Computing the System of Adjoint Plane Curves by Puiseux Expansion [1]

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Abstract

Let $\mathcal{C}$ be an irreducible plane algebraic curve. A curve $\mathcal{D}$ is called an adjoint curve of $\mathcal{C}$ if it meets certain conditions on the intersection multiplicities with curve-branches of $\mathcal{C}$. In this paper we show how these intersection multiplicities can be computed and how the conditions can be checked easily using the polar-curve of $\mathcal{C}$. This is then used to compute the linear system of adjoint curves of degree d of $\mathcal{C}$, for some $d \in \mathbb{N}$.

The main objects we deal with are curve-branches which we compute with the Newton-Puiseux algorithm and represent using their corresponding Puiseux-series.

Contents

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1. Introduction

Let $\mathcal{C} \subset \mathbf{P}^2_k$ be an irreducible plane algebraic curve. An important notion in the theory of plane algebraic curves is the notion of *adjoint curve*. Adjoint curves are not only of theoretical interest (see e.g. [Brieskorn and Knörrer, 1986] or [Keller, 1974]) but are also needed in the parametrization algorithm described by Sendra and Winkler in [Sendra and Winkler, 1991]. There are at least two different ways to describe adjoint curves. One way is with the help of the neighbourhood graph of a singular curve point, whereas another one is with the help of the intersection multiplicity with a curve-branch. In this paper we concentrate on the latter one.

The paper has the following structure. In section 2 we quote some very basic facts about algebraic curves and fix notation. Sections 3 introduces Puiseux-series which are then used in section 4 to define curve-branches. The definitions of standard resolution, multiplicity sequence and neighbourhood graph of a curve point are given in section 5. In the following section we define what we mean by an adjoint curve and we prove two short Lemmata which allow us to check if a curve $\mathcal{D}$ is adjoint to a given curve $\mathcal{C}$ in a very convenient way using a polar curve of $\mathcal{C}$. Section 7 shows how this can also be used to compute the whole system of adjoint curves of a given degree. Finally, section 8 consists of two examples where we actually show step by step how to compute the system of adjoint curves.

2. Plane Algebraic Curves

Throughout this paper, let k be a commutative, algebraically closed field with characteristic 0, e.g. the complex numbers.

2.1. Affine and Projective Space.

Definition 1. Let $n \in \mathbb{N}$. The set

$$\mathbf{A}^n_k = \{(a_1, \dots, a_n) \mid (a_1, \dots, a_n) \in k^n\}.$$

is called the *affine n-dimensional space over* k. The space $\mathbf{A}^2_k$ is also called the *affine plane over* k.

Definition 2. Let $f \in k[x_1, x_2]$ be a polynomial of degree d. The *affine algebraic plane curve $\mathcal{C}$ defined by* f is the set of zeros of f, i.e.

$$\mathcal{C} = \{(a_1, a_2) \in \mathbf{A}^2_k \mid f(a_1, a_2) = 0\}.$$

The polynomial f is called the *defining polynomial of $\mathcal{C}$*. The *degree of $\mathcal{C}$* is defined as the degree of its defining polynomial.

Definition 3. For $n \in \mathbb{N}$ let the equivalence relation $\sim_n$ on k^{n+1} be defined as

$$(a_0, \dots, a_n) \sim_n (b_0, \dots, b_n)$$
$$\text{iff}$$
$$(a_0, \dots, a_n) = (\lambda b_0, \dots, \lambda b_n)\ , \text{ for some } \lambda \in k.$$

The *projective n-dimensional space over* k, denoted by $\mathbf{P}^n_k$, is the set of equivalence classes of $k^{n+1} \setminus \{0\}$ w.r.t. $\sim_n$, i.e.

$$\mathbf{P}^n_k = \{(a_0, \dots, a_n) \mid (a_0, \dots, a_n) \in k^{n+1} \setminus \{0\}\}/\sim_n.$$

The 2-dimensional projective space over k is also called the *projective plane over* k.

Definition 4. Given a homogenous polynomial $f \in k[x_0, x_1, x_2]$. The *projective algebraic plane curve* $\mathcal{C}$ *defined by* f is the set of points

$$\mathcal{C} = \{(a_0, a_1, a_2) \in \mathbf{P}_k^2 \mid f(a_0, a_1, a_2) = 0\}.$$

As in the affine case, the polynomial f is called the *defining polynomial of* $\mathcal{C}$ and the *degree of* $\mathcal{C}$ is defined as the degree of the the polynomial f.

We will always assume that curves (affine or projective) are *irreducible*, i.e. their defining polynomial is irreducible in $k[x_1, x_2]$ or $k[x_0, x_1, x_2]$ respectively. There will be only one exception from this, namely in section 6 when defining adjoint curves.

2.2. **Homogenization and Dehomogenization.** In some situations (especially when dealing with local properties of curves) it is more convenient to work in affine space, in some other situations (when considering global properties) it is more natural to use projective space. For this reason we will show in this subsection how the two spaces can be related to each other. So we will be able to switch between affine and projective space whenever we find it useful.

The n-dimensional projective space can be seen as the completion of the n-dimensional affine space where the "points at infinity" are added. Let

$$W = \{(x_0, x_1, \ldots, x_n) \in \mathbf{P}_k^n \mid x_0 = 1\}$$

and

$$L = \{(x_0, x_1, \ldots, x_n) \in \mathbf{P}_k^n \mid x_0 = 0\}$$

be subsets of $\mathbf{P}_k^n$. It obviously holds that $\mathbf{P}_k^n = W \cup L$. As one can easily see the set W is isomorphic to $\mathbf{A}_k^n$. The subspace L of $\mathbf{P}_k^n$ is called the hyperplane at infinity.

Using this correspondence between affine and projective space we define for each affine curve $\mathcal{C}$ a corresponding projective curve $\mathcal{C}^*$, called the homogenized curve of $\mathcal{C}$ in the following way.

Definition 5. Let $f \in k[x_1, x_2]$ be of degree d defining an affine curve $\mathcal{C} \subset \mathbf{A}_k^2$. We can write f as $f = f_d + f_{d-1} + \cdots + f_0$, where each $f_i \in k[x_1, x_2]$ is homogenous and of degree i. The *homogenization of* f *w.r.t. the variable* x_0 is then defined as the polynomial $f^* \in k[x_0, x_1, x_2]$, where

$$f^*(x_0, x_1, x_2) = f_d(x_1, x_2) + x_0 f_{d-1}(x_1, x_2) + \cdots + x_0^d f_0(x_1, x_2).$$

The curve $\mathcal{C}^* \subset \mathbf{P}_k^2$ defined by f^* is called the *homogenization of* $\mathcal{C}$.

On the other hand, the n-dimensional projective space can be covered by $n+1$ overlapping n-dimensional affine spaces. Namely, let $W_i \subset \mathbf{P}_k^n$, for $0 \leq i \leq n$, be the subsets defined as

$$W_i = \{(a_0, \ldots, a_n) \in \mathbf{P}_k^n \mid a_i = 1\}.$$

Clearly, $\mathbf{P}_k^n = \bigcup_{i=0}^n W_i$ and each W_i is isomorphic to $\mathbf{A}_k^n$, for $0 \leq i \leq n$.

Applying this covering of the projective space to projective curves, we get the so called dehomogenization of projective curves.

Definition 6. Let $\mathcal{C} \subset \mathbf{P}^2_k$ be a projective curve defined by the homogenous polynomial $f \in k[x_0, x_1, x_2]$. The polynomial $f_{*,0} \in k[x_1, x_2]$ defined by

$$f_{*,0}(x_1, x_2) = f(1, x_1, x_2)$$

is called the *dehomogenization of* f *w.r.t. the variable* x_0. The curve $\mathcal{C}_{*,0} \subset \mathbf{A}^2_k$ defined by $f_{*,0}$ is called the *dehomogenization of* $\mathcal{C}$.

