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Computing the Intersection of Invariant Rings

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We propose an algorithm for computing the intersection of invariant rings R^{G_1} and R^{G_2} of finite groups $G_1, G_2 \leq GL_n(\mathbf{K})$, given in terms of algebra generators. The algorithm works in the case that the characteristic of $\mathbf{K}$ is either 0 or prime to the order of the group generated by G_1 and G_2, and calculates an algebra basis for the intersection $R^{G_1} \cap R^{G_2}$.

1. Introduction

Several algorithms have been proposed for computing the ring $\mathbf{K}[x_1, x_2, ..., x_n]^G$ of invariants of a finite matrix group $G \leq GL_n(\mathbf{K})$, see, e.g., (Sturmfels, 1993), (Kemper, 1997), (Kemper and Steel, 1997), and (Decker et. al, 1997). These algorithms usually proceed in two steps. In the first step a set of n algebraically independent invariants $\theta_1, \theta_2, ..., \theta_n$ is constructed, and the second step consists of finding a basis of $\mathbf{K}[x_1, x_2, ..., x_n]^G$ as a $\mathbf{K}[\theta_1, \theta_2, ..., \theta_n]-$module. Two exceptions are the algorithm of E. Noether, cf. (Noether, 1916), which works for groups G with $char(\mathbf{K}) > |G|!$, and Derksen's algorithm for linearly reductive groups, cf. (Derksen, 1997). Both algorithms compute a homogenous $\mathbf{K}-$algebra basis of the invariant ring $\mathbf{K}[x_1, x_2, ..., x_n]^G$. The elements of such a basis are called **fundamental invariants**.

We present an new algorithm for computing the intersection of invariant rings $\mathbf{K}[x_1, x_2, ..., x_n]^{G_1}$ and $\mathbf{K}[x_1, x_2, ..., x_n]^{G_2}$ of finite groups $G_1, G_2 \leq GL_n(\mathbf{K})$, which are given in terms of fundamental invariants. The result is a set of fundamental invariants for the ring $\mathbf{K}[x_1, x_2, ..., x_n]^{G_1} \cap \mathbf{K}[x_1, x_2, ..., x_n]^{G_2}$. This leads to a new approach for computing the invariant ring of a finite matrix group $G = \langle \sigma_1, ..., \sigma_k \rangle \leq GL_n(\mathbf{K})$ with $char(\mathbf{K})$ equal to 0 or prime to $|G|$. First one computes fundamental invariants for the invariant rings $\mathbf{K}[x_1, x_2, ..., x_n]^{\langle \sigma_i \rangle}$ for $1 \leq i \leq k$, and then the invariant ring equals

$$\mathbf{K}[x_1, x_2, .., x_n]^G = \bigcap_{i=1}^{k} \mathbf{K}[x_1, x_2, ..., x_n]^{\langle \sigma_i \rangle}.$$

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For the computation of fundamental invariants of $\mathbf{K}[x_1, x_2, ..., x_n]^{\langle\sigma_i\rangle}$ in the case $\mathbf{K} = \mathbf{C}$ we refet to (Bayer, 1998).

Also the algorithm can be used if the algebra bases of several invariant rings of some groups $G_1, G_2, ..., G_k$ are already known, and one wants to compute the invariant ring of the group $G = \langle G_1 \cup G_2 \cup ... \cup G_k\rangle$.

In Section 2 we give the fundamental definitions and present the finiteness theorem of D. Hilbert, because the intersection algorithm relies on some facts of the proof. In Section 3 we present the intersection algorithm with the correctness proof. With the exception of Lemma 3.1 the results seem to be new.

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2. Hilbert's Finiteness Theorem

Let $\mathbf{K}$ be a field, $G \leq GL_n(\mathbf{K})$ be a finite group and assume that $char(\mathbf{K}) = \mathbf{0}$ or $char(\mathbf{K}) \not| |G|$. The group G acts on the polynomial ring $\mathbf{K}[x_1, x_2, ..., x_n]$ via

$$\sigma \cdot f(x_1, x_2, ..., x_n) := f((x_1, x_2, ..., x_n) \cdot \sigma)$$

for $\sigma \in G$ and $f \in \mathbf{K}[x_1, x_2, ..., x_n]$, where the dot on the right-hand side denotes the ususal dot product from linear algebra.

