Technical Report

Automated Resolution of Singularities for Hypersurfaces

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We give a description of an algorithm for the resolution of singularities, in the case of hypersurfaces in characteristic zero. It is based on a constructive existence proof due to Villamayor. The algorithm is implemented in Maple.

1. Introduction

The problem of resolution of singularities is, given an algebraic variety X, construct a proper birational morphism $Y \rightarrow X$ such that Y is nonsingular. This problem has an impressive history. For curves, the existence of a resolution was already known in the 19th century. In the surface case, Walker gave an analytic proof (Walker, 1935), and Zariski gave an algebraic proof (Zariski, 1939). Five years later, Zariski proved resolution of 3-folds (Zariski, 1944). Then it took quite some time until Hironaka came up with his famous result of resolution for arbitrary dimension (Hironaka, 1964). Hironaka's result holds only for the characteristic zero case. In positive characteristic, there are some partial results (Abhyankar, 1965; Lipman, 1978), but the general case is still open. For an account of the difficulties in positive characteristic, the reader should consult (Hauser, 1998).

Hironaka's existence proof is not constructive. Later, (Villamayor, 1989; Villamayor, 1991; Encinas and Villamayor, 1999) and (Bierstone and Milman, 1991; Bierstone and Milman, 1997) have come up with more explicit strategies to find a resolution. Still, the main interest of these authors was to prove the existence of resolutions with particular properties (equivariance) and not actual computation.

In this paper, we describe an algorithm that was extracted from Villamayor's constructive proof. The algorithm is not new, since the main ideas are due to Villamayor; but this is the first time that a resolution algorithm has been formulated in such an explicit way. This is underlined by the fact that we could do a full implementation in Maple.

Even when taking the results of Villamayor and Bierstone-Milman into account, it is quite surprising that automated resolution is possible with nowaday's means. We are not aware of any other implementations of resolution, even in the relatively easy case of surfaces, except a program for cyclic surfaces (Castellanos *et al.*, 1998) (see also (Eisenbud, 1993), where resolution of surfaces is considered as an open problem in constructive algebraic geometry). Only for curves, there are some implementations, e.g. in (Tran and Winkler, 1997) or in MapleV release 5 (in the package "algcurves" by M. van Hoeij).

One of the main difficulties in automated resolution is the following. The resolution is obtained by *blowing up* subvarieties in the singular locus. Now, it is quite easy to blowup a point. But for more complicated blowing up centers, the construction is not so trivial. There are implementations for performing blowing ups in various computer algebra systems like (Greuel *et al.*, 1998) and (Bayer *et al.*, 1993) (both are using Gröbner bases, by the way). But these computations are time consuming and the output is a very cumbersome thing. In the resolution algorithm, we need to apply blowing up repeatedly, which would be rather difficult when using these procedures.

We solved this problem by working simultanuously with affine coordinates on the one side and regular parameters on the other side. Doing this, we can compute blowing ups without any computations more difficult than simple polynomial arithmetic and differentiation (no Gröbner bases). The blown up is represented via open coverings, using affine charts. The output of the whole resolution algorithm is a tree of charts, where

- the root node is $\mathbb{A}^n$, the ambient space of the given hypersurface,
- the children of each node are obtained from the parent node by blowing up or by some other operation described in section 4 (cover and exchange),
- the proper transform of the given hypersurface in the leaf nodes is nonsingular, and the total transform is a normal crossing divisor.

The output tree may be viewed by "clicking through the resolution" in an html hierarchy.

Gröbner bases (Buchberger, 1965; Buchberger, 1985; Winkler, 1988; Becker and Weispfenning, 1993) are, however, used in other places of the algorithm: testing whether a given ideal contains 1, solving linear equations over polynomials, computing quotient ideals.

The paper is structured as follows. In section 2, we give a precise description of the problem and introduce the necessary terminology. In section 3, we give an abstract description of the algorithm, using some strong macros like *BlowUp*. In section 4, we describe the data structures and basic operations concretely. In section 5, we discuss the main computational subtasks. Finally, we include an extended example in section 6.

The full program is available on the web site `http://www.risc.uni-linz.ac.at/-projects/basic/adjoints/blowup`. It has also been announced in the conference IMACS-ACA'98. The paper (Bodnár and Schicho, 1999) contains a discussion of some ideas in the algorithm.

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2. The Problem

In this section we formulate the problem of desingularization, we present the case we are dealing with and give the necessary terminology.

Henceforth k denotes a computable field of characteristic zero. We consider varieties over k. A *morphism* means a regular map. A morphism is called *proper* iff it is not a restriction of a map from some larger domain to an open dense subset. A morphism is *birational* if it has a rational inverse, i.e. an inverse map on some open dense subset of the image of the morphism, that can be given by rational functions.

Problem 1: Given a variety X, find a *desingularization* of it, which means a nonsingular variety Y together with a proper, birational morphism $\pi : Y \to X$.

At a nonsingular point p of a variety W of dimension n, $f_1, \ldots, f_n \in k[W]$ form a *regular system of parameters* iff they all vanish at p, and $df_1, \ldots, df_n$ generate the module of differential forms $\Omega(W)$ for some $U \subseteq W$ containing p.

