Building-In Hybrid Theories*

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Abstract

By a hybrid theory we mean a theory which is combined from different sub-theories. We present an approach to build-in hybrid theories into theorem provers. Our aim is to obtain a reasoner for a hybrid theory by a possibly simple combination of reasoners dedicated for its constituents. For this purpose we formulate sufficient criterions. This more detailed view on building-in theories is not covered by other general results [2, 3, 4, 7]. The technique described in [7] had to be refined. The method applies to different calculi. As an application we discuss the target language of the algebraic translation of multi-modal logic and extended multi-modal logic [5].

1 Introduction

Hybrid reasoning is usually understood as the cooperation of a foreground reasoner with a background reasoner. The foreground reasoner takes care of the general logical structure of a formula to be proved or refuted. The background reasoner is consulted whenever the meaning of special built-ins has to be considered. Many instances of this general scheme are known (see [9, 7, 2] for overviews). A number of general results [2, 3, 4, 7] form a framework for building-in theories into quit different theorem proving procedures. However, those approaches consider the built-in theory as homogeneous.

Nevertheless, in certain applications we have to take care of the internal structure of the built-in theory. Interesting case studies for this phenomenon are the translations of multi-modal logic (MML) or extended multi-modal logic (EML) into certain fragments of first-order logic with built-in theories following [5]. The obtained built-in theories are hybrids combining two sub-theories. One sub-theory is a definite theory which may, in the case of EML, contain equality. The other sub-theory is an equational theory. Each of the sub-theories is related to certain sublanguages of the target language of the translation. Using a general technique outlined in [7] for each of the sub-theories may be constructed theory reasoners that are complete for the related sublanguages. But the question was open whether a certain combination of the reasoners for the components of a hybrid theory is complete with respect to the whole target language. In this paper we present two sufficient criteria which generalise the both mentioned case studies.

Following this approach a reasoner for MML has been implemented. The prototypical implementation based on the connection method provides an automatic translation of a

*This work has partially been funded by the Deutsche Forschungsgemeinschaft under contract Pe 480/6-1 as well as by the Alfried Krupp von Bohlen und Halbach-Stiftung and the Alexander von Humboldt-Stiftung.

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$\mathcal{T}$		$\Re$	
$(X * Y) * Z = X * (Y * Z)$	(1)	$\neg k(a, X),\ k(a, f(\varepsilon!X))$	(6)
$1 * X = X$	(2)	$k(a, g(\varepsilon))$	(7)
$X * 1 = X$	(3)		
$\omega!(X * Y) = (\omega!X)!Y$	(4)		
$\omega!1 = \omega$	(5)		

Figure 1: A hybrid theory

given multi-modal formula into first-order clause logic and the construction of the corresponding hybrid theory. This paper is organised as follows. In section 2 we discuss a running example and merely illustrate and motivate some general notions. The algebraic translation of multi-modal logic to first-order logic and the resulting target language and theory give the background for this example. Section 3 is devoted to certain sufficient criteria for combining theories. Finally, in section 4 we suggest directions of further research and discuss related work. A long version of this paper will be available via the author's home page http://www.imn.htwk-leipzig.de/ uwe.

2 An example motivating hybrid theories

The idea of reasoning in hybrid theories will be illustrated by an example. Due to space restrictions the example is very simple. Nevertheless, it allows us to discuss characteristic properties of a wider class of examples. Let us consider the matrix M consisting of two clauses displayed 2. M is unsatisfiable in the theory displayed in figure 1.

$$\begin{bmatrix} \neg w(\varepsilon!X!Y!Z, a) & \neg k(a, Z) \\ w(\varepsilon!W!f(\varepsilon!W), a) & \neg k(a, W) \end{bmatrix}$$

$$\{X \mapsto e, Y, W \mapsto g(\varepsilon), Z \mapsto f(\varepsilon!g(\varepsilon))\}$$

