, this elimination must be bought by an increased "complexity" of the resulting formula. For example, the degrees of the parameters appearing in Hurwitz determinants are higher than the degrees of the parameters appearing in the polynomial f.

In this chapter, we will describe methods for *rewriting* the quantifier "$\forall z \in \mathbb{C}$". The resulting formulae do not contain this quantifier anymore, but comprise new (simpler) quantified subformulae.

4.1 The Hermite-Biehler Theorem

The so-called Hermite-Biehler Theorem reduces the stability of a polynomial, i.e. the condition that *all* roots of a polynomial must be in the open right halfplane, to a condition on the *real* roots of two related polynomials. It also provides a test for purely imaginary roots of a polynomial.

4.1.1 The Hermite-Biehler Theorem

We have no original references for the Hermite-Biehler theorem. The following definition and theorem are taken from [19].

Definition 4.1 (Positive pair)
Let $g, h \in \mathbb{R}[u]$ *with* $\deg h = k$ *and* $\deg g = k$ *or* $\deg g = k-1$. *Let* $v_1, \dots, v_k$ *be the roots of* h *and* $u_1, \dots, u_k$ *or* $u_1, \dots, u_{k-1}$ *be the roots of* g. *If the*

leading coefficients of g and h are of like sign and the roots of g and h are all real and alternate with

$$u_1 < v_1 < u_2 < v_2 < \ldots < u_k < v_k < 0$$

or

$$v_1 < u_1 < v_2 < \ldots < u_{k-1} < v_k < 0$$

then we say that g ***and*** *h* ***form a positive pair****.*

Theorem 4.1 (Hermite-Biehler[1])
Let $f \in \mathbb{R}[z]$ and $g, h \in \mathbb{R}[u]$ such that $f(z) = h(z^2) + zg(z^2)$. Then f is stable if and only if g and h form a positive pair.

Note that for simplicity this theorem is given in a parameter-free form. For a proof see [7] or [19]. The extension to robust stability of $f(z, q)$ with respect to Q is straightforward by requiring that the positive pair condition must hold for all $q \in Q$.

The following lemma is not directly related to the Hermite-Biehler theorem, although it exploits the same properties of polynomials. It can be regarded as a more specialized version of the Hurwitz test for purely imaginary roots (Lemma 3.6, page 26).

Lemma 4.2 (Hermite-Biehler test for purely imaginary roots)
Let $f \in \mathbb{R}[z]$ and $g, h \in \mathbb{R}[u]$ such that $f(z) = h(z^2) + zg(z^2)$. Then f has a pair of purely imaginary roots if and only if g and h have a common negative real root; furthermore, f has a root at the origin if and only if h has a root at zero.

The proof of this lemma follows immediately from the observation that the imaginary axis image with respect to f can be written as $f(j\omega) = h(-\omega^2) + j\omega g(-\omega^2)$.

4.1.2 The Stability Criterion

The Hermite-Biehler theorem can be directly interpreted as a stability criterion that does not contain a universally quantified complex variable. Nevertheless note that if the positive pair condition is formulated in the language of real closed fields the resulting formula (which we omit in the interest of readability) not contains a universally quantified real variable.

[1] According to [19] this version is a special case of the actual Hermite-Biehler theorem.

Theorem 4.3 (Parametric robust Hermite-Biehler stability criterion)

Let $P \subseteq \mathbb{R}^m$, $Q \subseteq \mathbb{R}^n$, $f = \sum_{i=0}^{k} a_i(p,q)z^i \in \mathbb{R}[z, p_1, \ldots, p_m, q_1, \ldots, q_n]$, $g, h \in \mathbb{R}[u, p_1, \ldots, p_m, q_1, \ldots, q_n]$ *such that* $f(z,p,q) = h(z^2,p,q) + zg(z^2,p,q)$.
Then

$$\begin{gathered} \forall p \in P : \forall q \in Q : \forall z \in \mathbb{C} : [f(z,p,q) = 0 \Rightarrow \mathrm{Re}\,(z) < 0] \\ \Leftrightarrow \\ \forall p \in P : \forall q \in Q : g(u,p,q) \textit{ and } h(u,p,q) \textit{ form a positive pair.} \end{gathered}$$

4.2 The Value Set Approach

Given a (parameterized) robust stability problem with $f(z,q)$ and Q, the idea of the value set approach is to consider the set $\{f(j\omega,q) | \omega \in \mathbb{R}, q \in Q\}$. The zero exclusion principle (Lemma A.4, page 68) gives conditions on this set that guarantee robust stability of f with respect to Q.

This section contains a short exposition of this idea presenting only the essential elements that we will use later on. A more detailed account can be found in [4].

4.2.1 The Stability Criterion

The zero exclusion principle (Lemma A.4, page 68) immediately leads to a formulation of a stability criterion that does not contain a universally quantified complex variable.

Theorem 4.4 (Parametric robust value set stability criterion)
Let $P \subseteq \mathbb{R}^m$, $Q \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[z, p_1, \ldots, p_m, q_1, \ldots, q_n]$. *If* $P \times Q$ *is connected and* $f(z,P,Q)$ *is of constant degree, then*

$$\begin{gathered} \forall p \in P : \forall q \in Q : \forall z \in \mathbb{C} : f(z,p,q) \textit{ is stable} \\ \Leftrightarrow \\ \exists \tilde{p} \in P : \exists \tilde{q} \in Q : f(z,\tilde{p},\tilde{q}) \textit{ is stable } \wedge \\ \forall p \in P : \forall q \in Q : \forall \omega \in \mathbb{R} : f(j\omega,p,q) \neq 0. \end{gathered}$$

This criterion says that — assuming $P \times Q$ is connected and f is of constant degree — $f(z,P,q)$ is robustly stable with respect to Q if and only if two condititions hold: (1) there exists a point $(\tilde{p},\tilde{q}) \in P \times Q$ for which f is stable and (2) the set $\{f(j\omega,p,q) | \omega \in \mathbb{R}, p \in P, q \in Q\}$ (i.e. the image of the imaginary axis under $f(z,P,Q)$) does not contain the origin.

Figures 4.1 and 4.2 illustrate this idea: In this example, $f(z,q_1,q_2) = z^2 + (q_1q_2 - q_1 - q_2 + \frac{3}{4})z + (-q_1 - \frac{q_2}{2} + \frac{3}{2})$ is robustly stable with respect to

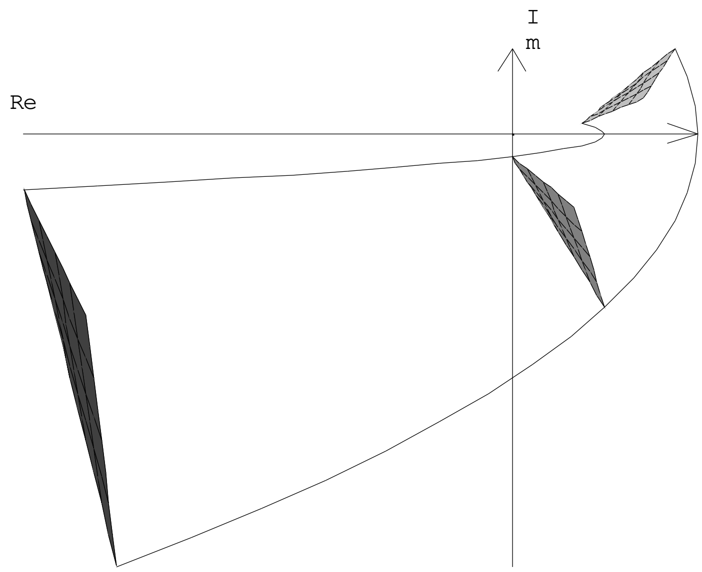

Figure 4.1: Illustration of the value set approach (stable case). *This figure shows the set* $\{f(j\omega, q) | \omega \in \mathbb{R}, q \in Q\}$ *for* $\omega \in \{-\frac{5}{2}, 1, \frac{1}{2}\}$, $f(z, q_1, q_2) = z^2 + (q_1 q_2 - q_1 - q_2 + \frac{3}{4})z + (-q_1 - \frac{q_2}{2} + \frac{3}{2})$ *and* $Q = [-\frac{1}{3}; \frac{1}{3}]^2$. *The set does not contain the origin, therefore* f *is robustly stable with respect to* Q.

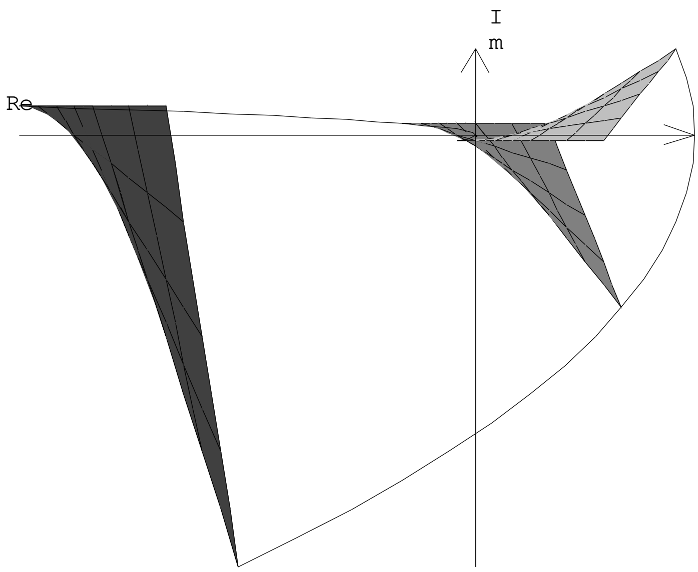

Figure 4.2: Illustration of the value set approach (instable case). *This figure shows the set* $\{f(j\omega, q) | \omega \in \mathbb{R}, q \in Q\}$ *for* $\omega \in \{-\frac{5}{2}, 1, \frac{1}{2}\}$, $f(z, q_1, q_2) = z^2 + (q_1 q_2 - q_1 - q_2 + \frac{3}{4})z + (-q_1 - \frac{q_2}{2} + \frac{3}{2})$ *and* $Q = [-1; 1]^2$. *The set contains the origin, therefore* f *is not robustly stable with respect to* Q.

$Q = [-\frac{1}{3}; \frac{1}{3}]^2$, but it is not robustly stable with respect to $Q = [-1; 1]^2$. In the first case the imaginary axis image with respect to $f(z, Q)$ excludes the origin, while in the second case the imaginary axis image passes through the origin (e.g. $f(z, \frac{1}{2}, \frac{1}{2})$ has the purely imaginary roots $\frac{\sqrt{3}}{2}j$ and $-\frac{\sqrt{3}}{2}j$).

4.2.2 Testing for Purely Imaginary Roots

As mentioned in the discussion of the zero exclusion principle in Section A.2.2, in most problems the only condition that is not trivially fulfilled is Condition (iv), which requires that f must not have any roots on the imaginary axis, or formally

$$\forall p \in P : \forall q \in Q : \forall \omega \in \mathbb{R} : f(j\omega, p, q) \neq 0.$$

If we require f to be of constant degree with respect to P and Q we can apply the Hurwitz test for purely imaginary roots[2] (Lemma 3.6, page 26), getting the formula

$$\forall p \in P : \forall q \in Q : D(p, q) \neq 0, \tag{4.1}$$

where $D(p, q)$ is the Hurwitz determinant of f.

If we alternatively apply the Hermite-Biehler test for purely imaginary roots (Lemma 4.2, page 34), we get the formula

$$\begin{aligned} \forall p \in P : \forall q \in Q : \quad & h(0, p, q) \neq 0 \wedge \\ & \forall u \in \mathbb{R}^- : [g(u, p, q) \neq 0 \vee h(u, p, q) \neq 0], \end{aligned} \tag{4.2}$$

where $g, h \in \mathbb{R}[u, p_1, \ldots, p_m, q_1, \ldots, q_n]$ such that $f(z, p, q) = h(z^2, p, q) + zg(z^2, p, q)$ and we require that the leading coefficients of g and h are of like sign for all $p \in P$ and all $q \in Q$.

The Hermite-Biehler condition gives rise to the following geometric intuition: As long as g and h form a positive pair (i.e. they have all negative real interleaving roots) the imaginary axis image with respect to f "rotates" around the origin with increasing ω; the alternation of the roots corresponds to the alternation of the axis crossings by the imaginary axis image. The set $\{f(j\omega, p, q) | \omega \in \mathbb{R}, p \in P, q \in Q\}$ does not contain the origin, therefore f is stable according to the zero exclusion principle. If any two roots of g and h "touch", then the imaginary axis image passes through the origin, and f is

[2] Note that in a general context this lemma only yields

$$D(p, q) \neq 0 \Rightarrow \forall \omega \in \mathbb{R} : f(j\omega, p, q) \neq 0.$$

The conditions of the zero exclusion principle guarantee equivalence.

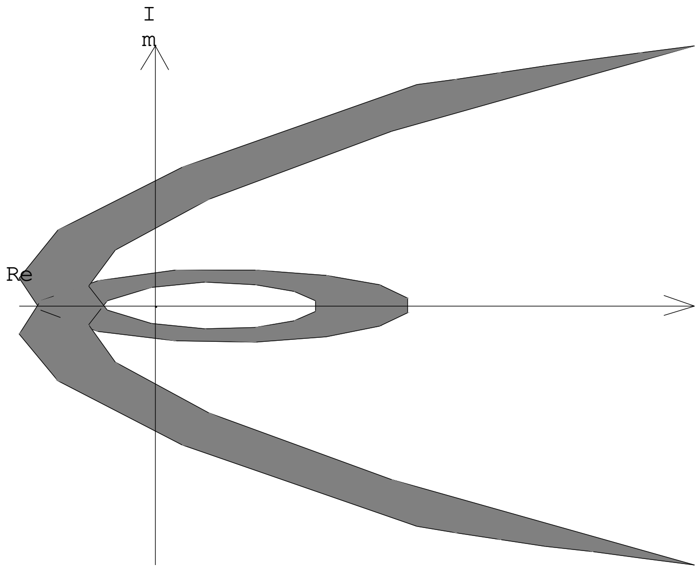

Figure 4.3: Illustration of the Hermite-Biehler theorem. *This figure shows the set* $\{f(j\omega, q) | \omega \in \mathbb{R}, q \in Q\}$ *for* $f(z,q) = z^4 + z^3 + (-\frac{5}{3}q^2 - \frac{15}{2}q + \frac{85}{6})z^2 + (-\frac{27}{8}q + \frac{45}{8})z - 30q + 34$ *and* $Q = [-\frac{1}{4}; \frac{1}{4}]$.

not stable. This interpretation is illustrated by the example shown in Figure 4.3, where $f(z,q) = z^4 + z^3 + (-\frac{5}{3}q^2 - \frac{15}{2}q + \frac{85}{6})z^2 + (-\frac{27}{8}q + \frac{45}{8})z - 30q + 34$ is robustly stable with respect to $Q = [-\frac{1}{4}; \frac{1}{4}]$.

4.2.3 Stability Margin Computation

The zero exclusion principle (Lemma A.4, page 68) can be applied to the computation of the stability margins SSM_f and PSM_f (Definition 2.5, page 11 and Definition 2.6, page 12). As a result we get a formula for the stability margin that does not contain a universally quantified complex variable.