Definition 2.1. *The polynomial f is* ***invariant*** *(w.r.t. G) if $\sigma \cdot f = f$ for all $\sigma \in G$. The ring*

$$\mathbf{K}[x_1, x_2, ..., x_n]^G := \{f \in \mathbf{K}[x_1, x_2, ..., x_n] | \forall \sigma \in G : \sigma \cdot f = f\}$$

is the ***invariant ring*** *of G.*

With $\langle \mathbf{K}[x_1, x_2, ..., x_n]^G_+ \rangle$ we denote the ideal in $\mathbf{K}[x_1, x_2, ..., x_n]$ which is generated by all homogenous invariants of $\mathbf{K}[x_1, ..., x_n]^G$ of positive degree.

Definition 2.2. *The Reynolds operator is the map*

$$^* : \mathbf{K}[x_1, x_2, ..., x_n] \to \mathbf{K}[x_1, x_2, ..., x_n]^G,$$

$$f^* := \frac{1}{|G|} \sum_{\sigma \in G} \sigma \cdot f.$$

Proposition 2.1. *Let $r \in \mathbf{K}[x_1, x_2, ..., x_n]$ and $f \in \mathbf{K}[x_1, x_2, ..., x_n]^G$. Then $(rf)^* = r^* f$ and $(r^*)^* = r^*$.*

Proof. See, e.g., (Sturmfels, 1993). $\square$

In 1890 D. Hilbert proved that the invariant ring of any group $G \leq GL_n(\mathbf{C})$, which admit a map satisfying the conclusion of Proposition 2.1, is finitely generated as a $\mathbf{C}-$algebra, cf. (Hilbert, 1890). The proof of this Theorem can be carried over to any field $\mathbf{K}$ with $char(\mathbf{K}) \not| |G|$, and implies that any ideal basis of $\langle \mathbf{K}[x_1, x_2, ..., x_n]^G_+ \rangle$, which consists of homogenous invariants, is already an algebra basis for $\mathbf{K}[x_1, x_2, ..., x_n]^G$. Since this property is the key for the intersection algorithm, presented in Section 2, we give it here.

THEOREM 2.1. *(Hilbert 1890) Let $\mathbf{K}$ be a field s.t. $char(\mathbf{K}) = 0$ or $char(\mathbf{K}) \not| |G|$. Then $\mathbf{K}[x_1, x_2, ..., x_n]^G$ is finitely generated as a $\mathbf{K}-$algebra.*

PROOF. Let $R = \mathbf{K}[x_1, x_2, ..., x_n]$ and R^G_+ denote the set of homogenous elements of R^G of positive degree. Hilbert's basis Theorem implies that the ideal $\langle R^G_+ \rangle \trianglelefteq R$ can be generated by finitely many homogenous invariant polynomials $h_1, h_2, ..., h_k$. We claim that $R^G = \mathbf{K}[h_1, h_2, ..., h_k]$. It is clear that $\mathbf{K}[h_1, h_2, ..., h_k] \subseteq R^G$. Now assume that this inclusion is strict and take a homogenous invariant $f \in R^G \backslash \mathbf{K}[h_1, h_2, ..., h_k]$ of minimal degree. Since $f \in \langle R^G_+ \rangle$ we have $f = \sum_{i=1}^k p_i h_i$ for some polynomials $p_i \in R$, The polynomial f is invariant, hence

$$f = f^* = \sum_{i=1}^{k} (p_i h_i)^* = \sum_{i=1}^{k} p_i^* h_i.$$

From $f \notin \mathbf{K}[h_1, h_2, ..., h_k]$ we conclude that for some $j \in \{1, 2, ..., k\}$ the polynomial p_j is not a constant and $p_j^* \neq 0$, so

$$\deg p_j = \deg p_j^* < \deg f,$$

but $p_j^* \in R^G$, a contradiction. □

3. The Intersection Algorithm

Let $R = \mathbf{K}[x_1, x_2, ..., x_n], G_1, G_2$ and G be finite subgroups of $GL_n(\mathbf{K})$ s.t. $G = \langle G_1 \cup G_2 \rangle$, where $\langle G_1 \cup G_2 \rangle$ denotes the group generated by the elements of G_1 and G_2, and assume $char(\mathbf{K}) = 0$ or $char(\mathbf{K}) \not| |G|$. Suppose that the two invariant rings $R^{G_1} = \mathbf{K}[f_1, f_2, ..., f_{m_1}]$ and $R^{G_2} = \mathbf{K}[g_1, g_2, ..., g_{m_2}]$ are given in terms of fundamental invariants. We present an algorithm which computes the algebra basis $\{h_1, h_2, ..., h_m\}$ of the intersection of R^{G_1} and R^{G_2}, i.e.

$$R^{G_1} \cap R^{G_2} = \mathbf{K}[h_1, h_2, ..., h_m] = R^G.$$

The restriction to the nonmodular case comes from the fact that algorithm is based on the proof of Hilbert's finiteness theorem and holds therefore only for a field $\mathbf{K}$ with $char(\mathbf{K}) \not| |G|$. The following theorem is the key to the algorithm.