The dehomogenizations of f (or $\mathcal{C}$) w.r.t. x_1 and x_2 are defined in the obvious way and are denoted by $f_{*,1}$ and $f_{*,2}$ (or $\mathcal{C}_{*,1}$ and $\mathcal{C}_{*,2}$) respectively.

Hence we can associate to each affine curve a uniquely defined projective curve and for each projective curve there are finitely many affine curves describing the projective curve in one of its affine subspaces.

From now on, we will use the same symbols for affine and projective curves since it will be clear from the context in which space the curves are defined.

2.3. **Multiplicity.**

Definition 7. Given two curves $\mathcal{C}$, $\mathcal{D} \subseteq \mathbf{P}^2_k$ defined by the polynomials f, $g \in k[x_0, x_1, x_2]$ having no common components. Let $q = (q_0, q_1, q_2) \in \mathcal{C} \cap \mathcal{D}$ be a common point of the curves and assume that the coordinates are chosen in such a way that $(1, 0, 0)$ does not lie on any line through two intersection points of $\mathcal{C}$ and $\mathcal{D}$. Let $r = \mathrm{res}_{x_0}(f, g) \in k[x_1, x_2]$ be the resultant of f and g w.r.t. x_0.

The *intersection multiplicity* $\nu_q(\mathcal{C}, \mathcal{D})$ *of* $\mathcal{C}$ *and* $\mathcal{D}$ *at* q is defined to be the multiplicity of the root of (q_1, q_2) in r.

Definition 8. For a given curve $\mathcal{C}$ the *multiplicity of a point* $q \in \mathcal{C}$ is defined to be the minimal intersection multiplicity of $\mathcal{C}$ and a line $\mathcal{L}$ through q, i.e.

$$\nu_q(\mathcal{C}) = \min_{\mathcal{L} \text{ line trough } q} (\nu_q(\mathcal{C}, \mathcal{L})).$$

Definition 9. Given a curve $\mathcal{C} \subset \mathbf{P}^2_k$, a point $q \in \mathcal{C}$ and a line $\mathcal{L}$ through q. The line $\mathcal{L}$ is called *tangent to* $\mathcal{C}$ *at* q iff $\nu_q(\mathcal{C}) < \nu_q(\mathcal{C}, \mathcal{L})$.

Definition 10. Let $\mathcal{C} \subset \mathbf{P}^2_k$ be a curve. All points of $\mathcal{C}$ with multiplicity greater than 1 are called *singular points*. A singular point q with multiplicity m is called an *ordinary singular point* if there are m different tangents to $\mathcal{C}$ at q. Otherwise, q is called a *non-ordinary singular point*.

3. Puiseux Series

Starting from the ring of formal power-series $k[[t]]$ we define the field of formal Laurent-series $k((t))$ and the algebraically closed field $k\langle\langle t\rangle\rangle$, the field of Puiseux-series.

Definition 11. A *formal power-series over* k *in the indeterminate* t is an expression of the form:

Let $r \in \mathbb{N}$,

$$A(t) = \sum_{i=r}^{\infty} a_i t^i \quad \text{, where } a_i \in k\text{, for } r \leq i.$$

The ring of all formal power-series is written as $k[[t]]$.

Definition 12. We define the field $k((t))$ of *formal Laurent-series* as the set of all $A(t)$ which can be written as:

Let $r \in \mathbb{Z}$,

$$A(t) = \sum_{i=r}^{\infty} a_i t^i \quad \text{, where } a_i \in k\text{, for } r \leq i.$$

Although we assumed that k is algebraically closed, the field $k((t))$ is not. Consider e.g. the polynomial $p(y) = y^2 - t \in k((t))[y]$ which is of degree 2 but has no root in $k((t))$. Formal roots of $p(y)$ would have to be of the form $y_1 = t^{\frac{1}{2}}$ and $y_2 = -t^{\frac{1}{2}}$.

Definition 13. Let $k\langle\langle t\rangle\rangle$ be the set of all $A(t)$ of the form:

Let $m \in \mathbb{N}$, $r \in \mathbb{Z}$,

$$A(t) = \sum_{i=r}^{\infty} a_i t^{\frac{i}{m}} \quad \text{, where } a_i \in k\text{, for } r \leq i.$$

We call the set $k\langle\langle t\rangle\rangle$ the set of *Puiseux-series over* k *in the indeterminate* t.

Proposition 1. *The set* $k\langle\langle t\rangle\rangle$ *of Puiseux-series is an algebraically closed field.*

Proof. For a proof we refer to [Walker, 1978] or [Keller, 1974]. □

Definition 14. A Puiseux-series $A(t) = \sum_{i=r}^{\infty} a_i t^{\frac{i}{m}}$ is of *order* $\frac{l}{m}$ if l is the smallest index for which $a_l \neq 0$. We will denote the order of $A(t)$ by $\mathbf{o}(A(t))$.

The representation of Puiseux-series is not unique as one can see from the following example: Let $A(t) = \sum_{i=r}^{\infty} a_i t^{\frac{i}{m}}$ and $B(t) = \sum_{i=s}^{\infty} b_i t^{\frac{i}{n}}$, where $s = 2r$, $n = 2m$ and $b_{2i} = a_i$, $b_{2i+1} = 0$, for $i \geq r$. Although in general $a_i \neq b_i$ it holds that $A(t) = B(t)$. To avoid such ambiguities we introduce a standard representation of Puiseux-series.

Definition 15. Let $A(t) = \sum_{i=r}^{\infty} a_i t^{\frac{i}{m}} \in k\langle\langle t\rangle\rangle$. We call $\sum_{i=r}^{\infty} a_i t^{\frac{i}{m}}$ a *standard representation of* $A(t)$ iff $\gcd(\{i \mid a_i \neq 0\}) = 1$. In this case m is called the *denominator of* $A(t)$.

Each Puiseux-series $A(t) = \sum_{i=r}^{\infty} a_i t^{\frac{i}{m}}$ has a unique standard representation. Assume that $d = \gcd(\{i \mid a_i \neq 0\}) > 1$. Then $A(t)$ can also be written as $\sum_{i=s}^{\infty} b_i t^{\frac{i}{n}}$, where $s = \frac{r}{d}$, $n = \frac{m}{d}$ and $b_i = a_{di}$, for $i \geq s$, which is now a standard representation of $A(t)$. The uniqueness of the standard representation is obvious.

Proposition 2. *Given a polynomial* $f \in k[x][y]$ *of* y*-degree* d*. The polynomial* f *can be factored into*

$$f(x, y) = c_0 x^j \prod_{i=0}^{d} (y - A_i(x)),$$

where $c_0 \in k$, $j \in \mathbb{N}_0$ *and* $A_i(x) \in k\langle\langle x \rangle\rangle$ *with* $\mathbf{o}\,(A_i(x)) \geq 0$*, for all* $1 \leq i \leq d$.

Proof. This follows basically from Proposition 1. A detailed proof can be found in [Keller, 1974]. □

Let $f \in k[x, y]$ be a polynomial of y-degree d defining a curve $\mathcal{C} \subset \mathbf{A}^2_k$. Assume that $f(0, y) \neq 0$. According to Proposition 2 the polynomial f can be factored into d linear factors $y - A_i(x)$, where $A_i(x) \in k\langle\langle x \rangle\rangle$, $\mathbf{o}\,(A_i(x)) \geq 0$, for all $1 \leq i \leq d$.

Since $f(x, A_i(x)) = 0$, for $1 \leq i \leq d$, we can associate with each Puiseux-series $A_i(x)$ a *(formal) local parametrization of* $\mathcal{C}$ *at* $(0, A_i(0))$ of the form

$$y = A_i(x).$$

Let $m_i \in \mathbb{N}$ be the denominator of $A_i(x)$. Substituting t^{m_i} for τ we get an equivalent local parametrization of $\mathcal{C}$ which is now of the form

$$\begin{aligned} x &= t^{m_i} \\ y &= A_i(t^{m_i}) = B_i(t), \text{ where } B_i(t) \in k[[t]]. \end{aligned}$$

We call such a local parametrization a *function-branch* of f at $(0, B_i(0))$.