Theorem 4.5 (Stability margin by purely imaginary roots)
Let $f \in \mathbb{R}[z, q_1, \ldots, q_n]$ *such that* $f(z, 0)$ *is stable. If* f *is of constant degree with respect to* $[-SSM_f; SSM_f]^n$ *then*

$$SSM_f = \sup\{e \in \mathbb{R}^+ | \forall q \in [-e; e]^n : \forall \omega \in \mathbb{R} : f(j\omega, q) \neq 0\};$$

if f *is of constant degree with respect to* $[0; PSM_f]^n$ *then*

$$PSM_f = \sup\{e \in \mathbb{R}^+ | \forall q \in [0; e]^n : \forall \omega \in \mathbb{R} : f(j\omega, q) \neq 0\}.$$

The intuition behind this theorem is as follows: For $q = 0$ we know that $f(z,q)$ is stable, i.e. that all roots of f are to the left of the imaginary axis. To compute the stability margin we increase e from 0 until the box $[-e;e]^n$ contains a parameter q for which $f(z,q)$ has a root on the imaginary axis. The first such e is the stability margin of f.

Proof: Let $S_1 := \{e \in \mathbb{R}^+ | \forall q \in [-e;e]^n : f(z,q) \text{ is stable}\}$ and $S_2 := \{e \in \mathbb{R}^+ | \forall q \in [-e;e]^n : \forall \omega \in \mathbb{R} : f(j\omega, q) \neq 0\}$. We will show that $S_1 = S_2$ and thus $\mathrm{SSM}_f = \sup S_1 = \sup S_2$.

"$\subseteq$": Let $\tilde{e} \in S_1$ and let $\tilde{q} \in [-\tilde{e};\tilde{e}]^n$ be arbitrary but fixed. $f(z,\tilde{q})$ is stable, therefore by definition there are no roots on the imaginary axis, i.e. $\forall \omega \in \mathbb{R} : f(j\omega, q) \neq 0$, which implies $\tilde{e} \in S_2$.

"$\supseteq$": Now let $\tilde{e} \in S_2$. The conditions of the zero exclusion principle (Lemma A.4, page 68) are fulfilled for f and $Q := [-\tilde{e};\tilde{e}]^n$, therefore f is robustly stable with respect to Q, which implies $\tilde{e} \in S_1$.

The proof for PSM_f is identical.

□

Theorem 4.5 combined with Formula 4.1, page 38 and Formula 4.2, page 38 immediately leads to the following corollaries.

Corollary 4.6 (Stability margin by Hurwitz determinants)
Let $f \in \mathbb{R}[z, q_1, \ldots, q_n]$ with $f(z,0)$ stable and let $D \in \mathbb{R}[q_1, \ldots, q_n]$ be the Hurwitz determinant of f. If $a_k(q)$, the leading coefficient of f with respect to z, is positive[3] for all $q \in [-SSM_f; SSM_f]^n$ then

$$SSM_f = \sup\{e \in \mathbb{R}^+ | \forall q \in [-e;e]^n : D(q) > 0\};$$

if $a_k(q)$ is positive for all $q \in [0; PSM_f]^n$ then

$$PSM_f = \sup\{e \in \mathbb{R}^+ | \forall q \in [0;e]^n : D(q) > 0\}.$$

In some sense this corollary can be regarded as a generalization of Theorem 2 of [8], where a similar statement is formulated for the linear, independent case. In this simpler case, which is described in Section 5.1.2, the quantifier "$\forall q \in [0;e]$" can be removed by applying Kharitonov's theorem (Theorem 5.1, page 43), and the problem is reduced to computing the smallest positive roots of four Hurwitz determinants depending on the single parameter e.

[3]If instead of a_k being positive we would require f being of constant degree, we would get $D(q) \neq 0$. We require positiveness to simplify the application of the positiveness bounds given in Section A.3.

Corollary 4.7 (Stability margin by Hermite-Biehler)
Let $f \in \mathbb{R}[z, q_1, \ldots, q_n]$ *with* $f(z, 0)$ *stable and let* $g, h \in \mathbb{R}[u, q_1, \ldots, q_n]$ *be such that* $f(z, q) = h(z^2, q) + zg(z^2, q)$*. If* f *is of constant degree with respect to* $[-SSM_f; SSM_f]^n$ *then*

$$SSM_f = \sup \left\{ e \in \mathbb{R}^+ \,\middle|\, \forall q \in [-e; e]^n : \begin{array}{l} h(0, q) \neq 0 \wedge \\ \forall u \in \mathbb{R}^- : [g(u, q) \neq 0 \vee h(u, q) \neq 0] \end{array} \right\};$$

if f *is of constant degree with respect to* $[0; PSM_f]^n$ *then*

$$PSM_f = \sup \left\{ e \in \mathbb{R}^+ \,\middle|\, \forall q \in [0; e]^n : \begin{array}{l} h(0, q) \neq 0 \wedge \\ \forall u \in \mathbb{R}^- : [g(u, q) \neq 0 \vee h(u, q) \neq 0] \end{array} \right\}.$$

Chapter 5

Eliminating Parametric Quantifiers: The Linear Case

In this chapter we are considering the easiest case of $f(z, q_1, \dots, q_n) = \sum_{i=0}^{k} a_i(q_1, \dots, q_n)z^i \in \mathbb{R}[z, q_1, \dots, q_n]$, where the a_i are linear in $q = (q_1, \dots, q_n)$.

All of the known material presented in this chapter is treated in depth in the textbooks [4] and [7], which I used as references.

5.1 The Linear Independent Case: Interval Polynomials

5.1.1 Kharitonov's Theorem

In 1978 Kharitonov presented a theorem that reduces the robust stability problem for an interval polynomial to the stability problem for only four polynomials [34, 35]. When this result found its way into western literature it initiated several research efforts into new approaches to the robustness problem[1].

Theorem 5.1 (Kharitonov [34, 35])
Let $Q \subseteq \mathbb{R}^n$ be of the form $Q = [\underline{q_1}; \overline{q_1}] \times \dots \times [\underline{q_n}; \overline{q_n}]$ with $0 \notin [\underline{q_n}; \overline{q_n}]$ and let $f \in \mathbb{R}[z, q_1, \dots, q_n]$ be of the form $f = q_1 + q_2 z + \dots + q_n z^{n-1}$. Then f is robustly stable with respect to Q if and only if the four ***Kharitonov***

[1] See [3] for an overview of the early history of this research.

polynomials

$$
\begin{array}{rcl}
k_1(z) & = & \underline{q_1} + \underline{q_2}z + \overline{q_3}z^2 + \overline{q_4}z^3 + \underline{q_5}z^4 + \underline{q_6}z^5 + \overline{q_7}z^6 + \dots \\
k_2(z) & = & \overline{q_1} + \overline{q_2}z + \underline{q_3}z^2 + \underline{q_4}z^3 + \overline{q_5}z^4 + \overline{q_6}z^5 + \underline{q_7}z^6 + \dots \\
k_3(z) & = & \overline{q_1} + \underline{q_2}z + \underline{q_3}z^2 + \overline{q_4}z^3 + \overline{q_5}z^4 + \underline{q_6}z^5 + \underline{q_7}z^6 + \dots \\
k_4(z) & = & \underline{q_1} + \overline{q_2}z + \overline{q_3}z^2 + \underline{q_4}z^3 + \underline{q_5}z^4 + \overline{q_6}z^5 + \overline{q_7}z^6 + \dots
\end{array}
$$

are stable.

Thus, for the special case of f being a polynomial whose coefficients are independent robustness parameters ranging over intervals, this theorem eliminates the quantifier "$\forall q \in Q$" from Problem 2.2 and reduces robust stability of a polynomial of arbitrary degree to stability of the four (parameter-free) Kharitonov polynomials k_1, k_2, k_3, and k_4. These four stability problems are of the form given as Problem 2.1, which can be solved by eliminating the universal quantifier "$\forall z \in \mathbb{C}$" by applying one of the methods presented in Chapter 3.

The original proof of the theorem is given in [34, 35]. Alternative proofs can be found in [12], [52], [9], and [39]. The following proof is new. It is based on the value set approach, i.e. Lemma A.4 (the zero exclusion principle) and Theorem 4.4 (the value set stability criterion).

Proof of Theorem 5.1: If $f(z,q)$ is stable for all $q \in Q$ then the four Kharitonov polynomials are obviously stable. Now assume that the four Kharitonov polynomials are stable. We will prove that $f(z,q)$ is stable for all $q \in Q$ by applying Lemma A.4. For this purpose we need to show that the four conditions of this lemma hold. Conditions (i) and (ii) follow immediately from the definition of Q and f; Condition (iii) follows from the stability of the Kharitonov polynomials. Thus it remains to show Condition (iv), i.e. $\forall w \in \mathbb{R} : \forall q \in Q : f(jw,q) \neq 0$.

We can write $f(jw,q)$ as $f_R(w,q) + jf_I(w,q)$ with

$$
\begin{array}{rcl}
f_R(w,q) & = & q_0 - q_2w^2 + q_4w^4 - q_6w^6 + \dots, \\
f_I(w,q) & = & q_1w - q_3w^3 + q_5w^5 - q_7w^7 + \dots
\end{array}
$$

and we see that $f(j\tilde{w},\tilde{q}) = 0$ is equivalent to both $f_R(\tilde{w},\tilde{q}) = 0$ and $f_I(\tilde{w},\tilde{q}) = 0$. Therefore we will show that if one of these polynomials is zero, the other must be non-zero. To do this we consider the **lower** and **upper bound**

polynomials of f_R and f_I: Defining

$$\begin{array}{rcl}
\underline{f_R}(w) & := & \underline{q_0} - \overline{q_2}w^2 + \underline{q_4}w^4 - \overline{q_6}w^6 + \dots\,, \\
\overline{f_R}(w) & := & \overline{q_0} - \underline{q_2}w^2 + \overline{q_4}w^4 - \underline{q_6}w^6 + \dots\,, \\
\underline{f_I}(w) & := & \underline{q_1}w - \overline{q_3}w^3 + \underline{q_5}w^5 - \overline{q_7}w^7 + \dots\,, \\
\overline{f_I}(w) & := & \overline{q_1}w - \underline{q_3}w^3 + \overline{q_5}w^5 - \underline{q_7}w^7 + \dots
\end{array}$$

we have for all $q \in Q$

$$\begin{array}{rcccl}
\underline{f_R}(w) & \leq & f_R(w,q) & \leq & \overline{f_R}(w), \\
\underline{f_I}(w) & \leq & f_I(w,q) & \leq & \overline{f_I}(w) \text{ for } 0 \leq w, \\
\overline{f_I}(w) & \leq & f_I(w,q) & \leq & \underline{f_I}(w) \text{ for } w \leq 0.
\end{array}$$

These lower and upper bound polynomials are related to the Kharitonov polynomials by

$$\begin{array}{rcl}
k_1(jw) & = & \underline{f_R}(w) + j\underline{f_I}(w) \\
k_2(jw) & = & \overline{f_R}(w) + j\overline{f_I}(w) \\
k_3(jw) & = & \overline{f_R}(w) + j\underline{f_I}(w) \\
k_4(jw) & = & \underline{f_R}(w) + j\overline{f_I}(w).
\end{array}$$

We see that for arbitrary but fixed w the set $\{f(jw,q) | \, q \in Q\}$ can be regarded as a rectangle parallel to the axes in the complex plane with the Kharitonov polynomials at its corners. This view was first presented by Dasgupta in [14]. Our goal is to show that due to this "bounding property" the Kharitonov polynomials force $f(jw,q)$ away from the origin. See Figure 5.1 for an illustration of this idea.

According to Lemma A.3 stability of the Kharitonov polynomials $k_i(z)$ implies that $\frac{d}{dw}\arg(k_i(jw)) > 0$. Applying Lemma A.2 yields

$$\underline{f_R}(w)\underline{f_I}'(w) \; > \; \underline{f_R}'(w)\underline{f_I}(w), \tag{5.1}$$
$$\overline{f_R}(w)\overline{f_I}'(w) \; > \; \overline{f_R}'(w)\overline{f_I}(w), \tag{5.2}$$
$$\overline{f_R}(w)\underline{f_I}'(w) \; > \; \overline{f_R}'(w)\underline{f_I}(w), \tag{5.3}$$
$$\underline{f_R}(w)\overline{f_I}'(w) \; > \; \underline{f_R}'(w)\overline{f_I}(w). \tag{5.4}$$

Now a case distinction provides the key argument of the proof:

- $\underline{f_R}(w) = 0$: (5.1) and (5.4) imply $\underline{f_R}'(w) \neq 0$.
  - $\underline{f_R}'(w) < 0$: (5.1) implies $0 < \underline{f_I}(w)$, (5.4) implies $0 < \overline{f_I}(w)$, and therefore $f_I(w,q) \neq 0$.

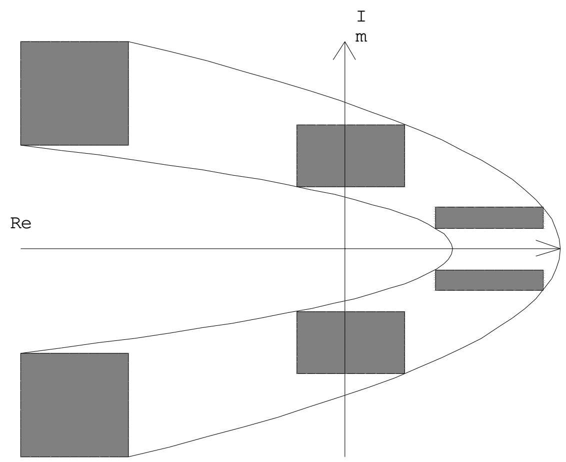

Figure 5.1: Intuition underlying Kharitonov's Theorem. *This figure shows* $\{f(jw, q_0, q_1) | (q_0, q_1) \in Q\}$ *for various values of* w *with* $f(z, q_0, q_1) = q_0 + q_1 z + z^2$, $Q = [1, 2]^2$. *The curves passing through the corners of these rectangles show the images of the imaginary axis generated by the Kharitonov polynomials* k_1 *and* k_2.

- $0 < \underline{f_R}'(w)$: (5.1) implies $\underline{f_I}(w) < 0$, (5.4) implies $\overline{f_I}(w) < 0$, and therefore $f_I(w, q) \neq 0$.

- $\overline{f_R}(w) = 0$: (5.2) and (5.3) imply $\overline{f_R}'(w) \neq 0$.
  - $\overline{f_R}'(w) < 0$: (5.3) implies $0 < \underline{f_I}(w)$, (5.2) implies $0 < \overline{f_I}(w)$, and therefore $f_I(w, q) \neq 0$.
  - $0 < \overline{f_R}'(w)$: (5.3) implies $\underline{f_I}(w) < 0$, (5.2) implies $\overline{f_I}(w) < 0$, and therefore $f_I(w, q) \neq 0$.

