# A Computer Program for the Resolution of Singularities

Gábor Bodnár and Josef Schicho
RISC-Linz, Univ. Linz, A-4040 Linz Austria

January 8, 1999

**Abstract**

Villamayor's constructive proof for the resolution of singularities implicitly contains an algorithm, which has been implemented by the authors. We describe some key features which had to be added as to make it work on the computer.

## 1 Introduction

Since recently we have constructive proofs for resolution of singularities in characteristic zero (see [Vil89, Vil91, BM91, BM97, EV]). These proofs do contain, at least implicitly, algorithms for computing the desingularization automatically. The authors have implemented an algorithm based on [EV] in Maple V R5.

The program takes a polynomial in $n$ variables as input. The output is a tree of charts, where

- the root node is $k^n$, the ambient space of the given hypersurface ($k$ is a field of algebraic numbers),
- the children of each node are obtained from the parent node by blowing up or by some other operation described below (cover and exchange),
- the proper transform of the given hypersurface in the leaf nodes is nonsingular, and the total transform is a normal crossing divisor.

The output tree may be viewed by "clicking through the resolution" in an html hierarchy.

Actually, there was a long way to go from the algorithm in [EV] to a working implementation, but space restrictions do not permit to go into technical details. Instead, we explain two typical computational problems and their solutions in the implementation.

First, it is technically difficult to represent blowups of arbitrary regular varieties in affine space. Using the general construction for blowing ups of arbitrary ideals, the blowup variety gets embedded in some high dimensional projective space (see [Har77], page 163). This is not convenient for performing the differentiating operations that are necessary in the algorithm. This problem is computationally mastered by working simultaneously with affine coordinates on the one side and global regular parameters on the other side. The concept of global regular parameters also appears in [BM97], p. 232, in another context.

Second, the constructive proofs are formulated in a local language, i.e. using power series. Theoretically, it is possible to treat the most difficult point in the next blowing up in a similar local fashion. However, a computer program has to resolve the variety as a whole (covering it by a finite number of charts), without making explicit reference to points.

There were also other problems which had to be solved (redundancy in the charts, growth of exponents), but these will not be treated here.

The full program including demo examples is available via the web site `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup`. This work was supported by the Austrian Science Fund (FWF) in the frame of the Adjoints project.

## 2 How to Blow Up a Circle

Suppose you want to resolve the surface $S$ in three-space with equation

$$z^2 + (x^2 + y^2 - 1)^2 y = 0.$$

The circle $x^2 + y^2 - 1 = z = 0$ is a double curve of $S$ which has to be blown up. Actually, the algorithm [EV] tells to first blowup the two points $(\pm 1, 0, 0)$. But for our purposes it suffices to consider what happens in the open chart $y \neq 0$. In this chart, the algorithm [EV] will blow up the circle.

Since the ideal of the circle has two generators, the blown up variety can be covered by two affine charts. One is isomorphic to the zero set of $u(x^2 + y^2 - 1) - z$ in $k^4$. The variable $z$ can be eliminated, so that this chart is actually isomorphic to $k^3$, with coordinates $x, y, u$. The blowing down map is given by $(x, y, u) \mapsto (x, y, u(x^2 + y^2 - 1))$.

The second chart $W$ is isomorphic to the zero set of $vz - (x^2 + y^2 - 1)$ in $k^4$. We do not know if this variety is isomorphic to $k^3$. In any case, such an isomorphism would be hard to find. Hence we have to continue computation with 4 variables, the ambient space $W$ being itself a hypersurface. The next step, computing the singularities of the proper transform of $S$ in $W$, is already much more difficult as in the case of charts isomorphic to $k^3$. One can imagine that these difficulties accumulate in cases where more blowing ups are necessary.

The problem is that we need to represent varieties in such a way that the blowing up can be covered by charts represented in the same way. On the other hand, this way of representing varieties should allow to perform various tasks (like computing multiplicities or coefficient ideals) easily and fast.

The solution is to use two different "coordinate systems", so that a chart is described by the following data.

1. A list of variables (VAR): For each chart $W$, we have the affine coordinates of the ambient affine space, say $x_1, \ldots, x_m$.

2. A list of polynomials (DEP): The chart $W$ is an algebraic subset of $k^m$. Its ideal is generated by a finite number of polynomials $D_1, \ldots, D_r \in k[x_1, \ldots, x_m]$, which we also call the *dependencies*, since they describe the algebraic relations between the coordinate functions restricted to $W$. In general, the number of ambient affine coordinates is larger than $\dim W$.