THEOREM 3.1. *Let $G \leq GL_n(\mathbf{K})$ be a finite group with subgroups G_1 and G_2 s.t. $G = \langle G_1 \cup G_2 \rangle$. Let $\langle R^G_+ \rangle \trianglelefteq R$ denote the ideal generated by all homogenous elements of positive degree from R^G. Let $I = \langle f_1, ..., f_{m_1} \rangle = \langle g_1, ..., g_{m_2} \rangle \trianglelefteq R$ be a proper ideal in R for homogenous elements $f_i \in R^{G_1}$ and $g_j \in R^{G_2}$,respectively, then*

$$I \supseteq \langle R^G_+ \rangle \Longrightarrow I = \langle R^G_+ \rangle.$$

Note : In general, I is strictly contained in $\langle R^G_+ \rangle$.

PROOF. If $G_1 = G_2$ then $G = G_1$ and $R^G = R^{G_1}$. If $I \supseteq \langle R^G_+ \rangle$ then $I = \langle R^G_+ \rangle$ because $\langle R^G_+ \rangle$ is the largest ideal which can be generated from the invariants of positive degree.

Therefore we may assume $R^{G_1} \neq R^{G_2}$. We denote the Reynolds operator of the groups G_k by ^{*_k} for $k = 1, 2$ and of the group G with *. Suppose I contains $\left\langle R_+^G \right\rangle$ properly and let $h \in I \backslash \left\langle R_+^G \right\rangle$ be a homogenous element of minimal degree. Then we can write h in the following form :

$$h = \sum_{i=1}^{m_1} p_i f_i$$

for some homogenous polynomials p_i with $\deg p_i = \deg h - \deg f_i$. At least one of the p_i 's is a nonzero constant, otherwise each f_i is contained in $\left\langle R_+^G \right\rangle$ because of the minimality of h. W.l.o.g. we may assume $h = f_1$. Since I is also generated by $g_1, ..., g_{m_2}$ we have

$$f_1 = \sum_{i=1}^{m_2} r_i g_i = \sum_{i=1}^{m_2} \lambda_i \underbrace{g_i}_{\notin < R_+^G >} + \sum_{i=1}^{m_2} s_i \underbrace{g_i}_{\in < R_+^G >}$$

for some homogenous polynomials r_i with $\deg r_i = \deg f_1 - \deg g_i$. Note that any g_i with $\deg(g_j) < \deg(f_1)$ is contained in $\left\langle R_+^G \right\rangle$ because of the minimality of the degree of f_1. It follows that all λ_i are contained in $\mathbf{K}$ and at least one is nonzero. An application of the Reynolds operator to f_1 yields $f_1^{*_2} = \sum_{i=1}^{m_2} \lambda_i g_i + \sum_{i=1}^{m_2} s_i^{*_2} g_i$ and we have

$$f_1 - f_1^{*_2} = \sum_{i=1}^{m_2} (s_i - s_i^{*_2}) g_i \in \left\langle R_+^G \right\rangle,$$

so $f_1 \equiv f_1^{*_2} \bmod \left\langle R_+^G \right\rangle$. Note that the ideal $\left\langle R_+^G \right\rangle$ is $G-$ invariant.
Now choose generators $G_1 = \langle \sigma_1 ..., \sigma_a \rangle$ and $G_2 = \langle \tau_1 ..., \tau_b \rangle$. We show via induction on the length l of products of the elements from $\{\sigma_1 ..., \sigma_a, \tau_1, ..., \tau_b\}$ that $f_1 \equiv \gamma \cdot f_1 \equiv f_1^{*_2} \equiv \gamma \cdot f_1^{*_2} \bmod \left\langle R_+^G \right\rangle$ for all $\gamma \in G$. The case $l = 0$ is trivial. So assume that $\gamma \in G$ is a product of $l - 1$ elements from the set $\{\sigma_1 ..., \sigma_a, \tau_1, ..., \tau_b\}$ and

$$f_1 \equiv \gamma \cdot f_1 \equiv f_1^{*_2} \equiv \gamma \cdot f_1^{*_2} \bmod \left\langle R_+^G \right\rangle. \tag{3.1}$$