We can apply a similar case distinction to the imaginary lower and upper bounds and thus show that $f(jw, q)$ cannot be zero if either f_R or f_I is equal to a lower or upper bound polynomial.

To conclude the proof we must show that $f(jw, q)$ cannot be zero for arbitrary f_R and f_I. We do this by reducing these "interior" cases to the boundary case: The leading coefficient of f is 1, therefore there exists a $\tilde{w} > 0$ such that $f(j\tilde{w}, q)$ is non-zero for all $q \in Q$. Continuity of f implies that if $f_R(w, q)$ and $f_I(w, q)$ have a common zero then there must exist a w such that $f_R(w, q)$ and $f_I(w, q)$ have a common zero and one of them is equal to a lower or upper bound polynomial.

□

5.1.2 Stability Margin Computation

Kharitonov's theorem immediately suggests a method for computing the stability margin (Definition 2.5, page 11)

$$\mathrm{SSM}_f = \sup\{e \in \mathbb{R}^+ \,|\, \forall q \in [-e; e]^n : \ f(z,q) \text{ is stable}\}.$$

By applying Kharitonovs theorem we can replace the formula "$\forall q \in [-e; e]^n$: $f(z,q)$ is stable" by "$k_1(z,e)$, $k_2(z,e)$, $k_3(z,e)$, and $k_4(z,e)$ are stable". Thus SSM_f is the supremum of all $e \in \mathbb{R}^+$ such that the four Kharitonov polynomials are stable. Applying the Routh-Hurwitz stability criterion (Theorem 3.4, page 24), we see that SSM_f must be the minimum of the smallest positive roots of the Hurwitz minors (which are polynomials in e) of $k_1, \dots, k_4$.

The method outlined above was first presented in [2][2] and a refined version (which reduces the computation of SSM_f to the computation of the minimum of the smallest positive roots of the Hurwitz determinants of $k_1, \dots, k_4$) is given in [8]. In Theorem 4.6, page 40 and Recipe 6.2, page 59 we generalize this approach to the non-linear dependent case.

Applying Hurwitz determinants introduces a significant computational cost if the degree of f (and thus the size of the Hurwitz matrix) is large. Furthermore, the determinant computation does not retain the independent linearity of the parameters. Therefore, it would be preferrable to have a method that avoids the computational effort and high degrees caused by Hurwitz determinant computations and exploits the independent linearity. These requirements are fulfilled by the method given in [47], which computes a lower bound[3] on the stability margin by applying the Hermite-Biehler theorem (Theorem 4.1, page 34).

In the following we develop a new method based on the Hermite-Biehler theorem for (exactly) computing the stability margin in the linear, independent case. More precisely, we will solve the following problem:

Problem 5.1 (Linear, independent stability margin)
Let $f(z, q_1, \dots, q_n) = \sum_{i=0}^{n-1} a_i(q_{i+1})z^i \in \mathbb{R}[z, q_1, \dots, q_n]$ such that $f(z,0)$ is stable, $a_i(q_{i+1}) = c_i + q_{i+1}$ with $c_i \in \mathbb{R}$, and f is of constant degree with respect to $[-SSM_f; SSM_f]^n$. Compute SSM_f, the stability margin for f.

[2]This paper seems to have been one of the first publications in western literature that gave an application of Kharitonov's theorem.

[3]The introduction to the paper somewhat misleadingly mentions a "method of computing the maximum allowable perturbations".

The assumption about the constant degree of f looks somehow artificial. Alternatively, we could require that $a_{n-1}(q_n)$ — the leading coefficient of f with respect to z — be a constant. To avoid both this restriction and complications stemming from a degree drop of f, we introduce a "compromise" assumption about the degree of f.

As in the Hermite-Biehler theorem we consider $g, h \in \mathbb{R}[u, q_1, \ldots, q_n]$ such that $f(z,q) = h(z^2,q) + zg(z^2,q)$. For now, assume that $n = 2l+1$ for $l \in \mathbb{N}$. Then we have

$$\begin{aligned} g(u,q) &= \sum_{i=0}^{l-1} a_{2i+1}(q_{2i+2})u^i, \\ h(u,q) &= \sum_{i=0}^{l} a_{2i}(q_{2i+1})u^i. \end{aligned}$$

For arbitrary but fixed $e \in \mathbb{R}^+$ define the lower and upper bound polynomials[4] for g and h by

$$\begin{aligned} \underline{g}(u,e) &= \sum_{i=0}^{l-1} a_{2i+1}((-1)^{l+1}e)u^i, \\ \overline{g}(u,e) &= \sum_{i=0}^{l-1} a_{2i+1}((-1)^{l}e)u^i, \\ \underline{h}(u,e) &= \sum_{i=0}^{l} a_{2i}((-1)^{l+1}e)u^i, \\ \overline{h}(u,e) &= \sum_{i=0}^{l} a_{2i}((-1)^{l}e)u^i. \end{aligned}$$

These polynomials are such that for all $q \in [-e;e]^n$ and all $u \in \mathbb{R}^-$ we have

$$\begin{aligned} \underline{g}(u,e) &\le g(u,q) \le \overline{g}(u,e), \text{ and} \\ \underline{h}(u,e) &\le h(u,q) \le \overline{h}(u,e). \end{aligned}$$

As noted earlier, we can apply Kharitonov's theorem to rewrite the formula "$\forall q \in [-e;e]^n : f(z,q)$ is stable" as "$k_1(z,e)$, $k_2(z,e)$, $k_3(z,e)$, and $k_4(z,e)$ are stable", where the k_i are the Kharitonov polynomials. This means that

$$\mathrm{SSM}_f = min\{\mathrm{PSM}_{k_1}, \ldots, \mathrm{PSM}_{k_4}\}.$$

For $e = 0$ we have $k_i(z,e) = f(z,0)$ and thus $k_i(z,0)$ is stable for $i = 1, \ldots, 4$. This means that the requirements for computing the stability margin by

[4] Compare these polynomials with the lower and upper bound polynomials of f_R and f_I defined in the proof of Kharitonov's theorem (page 45).

purely imaginary roots (Theorem 4.5, page 39) are fulfilled and thus PSM_{k_i} is the smallest $e \in \mathbb{R}^+$ such that $k_i(z, e)$ has a zero on the imaginary axis.

Now observe that we can write the Kharitonov polynomials in the form

$$\begin{aligned} k_1(z,e) &= \underline{h}(z^2,e) + z\underline{g}(z^2,e), \\ k_2(z,e) &= \overline{h}(z^2,e) + z\overline{g}(z^2,e), \\ k_3(z,e) &= \overline{h}(z^2,e) + z\underline{g}(z^2,e), \\ k_4(z,e) &= \underline{h}(z^2,e) + z\overline{g}(z^2,e). \end{aligned}$$

Consider k_1: We apply the Hermite-Biehler test for purely imaginary roots (Lemma 4.2, page 34) and see that PSM_{k_1} is the minimum of $e_{(i)}, e_{(ii)} \in \mathbb{R}^+$, where

i. $e_{(i)}$ is the smallest e such that $\underline{h}(0, e) = 0$, and

ii. $e_{(ii)}$ is the smallest e such that $\exists u \in \mathbb{R}^- : \underline{g}(u, e) = 0 \wedge \underline{h}(u, e) = 0$.

Condition (i) simply says $a_0(-e) = 0$, i.e. $e_{(i)} = c_0$.

Condition (ii) can be written as a system

$$\begin{aligned} \underline{g}(u,e) &= \underline{g}(u,1)e + g(u,0) &= 0 \\ \underline{h}(u,e) &= \underline{h}(u,1)e + h(u,0) &= 0 \end{aligned}$$

of linear equations in e. Eliminating e from these equations yields

$$\underline{h}(u,1)g(u,0) - \underline{g}(u,1)h(u,0) = 0,$$

which is an equation of at most degree n in u. Given a solution $\tilde{u} \in \mathbb{R}^-$ of this equation, $\tilde{e} = -\frac{g(\tilde{u},0)}{\underline{g}(\tilde{u},1)}$ yields a common zero of $\underline{g}(\tilde{u}, \tilde{e})$ and $\underline{h}(\tilde{u}, \tilde{e})$, if such an $\tilde{e} \in \mathbb{R}^+$ exists. Denoting the set of negative, real solutions of the above equation by U we have

$$e_{(ii)} = min \left\{ e \in \mathbb{R}^+ \;\middle|\; e = -\frac{g(u,0)}{\underline{g}(u,1)} \text{ for } u \in U \right\}.$$

Similar arguments can be used to handle the case where $n = 2l$ and to derive formulae for PSM_{k_2}, PSM_{k_3}, and PSM_{k_4}. In this context note that $\underline{g}(u, 1) = -\overline{g}(u, 1)$ and $\underline{h}(u, 1) = -\overline{h}(u, 1)$.

This analysis leads to the following recipe for solving the problem of stability margin computation in the linear, independent case.

Recipe 5.1 (Linear, independent stability margin)
Given *$f(z, q_1, \ldots, q_n) = \sum_{i=0}^{n-1} a_i(q_{i+1})z^i \in \mathbb{R}[z, q_1, \ldots, q_n]$ such that $f(z, 0)$ is stable, $a_i(q_{i+1}) = c_i + q_{i+1}$ with $c_i \in \mathbb{R}$, and f is of constant degree with respect to $[-SSM_f; SSM_f]^n$;*
compute *SSM_f, the stability margin for f.*

1. Let

$$\begin{aligned}
\underline{g}(u,e) &= \sum_{i=0}^{l-1} a_{2i+1}((-1)^{l+1}e)u^i, \\
\overline{g}(u,e) &= \sum_{i=0}^{l-1} a_{2i+1}((-1)^{l}e)u^i, \\
\underline{h}(u,e) &= \sum_{i=0}^{l} a_{2i}((-1)^{l+1}e)u^i, \\
\overline{h}(u,e) &= \sum_{i=0}^{l} a_{2i}((-1)^{l}e)u^i.
\end{aligned}$$

2. Let U be the set of negative real roots of

$$\underline{h}(u,1)g(u,0) - \underline{g}(u,1)h(u,0) = 0$$

and

$$\underline{h}(u,1)g(u,0) + \underline{g}(u,1)h(u,0) = 0.$$

3. Let

$$E := \{e \in \mathbb{R}^+ \,|\, e = \left|\frac{g(u,0)}{\underline{g}(u,1)}\right|, u \in U\}.$$

4. $SSM_f = \min E \cup \{|c_0|\}$.

To handle the a-priory restriction on the constant degree of f we can do the following: If $\min E \cup \{|c_0|\} \leq |c_{n-1}|$, then this minimum indeed is equal to the stability margin SSM_f. Otherwise, $|c_{n-1}|$ is a lower bound on SSM_f.

5.2 The Linear Dependent Case: Polytopes of Polynomials

Kharitonov's theorem inspired researchers to look into possible extensions to larger classes of polynomials. One obvious choice are polynomials whose coefficients are linear combinations of parameters.

If we identify a k-dimensional polynomial $f(z) = a_k z^k + a_{k-1} z^{k-1} + \ldots + a_0$ with the vector of its coefficients $(a_k, a_{k-1}, \ldots, a_0)$, then we can identify the set $f(z, Q)$ of parameterized polynomials of degree k with $Q \subseteq \mathbb{R}^n$ with a set $F \subseteq \mathbb{R}^{k+1}$. When Q is a box and the coefficients $a_i(q)$ of f are linear in q then F is a polytope. This observation motivates the following definition.

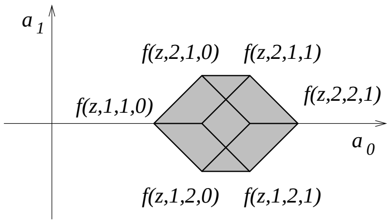

Figure 5.2: A polytope of polynomials. $f(z, q_1, q_2, q_3) = z^2 + a_1 z + a_0$ *with* $a_1 = q_1 - q_2$, $a_0 = q_1 + q_2 + q_3$, $q \in Q = [1; 2] \times [1; 2] \times [0; 1]$.

Definition 5.1 (Polytope of polynomials)
Let $Q \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[z, q_1, \ldots, q_n]$, $F \subseteq \mathbb{R}^{k+1}$ *the set of coefficient vectors of* $f(z, q)$ *for* $q \in Q$. *If* F *is a polytope (the convex hull of a finite number of points) then* $f(z, Q)$ *is called a* ***polytope of polynomials***.

From now on we will not distinguish between sets of polynomials of degree k and corresponding sets of $k+1$-dimensional vectors of coefficients, if there is no danger of confusion.

As mentioned earlier, if $Q \subseteq \mathbb{R}^n$ is of the form $Q = [\underline{q_1}; \overline{q_1}] \times \ldots \times [\underline{q_n}; \overline{q_n}]$ and $f \in \mathbb{R}[z, q_1, \ldots, q_n]$ is of the form $f(z) = f_0(z) + \sum_{i=1}^{n} q_i f_i(z)$, then $f(z, Q)$ is a polytope. This is the form that we most often encounter in applications. An example polytope is shown in Figure 5.2.

The following definition helps us to distinuish certain subsets of the set of points comprising a polytope.

Definition 5.2 (Faces, edges, and vertices)
Given hyperplanes H *that intersect only the boundary of a polytope* F, *i.e.* $F \cap H \subseteq \partial F$, *the two-dimensional intersections are called* ***faces***[5], *the one-dimensional intersections are called* ***edges***, *and the zero-dimensional intersections are called* ***vertices***.

The polynomials corresponding to faces, edges, and vertices are respectively called ***edge***, ***face***, *and* ***vertex polynomials***.

5.2.1 The Edge Theorem

The *Edge Theorem* [6] was inspired by Kharitonov's result. It shows that the stability problem for a polytope of polynomials can be reduced to checking

[5]In Computational Geometry the term *faces* is often used to denote polytope boundary elements of any dimension, whereas two-dimensional faces are denoted as *2-faces*. We do not require such flexibility of notation because we will not consider higher-dimensional faces.

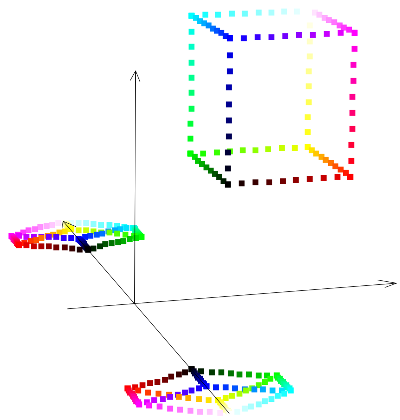

Figure 5.3: Edge roots of a polytope of polynomials. $f(z, q_1, q_2, q_3) = z^2 + a_1 z + a_0$ *with* $a_1 = q_1 - q_2$, $a_0 = q_1 + q_2 + q_3$. *The parameter set* $Q = [1; 2] \times [1; 2] \times [0; 1]$ *is sketched on the upper right and the roots of the polynomials corresponding to the edges of this cube are plotted on the lower left. The polytope* $f(z, Q)$ *is shown in Figure 5.2.*

the stability of the edge polynomials of this polytope.