A method to actually compute all function-branches at a given curve point is given by the *Newton-Puiseux algorithm* which is described e.g. in [Duval, 1989], [Walker, 1978], [Brieskorn and Knörrer, 1986] or [Keller, 1974].

Example 1. Let $f(x, y) = y^5 - 2x^2x^2 + 3yx^4 - x^6 \in k[x, y]$. Applying the Newton-Puiseux algorithm to f at we get

$$\begin{aligned} A_1(x) &= x^2 + x^6 + 3x^{10} + 13x^{14} + \cdots, \\ A_2(x) &= \frac{1}{2}x^2 - \frac{1}{32}x^6 + \frac{3}{256}x^{10} - \frac{13}{2048}x^{13} + \cdots, \\ A_3(x) &= 2\zeta_1^2 x^{\frac{2}{3}} - \frac{1}{2}x^{\frac{6}{3}} - \frac{1}{3}\zeta_1 x^{\frac{10}{3}} - \frac{5}{12}\zeta_1^2 x^{\frac{14}{3}} + \cdots, \\ A_4(x) &= 2\zeta_2^2 x^{\frac{2}{3}} - \frac{1}{2}x^{\frac{6}{3}} - \frac{1}{3}\zeta_2 x^{\frac{10}{3}} - \frac{5}{12}\zeta_2^2 x^{\frac{14}{3}} + \cdots, \\ A_5(x) &= 2\zeta_3^2 x^{\frac{2}{3}} - \frac{1}{2}x^{\frac{6}{3}} - \frac{1}{3}\zeta_3 x^{\frac{10}{3}} - \frac{5}{12}\zeta_3^2 x^{\frac{14}{3}} + \cdots, \end{aligned}$$

where ζ_1, ζ_2 and ζ_3 are roots of $\zeta^3 - \frac{1}{2} = 0$.

4. Branches

4.1. Function Branches.

Definition 16. A *projective plane function-branch* $\mathfrak{z}$ is a non-constant point of the projective plane $\mathbf{P}^2_k$ whose coordinates can be defined by formal Laurent-series. I.e. the function-branch $\mathfrak{z}$ can be written as $(A_0(t), A_1(t), A_2(t))$, where $A_j(t) \in k((t))$, $j = 0, 1, 2$ and the $A_j(t)$ are not proportional.

Let $\mathfrak{z} \subset \mathbf{P}^2_k$ be a function-branch which is defined by $(A_0(t), A_1(t), A_2(t))$, where $A_j(t) = \sum_{i=r}^{\infty} a_{j,i} t^i$, for $j = 0, 1, 2$. Let the matrices $M_i \in k_3^{i+1}$, for $r \leq i$, be defined as

$$M_i = \begin{pmatrix} a_{0,r} & a_{0,r+1} & \cdots & a_{0,i} \\ a_{1,r} & a_{1,r+1} & \cdots & a_{1,i} \\ a_{2,r} & a_{2,r+1} & \cdots & a_{2,i} \end{pmatrix}.$$

It is quite clear that $\mathrm{rk}(M_i) \leq \mathrm{rk}(M_{i+1}) \leq \mathrm{rk}(M_i) + 1$. We define the indices i_k, $k = 1, 2, 3$, as

$$i_k = \min\{i \mid \mathrm{rk}(M_i) = k\}.$$

The index i_1 is always defined since not all $A_j(t)$ are equal to 0. Also the index i_2 is always defined since we assumed that the $A_j(t)$ are non-proportional. On the other hand it is not guaranteed that i_3 exists.

Definition 17. Let the notation be as above. The *origin* q *of the function-branch* $\mathfrak{z}$ is defined as

$$q = \lim_{t \to 0} \left(\frac{A_0(t)}{t^{i_1}}, \frac{A_1(t)}{t^{i_1}}, \frac{A_2(t)}{t^{i_1}} \right).$$

Furthermore, let $\{(\lambda v_0, \lambda v_1, \lambda v_2) \mid \lambda \in k\}$ be the solution of the system of linear equations

$$a_{0,i} u_0 + a_{1,i} u_1 + a_{2,i} u_2 = 0 \text{ , for } 0 \leq i \leq i_2.$$

The projective line $\mathcal{L} \subset \mathbf{P}^2_k$ defined by $v_0 x_0 + v_1 x_1 + v_2 x_2 = 0$ is called the *tangent of the function-branch* $\mathfrak{z}$.

If the index i_3 does not exist we say that the function-branch $\mathfrak{z}$ is a *linear function-branch.*

Definition 18. Let $\mathcal{C} \subset \mathbf{P}^2_k$ be a curve defined by $f \in k[x_0, x_1, x_2]$ and $\mathfrak{z} \subset \mathbf{P}^2_k$ a function-branch defined by $(A_0(t), A_1(t), A_2(t))$. The function-branch $\mathfrak{z}$ is called a *function-branch of* $\mathcal{C}$ if $f(A_0(t), A_1(t), A_2(t)) = 0$.

The following Proposition states that function-branches can always be written in a so called normal-form.

Proposition 3. *A function-branch $\mathfrak{z} \subset \mathbf{P}^2_k$ can always be written in one of the following two ways:*

$$
\begin{aligned}
&A_0(t) = 1 \\
&A_1(t) = a_{1,0} + t^k \ , \ k > 0 \\
&A_2(t) = a_{2,h}t^h + a_{2,h+1}t^{h+1} + \cdots \quad , \ h > 0, \ h = 0 \ \textit{or} \ h < 0
\end{aligned}
$$

or

$$
\begin{aligned}
&A_0(t) = 1 \\
&A_1(t) = t^{-k} \ , \ k > 0 \\
&A_2(t) = a_{2,h}t^h + a_{2,h+1}t^{h+1} + \cdots \quad , \ h > 0, \ h = 0 \ \textit{or} \ h < 0,
\end{aligned}
$$

where $\gcd(\{i \mid a_{2,i} \neq 0\}) = 1$.

Proof. A proof of this Proposition can be found in [van der Waerden, 1973]. □

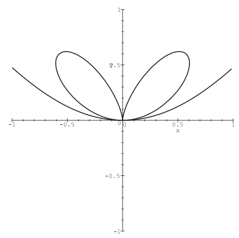
Definition 19. If a function-branch is given in one of the two forms of Proposition 3 we say that it is given in *normal-form*.

The question of finding all function-branches of $\mathcal{C}$ with a given origin q, w.l.o.g. we can assume that $q = (1, 0, 0)$, is hence equivalent to the problem of finding all local parametrizations of $\mathcal{C}$ at $(0, 0)$ which can be done by the Newton-Puiseux algorithm, as mentioned in the previous section.

4.2. **Curve Branches.** Let $\mathcal{C} \subset \mathbf{P}^2_k$ be a curve and $\mathfrak{z} \subset \mathbf{P}^2_k$ be a function-branch of $\mathcal{C}$ with normal-form $(A_0(t), A_1(t), A_2(t))$, where $A_1(t) = a_{1,0} + t^k$ or $A_1(t) = t^{-k}$. In both cases the k different function-branches which are obtained by replacing t by ζt, where $\zeta^k = 1$, are also function-branches of the curve $\mathcal{C}$. These function-branches are called conjugated function-branches of $\mathfrak{z}$.

Definition 20. Let $\mathcal{C} \subset \mathbf{P}^2_k$ be a curve. A family of conjugated function-branches of $\mathcal{C}$ is called a *curve-branch of* $\mathcal{C}$. A curve-branch is defined by defining one of its function-branches.

Example 2. Given the curve $\mathcal{C} \subset \mathbf{P}^2_k$ defined by $f(x_0, x_1, x_2) = x_0x_2^5 - 2x_0^2x_1^2x_2^2 + 3x_0x_1^4x_2 - x_1^6 \in k[x_0, x_1, x_2]$. The curve $\mathcal{C}$ has only one singular point namely $(1, 0, 0)$. Dehomogenizing f w.r.t. x_0 we get $f(x_1, x_2) = x_2^5 - 2x_1^2x_2^2 + 3x_1^4x_2 - x_1^6$.