For $\gamma\sigma_j$ we have $\gamma\sigma_j \cdot (f_1 - f_1^{*_2}) = \gamma \cdot f_1 - \gamma\sigma_j \cdot f_1^{*_2}$, so $\gamma\sigma_j \cdot f_1^{*_2} \equiv \gamma \cdot f_1$ and the induction hypothesis implies that $f_1 \equiv \gamma\sigma_j \cdot f_1 \equiv \gamma\sigma_j \cdot f_1^{*_2} \bmod \left\langle R_+^G \right\rangle$. Also for $\gamma\tau_k$ we have $\gamma\tau_k \cdot (f_i - f_i^{*_2}) = \gamma\tau_k \cdot f_i - \gamma \cdot f_i^{*_2}$, so $\gamma\tau_k \cdot f_i \equiv \gamma \cdot f_i^{*_2} \equiv \gamma \cdot f_i$. Thus we have

$$f_1^* = \frac{1}{|G|} \sum_{\sigma \in G} \sigma \cdot f_1 \equiv \frac{1}{|G|} \sum_{\sigma \in G} f_1 \bmod \left\langle R_+^G \right\rangle.$$

Hence

$$f_1 \equiv f_1^* \bmod \left\langle R_+^G \right\rangle.$$

The polynomial f_1^* is contained in R^G, in particular $f_1^* \in \left\langle R_+^G \right\rangle$ so $f_1 \equiv f_1^* \equiv 0 \bmod \left\langle R_+^G \right\rangle$ which implies $f_1 \in \left\langle R_+^G \right\rangle$, a contradiction. $\square$

It is sufficient to know an ideal basis for $\left\langle R_+^G \right\rangle$ containing only homogenous elements, because the Reynolds images of the generators are a set of fundamental invariants for G, which has been shown by H. Derksen, cf. (Derksen, 1997).

LEMMA 3.1. *Let $\langle h_1, h_2, ..., h_m \rangle$ be a set of arbitrary homogenous generators of the ideal $\left\langle R_+^G \right\rangle \trianglelefteq R$. Then $R^G = \mathbf{K}[h_1^*, h_2^*, ..., h_m^*]$.*

PROOF. It is sufficient to show that each homogenous invariant of positive degree is contained in $\langle h_1^*, ..., h_m^*\rangle$. Assume the converse and let $h = \sum_{i=1}^m p_i h_i$ be a homogenous invariant of minimal degree in $\langle h_1, h_2, ..., h_m\rangle \setminus \langle h_1^*, h_2^*, ..., h_m^*\rangle$ for some $p_i \in \mathbf{K}[x_1, x_2, ..., x_n]$. Note that for a homogenous $f \in R$ of with $\deg(f) < \deg(h)$ we have $f \in \langle h_1, h_2, ..., h_m\rangle \Leftrightarrow f \in \langle h_1^*, h_2^*, ..., h_m^*\rangle$. As in the above proof we write $h = \sum_{i=1}^m \lambda_i h_i + \sum_{i=1}^m s_i h_i$ for $\lambda_i \in \mathbf{K}$ and $s_i \in \mathbf{K}[x_1, x_2, ..., x_n]$ with $s_i = 0$ or $\deg(s_i) > 0$. Now the minimality of h implies that $\sum_{i=1}^m s_i h_i \in \langle h_1^*, h_2^*, ..., h_m^*\rangle$, so $\sum_{i=1}^m s_i h_i = \sum_{i=1}^m \widetilde{s}_i h_i^*$ for some $\widetilde{s}_i \in \mathbf{K}[x_1, x_2, ..., x_n]$. But

$$h = h^* = \sum_{i=1}^m \lambda_i h_i^* + \sum_{i=1}^m \widetilde{s}_i{}^* h_i^*,$$

a contradiction. Hence the ideal $\langle h_1^*, h_2^*, ..., h_m^*\rangle$ contains all homogenous invariants of positive degree, and we have $\langle h_1^*, h_2^*, ..., h_m^*\rangle = \left\langle R_+^G \right\rangle = \langle h_1, h_2, ..., h_m\rangle$. $\square$

For a different proof see (loc. cit), lemma 2.2 of ch. 1. We can now state the intersection algorithm. The algorithm computes a basis for the ideal $\left\langle R_+^G \right\rangle$ and applies the Reynolds operator to each basis element.