An algebraic set, contained in the nonsingular variety W, has at most *normal crossings*, iff for all points $x \in W$, there exists a system of regular parameters such that each irreducible component of the algebraic set is given by a subset of this parameter set locally at x.

Let W be a nonsingular variety, and $X \subset W$ be a variety of codimension 1. By abusing the language we call the pair (W, X) a *hypersurface*. If $W = \mathbb{A}^n$, (W, X) is an affine hypersurface, if $W = \mathbb{P}^n$, (W, X) is a projective hypersurface.

Problem 2: Given a hypersurface (W, X), find an *embedded desingularization* of it, which is a nonsingular variety W' together with a proper, birational morphism $\pi : W' \to W$, such that the *total transform*, $\pi^{-1}(X)$, has at most normal crossings.

An embedded desingularization of (W, X) provides indeed a desingularization of X. The rational inverse is defined on an open dense subset of X, and the closure of its image, say, X'—also called as the *proper transform* of X along π—is nonsingular in W' (it is one of the components of the preimage). Now X' together with the map $\pi|_{X'} : X' \to X$ is a desingularization of X.

Such a desingularization is usually constructed using the technique of blowing up, which we now describe.

Let X be an affine variety over k with $a_1, \ldots, a_r$ generators for its coordinate ring. Let $C \subset X$ be a variety with vanishing ideal $\mathbf{I}(C) = \langle g_0, \ldots, g_s \rangle$. The *blowing up* of X at C is defined as a proper, birational morphism $\pi : Y \to X$, where $Y \subset \mathbb{A}^r \times \mathbb{P}^s$ with ideal

$$\mathbf{I}(Y) = \{P \in k[x_1, \ldots, x_r, y_0, \ldots, y_s] \mid P(a_1, \ldots, a_r, tg_0, \ldots, tg_s) = 0 \text{ in } k[X][t]\}.$$

The morphism π is just the projection on the first r coordinates. The ideal $\mathbf{I}(Y)$ is homogeneous in the y's. The preimage of C in Y, denoted by E, is called the *exceptional divisor* of the blown up variety.

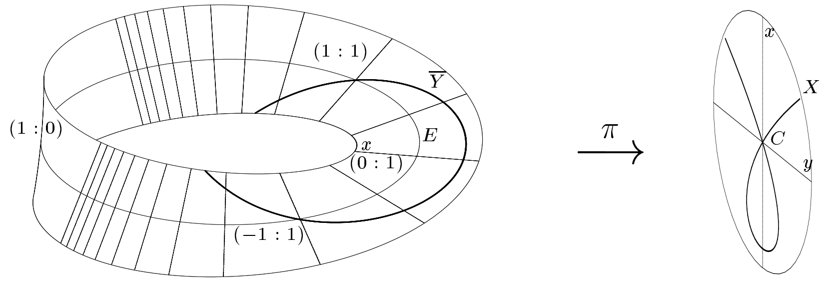

Figure 1. Blowing up $(\mathbb{A}^2, \mathbf{V}(y^2 - x^3 - x^2))$ at the origin.

Example 2.1. Let $X = \mathbb{A}^n$, $C = \mathbf{V}(x_1, \ldots, x_r)$, $r \leq n$. The blowing up of X along C is $\pi : Y \to X$, where $Y \subset \mathbb{A}^n \times \mathbb{P}^{r-1}$,

$$\mathbf{I}(Y) = \langle x_i y_j - x_j y_i \mid i, j = 1, \ldots, r \rangle_{k[x_1, \ldots, x_n, y_1, \ldots, y_r]},$$

π is the projection on the x_i's. The blown up variety can be covered by r affine charts of dimension n, covering $\mathbb{P}^{r-1}$ by its principal open subsets. So the first chart ($y_1 \neq 0$) is generated by $x_1, y_2, \ldots, y_r, x_{r+1}, \ldots, x_n$, where the x_j's were eliminated by the equations $x_1 y_j = x_j$, $j = 2, \ldots, r$.

For a hypersurface S containing C in its singular locus with multiplicity d, the total transform in the first chart is of the form $x_1^d \cdot g(x_1, y_2, \ldots, y_r, x_{r+1}, \ldots, x_n)$, where $x_1 = 0$ is the equation of E, the exceptional divisor, and $g = 0$ is of the proper transform.

3. An Abstract Description of the Algorithm

In this section, we describe the resolution algorithm extracted from Villamayor's constructive proof.

The desingularization of the given hypersurface is obtained by repeatedly blowing up the ambient space, as described in section 2. Villamayor's algorithm is a method to find the next center for blowing up.

The first restriction for the center is that it has to be a nonsingular variety. This ensures that the blown up varieties are nonsingular at any stage.

The second restriction is that the center has normal crossing with respect to the exceptional divisors constructed so far. This restriction ensures that the exceptional divisors have normal crossing at any stage.