3. A list of polynomials (IND): We require for each chart the existence of a global system of regular parameters, i.e. an $n$-tuple $(P_1, \ldots, P_n)$ of functions in $k[W]$, where $n = \dim W$, forming a regular system of parameters of each local ring of points in $W$. It is easy to see that every nonsingular variety can be covered by a finite number of charts which have such a global system of parameters (see also [BM97], page 232).

4. An $m \times n$ matrix (PDER): For each chart, we need to know the partial derivations with respect to the parameter system. Note that these are functions from $k[W]$ to $k[W]$. They are uniquely determined by their value at the generators $x_1, \ldots, x_m$. This allows to store the information of all partial derivatives in a finite matrix. Computing partial derivatives is easy (using this matrix), and we can now perform the various tasks mentioned above easily and fast.

During the resolution process, we need to do three operations of forming new charts from a given one. Sometimes, we need to *cover* a chart by smaller charts. Sometimes we need to *exchange* some function in IND. And sometimes, most notably, we need to *blowup* a chart. In the blowup operation, we require that the blowing up center is a variety defined by a subset of IND. Note that this implies that the center is nonsingular and has normal crossing with the hypersurfaces of $W$ defined by IND.

To proceed with our example surface, we first do a cover operation, producing a chart $C_1$ isomorphic to the open set $y \neq 0$:

1. VAR $= [x, y, z, v_1]$,

2. DEP $= [yv_1 - 1]$,

3. IND $= [x, y, z]$,

4. PDER $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -v_1^2 & 0 \end{pmatrix}$.

The entry of the last row of the matrix PDER is introduced by the cover operation and comes from the formula $\partial_y v_1 = \partial_y \frac{1}{y} = -\frac{1}{y^2} = -v_1^2$. The cover operation also produces another chart $C_2$, isomorphic to the open sets $y^2 - 1 \neq 0$, but we only treat $C_1$.

Next, we exchange $y$ by $x^2+y^2-1$ in IND. This is possible because $\partial_y(x^2+y^2-1)$ is invertible in $C_1$. We obtain the chart $E$:

1. VAR $= [x, y, z, v_1]$,
2. DEP $= [v_1 y - 1]$,
3. IND $= [x, x^2+y^2-1, z]$,
4. PDER $= \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & \frac{1}{2}v_1 & 0 \\ 0 & 0 & 1 \\ v_1^3 x & -\frac{1}{2}v_1^3 & 0 \end{pmatrix}$.

Now, we are ready for blowing up. We obtain two charts. The first, say, $B_1$ is given by

1. VAR $= [x, y, v_1, v_2]$,
2. DEP $= [v_1 y - 1]$,
3. IND $= [x, x^2+y^2-1, v_2]$,
4. PDER $= \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & \frac{1}{2}v_1 & 0 \\ v_1^3 x & -\frac{1}{2}v_1^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The variable $z$ has been eliminated. The proper transform of $S$ has equation $v_2^2+y$, hence it is nonsingular. The exceptional divisor is $x^2+y^2-1$, and one can show easily that the total transform is a normal crossing divisor.

The second chart $B_2$ is given by

1. VAR $= [x, y, z, v_1, v_2]$,
2. DEP $= [v_1 y - 1, v_2 z - x^2 - y^2 + 1]$,
3. IND $= [x, v_2, z]$,
4. PDER $= \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & -\frac{1}{2}v_1 z & -\frac{1}{2}v_2 v_1 \\ 0 & 0 & 1 \\ v_1^3 x & -\frac{1}{2}v_1^3 z & -\frac{1}{2}v_2 v_1^3 \\ 0 & 1 & 0 \end{pmatrix}$.

Again the proper transform $v_2^2 y+1$ is nonsingular and has normal crossing with the exceptional divisor $z$.