Theorem 5.2 (Edge Theorem [6])
Let $Q \subseteq \mathbb{R}^n$, $f \in \mathbb{R}[z, q_1, \ldots, q_n]$ *such that* $f(z, Q)$ *is a polytope of polynomials of constant degree. Then* f *is robustly stable with respect to* Q *if and only if the polynomials corresponding to the edges of* $f(z, Q)$ *are stable.*

This theorem is illustrated in Figure 5.3.

The main argument used in the proof of the Edge Theorem (see [6]) is the fact that the roots of a polynomial depend continuously on its coefficients. If the set of coefficients is a polytope, then robust stability is equivalent to the boundary of the set of the roots of $f(z, Q)$ being in the left open halfplane. Furthermore, we can prove that this boundary is a subset of the set of the edge polynomials' roots.

5.2.2 The Convex Directions Theorem

The Edge Theorem reduces robust stability of a polytope of polynomials to robust stability of its edge polynomials. The question is: Can we do even better? The *convex directions theorem* [42] defines conditions under which

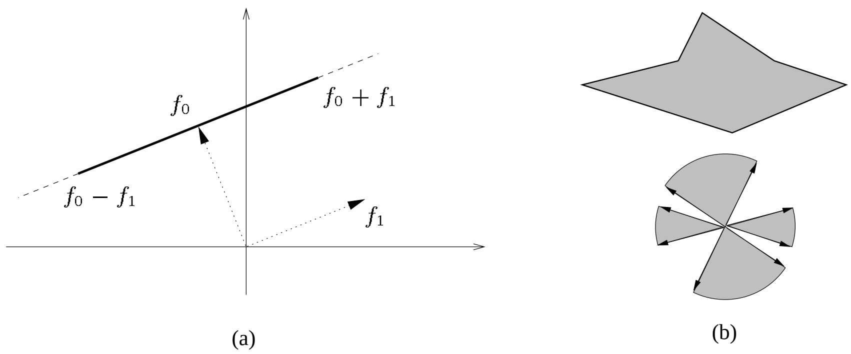

Figure 5.4: Convex directions. *(a) If f_1 is a convex direction and both $f_0 - f_1$ and $f_0 + f_1$ are stable, then all polynomials on the segment from $f_0 - f_1$ to $f_0 + f_1$ are stable. (b) The directions corresponding to the shaded sectors of the circle are convex directions for the area above.*

robust stability of edge polynomials reduces to stability of vertex polynomials.

The notion of convex directions describes the "directions" $f_1 \in \mathbb{R}[z]$ with respect to which the set of stable polynomials is convex: A polynomial f_1 is called a convex direction if for any polynomial f_0 stability of the "endpoints" $f_0(z) - f_1(z)$ and $f_0(z) + f_1(z)$ implies that all polynomials on the edge $f_0(z) + qf_1(z)$ between these endpoints are stable. This notion is illustrated in Figure 5.4 and formalized in the following definition.

Definition 5.3 (Convex direction)
Let $f(z,q) = f_0(z) + qf_1(z) \in \mathbb{R}[z,q]$, $Q = [-1;1]$, $f_1 \in \mathbb{R}[z]$, $4 \leq d \in \mathbb{N}$, $\deg(f_1) \leq d$. f_1 is called a ***convex direction*** *if*

$$\begin{aligned} \forall f_0 \in \mathbb{R}[z]: \quad & \forall q \in Q : [\deg(f_0 + qf_1) = d] \wedge \\ & f_0 - f_1 \textit{ is stable } \wedge f_0 + f_1 \textit{ is stable} \\ & \Rightarrow \\ & f_0 + qf_1 \textit{ is robustly stable with respect to } Q. \end{aligned}$$

The following theorem gives a condition that classifies convex directions.

Theorem 5.3 (Convex directions theorem [42])
$f_1 \in \mathbb{R}[z]$ is a convex direction if and only if the argument change (if defined) of the image of the imaginary axis under f_1 is bounded as specified by the following formula:

$$\forall \tilde{w} \in \mathbb{R}: \; f_1(j\tilde{w}) \neq 0 \Rightarrow \left[\frac{d}{dw}\arg(f_1(jw))\right]_{w=\tilde{w}} \leq \left|\frac{\sin(2\arg(f_1(j\tilde{w})))}{2\tilde{w}}\right|.$$

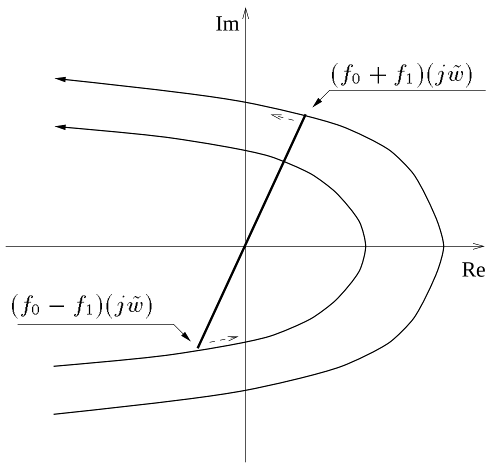

Figure 5.5: Intuition underlying the convex directions theorem. *This figure shows a situation where the segment from $f_0 - f_1$ to $f_0 + f_1$ passes through the origin. The end point polynomials are stable, therefore the argument change of the segment must be a linear combination of the argument change at the end points.*

The proof of Theorem 5.3 is based on the following observations:

- From Lemma A.4 (the Zero Exclusion Principle) we see that robust stability is ensured if $(f_0 + qf_1)(jw)$ is nonzero for any $q \in Q$ and $w \in \mathbb{R}$.
- From Lemma A.3 we see that stability of the polynomials at the ends of the segment $f_0 + qf_1$ implies that the arguments of $(f_0 - f_1)(jw)$ and $(f_0 + f_1)(jw)$ increase as w increases.[6]
- Combining the previous two observations with the graphical intuition given in Figure 5.5 shows that if the segment $(f_0 + qf_1)(jw)$ passes through the origin at any w then its argument change must be large. Thus if the argument change of f_1 is properly bounded[7] from above, the segment cannot pass through the origin and thus it is stable.

[6] This increasing argument for stable polynomials may seem contradictory to the *upper* bound on the argument of f_1 appearing in the theorem. The point is that the stability of $f = f_0 + qf_1$ is considered, not the stability of f_1. The latter need not be stable and it is indeed the case that most convex directions are instable. This is not surprising considering the fact that the convex directions theorem gives an upper bound on the argument change of convex directions, whereas Lemma A.3 gives a lower bound on the argument change of a stable polynomial.

[7] Lemma A.3 gives a bound of zero, which is correct for complex polynomials, but is

See [42] for a detailed proof.

The next theorem generalizes the convex directions theorem from a single edge to polytopes of polynomials.

Theorem 5.4 (Rantzer [42])
Let $Q \in \mathbb{R}^n$, $f_0 \in \mathbb{R}[z]$ and $f_1 \in \mathbb{R}[z, q_1, \ldots, q_n]$ such that $f_1(z, Q)$ is a polytope of polynomials, $4 \leq d \in \mathbb{N}$, $\deg(f_1, q) \leq d$ for $q \in Q$, $f(z, q) = f_0(z) + f_1(z, q)$, $T \subseteq Q$. Then the following conditions are equivalent:

1. *$f(z, t)$ is robustly stable with respect to T, where T is such that $f_1(jw, T)$ contains all vertices of $f_1(jw, Q)$ and all edges of $f_1(jw, Q)$ that are no convex directions.*
2. *If $f(z, Q)$ is of constant degree for all $q \in Q$ then $f(z, q)$ is robustly stable with respect to Q.*

See [42] for a proof.

Kharitonov's Theorem can be derived from this theorem because the edge polynomials of the Kharitonov rectangle contain only odd or even powers of z; it can be shown that all polynomials of this form are convex directions.

not sharp for real polynomials. The bound on the argument derivative given in the convex directions theorem is based on Lemma 11 of [42], which gives a sharp bound for real polynomials.

Chapter 6

Eliminating Parametric Quantifiers: The Nonlinear Case

In this chapter we are considering the most general case of $f(z, q_1, \ldots, q_n) = \sum_{i=0}^{k} a_i(q_1, \ldots, q_n)z^i \in \mathbb{R}[z, q_1, \ldots, q_n]$, where the a_i are arbitrary multivariate polynomials in $q = (q_1, \ldots, q_n)$.

Iterative methods for computing lower and upper bounds on the stability margin have been given for the multilinear case e.g. in [45] and [46] and for the nonlinear case e.g. in [17], [20], [45], and [50]. In Section 6.2 we give a new direct formula for computing a lower bound on the stability margin and in Section 6.2 we give a new exact method for computing the stability margin.

These methods (partially) solve the problem of computing stability margins (Definition 2.5, page 11 and Definition 2.6, page 12), which in turn leads to a (partial) solution[1] of the problem of robust stability (Problem 2.2, page 9).

6.1 Single-Parameter Stability Margin Computation

If f depends on only one parameter, i.e. $f \in \mathbb{R}[z, q_1]$, then from Corollary 4.6, page 40 we immediately see that we can compute SSM_f from the roots of the Hurwitz determinant D of f: From the assumption that $f(z, 0)$ is stable and

[1] I implemented Recipes 6.1, 6.2, and 6.3 in MuPAD [18]. Sources are available upon request from `andreas.neubacher@risc.uni-linz.ac.at`.

from the Routh-Hurwitz stability criterion (Theorem 3.4, page 24) it follows that $D(0) > 0$ if the leading coefficient of $f(z, 0)$ is positive. The observation that

$$\begin{aligned} \mathrm{SSM}_f &= \sup\{e \in \mathbb{R}^+ \,|\, \forall q_1 \in [-e; e] : D(q_1) > 0\} \quad \text{(Corollary 4.6)} \\ &= \min\{|e| \,|\, e \text{ is a real root of } D\} \end{aligned}$$

leads to the following recipe:

Recipe 6.1 (Single-parameter stability margin)
Given $f(z, q_1) = \sum_{i=0}^{k} a_i(q_1) z^i \in \mathbb{R}[z, q_1]$ *with* $f(z, 0)$ *stable and* $a_k(q_1) \neq 0$ *for* $q_1 \in [-SSM_f; SSM_f]$;
compute SSM_f.

1. *If* $a_k(0) < 0$ *then let* $g = -f$. *Otherwise, let* $g = f$.

2. *Let* $D \in \mathbb{R}[q_1]$ *be the Hurwitz determinant of* g.

3. *Let* $SSM_f = \min\{|e| \,|\, e$ *is a real root of* $D\}$.

If we cannot assume that $a_k(q_1) \neq 0$ for $q_1 \in [-\mathrm{SSM}_f; \mathrm{SSM}_f]$ then we can replace Step 3 by

3. *Let* $B_1 = \min\{|e| \,|\, e$ *is a real root of* $D\}$, $B_2 = \min\{|e| \,|\, e$ *is a real root of* $a_k\}$.

   If $B_1 \leq B_2$ *then* $SSM_f = B_1$. *Otherwise* $B_2 < SSM_f$, *i.e. we only get a lower bound on* SSM_f.

For computing PSM_f we can work through similar arguments and finally see that the positive stability margin is equal to the smallest positive root of D, i.e. $\mathrm{PSM}_f = \min\{e \,|\, e$ is a positive real root of $D\}$. Otherwise, the recipe for computing PSM_f is identical to that for computing SSM_f.

Note that in the single-parameter case we can completely solve the problems of robust and parametric stability by univariate root computation, because computing $\mathrm{PSM}_{f(z)}$ and $\mathrm{PSM}_{f(-z)}$ leads to the following observation: Let $D \in \mathbb{R}[q_1]$ be the Hurwitz determinant of f, let $\underline{Q}$ be the largest negative real root of D, and let $\overline{Q}$ be the smallest positive real root of D. Then

$$\{q_1 \in \mathbb{R} \,|\, f(z, q_1) \text{ is stable}\} =]\underline{Q}; \overline{Q}[\,,$$

which means we can exactly compute the set of stabilizing parameters for f.

6.2 Multi-Parameter Stability Margin Lower Bound Computation

If the parameter space is multidimensional, Corollary 4.6 does not allow us to reduce stability margin computation to root computation as in the single-parameter case. Nevertheless, we can compute a *lower bound* on PSM_f (and thereby on SSM_f) by applying the lower bound for positiveness given in Section A.3.2.

Recipe 6.2 (Lower bound on PSM_f)
Given $f(z, q_1, \ldots, q_n) = \sum_{i=0}^{k} a_i(q_1, \ldots, q_n) z^i \in \mathbb{R}[z, q_1, \ldots, q_n]$ *with* $f(z, 0)$ *stable;*
compute $B \in \mathbb{R}^+$ *such that* $B \leq PSM_f$.

1. *If* $a_k(0) < 0$ *then let* $g = -f$. *Otherwise, let* $g = f$.
2. *Let* $D \in \mathbb{R}[q_1, \ldots, q_n]$ *be the Hurwitz determinant of* g.
3. *Let* $B = \min\{\check{B}_{a_k}, \check{B}_D\}$ *by applying the lower bound on positiveness according to Theorem A.8, page 72.*

A lower bound for PSM_f can immediately be extended to a lower bound for SSM_f as follows: For $s \in \{-1, 1\}^n$ let $g_s(z, q_1, \ldots, q_n) = f(z, s_1 q_1, \ldots, s_n q_n)$. Then

$$\min\{B \in \mathbb{R}^+ \,|\, \exists s \in \{-1, 1\}^n : B \leq \mathrm{PSM}_{g_s}\} \leq \mathrm{SSM}_f.$$

Considering that this formula requires 2^n computations of PSM_{g_s} (one for each of the 2^n "hyperquadrants" of $\mathbb{R}^n$ represented by s), it may be preferrable not to take the minimum but to rather store PSM_{g_s} separately for each s, so that no information is lost.

6.3 Multi-Parameter Exact Stability Margin Computation

To compute the stability margin exactly, we consider the problem of stability margin computation as an optimization problem. To get differentiability we will switch from Definition 2.5, page 11, which uses the maximum norm, to the following definition based on the l_2 norm.

Definition 6.1 (Stability radius)
Let $f \in \mathbb{R}[z, q_1, \ldots, q_n]$ *such that* $f(z, 0)$ *is stable. Then*

$$SR_f := \sup\{e \in \mathbb{R}^+ \,|\, \forall q \in \mathbb{R}^n, \|q\|_2 \leq e : f(z, q) \text{ is stable}\}$$

is called ***the (symmetric) stability radius of*** f.

We immediately see that $\frac{1}{\sqrt{2}}\mathrm{SR}_f \leq \mathrm{SSM}_f \leq \mathrm{SR}_f$, so we can consider SR_f as a bound on the stability margin.

We derive the following lemma using similar arguments as when we derived Corollary 4.7, page 41 for SSM_f.