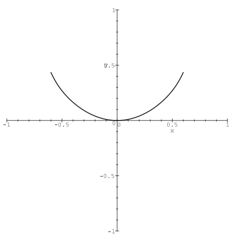


$\mathcal{C} : x_2^5 - 2x_1^2x_2^2 + 3x_1^4x_2 - x_1^6$

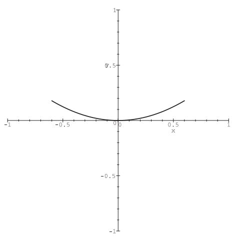
We already computed the function-branches of $f(x_1, x_2)$ in Example 1 and from there we see that $\mathcal{C}$ has three curve-branches with origin $(0, 0)$. The curve-branch $\mathfrak{z}_3$ consists of the three function-branches $A_3(x)$, $A_4(x)$ and $A_5(x)$.

Their normal-forms are given by

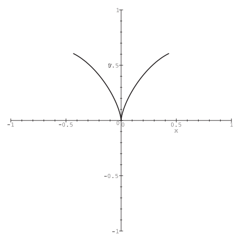
$$
\begin{aligned}
\mathfrak{z}_1 : A_0(t) &= 1 \\
A_1(t) &= t \\
A_2(t) &= t^2 + t^6 + 3t^{10} + 13t^{14} + \cdots, \\
\mathfrak{z}_2 : A_0(t) &= 1 \\
A_1(t) &= t \\
A_2(t) &= \frac{1}{2}t^2 - \frac{1}{32}t^6 + \frac{3}{256}t^{10} - \frac{13}{2048}t^{13} + \cdots, \\
\mathfrak{z}_3 : A_0(t) &= 1 \\
A_1(t) &= 2t^3 \\
A_2(t) &= 2t^2 - 2t^6 - \frac{8}{3}t^{10} - \frac{20}{3}t^{14} + \cdots.
\end{aligned}
$$



$\mathfrak{z}_1$



$\mathfrak{z}_2$



$\mathfrak{z}_3$

4.3. **Intersection Multiplicity.** As a generalization of the definition of the intersection multiplicity of two curves given in Definition 7 we define the intersection multiplicity of a curve-branch and a curve in the following way.

Definition 21. Given a curve $\mathcal{C} \subset \mathbf{P}^2_k$ defined by $f \in k[x_0, x_1, x_2]$ and a curve-branch $\mathfrak{z} \subset \mathbf{P}^2_k$ with normal-form $(A_0(t), A_1(t), A_2(t))$ and origin $q \in \mathbf{P}^2_k$.

The intersection multiplicity $\nu_q(\mathfrak{z}, \mathcal{C})$ of $\mathfrak{z}$ and $\mathcal{C}$ at q is equal to

$$\nu_q(\mathfrak{z}, \mathcal{C}) = \mathbf{o}\left(f(A_0(t), A_1(t), A_2(t))\right).$$

The Proposition below shows that this definition is indeed a generalization of Definition 7.

Proposition 4. *Given two curves $\mathcal{C}$, $\mathcal{D} \subset \mathbf{P}^2_k$ defined by f, $g \in k[x_0, x_1, x_2]$ having no common components. Let $q \in \mathcal{C} \cap \mathcal{D}$ be a point and let $\mathfrak{z}_j$, for $1 \le j \le s$, be the curve-branches of $\mathcal{C}$ which have q as their origin.*

The intersection multiplicity $\nu_q(\mathcal{C}, \mathcal{D})$ of the curves $\mathcal{C}$ and $\mathcal{D}$ at q is equal to

$$\nu_q(\mathcal{C}, \mathcal{D}) = \sum_{j=1}^{s} \nu_q(\mathfrak{z}_j, \mathcal{D}).$$

Proof. For a proof see [van der Waerden, 1973]. $\square$

5. Quadratic Transformations

Given a curve $\mathcal{C} \subset X = \mathbf{A}_k^2$. Assume that $q = (0,0) \in \mathbf{A}_k^2$ is a singular point of $\mathcal{C}$. In order to investigate the singularity at q in more detail, we replace the point q by the set all lines through q (or equivalently, the set of all directions in q). We call this new space X' and we ask what the curve $\mathcal{C}$ looks like within X'.

Let $X = \mathbf{A}_k^2$ be the affine plane and $q = (0,0) \in X$. As one can easily see, the set of all lines through q is isomorphic to the 1-dimensional projective space $\mathbf{P}_k^1$.

We define X' as

$$X' = \{(x,y) \in X \times \mathbf{P}_k^1 \mid x \in y\}$$

and it holds that for all $x \neq q$, there is exactly one line through x and q, whereas for $x = q$, all lines of $\mathbf{P}_k^1$ go through q.

Hence the projection

$$\begin{aligned} \pi : X' &\to X \\ (x,y) &\mapsto x. \end{aligned}$$

is an isomorphism for $X \setminus \{q\}$.

Now let $x = (x_1, x_2)$ and $y = (y_0, y_1)$. We can write X' as

$$X' = \{((x_1,x_2),(y_0,y_1)) \in X \times \mathbf{P}_k^1 \mid x_1 y_1 = x_2 y_0\}.$$

In the same way as the projective space $\mathbf{P}_k^1$ can be covered by two affine spaces $\mathbf{A}_k^1$ (see subsection 2.2.2), the space X' can be covered by two spaces $X^{(1)}$ and $X^{(2)}$ in the following way, $X' = X^{(1)} \cup X^{(2)}$:

$$\begin{aligned} X^{(1)} &= \{((x_1,x_2),(1,y_1)) \in X'\} = \{((x_1,x_1y_1),(1,y_1)) \in X \times \mathbf{P}_k^1\}, \\ X^{(2)} &= \{((x_1,x_2),(y_0,1)) \in X'\} = \{((x_2y_0,x_2),(y_0,1)) \in X \times \mathbf{P}_k^1\}. \end{aligned}$$

Introducing the local coordinates

$$x_1^{(1)} = y_1 \quad x_2^{(1)} = x_1$$

and

$$x_1^{(2)} = y_0 \quad x_2^{(2)} = x_2$$

we see that both spaces $X^{(1)}$ and $X^{(2)}$ are isomorphic to $\mathbf{A}_k^2$ and overlapping in the following way

$$\begin{aligned} x_1^{(2)} &= (x_1^{(1)})^{-1} \\ x_2^{(2)} &= x_1^{(1)} x_2^{(1)}. \end{aligned}$$

Applying this decomposition of X' to the projection $\pi : X' \to X$, we can write π in local coordinates of $X^{(1)}$ and $X^{(2)}$ as

$$\begin{aligned} \pi|_{X^{(1)}} : X^{(1)} &\to X \\ (x_1^{(1)}, x_2^{(1)}) &\mapsto (x_1^{(1)}, x_1^{(1)} x_2^{(1)}) \end{aligned}$$

and

$$\begin{aligned} \pi|_{X^{(2)}} : X^{(2)} &\to X \\ (x_1^{(2)}, x_2^{(2)}) &\mapsto (x_1^{(2)} x_2^{(2)}, x_2^{(2)}) \end{aligned}$$

respectively.

Definition 22. The above process of passing from X to X' by $\pi : X' \to X$ is called a *σ-process at* $(0,0)$ or *quadratic transformation at* $(0,0)$ or *blowing-up at* $(0,0)$.

Since a σ-process at $(0,0)$ is a strict local process, we can define a *σ-process at* q, for $q \in \mathbf{A}^2_k$, by first moving the point q by a linear transformation to the origin and then applying a σ-process at $(0,0)$.

Definition 23. A *σ-process at* q for a point $q \in \mathbf{A}^2_k$ is defined by $\pi \circ T$, where $T : \mathbf{A}^2_k \to \mathbf{A}^2_k$ is a linear transformation moving q to $(0,0)$ and π is a σ-process at $(0,0)$.