Here are Maple programs for the operations blowup, cover, and exchange (actually these are not the subprograms of the downloadable package mentioned above, but simplified variants written for demonstration purposes).

```
#Input:  VAR, DEP, IND, PDER: corresponding data of a chart,
#        C:   center--list of indices to IND.
#Output: OUT: a list of new charts generated by the blowing-up
blowup := proc(VAR, DEP, IND, PDER, C)
local i, j, k, VAR1, DEP1, IND1, PDER1, OUT;
OUT := [];
for i from 1 to nops(C) do
  VAR1:= VAR; DEP1:= DEP; IND1:= IND; PDER1:= PDER;
  for j from 1 to nops(C) do
    if j<>i then
      VAR1:= [op(VAR1), v.(nops(VAR1)+1)];
      DEP1:= [op(DEP1), VAR1[nops(VAR1)]*IND[C[i]]-IND[C[j]]];
      IND1[C[j]]:= VAR1[nops(VAR1)];
      for k from 1 to nops(VAR) do
        PDER1[k][C[i]]:= PDER1[k][C[i]]+VAR1[nops(VAR1)]*PDER1[k][C[j]];
        PDER1[k][C[j]]:= PDER1[k][C[j]]*IND[C[i]];
      od;
      PDER1:= [op(PDER1), [seq(0, k=1..nops(IND))]];
      PDER1[nops(PDER1)][C[j]]:= 1;
    fi;
  od;
  OUT:= [op(OUT), [VAR1, DEP1, IND1, PDER1]];
od;
RETURN(OUT);
end:

#Input:  VAR, DEP, IND, PDER: corresponding data of a chart,
#        F:   a list of polynomials
#Output: OUT: a list of new charts
cover := proc(VAR, DEP, IND, PDER, F)
local i, j, VAR1, DEP1, PDER1, OUT;
OUT := [];
for i from 1 to nops(F) do
  VAR1:= [op(VAR), v.(nops(VAR)+1)];
  DEP1:= [op(DEP), VAR1[nops(VAR1)]*F[i]-1];
  PDER1:= [op(PDER), [seq(0, j=1..nops(IND))]];
  for j from 1 to nops(IND) do
    PDER1[nops(VAR1)][j]:= -VAR1[nops(VAR1)]^2*pdiff(F[i], j, VAR, PDER);
  od;
  OUT:= [op(OUT), [VAR1, DEP1, IND, PDER1]];
od;
RETURN(OUT);
end:

#Input:  VAR, DEP, IND, PDER: corresponding data of a chart,
#        F:   a polynomial,   p:   an index to IND
#        a:   the inverse of pdiff(F, p, VAR, PDER)
#Output: the chart with IND[p] and F exchanged
exchange := proc(VAR, DEP, IND, PDER, F, p, a)
local i, j, IND1, PDER1;
IND1:= IND;  IND1[p]:= F;  PDER1:= PDER;
for i from 1 to nops(VAR) do
  PDER1[i][p]:= a*PDER[i][p];
  for j from 1 to nops(IND) do
    if j<>p then
      PDER1[i][j]:= PDER[i][j]-a*PDER[i][p]*pdiff(F, j, VAR, PDER);
    fi;
  od;
```

```
od;
RETURN([VAR, DEP, IND1, PDER1]);
end:

#Input:  VAR, PDER: corresponding data of a chart,
#        F:   a polynomial,   p:   an index to IND
#Output: dF/dIND[p]
pdiff := proc(F, p, VAR, PDER)
local i;
RETURN(add(diff(F,VAR[i])*PDER[i][p], i=1..nops(VAR)));
end:
```

# 3 How to Compute the Blowup Centers

The most natural idea for computing the centers is to use the upper-semicontinuous function that defines the stratification (see [EV]). One would have to express the constraints of lying in the maximum stratum by algebraic equations, and by computing the radical of the ideal generated by these equations, one would obtain the ideal of the center. Of course, in order to apply the efficient blowup procedure described above, one would then need some cover and exchange operations as to make this ideal generated by parameters. All these tasks can be done algorithmically, but the computations are rather costly.

In our implementation, we ignore the upper semicontinuous function of [EV], and we use the inductive construction directly. Adjusting the charts by cover and exchange operations as described below, we achieve that the blowing up center arises as a parameter variety almost automatically.

In the algorithm [EV], the blowing up centers arise in two possible ways: as a hypersurface component of the singular locus or as an intersection of exceptional divisors (in the "monomial case", see [EV]). Normally, a blowing up operation happens after applying some induction steps that reduce the dimension. To get the center of the blowing up operation which effects the given hypersurface, one has to intersect with all the intermediate hypersurfaces in the inductive steps. Therefore, when we want to compute blowing up centers, we have to perform three tasks:

1. find a hypersurface to reduce the dimension (i.e. a hypersurface of "maximal contact", see [EV]);

2. find the hypersurface components of the singular locus;

3. intersect exceptional divisors or/and previously found hypersurfaces of maximal contact.