Lemma 6.1 (Stability radius by Hermite-Biehler)
$f \in \mathbb{R}[z, q_1, \dots, q_n]$ with $f(z, 0)$ stable and let $g, h \in \mathbb{R}[u, q_1, \dots, q_n]$ be such that $f(z, q) = h(z^2, q) + zg(z^2, q)$. If f is of constant degree with respect to the ball with radius SR_f then

$$SR_f = \min\{\|q\|_2 \,|\, q \in \mathbb{R}^n \,\wedge\, (h(0, q) = 0 \vee \exists u \in \mathbb{R}^n : g(u, q) = h(u, q) = 0)\}.$$

Thus we arrive at the smooth, convex, nonlinear programs

$$\begin{array}{l} \min \|q\|_2^2 \\ \text{for } q \in \mathbb{R}^n \text{ subject to} \\ h(0, q) = 0, \end{array} \tag{6.1}$$

and

$$\begin{array}{l} \min \|q\|_2^2 \\ \text{for } q \in \mathbb{R}^n \text{ subject to} \\ g(u, q) = 0 \,\wedge\, h(u, q) = 0, \end{array} \tag{6.2}$$

where $\|q\|_2^2 = \sum_{i=1}^n q_i^2$. If $q^{(6.1)}$ and $q^{(6.2)}$ are solutions to these programs then we have

$$\mathrm{SR}_f = \min\{\sqrt{q^{(6.1)}}, \sqrt{q^{(6.2)}}\}.$$

Introducing Laplace multipliers we get the following conditions on solutions to these programs: If $\tilde{q}$ is a solution to Program 6.1, then there exists a non-zero $\lambda_h \in \mathbb{R}$ such that

$$\begin{aligned} h(0, \tilde{q}) &= 0 \,\wedge \\ \nabla \|q\|_2^2 - \lambda_h \nabla h(0, q) &= 0, \end{aligned}$$

or

$$\begin{aligned} h(0, \tilde{q}) &= 0 \,\wedge \\ \lambda_h \nabla h(0, q) &= 0. \end{aligned}$$

If $\tilde{q}$ is a solution to Program 6.2, then there exist non-zero $\lambda_g, \lambda_h \in \mathbb{R}$ such that

$$\begin{aligned} g(u, \tilde{q}) &= 0 \,\wedge \\ h(u, \tilde{q}) &= 0 \,\wedge \\ \nabla \|q\|_2^2 - \lambda_g \nabla g(u, q) - \lambda_h \nabla h(u, q) &= 0, \end{aligned}$$

or

$$\begin{aligned} g(u,\tilde{q}) &= 0 \wedge \\ h(u,\tilde{q}) &= 0 \wedge \\ -\lambda_g \nabla g(u,q) - \lambda_h \nabla h(u,q) &= 0. \end{aligned}$$

These four conditions are systems of polynomial equations, which we can solve by computing these systems' Gröbner bases with respect to the lexicographic ordering (see Section A.4).

A non-zero stability radius always exists, therefore at least one of these systems must have a solution. If there are infinitely many solutions, then due to convexity $\|q\|_2^2$ must be the same for each connected set of solutions. Therefore, a finite number of univariate root computations yields a finite set of points that contains the minimum.

We summarize the method outlined above in the following recipe.

Recipe 6.3 (Stability radius computation)
Given *$f \in \mathbb{R}[z, q_1, \ldots, q_n]$ such that $f(z,0)$ is stable;*
compute *SR_f.*

1. *Let $g, h \in \mathbb{R}[u, q_1, \ldots, q_n]$ be such that $f(z,q) = h(z^2,q) + zg(z^2,q)$.*

2. *Let*

$$\begin{aligned} F_1 &:= \{h(0,q)\} \cup \{2q_i - \lambda_h \frac{d}{dq_i} h(0,q) | \, i = 1,\ldots,n\} \\ F_2 &:= \{h(0,q)\} \cup \{\frac{d}{dq_i} h(0,q) | \, i = 1,\ldots,n\} \\ F_3 &:= \{g(u,q), h(u,q), \lambda_g \frac{d}{du} g(u,q) + \lambda_h \frac{d}{du} h(u,q)\} \cup \\ & \{2q_i - \lambda_g \frac{d}{dq_i} g(u,q) - \lambda_h \frac{d}{dq_i} h(u,q) | \, i = 1,\ldots,n\} \\ F_4 &:= \{g(u,q), h(u,q), \lambda_g \frac{d}{du} g(u,q) + \lambda_h \frac{d}{du} h(u,q)\} \cup \\ & \{\lambda_g \frac{d}{dq_i} g(u,q) + \lambda_h \frac{d}{dq_i} h(u,q) | \, i = 1,\ldots,n\} \end{aligned}$$

3. *Compute the Gröbner bases $G_1, \ldots, G_4$ with respect to the lexicographic ordering for $F_1, \ldots, F_4$.*

4. *Compute the sets of common zeros $Z_1, \ldots, Z_4$ for $G_1, \ldots, G_4$.*

5. *$SR_f = \min\{\tilde{q}_1, \ldots, \tilde{q}_4\}$, where the $\tilde{q}_i$ are such that $\|\tilde{q}_i\| = \min_{q \in Z_i} \|q\|_2$. If Z_i is empty, then $\tilde{q}_i := \infty$.*

Chapter 7

Conclusions

I feel that the approach taken in this thesis shows that putting problems into a general quantifier elimination framework makes "goal driven" problem solving easier. For example, finding a solution to the general problem of parametric robust stability (Problem 2.4, page 10) can be viewed as the elimination of various quantifiers. Such a formulation makes it easy to split off sub-problems (such as eliminating the non-parametric universal quantifier) while keeping a clear picture of the exact requirements remaining after these sub-problems have been solved.

In this thesis we mostly concentrated on stability margin computation. In this context the exact method based on Gröbner bases (Recipe 6.3, page 61) could probably be speeded up in the multilinear case (which we did not treat separately in this thesis) by working with a special implementation of the Gröbner bases method. The quality of the method for computing a lower bound on the stability margin (Recipe 6.2, page 59) should be investigated. Alternatively, a related upper bound should be devised so that the exact value of the stability margin can be bounded from below and above.

A promising area that is not covered in this thesis is computation of the quality of stability, either by computing the damping ratio or the stability measure in root space according to Definitions 2.7 and 2.8, page 13. It might also be of interest to solve design problems where one is looking for parameters that guarantee a certain damping ratio. For this purpose one could reformulate the stability margin by requiring that the nominal solution at least fulfill the required damping ratio and that all solutions in the box defined by the stability margin fulfill it, too.

To enhance our understanding of the behavior of stable polynomials more results in root space would be useful. For example, one could prove the proposition[1] that in the stable linear independent case and for stable polynomials on an edge along a convex direction the root closest to the imaginary axis must be a root of a vertex polynomial.

The method for approximate quantifier elimination presented in Appendix B is not sufficiently efficient to be used for large problems. Much work both in data structuring and algorithm development remains to be done in this direction.

[1] Proposed by Hoon Hong in private correspondance.

Appendix A

Algebraic Tools

This chapter gives an overview of several results that are not directly related to stability. These results are used as fundamental tools in the context of solving the problem of parametric robust stability.

A.1 Resultants

The following notions and result from resultant theory are taken from [38]; most textbooks on algebra or symbolic computation cover this topic.

Definition A.1 (Sylvester matrix)
*Let $g(u) = \sum_{i=0}^{m} b_i u^i \in \mathbb{R}[u]$ and $h(u) = \sum_{i=0}^{n} c_i u^i \in \mathbb{R}[u]$. The **Sylvester matrix of** g **and** h is defined to be*

$$\begin{pmatrix} b_m & b_{m-1} & \dots & b_0 & & & & \\ & b_m & b_{m-1} & \dots & b_0 & & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & b_m & b_{m-1} & & \dots & b_0 \\ c_n & c_{n-1} & \dots & c_0 & & & & \\ & c_n & c_{n-1} & \dots & c_0 & & & \\ & & \ddots & \ddots & & & \ddots & \\ & & & c_n & c_{n-1} & & \dots & c_0 \end{pmatrix},$$

with n rows of coefficients of g and m rows of coefficients of h. Entries left empty are defined to be zero.

Definition A.2 (Resultant)
*Let $g, h \in \mathbb{R}[u]$. The **resultant** $\mathrm{res}(g, h)$ **of** g **and** h is defined to be the determinant of the **Sylvester matrix** of g and h.*

Theorem A.1 (Fundamental theorem of resultants)
Let $g, h \in \mathbb{R}[u]$. The resultant of g and h is zero if and only if g and h have a common zero.

A.2 Some Properties of Real and Complex Polynomials

In this section we list several well-known elementary results about polynomials. We give proofs for these lemmata if we find that they give valuable intuition about polynomials in the context of robust stability.

A.2.1 Argument Change

Stability is related to the location of the zeros of a polynomial, therefore the location of a polynomial's values with respect to the origin is of interest. One aspect of the location of a complex number is its argument. The following two lemmas show properties of the argument derivative of polynomials and relate it to stability.

Lemma A.2 (Argument derivative)
Let $f \in \mathbb{C}[w]$, $f_R, f_I \in \mathbb{R}[w]$ with $f(w) = f_R(w) + j f_I(w)$. Then

$$\frac{d}{dw} \arg(f(w)) = \operatorname{Im}\left(\frac{f'(w)}{f(w)}\right) = \frac{f_R(w) f_I'(w) - f_R'(w) f_I(w)}{f_R^2(w) + f_I^2(w)}.$$

This lemma follows immediately from $\arg(f(w)) = \operatorname{Im}(\log(f(w)))$.

Lemma A.3 (Increasing argument)
Let $f \in \mathbb{R}[z]$. If f is stable, then

$$\forall \tilde{w} \in \mathbb{R} : \left[\frac{d}{dw} \arg(f(jw))\right]_{w=\tilde{w}} > 0.$$

We can interpret the statement of this lemma as saying that the origin is to the "left" of the image of the imaginary axis (see Figure A.1). Zero is a sharp lower bound only for complex polynomials. Lemma 11 of [42] provides a sharp lower bound for real polynomials (which is greater than zero) based on an inductive proof; [33] proves the correctness of this bound directly by applying the Hermite-Biehler theorem (Theorem 4.1, page 34).

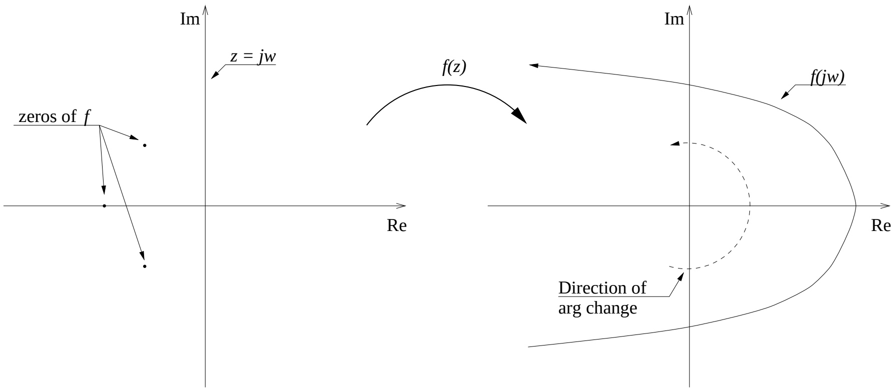

Figure A.1: Argument change of the imaginary axis image. *The zeros of a stable polynomial f are to the left of the imaginary axis jw, therefore the origin must be to the left of the image $f(jw)$ of the imaginary axis and thus the argument must increase as w increases.*

Proof of Lemma A.3: Let $f(z) = \prod(z - z_i)$. For the zeros z_i of f stability of f implies $\mathrm{Re}\,(z_i) < 0$. Furthermore $\arg(f(z)) = \sum \arg(z - z_i)$ and $\arg'(f(z)) = \sum \arg'(z - z_i)$. Using Lemma A.2 we have for any z_i

$$\begin{aligned}
\frac{d}{dw}\arg(jw - z_i) &= \mathrm{Im}\left(\frac{j}{jw - z_i}\right) \\
&= \mathrm{Re}\left(-\frac{jw + \overline{z_i}}{w^2 + |z_i|^2}\right) \\
&= -\frac{1}{w^2 + |z_i|^2}\mathrm{Re}\,(z_i) \\
&> 0,
\end{aligned}$$

and thus $0 < \sum \frac{d}{dw}\arg(jw - z_i) = \frac{d}{dw}\arg(f(jw))$.

□

Note that positive argument derivative is not sufficient for stability. A counterexample is $f(z) = (z + 2)(z - 1)$, which has positive argument derivative but is not stable, because it has a zero in the right half plane.

A.2.2 The Zero Exclusion Principle

The **zero exclusion principle**[1] gives conditions under which the robust stability of a polynomial can be reduced to the existence of roots on the

[1] Also known as "zero exclusion condition" or "zero elimination property"

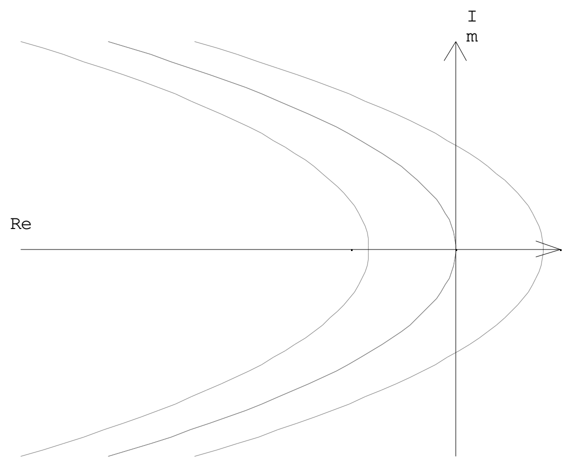

Figure A.2: Illustration of the zero exclusion principle. *This figure shows the imaginary axis images with respect to* $f(z, q_1) = z^2 + z + (q_1 - 1)$ *for* $q_1 \in \{0, 1, 2\}$*. The leftmost curve (instable case) corresponds to* $f(j\omega, 0)$*, the middle curve (instable boundary case) corresponds to* $f(j\omega, 1)$*, and the rightmost curve (stable case) corresponds to* $f(j\omega, 2)$*.*

imaginary axis.

Lemma A.4 (Zero exclusion principle)
Let $f \in \mathbb{R}[z, q_1, \ldots, q_n]$*,* $Q \subseteq \mathbb{R}^n$*. If*

i. Q *is connected, and*

ii. $f(z, Q)$ *is of constant degree, and*

iii. $\exists \tilde{q} \in Q : f(z, \tilde{q})$ *is stable, and*

iv. $\forall \omega \in \mathbb{R} : \forall q \in Q : f(j\omega, q) \neq 0$,

then $f(z, q)$ *is robustly stable with respect to* Q*.*

Figure A.2 illustrates the intuition behind the zero exclusion principle: In this example we have $f(z, q_1) = z^2 + z + (q_1 - 1)$. n is 1, therefore Condition (i) is fulfilled for any Q that is an interval $[\underline{q}; \overline{q}]$; q_1 does not appear in the leading coefficient of f, therefore Condition (ii) is fulfilled for any $q_1 \in \mathbb{R}$. Now we can distinguish three cases:

- If all elements of Q are greater than 1, f is robustly stable with respect to Q.

- If all elements of Q are less than 1, Condition (iii) is violated, i.e. no $f(z, q_1)$ is stable.