A *sigma-process at* Q, where $Q \subset \mathbf{A}^2_k$ is a finite set of points is defined as the composition of all σ-processes at q, for $q \in Q$.

Definition 24. Let $\mathcal{C} \subset \mathbf{A}^2_k$ be a curve, $q \in \mathcal{C}$ a point on it and $\pi : X' \to X$ a σ-process at q.

The curve $\mathcal{C}' = \pi^{-1}(\mathcal{C})$ is called the *total transform of* $\mathcal{C}$, the line $E = \pi^{-1}(q)$ is called the *exceptional divisor* and the curve $\tilde{\mathcal{C}} = \overline{\mathcal{C}' \setminus E}$ is called the *proper transform of* $\mathcal{C}$.

Definition 25. Given a curve $\mathcal{C} \subset X = \mathbf{A}^2_k$ and a point $q \in \mathcal{C}$. A *finite sequence of σ-processes*

$$\Psi_i : X = X_0 \xleftarrow{\Phi_1} X_1 \xleftarrow{\Phi_2} X_2 \xleftarrow{\Phi_3} \cdots \xleftarrow{\Phi_{i-1}} X_{i-1} \xleftarrow{\Phi_i} X_i$$

is defined as:

The mapping Φ_1 is a σ-process at q where $\mathcal{C}'_1$, E_1 and $\tilde{\mathcal{C}}_1$ denote the total transform, the exceptional divisor and the proper transform w.r.t. Φ_1.

For $2 \le j \le i$, we define Φ_j as the σ-process at $\mathcal{C}'_{j-1} \cap E_{j-1}$. Let $\mathcal{C}'_j = \Phi_j^{-1}(\mathcal{C}'_{j-1})$, $E_j = \Phi_j^{-1}(E_{j-1})$ and $\tilde{\mathcal{C}}_j = \overline{\mathcal{C}'_j \setminus E_j}$.

The points in $E_j \cap \tilde{\mathcal{C}}_j$ are called the *infinitely near neighbouring points of* q *in* X_j. For our convenience we say q is also a neighbouring point of q.

We are now able to state, what we mean by *resolving* a singularity of a plane algebraic curve.

Proposition 5. *Each singularity* q *of a plane algebraic curve* $\mathcal{C}$ *can be transformed by a finite sequence of σ-processes such that the proper transform of* $\mathcal{C}$ *cuts the exceptional divisor transversally in regular points only.*

Proof. For a proof we refer to [Brieskorn and Knörrer, 1986]. □

Such a resolution is not unique. For example, blowing-up a regular point does not affect the resolution process. But as one can show, for a given curve $\mathcal{C}$ there always exists a "smallest" resolution with normal crossings (the proper transform of the curve cuts the exceptional divisor transversely in regular points only).

Definition 26. For a given curve $\mathcal{C}$ the smallest possible resolution of $q \in \mathcal{C}$ with normal crossings is called the *standard resolution of $\mathcal{C}$ at* q.

5.1. **Multiplicity Sequence and Neighbourhood Graph.** There are different ways to describe the resolution process. One possibility is to use *multiplicity sequences* and *neighbourhood graphs.*

Given a curve $\mathcal{C}$ and a point $q \in \mathcal{C}$. Let us start with the additional assumption that there is only one curve-branch $\mathfrak{z}$ of $\mathcal{C}$ with origin q.

Note that in this situation after a applying a σ-process at q we get exactly one infinitely near neighbouring point of q, i.e. the set $E_1 \cap \tilde{\mathcal{C}}_1$ consists only of one point q_1. Applying another σ-process at q_1 we again get only one infinitely near neighbouring point q_2. As one can easily see this situation stays true after applying further σ-processes at the infinitely near neighbouring points. For details we refer to [Brieskorn and Knörrer, 1986].

Let m_0 be the multiplicity of q and $m_1, \ldots, m_N$ be the multiplicities of the infinitely near points of q occurring in the standard resolution of q.

Definition 27. The sequence $(m_0, m_1, \ldots, m_N)$ is called the *multiplicity sequence of $\mathcal{C}$ at the point* q.

The situation becomes somewhat more difficult if we consider the case, where the singular point q is the origin of more than one curve-branch of $\mathcal{C}$. Here we also have to take into account after which σ-processes in the standard-resolution the curve-branches get separated.

Assume that there are s different curve-branches $\mathfrak{z}_j$ of $\mathcal{C}$ with origin q. Resolving the curve $\mathcal{C}$ at q we obtain s multiplicity sequences $(m_{j,0}, m_{j,1}, \ldots, m_{j,N})$ together with the information which multiplicities $m_{j,k}$ belong to the same infinitely near neighbouring point of q. Thus we get partitions P_k, for $1 \leq k \leq N$, of the set $\{1, \ldots, s\}$ where j and j' are in the same subset of P_k if $m_{j,k} = m_{j',k}$ belong to the same infinitely near point.

Definition 28. The multiplicity sequences

$$(m_{j,0}, m_{j,1}, \ldots, m_{j,N}) \quad , \text{for } j = 1, \ldots, s$$

of the curve-branches $\mathfrak{z}_j$ of $\mathcal{C}$ at q together with the partitions P_k, for $1 \leq k \leq N$ are called the *system of multiplicity sequences of $\mathcal{C}$ at* q or the *neighbourhood graph of $\mathcal{C}$ at* q.

Definition 29. Given a curve $\mathcal{C} \subset \mathbf{A}^2_k$ and a point q on it. Let the neighbourhood graph of $\mathcal{C}$ at q be given by $(m_{j,0}, m_{j,1}, \ldots, m_{j,N})$, for $j = 1, \ldots, s$, and P_k, for $1 \leq k \leq N$, where s is the number of curve-branches and N is the length of the standard resolution. Furthermore let q' be a neighbouring point of q and let m_{j_i,k_0}, for $1 \leq i \leq r$ and $k_0 \in \{1, \ldots, N\}$, be the multiplicities belonging to q'.

The *multiplicity of* q' *w.r.t. the curve-branch* $\mathfrak{z}_{j_i}$ is defined as m_{j_i,k_0} and the *multiplicity of* q' *w.r.t. the curve* $\mathcal{C}$ is defined as $\sum_{i=1}^{r} m_{j_i,k_0}$.

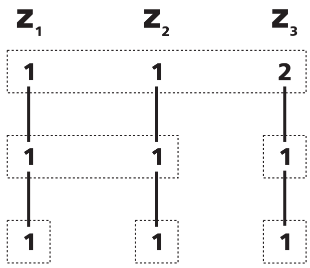
Example 3. Let $\mathcal{C} \subset \mathbf{P}_k^2$ be the curve defined by $f(x_0, x_1, x_2) = x_0x_2^5 - 2x_0^2x_1^2x_2^2 + 3x_0x_1^4x_2 - x_1^6 \in k[x_0, x_1, x_2]$. As we have already seen (see Example 2), the singularity at $(1,0,0)$ decomposes into three curve-branches $\mathfrak{z}_1$, $\mathfrak{z}_2$ and $\mathfrak{z}_3$. The multiplicity sequences of the curve-branches are given by

$$\begin{aligned} \mathfrak{z}_1 &: (1,1,1) \\ \mathfrak{z}_2 &: (1,1,1) \\ \mathfrak{z}_3 &: (2,1,1) \end{aligned}$$

and the partitions are given by

$$\begin{aligned} P_1 &= \{\{1,2,3\}\} \\ P_2 &= \{\{1,2\},\{3\}\} \\ P_3 &= \{\{1\},\{2\},\{3\}\}. \end{aligned}$$

We can also visualize the neighbourhood graph of $\mathcal{C}$ at $(1,0,0)$ in the following way



5.2. **Intersection Multiplicity.** The following Proposition shows how one can compute the intersection multiplicity of two curve-branches, if the neighbourhood graph of their union is known.