ALGORITHM 3.1. **Intersection**$(\langle f_1, f_2, ..., f_{m_1}\rangle, \langle g_1, g_2, ..., g_{m_2}\rangle, *)$
In: Algebra generators $f_1, f_2, ..., f_{m_1}$ *for* R^{G_1} *and* $g_1, g_2, ..., g_{m_2}$ *for* R^{G_2}.
The Reynolds operator * *for the group* $\langle G_1 \cup G_2\rangle$.
Out: Algebra generators $h_1, h_2, ..., h_m$ *for* $R^{\langle G_1 \cup G_2\rangle}$.
begin
$I_1 := \langle f_1, f_2, ..., f_{m_1}\rangle$;
$I_2 := \langle g_1, g_2, ..., g_{m_2}\rangle$;
while $I_1 \neq I_2$ ***do***
. $I := I_1 \cap I_2$;
. $I_1 := \left\langle I \cap R^{G_1}\right\rangle$; *// Intersection of* I *with the subring* R^{G_1}.
. $I_2 := \left\langle I \cap R^{G_2}\right\rangle$; *// Intersection of* I *with the subring* R^{G_2}.
end;
return(I_1^*); *// Apply the Reynolds operator to each basis element.*
end Intersection.

The ideal operations in the algorithm are based on Gröbner bases, see (Buchberger, 1985), (Becker, 1993) or (Cox et. al., 1992) for an introduction. For comparing two ideals and for the computation of their intersection we refer, e.g., to (Becker, 1993) or (Cox et. al., 1992). The computation of the intersection of an ideal with an invariant ring is described in section 2.6 of (Sturmfels, 1993).

The termination and correctness of the algorithm follows from the next two theorems.

LEMMA 3.2. *In each iteration of the while loop the ideal* $\left\langle R_+^G \right\rangle$ *is contained in* $I_1 \cap I_2$.

PROOF. Follows from the inclusion of ideals $\langle f_1, f_2, ..., f_{m_1}\rangle \supseteq \left\langle R_+^G \right\rangle$ and $\langle g_1, g_2, ..., g_{m_2}\rangle \supseteq \left\langle R_+^G \right\rangle$ and the equality $R^G = R^{G_1} \cap R^{G_2}$. $\square$

THEOREM 3.2. *The algorithm* **Intersection** *terminates.*

PROOF. Consider the residue class ring $\overline{R} = R/\langle R_+^G \rangle$. Since $\overline{R}$ is zero-dimensional it is Artinian, hence each strictly descending sequence $\{\overline{J}_k\}$ of ideals must stabilize, i.e. $\overline{J}_N = \overline{J}_{N+k}$ for some $N \in \mathbf{N}$ and all $k \in \mathbf{N}$. Let J_n be the intersection $I_1 \cap I_2$ in the $n-$th iteration of the while loop and denote the image of J_n in $\overline{R}$ with $\overline{J}_n$. We obtain a strictly descending sequence of ideals in $\overline{R}$, hence after a finite number of steps the sequence stabilizes. The proof follows from the fact, that the ideals in $\overline{R}$ correspond uniquely to ideals in R containing $\langle R_+^G \rangle$.□

THEOREM 3.3. *The algorithm* **Intersection** *is correct.*

PROOF. It follows from Theorem 3.2 that $I_1 = I_2$ after a finite number of iterations of the while loop. Now Lemma 3.2 implies that I_1 and I_2 satisfy the condition of Theorem 3.1. Let $h_1, h_2, ..., h_m \in R^{G_1}$ be a set of homogenous generators for the ideal I_1, then we know from Lemma 3.1 that $R^G = \mathbf{K}[h_1^*, h_2^*, ..., h_m^*]$. □

EXAMPLE 3.1. *Let* $\mathbf{K} = \mathbf{C}$, $\sigma_1 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ *and* $\sigma_2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. *Fundamental invariants for* $G_1 = \langle\sigma_1\rangle$ *and* $G_2 = \langle\sigma_2\rangle$ *are given by*

$$F_1 = \{x^2 + y^2, z^2, xyz, x^2z - y^2z, x^2y^2, x^3y - xy^3\}$$

and

$$F_2 = \{x + y + z, xy + xz + yz, xyz, x^2y + xz^2 + y^2z\}$$

respectively. We set $G = \langle\sigma_1, \sigma_2\rangle$ *and call* **Intersection**$(F_1, F_2, ^*)$ *and obtain*

$$\begin{aligned} h_1 &= x^2 + y^2 + z^2, \\ h_2 &= xyz, \\ h_3 &= 2x^4 + x^2y^2 + 2y^4 + x^2z^2 + y^2z^2 + 2z^4. \end{aligned}$$

It follows from the previous results that $\mathbf{K}[x, y, z]^G = \mathbf{K}[h_1, h_2, h_3]$.