- If Q contains 1, Condition (iv) is violated, i.e. the image of the imaginary axis passes through the origin as $q_1 \in Q$ crosses the boundary from the stabilizing region (> 1) into the non-stabilizing region (< 1).

Most interesting problems fulfill Conditions (i) and (ii). Thus Lemma A.4 means that if we can guarantee the existence of a single stabilizing parameter in Q (so that Condition (iii) is fulfilled), then the robust stability property (i.e. all zeros of $f(z, Q)$ are in the open left halfplane) is reduced to checking whether no zero of $f(z, Q)$ is on the imaginary axis. Condition (iii) is always fulfilled for stability margin problems (Section 2.2.2), so we will use the zero exclusion principle as the main tool for computing stability margins.

The proof of Lemma A.4 is based on the continuous dependency of the zeros of a polynomial on its coefficients and the fact that the imaginary axis is the boundary between the left and the right halfplane, i.e. the regions where respectively all and no zeros must be located to ensure stability.

Proof of Lemma A.4: We will prove the implication directly, therefore assume that (i), ... , (iv) hold.

Let $Q = Q_0 \cup \ldots \cup Q_d$ where $q \in Q_i$ if $f(z, q)$ has exactly i zeros in the open left halfplane. Due to (iii) we know that Q_d is not empty. Now we distinguish among two cases:

- If $Q_0, \ldots, Q_{d-1}$ are empty, then $Q = Q_d$. In this case $f(z, q)$ is stable for all $q \in Q$; qed.

- Otherwise, there exists at least one non-empty Q_i with $i < d$ (i.e. f is not stable). From (i) we see that there must be a continuous path in Q from any element in Q_d to any element in Q_i. The zeros of $f(z, q)$ are all in the open left halfplane for $q \in Q_d$, (ii) guarantees that there is at least one zero of $f(z, q)$ outside the open left halfplane for $q \in Q_i$, and the zeros depend continuously on the q. Therefore there must be a $\tilde{q}$ on the path s.t. $f(z, \tilde{q})$ has at least one zero on the imaginary axis. This is a contradiction to (iv).

$\square$

A.2.3 Real Roots Bounded by Intervals

The Hermite-Biehler theorem (Theorem 4.1, page 34) associates stability with the locations of the real roots of polynomials. The following theorem gives conditions on the coefficients of a polynomial which guarantee that the roots of the polynomial lie within given isolating intervals.

Theorem A.5 (Real root isolation conditions [47])
Let $g(u) = u^k + b_{k-1}u^{k-1} + \ldots + b_0 \in \mathbb{R}[u]$ *and let* $\underline{u_1} < \overline{u_1} \leq \underline{u_2} < \overline{u_2} \leq \ldots \leq \underline{u_k} < \overline{u_k} \in \mathbb{R}$. *Then for* $i = 1, \ldots, k$ *each open interval* $]\underline{u_i}; \overline{u_i}[$ *contains exactly one root of* g *if and only if the inequalities*

$$\begin{array}{rcl} (-1)^{k-i+1} g(\underline{u_k}) & > & 0 \text{ and} \\ (-1)^{k-i} g(\overline{u_i}) & > & 0 \end{array} \tag{A.1}$$

hold for all $i = 1, \ldots, k$.

This theorem is essentially similar to Theorem 1 of [47]. We slightly modified the conditions so that they are better adapted to our purposes. The following proof follows the one given in [47].

Proof: Assume that for $i = 1, \ldots, k$ each open interval $]\underline{u_i}; \overline{u_i}[$ contains exactly one root of g. The leading coefficient of g is assumed to be positive, therefore if $\tilde{u}$ is not a root of g and there are l roots in the interval $[\tilde{u}; \infty[$ then the sign of $g(\tilde{u})$ is $(-1)^l$. There are $k - i + 1$ roots of g to the right of $\underline{u_i}$ and $k - i$ roots of g to the right of $\overline{u_i}$, from which the inequalities A.1 immediately follow.

Now assume that the inequalities A.1 hold. For $i = 1, \ldots, k$ g changes its sign in each interval $]\underline{u_i}; \overline{u_i}[$, therefore there must be at least one root of g in each of these interval. There are k intervals, therefore each root of g is contained in an interval.

□

A.3 Multivariate Positiveness Bounds

Given a multivariate polynomial $f \in \mathbb{R}[x_1, \ldots, x_n]$ with $f(0) > 0$ we are interested in the connected set containing 0 where f is positive.

The notion of absolute positiveness proposed in [25] and the bound for absolute positiveness given in [23] provide an excellent tool for partially solving this problem.

First some notation adapted from [23] for working with multivariate polynomials:

Notation:

$$
\begin{array}{rcl}
\mu & := & (\mu_1, \ldots, \mu_n) \in \mathbb{N}^n \\
|\mu| & := & \mu_1 + \cdots + \mu_n \\
\nu - \mu & := & (\nu_1 - \mu_1, \ldots, \nu_n - \mu_n) \\
\nu > \mu & := & \nu_1 \geq \mu_1 \wedge \ldots \wedge \nu_n \geq \mu_n \wedge \nu \neq \mu \\
x & := & (x_1, \ldots, x_n) \\
1/x & := & (1/x_1, \ldots, 1/x_n) \\
\forall x \geq B & := & \forall x_1 \geq B \cdots \forall x_n \geq B
\end{array}
$$

□

A.3.1 A Bound for Absolute Positiveness of Multivariate Polynomials

In this section we give a short summary of the main results of [23].

Definition A.3 (Absolute positiveness)
Let $f \in \mathbb{R}[x]$ and let $B \in \mathbb{R}$. f is called ***absolutely positive from*** *B iff*

i. $\forall x \geq B : f(z) > 0$*, and*

ii. $\forall x \geq B : g(z) > 0$ *for every non-zero partial derivative g of f.*

A B from which f is absolutely positive is called a ***bound for absolute positiveness of*** *f. The infimum over all such B is called the* ***absolute positiveness margin of*** *f.*

Theorem A.6 (Bound for absolute positiveness, Thm. 2.2 of [23])
Let $f = \sum_\mu a_\mu x^\mu \in \mathbb{R}[x]$ such that for every μ with $a_\mu < 0$ there exists a $\nu > \mu$ with $a_\nu > 0$. Then

$$
\hat{B}_f := \frac{1}{1 - \sqrt[n]{\frac{1}{2}}} \max_{a_\mu < 0} \min_{a_\nu > 0, \nu > \mu} \left| \frac{a_\mu}{a_\nu} \right|^{\frac{1}{|\nu - \mu|}}
$$

is a bound for the absolute positiveness of f. If there are no $a_\mu < 0$ then $\hat{B}_f = 0$.

Theorem A.7 (Quality of the bound, Theorem 2.3 of [23])
Let f and B be as in Theorem A.6 and let the A be the absolute positiveness margin of f. Then

$$
\frac{\hat{B}_f}{A} \leq \frac{1}{1 - \sqrt[n]{\frac{1}{2}}} \frac{\deg_{x_1} f + \cdots + \deg_{x_n} f}{\ln(2)}.
$$

A.3.2 A Lower Bound for Positiveness of Multivariate Polynomials

From the bound given in Section A.3.1 we can straightforwardly derive a lower bound on positiveness of f.

Definition A.4 (Lower bound for positiveness)
Let $f \in \mathbb{R}[x]$ with $f(0) > 0$ and let $B \in \mathbb{R}$. B is called a ***lower bound for the positiveness of*** *f iff $\forall 0 \le x \le B : f(x) > 0$.*

Theorem A.8 (Lower bound for positiveness)
Let $f = \sum_\mu a_\mu x^\mu \in \mathbb{R}[x]$ with $f(0) > 0$. Then

$$\check{B}_f := (1 - \sqrt[n]{\tfrac{1}{2}}) \min_{a_\mu < 0} \max_{a_\nu > 0, \nu < \mu} \left| \frac{a_\nu}{a_\mu} \right|^{\frac{1}{|\mu - \nu|}}$$

is a lower bound for the positiveness of f.

Proof: If there are no $a_\mu < 0$ then $f(x) > 0$ for all $x \ge 0$ and $\check{B}_f = \infty$.

Now assume there exists at least one $a_\mu < 0$. Let $d = (\deg_{x_1} f, \ldots, \deg_{x_n} f)$, $\tilde{f}(y) = y^d f(\frac{1}{y}) = \sum_\mu a_\mu y^{d-\mu}$. $\tilde{f}$ fulfills the conditions of Theorem A.6, therefore $\hat{B}_{\tilde{f}}$ exists. $\check{B}_f$ is defined as $1/\hat{B}_{\tilde{f}}$ so that we have

$$\begin{aligned} \forall y \ge \hat{B}_{\tilde{f}} : \tilde{f}(y) > 0 \quad &\Leftrightarrow \quad \forall \tfrac{1}{y} \le \tfrac{1}{\hat{B}_{\tilde{f}}} : f(\tfrac{1}{y}) > 0 \\ &\Leftrightarrow \quad \forall x \le \check{B}_f : f(x) > 0. \end{aligned}$$

□

A.4 Polynomial Equation Solving by Gröbner Bases

The overview given in this section was mostly taken from [51]. This textbook and also [11] contain an introduction to Gröbner bases that will not be duplicated here. In this section we will only show how Gröbner bases can be used for solving systems of polynomial equations.

Consider

$$f_1(x_1, \ldots, x_n) = 0, \tag{A.2}$$

$$\vdots \tag{A.3}$$

$$f_m(x_1, \ldots, x_n) = 0 \tag{A.4}$$

with $f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_n]$. Such a system is called a system of polynomial or algebraic equations. The following theorem provides a test for the existence of solutions of such a system.

Theorem A.9 (Solvability test for a system of equations [10])
Let G be a normed Gröbner basis of the ideal of $f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_n]$. Then the system $f_1(x) = 0, \ldots, f_m(x) = 0$ is solvable if and only if $1 \notin G$.

The next theorem shows how Gröbner bases can be used to triangularize a system of polynomial equations.

Theorem A.10 (Elimination property of Gröbner bases [49])
Let G be a Gröbner basis of the ideal I of $f_1, \ldots, f_m \in \mathbb{R}[x_1, \ldots, x_n]$ with respect to the lexicographic ordering $x_1 < \ldots < x_n$. Then for $i = 1, \ldots, n$ the set $G \cap K[x_1, \ldots, x_i]$ is a basis for $I \cap K[x_1, \ldots, x_i]$.

If the system of polynomial equations has finitely many solutions, this theorem says that we can compute the solutions by first computing the roots of the polynomial(s) in G depending only on x_1. Then we can substitute these roots in the polynomial(s) in G depending only on x_1 and x_2, compute the roots of the resulting polynomials, which are univariate in x_2; and so on.

Appendix B

Computing Nominal Solutions by Approximate Quantifier Elimination

When dealing with the problem of computing the stability margin according to Definition 2.5, page 11, we always assume that we have given a so-called *nominal* solution of the stability problem. This means that in addition to the polynomial $f \in \mathbb{R}[z, p_1, \ldots, p_m, q_1, \ldots, q_n]$ and the robust stability region $Q \subseteq \mathbb{R}^n$ we also require a-priory knowledge of $\tilde{p} \in \mathbb{R}^m$ and $\tilde{q} \in \mathbb{R}^n$ such that $f(z, \tilde{p}, \tilde{q})$ is stable.

In this chapter we present a search method for computing such values $\tilde{p}$ and $\tilde{q}$. This method is sufficiently general to provide solutions to the solve the problem of robust stability, stability margin computation, etc. Due to its generality, it does not exploit special knowledge about the structure of these problems, and therefore should only serve as a fallback if the specialized methods presented in earlier chapters are not applicable. This method was proposed by Hoon Hong and first introduced in [28].

B.1 What Is Approximate Quantifier Elimination?

As we have seen in Section 2.2, the problems we are dealing with are problems of quantifier elimination. For example the problem of parametric robust stability (Problem 2.4, page 10) consists of finding the set P such that for all $p \in P$ the formula

$$\forall q \in Q : \forall z \in \mathbb{C} : [f(z, p, q) = 0 \Rightarrow \mathrm{Re}\,(z) < 0]$$

is true.

P may be of a complicated structure, so that we need to consider ways of giving a quantifier-free description of P that is not overly complex. The approach taken by approximate quantifier elimination is to approximate P from the inside and from the outside by boxes until the volume of the area between these approximations is smaller than a specified error bound.

B.1.1 Boxes and Box-Sets

From now on let $\mathbb{F}$ be a subset of $\mathbb{R}$. By using this set we keep the set of numbers used for computation variable. In concrete implementations, $\mathbb{F}$ will be a set of finite-precision floating point number or the set of algebraic numbers.

Definition B.1 (Box)
*An m-**dimensional box** given by a vector of intervals $(B_1, \ldots, B_m)$ with $B_i \in \mathbb{IF}$ is defined to be the set $B := \{p \in \mathbb{F}^m \mid p_i \in B_i\}$.*

*The B_i are called the **component intervals** or **components** of B. These intervals uniquely define a box, therefore the set of m-dimensional boxes is isomorphic to $\mathbb{IF}^m$, the m-dimensional space of interval vectors.*

Note that in case of $m = 0$ the set $\mathbb{F}^0$ contains only a single point, which will be denoted by 0. Therefore there exists only a single 0-dimensional box, which must be equal to $\mathbb{F}^0$.

Definition B.2 (Boundary of a box)
*The **boundary of a box** B is defined to be the set $\{p \in \mathbb{F}^m \mid \exists i : p_i = \underline{B_i} \vee p_i = \overline{B_i}\}$.*

Definition B.3 (Box-set)
*An m-**dimensional box-set** is defined to be a finite set of m-dimensional boxes. $\mathcal{B}$ is called a **strict box-set** iff any two boxes in $\mathcal{B}$ either are disjoint or intersect only on their boundaries.*

Remarks:

- Every box-set $\mathcal{B}$ defines a unique point-set $\bar{\mathcal{B}} := \{p \in \mathbb{F}^m \mid \exists B \in \mathcal{B} : p \in B\}$. If there is no danger of confusion we will identify box-sets with their associated point-sets and write e.g. $\mathcal{B} \subseteq P \subset \mathbb{F}^m$.
- For every box-set $\mathcal{B}$ there exists a strict box-set $\mathcal{C}$ such that $\bar{\mathcal{B}} = \bar{\mathcal{C}}$.

Definition B.4 (Volume)
*The **volume** of a box B is defined to be* $\mathrm{Volume}(B) := \prod_i |B_i|$. *Note that if at least one of the B_i is a single-point interval the volume of B is 0.*

The volume $\mathrm{Volume}(\mathcal{C})$ *of a strict box-set $\mathcal{C}$ is defined to be* $\sum_{C \in \mathcal{C}} \mathrm{Volume}(C)$. *The volume of a box-set $\mathcal{B}$ is defined to be the volume of a strict box-set $\mathcal{C}$ with $\bar{\mathcal{B}} = \bar{\mathcal{C}}$.*

B.1.2 Constraint Syntax and Semantics

The following abstract syntax describes the general structure of constraints[1] for which the problem of approximate quantifier elimination will be defined.