Proposition 6. *Let $\mathfrak{z}_0$, $\mathfrak{z}_1 \subset \mathbf{A}_k^2$ be two distinct curve-branches with origin q. Let the multiplicity-sequences of their union be*

$$(m_{0,0}, m_{0,1}, \ldots, m_{0,N}) \text{ and}$$
$$(m_{1,0}, m_{1,1}, \ldots, m_{1,N}),$$

where, for $0 \leq k \leq l$, the multiplicities $m_{0,k}$ and $m_{1,k}$ belong to the same infinitely-near point in the standard resolution of $\mathfrak{z}_0 \cup \mathfrak{z}_1$, and for $k > l$ they do not.

Then the intersection multiplicity of $\mathfrak{z}_0$ and $\mathfrak{z}_1$ at q satisfies

$$\nu_q(\mathfrak{z}_0, \mathfrak{z}_1) = \sum_{k=0}^{l} m_{0,k} m_{1,k}.$$

Proof. A proof can be found e.g. in [Brieskorn and Knörrer, 1986]. □

6. Adjoints

Let $\mathcal{C} \subset \mathbf{P}^2_k$ be an irreducible curve with r singular points q_i, for $1 \leq i \leq r$. For each singularity q_i let $\mathfrak{z}_{i,j}$, for $1 \leq j \leq s_i$, be the curve-branches of $\mathcal{C}$ which have q_i as their origin.

Furthermore we define $m_{i,j,k}$ and $n_{i,j,k}$, for $1 \leq k \leq N_i$, as the multiplicities of the infinitely near points of q_i w.r.t. to the curve-branch $\mathfrak{z}_{i,j}$ and the curve $\mathcal{C}$, respectively.

The following definition can be found in [Keller, 1974].

Definition 30. Let the notation be as above. For each curve-branch $\mathfrak{z}_{i,j}$ we define a natural number $d_{i,j}$ in the following way.

$$d_{i,j} = \sum_{k=0}^{N_i} m_{i,j,k}(n_{i,j,k} - 1)$$

A curve $\mathcal{D}$ (which may also be reducible) is called an *adjoint curve of* $\mathcal{C}$ if the intersection multiplicity of each curve-branch $\mathfrak{z}_{i,j}$ of $\mathcal{C}$ and the curve $\mathcal{D}$ at all singular points q_i is at least $d_{i,j}$, i.e.

$$\nu_{q_i}(\mathfrak{z}_{i,j}, \mathcal{D}) \geq d_{i,j} \ ,$$

for $1 \leq j \leq s_i$, $1 \leq i \leq r$.

6.1. **The Polar Curve.** In order to be able to give a more convenient description of the adjoint curves we define the polar curve of a curve w.r.t. to some point.

Definition 31 (Polar Curve)**.** Given a curve $\mathcal{C} \subset \mathbf{P}^2_k$ with defining polynomial $f(x_0, x_1, x_2) \in k[x_0, x_1, x_2]$ and a point $p = (p_0, p_1, p_2) \in \mathbf{P}^2_k$.

The *polar-curve* $P_{\mathcal{C},p}$ *of* $\mathcal{C}$ *with respect to* p is defined by the polynomial

$$P_{\mathcal{C},p} : p(x_0, x_1, x_2) = \sum_{i=0}^{2} p_i \frac{\partial}{\partial x_i} f(x_0, x_1, x_2).$$

In the following we will prove two short Lemmata which deal with the intersection-multiplicity of a curve-branch and the polar curve. (We also reproduce the proofs here, although we are quite sure that these Lemmata are well known since we could not find suitable proofs in literature.)

Lemma 1. *Let* $\mathcal{C} \subset \mathbf{P}^2_k$ *be a curve,* $q \in \mathbf{P}^2_k$ *a point on* $\mathcal{C}$ *and* $\mathfrak{z}$ *a curve-branch of* $\mathcal{C}$ *with origin* q*. Further let* p_1*,* $p_2 \in \mathbf{P}^2_k$ *both not equal to* q *and* $\mathcal{P}_1 = P_{\mathcal{C},p_1}, \mathcal{P}_2 = P_{\mathcal{C},p_2}$ *and* $\mathcal{L}_1 = \overline{q, p_1}, \mathcal{L}_2 = \overline{q, p_2}$*.*

Then

$$\nu_q(\mathfrak{z}, \mathcal{P}_1) - \nu_q(\mathfrak{z}, \mathcal{L}_1) = \nu_q(\mathfrak{z}, \mathcal{P}_2) - \nu_q(\mathfrak{z}, \mathcal{L}_2).$$

Proof. W.l.o.g. we can assume that $q = (1, 0, 0)$ and that q, p_1, p_2 do not lie on a line. So we can assume that $p_1 = (0, 0, 1)$ and $p_2 = (0, 1, 0)$.

This yields then:

$$\mathcal{L}_1 : x_1 = 0$$
$$\mathcal{L}_2 : x_2 = 0$$

The polar curves $\mathcal{P}_1$ and $\mathcal{P}_2$ are then of the form

$$\mathcal{P}_1 : \frac{\partial}{\partial x_2} f(x_0, x_1, x_2) = 0,$$
$$\mathcal{P}_2 : \frac{\partial}{\partial x_2} f(x_0, x_1, x_2) = 0.$$

Let $(A_0(t), A_1(t), A_2(t)$ be the normal form of $\mathfrak{z}$. Since $\mathfrak{z}$ is a curve-branch of $\mathcal{C}$ it clearly holds by definition that $f(A_0(t), A_1(t), A_2(t)) = 0$ and hence

$$\frac{d}{dt} f(A_0(t), A_1(t), A_2(t)) = 0$$

But this is equal to

$$\frac{\partial}{\partial x_1} f(A_0(t), A_1(t), A_2(t)) \frac{d}{dt} A_1(t) + \frac{\partial}{\partial x_2} f(A_0(t), A_1(t), A_2(t)) \frac{d}{dt} A_2(t) = 0$$

from which follows that

$$\mathbf{o}\left(\frac{\partial}{\partial x_1} f(A_0(t), A_1(t), A_2(t))\right) + \mathbf{o}\left(\frac{d}{dt} A_1(t)\right) =$$
$$= \mathbf{o}\left(\frac{\partial}{\partial x_2} f(A_0(t), A_1(t), A_2(t))\right) + \mathbf{o}\left(\frac{d}{dt} A_2(t)\right).$$

So we finally get (see Definition 21)

$$\nu_q(\mathfrak{z}, \mathcal{P}_1) - \nu_q(\mathfrak{z}, \mathcal{L}_1) = \nu_q(\mathfrak{z}, \mathcal{P}_2) - \nu_q(\mathfrak{z}, \mathcal{L}_2).$$

□

Lemma 2. *Let $\mathcal{C} \subset \mathbf{P}^2_k$ be a curve, $q \in \mathbf{P}^2_k$ a point on $\mathcal{C}$ and $\mathfrak{z}$ a curve-branch of $\mathcal{C}$ with origin q. Furthermore let $p \neq q \in \mathbf{P}^2_k$ and $\mathcal{P} = P_{\mathcal{C},p}$ and $\mathcal{L} = \overline{q,p}$. We define m_k and n_k, for $1 \leq k \leq N$, as the multiplicities of the infinitely near points of q w.r.t. to the curve-branch $\mathfrak{z}$ and the curve $\mathcal{C}$.*

Then

$$\nu_q(\mathfrak{z}, \mathcal{P}) - \nu_q(\mathfrak{z}, \mathcal{L}) = \sum_{k=0}^{N} m_k(n_k - 1) - 1.$$

Proof. W.l.o.g. we can assume that $q = (1, 0, 0)$ and $p = (0, 0, 1)$. Hence we get

$$\mathcal{L} : x_1 = 0$$

and

$$\mathcal{P} : \frac{\partial}{\partial x_2} f(x_0, x_1, x_2) = 0.$$

Using Lemma 1 we can further assume that the point p is in general position w.r.t. to the curve $\mathcal{C}$ which implies that $\nu_q(\mathcal{C}) = \nu_q(\mathcal{C}, \mathcal{L}) = n_0$ and $\nu_q(\mathfrak{z}) = \nu_q(\mathfrak{z}, \mathcal{L}) = m_0$.