Figure 2: A matrix with spanning hybrid theory mating and simultaneous unifier

The considered theory consists of two theories $\mathcal{T}$ and $\Re$. The union $\mathcal{T} \cup \Re$ is the hybrid theory we would like to build-in into a theorem prover. The set of clauses M is written in matrix form, each clause forms a row consisting of two literals. Lower case letters denote predicate symbols (like w and k) or constants (like ε and e) and function symbols (like f and g). Some binary function symbols have been used in infix notation (i.g. $\star$ and !). Capital letters represent variables (here: X, Y, Z and W). A clause represents the universally closed disjunction of its literals, a matrix the conjunction of its clauses. What is the general procedure for proving the theory unsatisfiability of a matrix? It is sufficient to find a ground instance of a set of copies of clauses of the matrix (of a so called amplification of M) which is unsatisfiable in the considered theory. For open (i.e. quantifier free) theories this condition is also necessary due the theory version of the Herbrand theorem which holds for those theories. Such an instantiation may be given by a substitution like that in the lower half of figure 2. The amplification used in this figure is trivial. Just one copy occurs for each clause.

How the theory unsatisfiability of a ground instance of the amplification of a matrix may be proved? For this purpose it would be helpful to imagine the disjunctive normal form of the ground instance. Every disjunct of this normal form is the conjunction of its literals and must be theory unsatisfiable. In order to prove this we just have to make sure that each of those conjunctions contains a theory unsatisfiable subset of literals, a so called theory connection. For efficiency reasons we are interested in indicating minimal theory connections. A set of theory connections with this property is called spanning theory mating. The property to be proved is that there is a spanning theory mating which has a simultaneous theory unifier. In figure 2 the elements of a spanning theory mating are indicated by arcs.

There are different procedures checking sufficient conditions for the existence of such a theory mating - saturation based (like theory resolution) as well goal oriented (like different theory connection calculi). The key capabilities needed in any case are first of all to supply sufficiently many theory connections in order to find all necessary spanning theory matings and at second to construct algorithms for finding sufficiently many theory unifiers. Here is important to take in account the set of formulas which may appear as input for the theorem proving procedure. This set will be called a *query language*. The first property has been formalised by the notion of a set of theory connections complete with respect to a certain query language. The second notion is that of a solvable unification problem within a set of theory connections (see [7]). Returning to our running example: if we assume that equality does not appear in queries and that the predicate symbols of the sub-theory $\Re$ will occur only in negative literals then we get the following simple situation: either we have to consider a pair of literals $w(t_1, t_2), \neg w(r_1, r_2)$ such that the terms $w(t_1, t_2)$ $w(r_1, r_2)$ are $\mathcal{T}$-unifiable or we have a literal $\neg k(t_1, k_2)$ which is $\Re$-unifiable. If in the running example the equality sign would be admitted in queries then $\mathcal{T}$-connections would be of a much more difficult form. They could contain unpredictably many equality literals and the simultaneous unification problem became undecidable (see [6] for a dicussion).

3 Combining theories

In section 2 we have discussed an example motivating the treatment of the background reasoner as a hybrid system. Let us now forge precise notions from the observations made for the target logic of the algebraic translation of multi-modal reasoning. Our goal is to be able to construct a $\mathcal{T}\cup\Re$-reasoner from given $\mathcal{T}$- and $\Re$-reasoners. To be more precise. A formula will be considered as consisting of a $\mathcal{T}$-layer and an $\Re$-layer. There sets of theory connections $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_\Re$ for the sublanguages. The intended $\mathcal{T}$-reasoner should try to find an $\mathcal{U}_\mathcal{T}$-connection if the current goal is in the $\mathcal{T}$-layer and an $\mathcal{U}_\Re$-connection if the current goal is in the $\Re$-layer. Under which circumstances $\mathcal{U}_\mathcal{T} \cup \mathcal{U}_\Re$ is a complete set of $\mathcal{T} \cup \Re$-connections for $\mathcal{Q}$ if so are $\mathcal{U}_\mathcal{T}$ for $\mathcal{Q}_\mathcal{T}$ and $\mathcal{U}_\Re$ for $\mathcal{Q}_\Re$? Will the theory unification problems in $\mathcal{U}_\mathcal{T}$ and $\mathcal{U}_\Re$ interfere or not. The last alternative would us allows to use the unification algorithms for the connections belonging to one of both layers without change.

Definition 3.1 Let the theories $\mathcal{T}$ and $\Re$ form a hybrid theory in the union $\Sigma \cup \Delta$ of their signatures and let $\mathcal{Q}$ be a query language formulated in the signature $\Sigma \cup \Delta$.