Definition B.5 (Constraint syntax)

Constraint :=
  quantified constraint |
  boolean constraint |
  atomic constraint

quantified constraint :=
  FORALL(variable list, boxset, constraint) |
  EXISTS(variable list, boxset, constraint)

boolean constraint :=
  (n-ary boolean function, n elementary constraints)

atomic constraint :=
  term comparison-operator 0

boolean function :=
  $f : \mathbb{B}^n \to \mathbb{B}$

elementary constraint :=
  quantified constraint |
  atomic constraint

term :=
  $f : \mathbb{F}^n \to \mathbb{F}$

[1] In Logic the term "formula" is used rather than "constraint". We mainly consider problems where formulae are interpreted as constraints that must be fulfilled by certain systems, therefore we prefer the term "constraint".

comparison-operator :=

$=$ |
$\neq$ |
$<$ |
$\leq$ |
$>$ |
$\geq$

Remarks: To simplify the definition of the semantics of constraints we will assume that the following restrictions always hold:

- In quantified constraints, the dimension of the box-set must always be equal to the number of variables in the variable list.
- There is no nesting of quantified constraints of the same kind, i.e. "FORALL(... , ... , FORALL(...))" is not allowed. Such constraints are represented by a single "FORALL" with a larger number of variables in the variable list.
- The identity function $\mathrm{id}(x) = x$ must not appear in boolean constraints. A boolean constraint "$(\mathrm{id}(x),\ F)$" can be replaced by "F".

Definition B.6 (Free variable)
Let F be a constraint. Any variable appearing in a term of F that is not contained in the variable list of a quantified constraint enclosing this term is called ***free*** *in F.*

Definition B.7 (Constraint semantics)
Let $\mathbb{B}$ be the set of boolean values $\{\text{TRUE}, \text{FALSE}\}$. Let $F(x)$ be a constraint with m free variables, $p \in \mathbb{F}^m$. The interpretation $F(p)$ of F wrt. p is defined by recursion over the syntax of F as follows:

- **Quantified constraint**: *The interpretation $F(p)$ of a quantified constraint $F(x) = \text{FORALL}(y, \mathcal{B}, G(x, y))$ is*
  - TRUE, *if for all $q \in \bar{\mathcal{B}}$ the interpretation $G(p, q)$ is* TRUE;
  - FALSE, *if there exists a $q \in \bar{\mathcal{B}}$ such that the interpretation $G(p, q)$ is* FALSE;
  - *undefined, otherwise.*

  The interpretation $F(p)$ of a quantified constraint $F(x)$ of the form EXISTS$(y, \mathcal{B}, G(x, y))$ is

- TRUE, *if there exists a* $q \in \bar{\mathcal{B}}$ *such that the interpretation* $G(p,q)$ *is* TRUE*;*
- FALSE, *if for all* $q \in \bar{\mathcal{B}}$ *the interpretation* $G(p,q)$ *is* FALSE*;*
- *undefined, otherwise.*

- **Boolean constraint***: Define* $T(p, F_1, \ldots, F_n)$ *as the set* $\{t \in \mathbb{B} | \forall i : [F_i \in \mathbb{B} \Rightarrow t_i = F_i(p)]\}$.

  The interpretation $F(p)$ *of a boolean constraint* $F(x) = (f, F_1, \ldots, F_n)$ *is*

  - TRUE, *if* $\forall t \in T(p, F_1, \ldots, F_n) : [f(t) = \text{TRUE}]$*;*
  - FALSE, *if* $\forall t \in T(p, F_1, \ldots, F_n) : [f(t) = \text{FALSE}]$*;*
  - *undefined, otherwise.*

  The rationale behind this definition is that if any F_i *is undefined the value of* $f(\ldots, F_{i-1}(p), \text{TRUE}, F_{i+1}(p), \ldots)$ *must be equal to the value of* $f(\ldots, F_{i-1}(p), \text{FALSE}, F_{i+1}(p), \ldots)$ *or* $F(p)$ *is undefined.*

- **Atomic constraint***: The interpretation* $F(p)$ *of an atomic constraint* $F(x) = f(x)$ OP 0 *with* OP $\in \{=, \neq, <, \leq, >, \geq\}$ *is*

  - TRUE, *if* $f(p)$ OP 0 *is* TRUE*;*
  - FALSE, *if* $f(p)$ OP 0 *is* FALSE.

B.1.3 The Problem of Approximate Quantifier Elimination

The general problem of approximate quantifier elimination is defined as follows:

Problem B.1 (Approximate quantifier elimination)
Given

- *a constraint* F *with* m *free variables,*
- *an* m*-dimensional box-set* $\mathcal{B}$*, and*
- *a precision* $\epsilon \in [0; 1]$*;*

find

- *a box-set* $\mathcal{B}_y$ *such that* $\bar{\mathcal{B}}_y \subseteq \bar{\mathcal{B}}$ *and* $\forall p \in \bar{\mathcal{B}}_y : [F(p) = \text{TRUE}]$,

- *a box-set* $\mathcal{B}_n$ *such that* $\bar{\mathcal{B}}_n \subseteq \bar{\mathcal{B}}$ *and* $\forall p \in \bar{\mathcal{B}}_n : [F(p) = \text{FALSE}]$,
- *a box-set* $\mathcal{B}_u$ *such that* $\bar{\mathcal{B}}_u \subseteq \bar{\mathcal{B}}$ *and*
  - $\bar{\mathcal{B}}_u$ *is the closure of the set* $\bar{\mathcal{B}} - \bar{\mathcal{B}}_y - \bar{\mathcal{B}}_n$,
  - $\dfrac{\text{Volume}(\mathcal{B}_u)}{\text{Volume}(\mathcal{B})} \leq \epsilon$.

The solution to this problem allows the computation of approximations to the solution set of a constraint F and its complement up to an arbitrary precision.

For computing a nominal solution to the problem of parameterized robust stability we do not need to approximate the solution set but only need to find a (single-point) box contained in the solution set. This problem can be formulated as follows:

Problem B.2 (Single solution box)
Given

- *a constraint* F *with* m *free variables and*
- *an* m*-dimensional box-set* $\mathcal{B}$;

find

- *a non-empty box* $Y \subseteq \mathcal{B}$ *such that* $\forall p \in Y : [F(p) = \text{TRUE}]$;
- *or prove that such a box does not exist.*

This problem is related to the problem of (approximately) computing the stability margin: If F describes the stability margin, a lower bound ϵ on the distance of the points in Y and the boundary of the solution set of F is given, and Y is required to be symmetric, then the width of Y is a lower bound of the stability margin.

B.2 An Algorithmic Framework

In this section a general framework (a top-level algorithm) for solving the problems specified in Section B.1.3 is presented. The framework is sufficiently general to be used for both the single solution box problem and the approximate quantifier elimination problem. As only an algorithmic *framework* is given, we cannot fully consider correctness and termination here. These issues will be taken care of in Section B.3, where the missing parts are filled in, resulting in a straighforward solution to the problem of approximate quantifier elimination.

The approach for solving the problem is based on the notion of **refinement**. By this we mean the process of computing successively larger sets over which the truth value of a constraint is known, thereby reducing the area over which the truth value of the constraint is unknown.

The main element that allows for variation is the **strategy** `S`, which is given as an argument to each algorithm. This is a formal means for influencing the choices that can be made in the algorithms. There is no fundamental difference between saying "call algorithm `Refine1` or algorithm `Refine2`" or saying "call algorithm `Refine` with strategy `S1` or strategy `S2`", but the latter style makes talking about different solution strategies easier.

B.2.1 Top Level

The top level structure of the method follows the constraint syntax, as can be seen by the top algorithm `Refine`.

Correctness and Termination: The correctness and termination of this algorithm directly depends on the correctness and termination of its subalgorithms `RefineAtomic`, `RefineBoolean`, and `RefineQuantified`.

B.2.2 Atomic Constraints

The algorithm `RefineAtomic` is based upon an algorithm `RefineFunction`, which is regarded as a black-box method and whose specifications are as follows:

Inputs:

f : A function from $\mathbb{F}^m$ to $\mathbb{F}$.

$\textsc{op}$: An element of $\{=, \neq, <, \leq, >, \geq\}$.

U : A non-empty box.

`v` : Variables $(x_{i_1}, \ldots, x_{i_k})$.

Outputs:

$\mathcal{U}_y$: A box-set with $\mathcal{U}_y \subseteq U$ and $\forall p \in \mathcal{U}_y : [f(p)\ \textsc{op}\ 0]$.

$\mathcal{U}_n$: A box-set with $\mathcal{U}_n \subseteq U$ and $\forall p \in \mathcal{U}_n : [\neg(f(p)\ \textsc{op}\ 0)]$.

$\mathcal{U}_u$: The closure of $U - \mathcal{U}_y - \mathcal{U}_n$. Either $\mathcal{U}_y$ or $\mathcal{U}_n$ must be non-empty.

The boxes in $\mathcal{U}_y$, $\mathcal{U}_n$, and $\mathcal{U}_u$ must be such that components not corresponding to variables in `v` must be equal to the corresponding components of U. In other words, U may only be broken up w.r.t. the variables in `v`.

The existence of such an algorithm is a necessary requirement we need to put upon atomic constraints.

Correctness: The correctness of this algorithm directly depends on the correctness of its subalgorithm `RefineFunction`, and upon the termination condition, which may introduce additional requirements on the output.

Termination: The termination of this algorithm depends on the termination of its subalgorithm `RefineFunction`, and upon the termination condition, which is not yet specified.

B.2.3 Boolean Constraints

The algorithm `RefineBoolean` takes a "lazy" approach by separetely evaluating boxes over the elementary formulas until a sufficient number of truth values has been computed.

The notation used in Step 2.5 of algorithm `RefineBoolean` implies that f is evaluated over undefined values. The exact semantics for this shorthand notation is given in Definition B.7.

Correctness: The correctness of this algorithm directly depends (recursively) on the correctness of the algorithm `Refine`, and upon the termination condition, which may introduce additional requirements on the output.

Termination: The termination of this algorithm depends (recursively) on the termination of the algorithm `Refine`, and upon the termination condition, which is not yet specified.

B.2.4 Quantified Constraints

The algorithm `RefineQuantified` handles existential and all quantifiers in a similar manner. Although the equivalence $\exists x : [F(x)] \Leftrightarrow \neg\forall x : [\neg F(x)]$ could be used to reduce existential quantifiers to all quantifiers, we prefer treating both types of quantifiers separately. This will allow optimizing the handling of series of quantifier changes such as $\forall x : \exists y : \forall z : \ldots$ later on.

The notations $\hat{U} \leftarrow \mathcal{U} \bigtriangleup C$ used in Step 2.3 and $\mathcal{U}_y \leftarrow \bigtriangledown^m \hat{\mathcal{U}}_y$, etc. used in Step 2.4 are the canonical extension to box-sets of

$$\begin{array}{rccc} \bigtriangleup : & \mathbb{F}^m \times \mathbb{F}^n & \rightarrow & \mathbb{F}^{m+n} \\ & p, q & \mapsto & (p, q) \end{array}$$

(i.e. an embedding operation) and its inverses

$$\begin{array}{rrcc} \nabla^m : & \mathbb{F}^k & \to & \mathbb{F}^m \\ & p & \mapsto & (p_1, \ldots, p_m) \end{array}$$

and

$$\begin{array}{rrcc} \nabla_n : & \mathbb{F}^k & \to & \mathbb{F}^n \\ & p & \mapsto & (p_{k-n+1}, \ldots, p_k) \end{array}$$

(i.e. projection operations).

Correctness: The correctness of this algorithm directly depends (recursively) on the correctness of the algorithm `Refine`, and upon the termination condition, which may introduce additional requirements on the output.

Termination: The termination of this algorithm depends (recursively) on the termination of the algorithm `Refine`, and upon the termination condition, which is not yet specified.

B.2.5 Algorithms

$(\mathcal{B}_y, \mathcal{B}_n, \mathcal{B}_u) \leftarrow$ `Refine` $(F, \mathcal{B}, \mathtt{S})$

Refine a constraint over a box.

Inputs:

F : A constraint with m free variables.

$\mathcal{B}$: An m-dimensional box-set.

`S` : The strategy, which implies the termination condition.

Outputs:

$\mathcal{B}_y$: A box-set such that $\bar{\mathcal{B}}_y \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_y : [F(p) = \text{TRUE}]$.

$\mathcal{B}_n$: A box-set such that $\bar{\mathcal{B}}_n \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_n : [F(p) = \text{FALSE}]$.

$\mathcal{B}_u$: A box-set such that $\bar{\mathcal{U}}_u \subseteq \bar{\mathcal{B}}$ and $\bar{\mathcal{U}}_u$ is the closure of the set $\bar{\mathcal{B}} - \bar{\mathcal{B}}_y - \bar{\mathcal{B}}_n$.

When the algorithm terminates, the termination condition implied by `S` is fulfilled.

Begin

Step 1: *Atomic constraint case*
  if F is an atomic constraint **then**
    $(\mathcal{B}_y, \mathcal{B}_n, \mathcal{B}_u) \leftarrow$ `RefineAtomic`$(F, \mathcal{B},$ `S`$)$
  endif
Step 2: *Boolean constraint case*
  if F is a boolean constraint **then**
    $(\mathcal{B}_y, \mathcal{B}_n, \mathcal{B}_u) \leftarrow$ `RefineBoolean`$(F, \mathcal{B},$ `S`$)$
  endif
Step 3: *Quantified constraint case*
  if F is a quantified constraint **then**
    $(\mathcal{B}_y, \mathcal{B}_n, \mathcal{B}_u) \leftarrow$ `RefineQuantified`$(F, \mathcal{B},$ `S`$)$
  endif
End

Algorithm 1: `Refine`

$(\mathcal{B}_y, \mathcal{B}_n, \mathcal{B}_u) \leftarrow$ `RefineAtomic` $(F, \mathcal{B},$ `S`$)$

Refine a constraint over a box.

Inputs:
F : An atomic constraint $f(x)$ OP 0 with $f : \mathbb{F}^m \to \mathbb{F}$ and OP $\in \{=, \neq, <, \leq, >, \geq\}$.
$\mathcal{B}$: An m-dimensional box-set.
`S` : The strategy, which implies the termination condition.

Outputs:
$\mathcal{B}_y$: A box-set such that $\bar{\mathcal{B}}_y \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_y : [F(p) = \text{TRUE}]$.
$\mathcal{B}_n$: A box-set such that $\bar{\mathcal{B}}_n \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_n : [F(p) = \text{FALSE}]$.
$\mathcal{B}_u$: A box-set such that $\bar{\mathcal{U}}_u \subseteq \bar{\mathcal{B}}$ and $\bar{\mathcal{U}}_u$ is the closure of the set $\bar{\mathcal{B}} - \bar{\mathcal{B}}_y - \bar{\mathcal{B}}_n$.

When the algorithm terminates, the termination condition implied by `S` is fulfilled.

Begin
Step 1: *Initialize*
  $\mathcal{B}_y \leftarrow \{\}$
  $\mathcal{B}_n \leftarrow \{\}$

$\mathcal{B}_u \leftarrow \mathcal{B}$

Initialize the global state if necessary.