In task 1, we also want to perform an exchange operation in order to have a local equation of the hypersurface of maximal contact in IND. Doing this, we observe that task 3 requires no real computation at all (i.e. it is purely combinatorial), because the blowing up center is already defined by some subset of IND.

Suppose we have given a chart $W$ and an ideal in $J \subset k[W]$ of order $b > 0$, generated by a finite set $B$ of elements in $k[W]$. Let $B'$ be the set of partial derivatives of elements in $B$ up to order $b-1$. (The set of common zeroes of $B'$ is the singular locus of $(J,b)$.) Then the partial derivatives of $B'$ do not have any common zeroes. Hence we can cover $W$ by the subsets $\frac{\partial F}{\partial P}\neq 0$, $F\in B'$, $P\in \mathrm{IND}$. For any of these subsets, one can show that $(\mathrm{IND}-\{P\})\cup\{F\}$ is a global parameter system: we can exchange $P$ by $F$. Moreover, $F$ defines a hypersurface of maximal contact. Hence task 1 can be solved by cover and exchange operations. The theory also tells us that the cover operation is necessary because there might not exist a global element of order 1 in the ideal generated by $B'$ (see [EV]).

Task 2 is the most difficult one. Let us assume that we have to resolve an ideal $J\subset k[U]$ of order $b$. First, we have to perform cover and exchange operations to achieve that $P\in\mathrm{IND}$ is of maximal contact, i.e. $P$ is in the ideal $J'$ generated by the partial derivatives of $J$ up to order $b-1$. Assume that this is done. Because the singular locus of $(J,b)$ is contained in $V:=\mathrm{ZeroSet}(P)$, any hypersurface component of the singular locus is a component of $V$. Note that $V$ might be reducible, even though it is a "parameter hyperplane": we only know that it is locally irreducible, but it may have several connected components. In this situation, task 2 requires a cover that separates the components of $V$ contained in the singular locus from the remaining components.

In order to do a cover operation, we need a finite set of elements that do not have a common zero. Here is the idea for finding such a set.

Let $I\subset k[V]$ be the coefficient ideal of $(J,b)$ in $V$. (For the precise definition of the coefficient ideal, we refer to [EV]. It should be mentioned that the singular locus of the coefficient ideal coincides with the singular locus of $(J,b)$.) In any component of $V$ in the singular locus, the stalk of $I$ is zero, and the stalk of the annihilator $\mathrm{Ann}(I)$ contains 1. On the other hand, there is a number $b'$, such that the derivatives of $I$ of order up to $b'$ do not have common zeroes in all other components. (The smallest such number is the order of the coefficient ideal.) Hence these derivatives together with a generating set for $\mathrm{Ann}(I)$ do not have common zeroes.

It should be noted that the algorithm requires only a very small toolbox of algebro-geometric algorithms: deciding if a finite set of elements has a common zero, computing a generating set for the annihilator, and computing the quotient of two elements. This toolbox is provided by standard techniques using Gröbner bases (see [Buc85, BW93]).

# References


[BM91] E. Bierstone and P. Milman. A simple constructive proof of canonical resolution of singularities. In T. Mora and C. Traverso, editors, *Effective methods in algebraic geometry*, pages 11–30. Birkhäuser, 1991.

[BM97] E. Bierstone and P. Milman. Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant. *Invent. math.*, 128:207–302, 1997.

[Buc85] B. Buchberger. Gröbner Bases: An Algorithmic Method in Polynomial Ideal Theory. In N. K. Bose, editor, *Recent Trends in Multidimensional Systems Theory*, chapter 6. D. Riedel Publ. Comp., 1985.

[BW93] T. Becker and V. Weispfenning. *Gröbner bases - a computational approach to commutative algebra.* Graduate Texts in Mathematics. Springer Verlag, 1993.

[EV] S. Encinas and O. Villamayor. A course on constructive desingularization and equivariance. This volume.

[Har77] R. Hartshorne. *Algebraic Geometry.* Springer–Verlag, 1977.

[Vil89] O. Villamayor. Constructiveness of Hironaka's resolution. *Ann. Scient. Ecole Norm. Sup. 4*, 22:1–32, 1989.

[Vil91] O. Villamayor. Introduction to the algorithm of resolution. In T. Mora and C. Traverso, editors, *Algebraic geometry and singularities*, pages 123–154. Birkhäuser, 1991.