Then every clause C in a matrix $M \in \mathcal{Q}$ contains two sub-clauses $C_\mathcal{T}$ and $C_\Re$ consisting of literals L expressed in signature Σ (respectively L' expressed in signature Δ). The set of nonempty sub-clauses $C_\mathcal{T}$ of M will be called the *$\mathcal{T}$-layer* of M. Analogously will be defined the *$\Re$-layer* of M. By $\mathcal{Q}_\mathcal{T}$ (analogously $\mathcal{Q}_\Re$) will be denoted the set of all matrices being the $\mathcal{T}$-layer (respectively the $\Re$-layer) of a query from $\mathcal{Q}$. $\mathcal{Q}_{\mathcal{T}}$ (analogously $\mathcal{Q}_{\Re}$) will be called the *$\mathcal{T}$-layer* (respectively the *$\Re$-layer*) of $\mathcal{Q}$.

If for a matrix $M \in \mathcal{Q}$ every of its clauses is the union of its $\mathcal{T}$- and $\Re$-layers then M will be said to be *covered by its $\mathcal{T}$- and $\Re$-layers.* If every matrix $M \in \mathcal{Q}$ is covered by its $\mathcal{T}$- and $\Re$-layers then query language $\mathcal{Q}$ is said to be *covered by its $\mathcal{T}$- and $\Re$-layers.* $\Box$

Definition 3.2 Let $\mathcal{T}$ and $\Re$ form a hybrid theory in the union $\Sigma \cup \Delta$ of signatures. Let $\mathcal{Q}$ be a query language formulated in a signature containing both signatures Σ and Δ. Moreover, let $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$ be sets of $\mathcal{T}$-connections and of $\Re$-connections.

We say that $\mathcal{U}_{\mathcal{T}}$ and $\mathcal{U}_{\Re}$ are *separated w.r.t.* $\mathcal{Q}$ if and only if there does not exist connections $u \in \mathcal{U}_{\mathcal{T}}$ and $u' \in \mathcal{U}_{\Re}$ with $L \in u \cap u'$. $\Box$

The following propositions 3.1 and 3.2 give sufficient criteria for the theory completeness of the union of sets of theory connections that are theory complete with respect to the constituent sub-theories of a hybrid theory. The more restricted case of the target logic of the multi-modal logic will be covered by proposition 3.1. The other criterion covers the case of the target logic of the algebraic translation of extended multi-modal logic. Finally we have to formalise a property saying that the unification problems in the sub-theories do not interfere. Due to lack of space we omit this here.

Proposition 3.1 *Let theories $\mathcal{T}$ and $\Re$ expressed in the signatures Σ and Δ respectively form a hybrid theory such that $\mathcal{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover, suppose that: (1) The sets of $\mathcal{T}$-connections $\mathcal{U}_{\mathcal{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ are complete w.r.t. $\mathcal{Q}_{\mathcal{T}}$ and $\mathcal{Q}_{\Re}$ respectively. (2) In $\mathcal{Q}$ equality literals occur only negative. (3) In both theories positive equality literals may occur only within conditional equations[1]. (4) The sets of predicate symbols occurring in $\mathcal{T} \cup \mathcal{Q}_{\mathcal{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint. (5) If equality occurs in $\mathcal{T} \cup \Re$ then let $\mathcal{T}_1$ be that of the sub-theories $\mathcal{T}$ and $\Re$ that does not contain equality and $\mathcal{U}_1$ be the set of theory connections for that sub-theory. Moreover let $\mathcal{E}$ be the set of equational axioms in $\mathcal{T} \cup \Re$. For every $u \in \mathcal{U}_1$ and substitution σ holds $\mathcal{E} \cup \mathcal{T}_1 \models \sigma(\bigvee \bar{u})$ if and only if $\mathcal{T}_1 \models \sigma(\bigvee \bar{u})$. Then the sets of $\mathcal{T}$-connections $\mathcal{U}_{\mathcal{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ are separated with respect to $\mathcal{Q}$. Moreover, $\mathcal{U}_{\mathcal{T}} \cup \mathcal{U}_{\Re}$ is $\mathcal{T}, \Re$-complete with respect to $\mathcal{Q}$.*