Step 2: *Approximation loop*

while the termination condition implied by `S` is not satisfied **do**

[*2.1: Box choice*]

Choose a box U from $\mathcal{B}_u$; $\mathcal{B}_u \leftarrow \mathcal{B}_u - U$.

[*2.2: Variable choice*]

Choose variables $v \leftarrow \{x_{i_1}, \ldots, x_{i_k}\}$ with $1 \leq i_1, \ldots, i_k \leq m$.

[*2.3: Function refinement*]

$(\mathcal{U}_y, \mathcal{U}_n, \mathcal{U}_u) \leftarrow$ `RefineFunction`$(f,\ $ OP $, U, v)$

[*2.4: State update*]

$\mathcal{B}_y \leftarrow \mathcal{B}_y \cup \mathcal{U}_y$

$\mathcal{B}_n \leftarrow \mathcal{B}_n \cup \mathcal{U}_n$

$\mathcal{B}_u \leftarrow \mathcal{B}_u \cup \mathcal{U}_u$

Update the global state if necessary.

endwhile

End

Algorithm 2: `RefineAtomic`

$(\mathcal{B}_y, \mathcal{B}_n, \mathcal{B}_u) \leftarrow$ `RefineBoolean` $(F, \mathcal{B},$ `S`$)$

Refine a constraint over a box.

Inputs:

F : A boolean constraint $(f, F_1, \ldots, F_n)$ with $f : \mathbb{B}^n \rightarrow \mathbb{B}$ and F_i elementary constraints with at most m free variables.

$\mathcal{B}$: An m-dimensional box-set.

`S` : The strategy, which implies the termination condition.

Outputs:

$\mathcal{B}_y$: A box-set such that $\bar{\mathcal{B}}_y \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_y : [F(p) = \text{TRUE}]$.

$\mathcal{B}_n$: A box-set such that $\bar{\mathcal{B}}_n \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_n : [F(p) = \text{FALSE}]$.

$\mathcal{B}_u$: A box-set such that $\bar{\mathcal{U}}_u \subseteq \bar{\mathcal{B}}$ and $\bar{\mathcal{U}}_u$ is the closure of the set $\bar{\mathcal{B}} - \bar{\mathcal{B}}_y - \bar{\mathcal{B}}_n$.

When the algorithm terminates, the termination condition implied by `S` is fulfilled.

Begin

Step 1: *Initialize*

$\mathcal{B}_y \leftarrow \{\}$
$\mathcal{B}_n \leftarrow \{\}$
$\mathcal{B}_u \leftarrow \mathcal{B}$
Initialize the global state if necessary.

Step 2: *Approximation loop*

while the termination condition implied by `S` is not satisfied **do**
[*2.1: Box choice*]
Choose a box U from $\mathcal{B}_u$; $\mathcal{B}_u \leftarrow \mathcal{B}_u - \{U\}$.
[*2.2: Constraint choice*]
Choose an elementary constraint F_i with $1 \leq i \leq n$.
[*2.3: Termination choice*]
Choose a strategy `s` for the elementary constraint refinement.
[*2.4: Elementary constraint refinement*]
$(\mathcal{U}_y, \mathcal{U}_n, \mathcal{U}_u) \leftarrow$ `Refine`$(F_i, \{U\}, \texttt{s})$
[*2.5: State update*]
Remember that $F_i = \text{TRUE}$ on $\mathcal{B}_y$ and $F_i = \text{FALSE}$ on $\mathcal{B}_n$
if $f(F_1(U), \ldots, F_{i-1}(U), \text{TRUE}, F_{i+1}(U), \ldots, F_n(U)) = \text{TRUE}$ **then**
$\mathcal{B}_y \leftarrow \mathcal{B}_y \cup \mathcal{U}_y$
else if the value is FALSE **then**
$\mathcal{B}_n \leftarrow \mathcal{B}_n \cup \mathcal{U}_y$
else
$\mathcal{B}_u \leftarrow \mathcal{B}_u \cup \mathcal{U}_y$
endif
if $f(F_1(U), \ldots, F_{i-1}(U), \text{FALSE}, F_{i+1}(U), \ldots, F_n(U)) = \text{TRUE}$ **then**
$\mathcal{B}_y \leftarrow \mathcal{B}_y \cup \mathcal{U}_n$
else if the value is FALSE **then**
$\mathcal{B}_n \leftarrow \mathcal{B}_n \cup \mathcal{U}_n$
else
$\mathcal{B}_u \leftarrow \mathcal{B}_u \cup \mathcal{U}_n$
endif
$\mathcal{B}_u \leftarrow \mathcal{B}_u \cup \mathcal{U}_u$
Update the global state if necessary.
endwhile

End

Algorithm 3: `RefineBoolean`

$(\mathcal{B}_y, \mathcal{B}_n, \mathcal{B}_u) \leftarrow$ `RefineQuantified` $(F, \mathcal{B}, \texttt{S})$

Refine a constraint over a box.

Inputs:

F : A quantified constraint $\forall y \in C : [G(x, y)]$ with C an n-dimensional box and G a constraint with at most $m + n$ free variables.

$\mathcal{B}$: An m-dimensional box.

`S` : The strategy, which implies the termination condition.

Outputs:

$\mathcal{B}_y$: A box-set such that $\bar{\mathcal{B}}_y \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_y : [F(p) = \text{TRUE}]$.

$\mathcal{B}_n$: A box-set such that $\bar{\mathcal{B}}_n \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_n : [F(p) = \text{FALSE}]$.

$\mathcal{B}_u$: A box-set such that $\bar{\mathcal{U}}_u \subseteq \bar{\mathcal{B}}$ and $\bar{\mathcal{U}}_u$ is the closure of the set $\bar{\mathcal{B}} - \bar{\mathcal{B}}_y - \bar{\mathcal{B}}_n$.

When the algorithm terminates, the termination condition implied by `S` is fulfilled.

Begin

Step 1: *Initialize*

$\mathcal{B}_y \leftarrow \{\}$
$\mathcal{B}_n \leftarrow \{\}$
$\mathcal{B}_u \leftarrow \mathcal{B}$
Initialize the global state if necessary.

Step 2: *Approximation loop*

while the termination condition implied by `S` is not satisfied **do**

[*2.1: Box choice*]
Choose a box-set $\mathcal{U}$ from $\mathcal{B}_u$; $\mathcal{B}_u \leftarrow \mathcal{B}_u - \mathcal{U}$.

[*2.2: Termination choice*]
Choose a strategy `s` for the refinement of G.

[*2.3: Recursive constraint refinement*]
$\hat{\mathcal{U}} \leftarrow \mathcal{U} \bigtriangleup C$
$(\hat{\mathcal{U}}_y, \hat{\mathcal{U}}_n, \hat{\mathcal{U}}_u) \leftarrow$ `Refine`$(G, \hat{\mathcal{U}}, $ `s`$)$

[*2.4: State update*]
if F is all quantified **then**
$\mathcal{U}_y$ is the largest subset of $\bigtriangledown^m \hat{\mathcal{U}}_y$ such that for all $p \in \mathcal{U}_y \bigtriangleup C$ we also have $p \in \hat{\mathcal{U}}_y$
$\mathcal{U}_n \leftarrow \bigtriangledown^m \hat{\mathcal{U}}_n$
else
[*F is existentially quantified*]
$\mathcal{U}_y \leftarrow \bigtriangledown^m \hat{\mathcal{U}}_y$

$\mathcal{U}_n$ is the largest subset of $\bigtriangledown^m \hat{\mathcal{U}}_n$ such that for all $p \in \mathcal{U}_n \triangle C$ we also have $p \in \hat{\mathcal{U}}_n$
endif
$\mathcal{U}_u \leftarrow$ the closure of $\mathcal{U} - \mathcal{U}_y - \mathcal{U}_n$
$\mathcal{B}_y \leftarrow \mathcal{B}_y \cup \mathcal{U}_y$
$\mathcal{B}_n \leftarrow \mathcal{B}_n \cup \mathcal{U}_n$
$\mathcal{B}_u \leftarrow \mathcal{B}_u \cup \mathcal{U}_u$
Update the global state if necessary.
endwhile
End

Algorithm 4: `RefineQuantified`

B.3 A Straightforward Solution

In this section we give a straightforward solution[2] filling in the missing parts from the framework presented in Section B.2. Its main aspects are:

- Restriction to a finite set of floating point numbers.
- Restriction of functions in atomic constraints to polynomials.
- Always choose the largest unknown box for processing and only compute a single box of new information in every iteration.
- Use straightforward strategies for other choices.

B.3.1 The Restricted Problem

For reasons of efficiency, we will prefer to compute with floating point numbers whenever possible. Furthermore, for simplicity we want to avoid transcendental functions. This leads to the following restrictions to the specifications given in Section B.1:

- $\mathbb{F}$ is a finite subset of $\mathbb{R}$.

  As a consequence $\mathbb{IF}$ contains only a finite number of intervals with all intervals being subsets of $[\min(\mathbb{F}); \max(\mathbb{F})]$. This implies that there is a finite number of m-dimensional boxes.

[2]I implemented this and several alternative solutions in C++ based on the STURM C++ library for symbolic computation [24, 29]. Sources are available upon request from `andreas.neubacher@risc.uni-linz.ac.at`.

- Atomic constraints $F(x) = f(x)$ OP 0 are defined over *polynomial* functions $f : \mathbb{F}^n \to \mathbb{F}$.

In this section we consider a straightforward method for producing as much information as possible under these conditions. The restriction to a finite set of numbers means that we cannot make statements about arbitrary intervals in $\mathbb{R}$, therefore we cannot solve the general problems specified in Section B.1.3. We will consider the following restricted problems:

Problem B.3 (Approximate quantifier elimination (finite domain))

Given

- *a constraint F with m free variables,*
- *an m-dimensional box-set $\mathcal{B}$, and*
- *a precision $\epsilon \in [0; 1]$;*

find

- *a box-set $\mathcal{B}_y$ such that $\bar{\mathcal{B}}_y \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_y : [F(p) = \text{TRUE}]$,*
- *a box-set $\mathcal{B}_n$ such that $\bar{\mathcal{B}}_n \subseteq \bar{\mathcal{B}}$ and $\forall p \in \bar{\mathcal{B}}_n : [F(p) = \text{FALSE}]$,*
- *a box-set $\mathcal{B}_u$ such that $\bar{\mathcal{B}}_u \subseteq \bar{\mathcal{B}}$ and*
  - *$\bar{\mathcal{B}}_u$ is the closure of the set $\bar{\mathcal{B}} - \bar{\mathcal{B}}_y - \bar{\mathcal{B}}_n$,*
  - $\dfrac{\text{Volume}(\mathcal{B}_u)}{\text{Volume}(\mathcal{B})} \leq \epsilon$.
- *or show that $\mathbb{F}$ does not suffice for this computation.*

Problem B.4 (Single solution box (finite domain))
Given

- *a constraint F with m free variables and*
- *an m-dimensional box-set $\mathcal{B}$;*

find

- *a non-empty box $Y \subseteq \bar{\mathcal{B}}$ such that $\forall p \in Y : [F(p) = \text{TRUE}]$;*
- *or prove that such a box does not exist;*
- *or show that $\mathbb{F}$ does not suffice for this computation.*

The difference to the problems given in Section B.1.3 is that it may happen that the range of $\mathbb{F}$ does not suffice for computing solutions to the original problems. This case must be regarded as an additional result of the computation.

B.3.2 Atomic Constraints

The algorithm `RefineFunction`(f, OP , U, v) for polynomial functions f is based on the tightening methods described in [48]. It proceeds as follows:

1. Evaluate f w.r.t. the components of U not in v, getting a function $\bar{f} : \mathbb{F}^{n-k} \rightarrow \mathbb{IF}$, where n is the dimension of U and k is the number of variables in v.
2. Approximate the areas where $\bar{f}$ is positive, negative, and zero.
3. Compute $\mathcal{U}_y$, $\mathcal{U}_n$, $\mathcal{U}_u$ from these areas and OP.
4. If the approximation is not possible due to the finiteness of $\mathbb{F}$ this fact must also be returned to the caller of `RefineFunction`.

The choices in the algorithm `RefineAtomic` are:

Box choice: Choose the largest box U in $\mathcal{U}$.

Variable choice: Choose the variable w.r.t. the largest component of U; choose the variable with smallest index for breaking ties.

B.3.3 Boolean Constraints

The choices in the algorithm `RefineBoolean` are:

Box choice: Choose the largest box U in $\mathcal{U}$ among those over which a minimal number of F_i is undefined.

Constraint choice: Choose the constraint with smallest index among those whose evaluation over U is unknown.

Termination choice: Have elementary constraint refinement solve the single box problem (i.e. compute until at least one of $\mathcal{U}_y$ or $\mathcal{U}_n$ is nonempty).

B.3.4 Quantified Constraints

The choices in the algorithm `RefineQuantified` are:

Box choice: Choose $\mathcal{U} = \{u\}$ such that U is the largest box in $\mathcal{B}_u$.

Termination choice: Have recursive constraint refinement solve

- the single solution box problem (i.e. compute until either $\mathcal{U}_y$ is non-empty or until $\mathcal{U}_u$ is empty) in case of existential quantification, and
- the inverse single solution box problem (i.e. compute until either $\mathcal{U}_n$ is non-empty or until $\mathcal{U}_u$ is empty) in case of all quantification.

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Personal Data

Born September 1, 1968 in Bad Ischl, Austria
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Education

1974–1978 Primary school.

1978–1986 BRG Maderspergerstraße and Mathematisches Realgymnasium Landwiedstraße in Linz; final exam with honors.

1986–1992 Student at the Johannes Kepler Universität in Linz, Mathematics with emphasis on information processing ("Studienrichtung Technische Mathematik, Studienzweig Daten- und Informationsverarbeitung").

Graduated as Dipl.-Ing. (equivalent to M.S. in mathematics with minor in computer science) November 19, 1992; diploma thesis: "An Introduction to the Symbolic Integration of Elementary Functions".

1992–1997 Graduate student at the Research Institute for Symbolic Computation (RISC-Linz), Johannes Kepler Universität Linz, Austria (PhD advisor Univ.-Doz. Dr. Hoon Hong).

Professional Experience

1984–1988 Summer jobs in construction.

Summer 1988 System management (MS-DOS) and training of employees in basic computer skills at Fa. Grömer Engineering.

1989 Applications programmer at Fa. Grömer Engineering (programming language C).

Spring 1990 and 1992 Programmer at Fa. Mango Data (Development of an automatic warehousemanagement and controlsystem (programming language C).

1991–1993 Maintainer of the computer algebra system SACLIB at RISC-Linz (programming language C).

1992–1993 Systemmanagement for the UNIX environments and the intranet at RISC-Linz.

1993-1996 Research assistant (assistant to the project manager, software and algorithm development for the STURM library, programming language C++) at RISC-Linz in the frame of the ACCLAIM project sponsored by the European Community Basic Research Action (ESPRIT 7195) and the Austrian Science Foundation (FWF Project No. P9374-PHY).

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