After introducing affine coordinates

$$x = \frac{x_1}{x_0}$$
$$y = \frac{x_2}{x_0}$$

the curve $\mathcal{C}$ has equation $f(x, y) = f(1, x_1, x_2)$ and the polar curve $\mathcal{P}$ and the line $\mathcal{L}$ are defined by

$$\mathcal{L} : x = 0$$

and

$$\mathcal{P} : \frac{\partial}{\partial y} f(x,y) = 0.$$

We will prove the Lemma by induction on the length z of the standard resolution of q.

$z = 0$: In this case q is a regular point. So we get $\nu_q(\mathfrak{z}, \mathcal{P}) - \nu_q(\mathfrak{z}, \mathcal{L}) = 0 - 1 = -1$ which is equal to $1(1-1) - 1 = -1$.

$z \to z+1$: We apply the σ-process

$$x = v$$
$$y = uv$$

which has as exceptional divisor $v = 0$. Let $\tilde{\mathcal{C}}$, $\tilde{\mathfrak{z}}$ and $\tilde{\mathcal{P}}$ be the strict transforms of $\mathcal{C}$, $\mathfrak{z}$ and $\mathcal{P}$ w.r.t. this σ-process.

As one can easily check $\tilde{\mathcal{P}}$ is the polar curve of $\tilde{\mathcal{C}}$ w.r.t. $(0,1,0)$ (embedded in the projective space) since we assumed that p is in general position. The line through an infinitely near point of q and $(0,1,0)$ is therefore defined by $v = 0$. So we are in a situation where we can apply our induction hypothesis

$$\nu_{\tilde{q}}(\tilde{\mathfrak{z}}, \tilde{\mathcal{P}}) - \nu_{\tilde{q}}(\tilde{\mathfrak{z}}, \tilde{\mathcal{L}}) = \sum_{k=1}^{N} m_k(n_k - 1) - 1.$$

Since $\tilde{\mathcal{L}}$ is exactly the exceptional divisor it holds that $\nu_{\tilde{q}}(\tilde{\mathfrak{z}}, \tilde{\mathcal{L}}) = m_0$.

Using Proposition 6 we get

$$\begin{aligned}
\nu_q(\mathfrak{z}, \mathcal{P}) &= m_0(n_0 - 1) + \nu_{\tilde{q}}(\tilde{\mathfrak{z}}, \tilde{\mathcal{P}}) = \\
&= m_0(n_0 - 1) + \sum_{k=1}^{N} m_k(n_k - 1) - 1 + m_0 = \\
&= \sum_{k=0}^{N} m_k(n_k - 1) - 1 + m_0.
\end{aligned}$$

Again using the fact that p is a general point we get

$$\nu_q(\mathfrak{z}, \mathcal{P}) - \nu_q(\mathfrak{z}, \mathcal{L}) = \sum_{k=0}^{N} n_k(m_k - 1) - 1.$$

□

Using Lemma 2 we get the following characterization of adjoint curves.

Corollary 1. *Let $\mathcal{C} \subset \mathbf{P}^2_k$ be a curve and $\mathcal{P} \subset \mathbf{P}^2_k$ the polar-curve of $\mathcal{C}$ w.r.t. some point p not lying on $\mathcal{C}$. A curve $\mathcal{D} \subset \mathbf{P}^2_k$ is adjoint to $\mathcal{C}$ if*

$$\nu_q(\mathfrak{z}, \mathcal{D}) \geq \nu_q(\mathfrak{z}, \mathcal{P}) - \nu_q(\mathfrak{z}, \overline{q,p}) + 1$$

for all singular points q of $\mathcal{C}$ and all curve-branches $\mathfrak{z}$ of $\mathcal{C}$ with origin q.

7. Computing the System of Adjoint Curves

Given a curve $\mathcal{C}$ and a natural number d. We will show in this section how to compute the system of adjoint curves of $\mathcal{C}$ of degree d.

7.1. Linear System of Curves. A curve $\mathcal{C} \subset \mathbf{P}^2_k$ of degree d is defined by a polynomial

$$f(x_0, x_1, x_2) = \sum_{i+j+k=d} a_{i,j,k} x_0^i x_1^j x_2^k.$$

So we can set up correspondence between the curves of degree d and the coefficients $a_{i,j,k}$ (which is an isomorphism up to a common factor of the $a_{i,j,k}$'s).

Hence the space of all curves of degree d can be identified with the projective space $\mathbf{P}_k^{N_d}$, where $N_d = \frac{d(d+3)}{2}$.

Definition 32. A linear subspace L of $\mathbf{P}^{N_d}$ is called *a linear system of curves of degree* d. If a point p is a point of multiplicity greater or equal to m for every curve in the linear system L, p is called a *base point of multiplicity* m *of the linear system* L.

Definition 33. Given a curve $\mathcal{C} \subset \mathbf{P}^2_k$ and $d \in \mathbb{N}$. The linear system of curves of degree d which consists of all adjoint curves of $\mathcal{C}$ is called the *(linear) system of adjoint curves of degree* d *of* $\mathcal{C}$.

7.2. Computing the System of Adjoint Curves. Given a curve $\mathcal{C}$ defined by $f(x_0, x_1, x_2) \in k[x_0, x_1, x_2]$ and a natural number d. We want to compute the system of adjoint curves of $\mathcal{C}$ of degree d.

Let $g(x_0, x_1, x_2) \in k[x_0, x_1, x_2]$ be a form of degree d with undetermined coefficients $a_{i,j,k}$ defining a curve $\mathcal{D}$, i.e.

$$g(x_0, x_1, x_2) = \sum_{i+j+k=d} a_{i,j,k} x_0^i x_1^j x_2^k.$$

Let R be the number of singular points of $\mathcal{C}$. For each singular point q_i, for $1 \leq i \leq R$, we define S_i as the number of curve-branches of $\mathcal{C}$ with origin q_i. By $\mathfrak{z}_{i,j} = (x_{i,j}(t), y_{i,j}(t), 1)$, for $1 \leq j \leq S_i$, we denote the curve-branches of $\mathcal{C}$ which have q_i as their origin.

We fix some point $p \notin \mathcal{C}$ and compute the polar curve $\mathcal{P} = P_{\mathcal{C},p}$ of $\mathcal{C}$ w.r.t. p. Let $\mathcal{L}_i$ be the line through p and q_i.

Next we compute

$$d_{i,j} = \nu_{q_i}(\mathfrak{z}_{i,j}, \mathcal{P}) - \nu_{q_i}(\mathfrak{z}_{i,j}, \mathcal{L}_i) + 1.$$

Requiring that

$$\nu_{q_i}(\mathfrak{z}_{i,j}, \mathcal{D}) \geq d_{i,j}$$

which is equivalent to

$$o\left(g(x_{i,j}(t), y_{i,j}(t), 1)\right) \geq d_{i,j}$$

we get linear conditions on the indeterminates $a_{i,j,k}$.

Solving the resulting linear system we get the system of adjoint curves of degree d of $\mathcal{C}$.