Proposition 3.2 *Let theories $\mathcal{T}$ and $\Re$ expressed in the signatures Σ and Δ respectively form a hybrid theory such that $\mathcal{T} \cup \Re$ is consistent. The query language $\mathcal{Q}$ is formulated in the union $\Sigma \cup \Delta$ of signatures. Moreover, suppose that: (1) The sets of $\mathcal{T}$-connections $\mathcal{U}_{\mathcal{T}}$ and of $\Re$-connections $\mathcal{U}_{\Re}$ are complete w.r.t. $\mathcal{Q}_{\mathcal{T}}$ and $\mathcal{Q}_{\Re}$ respectively. (2) In $\mathcal{Q}$ equality literals occur only negative. (3) In both theories positive equality literals may occur only within conditional equations. (4) The sets of non-equational predicate symbols occurring in $\mathcal{T} \cup \mathcal{Q}_{\mathcal{T}}$ and $\Re \cup \mathcal{Q}_{\Re}$ are disjoint. (5) If $\mathcal{T}_{=+}$ and $\Re_{=+}$ are the sets of non-negative equational clauses[2] in $\mathcal{T}$ (and $\Re$ respectively) then hold $\mathcal{T} \models \Re_{=+}$ and $\Re \models \mathcal{T}_{=+}$. Then $\mathcal{U}_{\mathcal{T}} \cup \mathcal{U}_{\Re}$ is $\mathcal{T}, \Re$-complete with respect to $\mathcal{Q}$.*

4 Further and related work

Directions of further work. The algebraic translation of (extended) multi-modal logic into fragments of first-order logic gave the motivation to study reasoning in hybrid

[1] Definite clauses not containing predicate symbols different from equality.

[2] Clauses not containing predicate symbols different from equality.

theories. We formulated criteria for obtaining complete reasoners for a hybrid theory from complete reasoners for its components. The considered theories are given syntactically. It would be interesting to apply analogous investigations to theories given semantically, i.e. by classes of models. A recent case study [8] points out that theorem proving in large theories is a hard task for theorem provers. Perhaps the view of hybrid theories may help to make proving in those theories easier. Further question is, what can we learn for proving in hybrid theories from combination techniques for unification algorithms (see [1]).

Related work. M. Stickel extended resolution to theory resolution and showed many improvements of resolution as special kinds of theory resolution [9]. [2] presents an alternative view exploring possibilities of computing sets of theory connections. An approach to theories given by classes of models has been presented by H.-J. Bürckert [4].

References

[1] Franz Baader and Klaus U. Schulz. Combination techniques and decision problems for disunification. Research Report CIS-Rep-93-65, Centre for Language and Information Processing (CIS), Leopoldstr. 139, D-8000 Munich 40, Germany, jan 1993. The paper will appear in a special issue of the Journal of Theoretical Computer Science containing selected papers of RTA 93 (C. Kirchner, Editor).

[2] Peter Baumgartner. *Theory Reasoning in Connection Calculi and the Linearizing Completion Approach.* PhD thesis, Universität Koblenz-Landau, 1996. (Accepted).

[3] Peter Baumgartner and Frieder Stolzenburg. Constraint model elimination and a pttp-implementation. In *Proceedings Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, Lecture Notes in Artificial Intelligence, pages 201–216. Springer, 1995.

[4] Hans-Jürgen Bürckert. A resolution principle for constrained logics. *Artificial Intelligence*, 66:235–271, 1994.

[5] Francoise Debart, Patrice Enjalbert, and Madeleine Lescot. Multi Modal Logic Programming Using Equational and Order-Sorted Logic. In M. Okada and S. Kaplan, editors, *Proc. 2nd Conf. on Conditional and Typed Rewriting Systems.* Springer, 1990. LNCS.

[6] Anatoli Degtyarev and Andrei Voronkov. Equality elimination for semantic tableaux. In *Proceedings Workshop on Theorem Proving with Analytic Tableaux and Related Methods*, 1995.

[7] Uwe Petermann. Completeness of the pool calculus with an open built in theory. In Georg Gottlob, Alexander Leitsch, and Daniele Mundici, editors, *3rd Kurt Gödel Colloquium '93*, volume 713 of *Lecture Notes in Computer Science.* Springer-Verlag, 1993.

[8] Wolfgang Reif and Gerhard Schellhorn. Theorem proving in large theories. In *Proceedings of the 14-th International Conference on Automated Deduction (CADE-14).* Springer-Verlag, 1997. LNAI.

[9] M.E. Stickel. Automated deduction by theory resolution. *J. of Automated Reasoning*, 4(1):333–356, 1985.