EXAMPLE 3.2. *Let* $\mathbf{K} = \mathbf{F_5}, \sigma_1 = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & 3 \\ 1 & 2 \end{pmatrix}$ *and* $G = \langle\sigma_1, \sigma_2\rangle$. *Note that* $|G| = 96$. *The subgroup* $\langle\sigma_1\rangle$ *has order* 8 *and* $R^{\langle\sigma_1\rangle}$ *is generated by the fundamental invariants*

$$\begin{aligned} F_1 = \{&x^4 + x^2y^2 + y^4, x^3y + 4x^2y^2 + 4xy^3 + 3y^4, \\ &x^8 + 2x^3y^5 + 4x^2y^6 + 3xy^7 + 2y^8, x^7y + 2x^3y^5 + 2x^2y^6 + 2xy^7 + 4y^8\} \end{aligned}$$

The subgroup $\langle\sigma_2\rangle$ *has order* 4 *and* $R^{\langle\sigma_1\rangle}$ *is generated by the fundamental invariants*

$$F_2 = \{x^2 + 3xy + y^2, x^3 + 3xy^2 + y^3, x^4 + 3x^2y^2 + 4xy^3 + y^4\}$$

We call **Intersection**$(F_1, F_2, ^*)$ *and obtain fundamental invariants for* R^G,*namely*

$$\begin{aligned} h_1 &= 3x^{12} + x^{11}y + 3x^{10}y^2 + x^8y^4 + 4x^7y^5 + 2x^5y^7 + x^4y^8 + 2x^2y^{10} + 3xy^{11} + 3y^{12}, \\ h_2 &= 2x^8 + x^7y + x^6y^2 + 2x^5y^3 + 3x^4y^4 + 4x^3y^5 + 4x^2y^6 + 3xy^7 + 2y^8. \end{aligned}$$

Hence

$$\mathbf{F_5}[x, y]^G = \mathbf{F_5}[h_1, h_2].$$

It is an interesting question whether the algorithm **Intersection** can be generalized to linear reductive groups. An implementation of the algorithm in Mathematica or Singular (cf. (GPS, 1997)) can be found at the homepage of the author.

References

Bayer, T.*Algorithmic Aspects of Invariant Theory.* Diploma thesis, RISC report (to appear).

Becker, T., Weispfennig, V., *Gröbner Bases.* Springer Verlag New York (1993).

Buchberger, B., *Gröbner Bases - an Algorithmic Method in Polynomial Ideal Theory.* In : Bose, N.K.(ed) : Multidimensional System Theory. D. Reidel, Dordrecht 1985 (pp184 - 232).

Cox, D., Little, J., O'Shea, D., *Ideals, Varieties, and Algorithms.* Springer Verlag New York (1992).

Decker, W., Heydtmann, A.E., Schreyer, F., *Generating a Noetherian Normalization of the Invariant Ring of a Finite Group*,J. Symb. Comp (to appear).

Derksen, H., *Constructive Invariant Theory and the Linearization Problem.* PhD thesis, University of Basel (1997).

G.-M. Greuel, G. Pfister, and H. Schönemann. Singular Reference Manual. In *Reports On Computer Algebra*, number 12. Centre for Computer Algebra, University of Kaiserslautern, May 1997. `http://www.mathematik.uni-kl.de/~zca/Singular`

Kemper, G., *Calculating Optimal Homogenous Systems of Parameters.* IWR Preprint **97-08**, Heidelberg (1997).

Kemper, G., Steel, A., *Some Algorithms in Invariant Theory of Finite Groups.* (1997)

Hilbert, D., *Über die Theorie der Algebraischen Formen.* Collected Papers, Volume II, 2nd ed. Springer Verlag Berlin Heidelberg New York (1970).

Noether, E., *Der Endlichkeitssatz der Invarianten endlicher Gruppen.* Math. Ann. 77, (pp 89 - 92) (1916).

Sturmfels, B., *Algorithms in Invariant Theory.* Springer Verlag Wien New York (1993).