8. Examples

8.1. **Example 1.** Let $\mathcal{C} \subset \mathbf{P}^2_k$ be the curve defined by (see also Example 2 and 3)

$$x_0x_2^5 - 2x_0^2x_1^2x_2^2 + 3x_0x_1^4x_2 - x_1^6.$$

The polar curve $\mathcal{P}$ of $\mathcal{C}$ w.r.t. $p = (0,1,0) \notin \mathcal{C}$ is then defined by

$$-4x_0^2x_1x_2^2 + 12x_0x_1^3x_2 - 6x_1^5.$$

The curve $\mathcal{C}$ has only one singularity at $q = (1,0,0)$ and $\mathcal{C}$ decomposes at q into three branches

$$\begin{aligned}
\mathfrak{z}_1 : A_0(t) &= 1 \\
A_1(t) &= t \\
A_2(t) &= t^2 + t^6 + 3t^{10} + 13t^{14} + \cdots, \\
\mathfrak{z}_2 : A_0(t) &= 1 \\
A_1(t) &= t \\
A_2(t) &= \frac{1}{2}t^2 - \frac{1}{32}t^6 + \frac{3}{256}t^{10} - \frac{13}{2048}t^{13} + \cdots, \\
\mathfrak{z}_3 : A_0(t) &= 1 \\
A_1(t) &= 2t^3 \\
A_2(t) &= 2t^2 - 2t^6 - \frac{8}{3}t^{10} - \frac{20}{3}t^{14} + \cdots.
\end{aligned}$$

Next we compute

$$\begin{aligned}
d_1 &= \nu_q(\mathfrak{z}_1, \mathcal{P}) - \nu_q(\mathfrak{z}_1, \overline{q,p}) + 1 = 5 - 2 + 1 = 4, \\
d_2 &= \nu_q(\mathfrak{z}_2, \mathcal{P}) - \nu_q(\mathfrak{z}_2, \overline{q,p}) + 1 = 5 - 2 + 1 = 4, \\
d_3 &= \nu_q(\mathfrak{z}_3, \mathcal{P}) - \nu_q(\mathfrak{z}_3, \overline{q,p}) + 1 = 7 - 2 + 1 = 6.
\end{aligned}$$

Let $\mathcal{D}$ be a curve of degree 3 with defining polynomial (written in affine coordinates)

$$\begin{aligned}
g(x_0, x_1, x_2) = {} & a_{3,0,0}x_0^3 + \\
& a_{2,1,0}x_0^2x_1 + a_{2,0,1}x_0^2x_2 + \\
& a_{1,2,0}x_0x_1^2 + a_{1,1,1}x_0x_1x_2 + a_{1,0,2}x_0x_2^2 + \\
& a_{0,3,0}x_1^3 + a_{0,2,1}x_1^2x_2 + a_{0,1,2}x_1x_2^2 + a_{0,0,3}x_2^3.
\end{aligned}$$

According to Corollary 1 the curve $\mathcal{D}$ is adjoint to $\mathcal{C}$ iff

$$\nu_q(\mathfrak{z}_1, \mathcal{D}) = \mathbf{o}\left(g(A_0(t), A_1(t), A_2(t))\right) =$$
$$\mathbf{o}\left(a_{3,0,0} + a_{2,1,0}t + (a_{2,0,1} + a_{1,2,0})t^2 + (a_{1,1,1} + a_{0,3,0})t^3 + \cdots\right) \geq 4,$$

$$\nu_q(\mathfrak{z}_2, \mathcal{D}) = \mathbf{o}\left(g(A_0(t), A_1(t), A_2(t))\right) =$$
$$\mathbf{o}\left(a_{3,0,0} + a_{2,1,0}t + (\frac{1}{2}a_{2,0,1} + a_{1,2,0})t^2 + (\frac{1}{2}a_{1,1,1} + a_{0,3,0})t^3 + \cdots\right) \geq 4,$$

$$\nu_q(\mathfrak{z}_3, \mathcal{D}) = \mathbf{o}\left(g(A_0(t), A_1(t), A_2(t))\right) =$$
$$\mathbf{o}\left(a_{3,0,0} + 2a_{2,0,1}t^2 + 2a_{2,1,0}t^3 + 4a_{1,0,2}t^4 + 4a_{1,1,1}t^5 + \cdots\right) \geq 6.$$

Solving the resulting linear system we get

$$a_{3,0,0} = a_{2,1,0} = a_{2,0,1} = a_{1,2,0} = a_{1,1,1} = a_{0,3,0} = 0.$$

Hence the system of adjoint curves of degree 3 of $\mathcal{C}$ is of the form

$$a_{1,0,2}x_0x_2^2 + a_{0,2,1}x_1^2x_2 + a_{0,1,2}x_1x_2^2 + a_{0,0,3}x_2^3.$$

8.2. **Example 2.** If a curve has more than one singularity we treat each singular point separately and combine the linear constraints we get for the indeterminates $a_{i,j,k}$.

We want to compute the system of adjoint curves of degree 6 of the curve $\mathcal{C} \subset \mathbf{P}_k^2$ defined by

$$x_0^6x1^4 - 23x_0^4x_1^5x_2 + x_0^7x_1x_2^2 - 2x_1^3x_2^7 + 3x_0^7x_2^3 + 5x_0^{10}.$$

The curve $\mathcal{C}$ has two singular points $q_1 = (0,1,0)$, $q_2 = (0,0,1) \in \mathbf{P}_k^2$. Let $\mathcal{P} \subset \mathbf{P}_k^2$ be the polar-curve of $\mathcal{C}$ w.r.t. $p = (1,0,0) \notin \mathcal{C}$, i.e.

$$6x_0^5x_1^4 - 92x_0^3x_1^5x_2 + 7x_0^6x_1x_2^2 + 21x_0^6x_2^3 + 50x_0^9.$$

The singularity at q_1 consists of three curve-branches

$$\begin{aligned}
\mathfrak{z}_1 : A_0(t) &= t\\
A_1(t) &= 1\\
A_2(t) &= \frac{1}{23}t^2 + \frac{5}{23}t^6 + \cdots,\\
\mathfrak{z}_2 : A_0(t) &= \zeta_1 t^3\\
A_1(t) &= 1\\
A_2(t) &= \zeta_1 t^2 + \frac{1}{12}t^6 + \frac{7}{3312}\zeta_1 t^{10} - \frac{23}{24}\zeta_1 t^{13} + \cdots,\\
\mathfrak{z}_3 : A_0(t) &= \zeta_2 t^3\\
A_1(t) &= 1\\
A_2(t) &= \zeta_2 t^2 + \frac{1}{12}t^6 + \frac{7}{3312}\zeta_2 t^{10} - \frac{23}{24}\zeta_2 t^{13} + \cdots,
\end{aligned}$$

where ζ_1, ζ_2 are a roots of $2\zeta^2 + 23$. Whereas there is only one curve-branch of $\mathcal{C}$ with origin q_2, namely

$$\begin{aligned}
\mathfrak{z}_4 : A_0(t) &= \frac{2}{3}t^3\\
A_1(t) &= \frac{4}{9}t^7 + \frac{16}{729}t^{14} + \frac{160}{2187}t^{16} - \frac{640}{177147}t^{23} + \cdots\\
A_2(t) &= 1.
\end{aligned}$$

For the intersection multiplicities d_i, $1 \leq i \leq 4$, we get

$$\begin{aligned}
d_1 &= \nu_{q_1}(\mathfrak{z}_1, \mathcal{P}) - \nu_{q_1}(\mathfrak{z}_1, \overline{q_1, p}) + 1 = 5 - 2 + 1 = 4,\\
d_2 &= \nu_{q_1}(\mathfrak{z}_1, \mathcal{P}) - \nu_{q_1}(\mathfrak{z}_2, \overline{q_1, p}) + 1 = 11 - 2 + 1 = 10,\\
d_3 &= \nu_{q_1}(\mathfrak{z}_1, \mathcal{P}) - \nu_{q_1}(\mathfrak{z}_3, \overline{q_1, p}) + 1 = 11 - 2 + 1 = 10,\\
d_4 &= \nu_{q_2}(\mathfrak{z}_1, \mathcal{P}) - \nu_{q_2}(\mathfrak{z}_4, \overline{q_2, p}) + 1 = 18 - 7 + 1 = 12.
\end{aligned}$$

The system of adjoint curves of degree 6 of $\mathcal{C}$ is given by

$$a_{3,1,2}x_0^3x_1x_2^2 + a_{5,1,0}x_0^5x_1 + a_{4,1,1}x_0^4x_1x_2 + a_{4,0,2}x_0^4x_2^2 + a_{5,0,1}x_0^5x_2 + a_{6,0,0}x_0^6 + a_{2,2,2}x_0^2x_1^2x_2^2 + a_{3,2,1}x_0^3x_1^2x_2 + a_{4,2,0}x_0^4x_1^2 + a_{2,1,3}x_0^2x_1x_2^3.$$

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