The method of Villamayor is based on a intermixed induction on the multiplicity and the dimension, also involving the exceptional divisors that have arised from previous blowing ups. He introduces *basic objects* of the kind (W, J, b, E), where

- W is a regular variety,
- J is a nonzero ideal in $k[W]$ (initially the ideal of the hypersurface),
- b is a positive integer,
- E is a finite sequence of regular hypersurfaces of W with normal crossing (the exceptional divisors up to the current stage).

We sometimes omit components of basic objects when they are not needed.

Remark 3.1. Since decomposing into irreducible varieties is computationally hard, we will later drop the condition that W is irreducible (i.e. we work with algebraic sets instead of varieties).

In the following, we use "partial derivatives". If $W = \mathbb{A}^n$, these have the usual meaning. Otherwise, we can define the partial derivatives only locally with respect to a system of parameters. For charts, a precise definition will be given in section 4.

The *singular locus* of a basic object (W, J, b, E) is defined by

$$\mathrm{Sing}(J, b) := \{x \mid \mathrm{ord}_x(J) \geq b\},$$

where

$$\mathrm{ord}_x(J) := \min\{\mathrm{ord}_x(f) \mid f \in J\},$$

and $\mathrm{ord}_x(f)$ is the smallest integer m such that there exists a partial derivative of f of order m not vanishing at x. The order of J at x is positive iff x is in the zero set of J.

A basic object is resolved iff its singular locus is empty. When trying to resolve a basic object (W, J, b, E), we do not want to change anything outside its singular locus. This motivates the third restriction: the blowing up center must be contained in the singular locus. It ensures that the total transform of J (i.e. the ideal that is generated by the pullbacks of elements in J) can be written as $\mathbf{I}(F)^b \cdot J'$, where F is the exceptional divisor.

Let W' be the blown up. Let E' be the sequence of all proper transforms of divisors in E, together with the exceptional divisor E. Let J' be as above. Then the basic object (W', J', b, E') is called the *transform* of (W, J, b, E). In the algorithm, this is going to be the next basic object to be resolved.

We say that a blowing up operation is *permissible* with respect to the basic object (W, J, b, E) iff the blowing up center fulfills the three restrictions above: nonsingular, normal crossing with E, contained in the singular locus.

Here is a technical lemma which is used several times in the correctness proof.

LEMMA 3.1. *Let (W, J, b, E) and (W, I, b, E) be basic object. Let Y be a permissible blowing up center, and let (W', J', b, E') and (W', I', b, E') be the transforms. Then the following hold.*

1. *$Sing(J, b) = Sing(J^n, nb)$ for $n \geq 1$.*
2. *$Sing(J + I, b) = Sing(J, b) \cap Sing(I, b)$.*
3. *The transform of (W, J^n, nb, E) is equal to $(W', (J')^n, nb, E')$ for $n \geq 1$.*
4. *The transform of $(W, J + I, b, E)$ is equal to $(W', J' + I', b, E')$.*

PROOF. (1): follows immediately from the relation $\mathrm{ord}_x(f^n) = n \cdot \mathrm{ord}_x(f)$.

(2): follows from the fact that in the definition of $\mathrm{ord}_x(J)$, it suffices to take the minimum over a fixed set of generators.

(3): let $\pi : W' \to W$ be the blowing down map, and let π^* be the pullback homomorphism. Then both sides are generated by

$$\left\{ \frac{(\pi^* f)^n}{h^{nb}} \;\middle|\; f \in J \right\} = \left\{ \left(\frac{\pi^* f}{h^b} \right)^n \;\middle|\; f \in J \right\},$$

where h is a local equation of the exceptional divisor.

(4): both sides are generated by

$$\left\{ \frac{\pi^* f}{h^b} \;\middle|\; f \in J \text{ or } f \in I \right\}.$$

□

Basic objects come in three variants: weighted objects, tagged objects, and balanced objects. We have resolution algorithms for these three types, which call each other recursively.

3.1. WEIGHTED OBJECTS

A *weighted object* is a basic object (W, J, b, E), together with a function $a : E \to \mathbb{N}$, such that the following holds. Let $E = (H_1, \ldots, H_r)$, and let $(a_1, \ldots, a_r)$ be the sequence of values of a. Define

$$M = \mathbf{I}(H_1)^{a_1} \cdot \ldots \mathbf{I}(H_r)^{a_r}$$

(locally, this ideal is generated by a monomial in a system of parameters). Then the ideal J can be written as $J = M \cdot \overline{J}$. We call M the monomial part and $\overline{J}$ the proper part of J. The largest order of $\overline{J}$ is called the *multiplicity.*

The transform of the new component a is done as follows. For any hypersurface $H \in E$, the value of the proper transform of a is defined to be $a(H)$, i.e. old hypersurfaces remain with old values. The value of a at the new exceptional divisor F is defined to be $w - b$, where w is the minimal order of J at points in the blowing up center. It is easy to check that an equation $M' \cdot \overline{J}' = J'$ holds for the transform, for some $\overline{J}'$.

A *simple object* is a basic object (W, J, b, E) satisfying $\mathrm{ord}_x(J) = b$ for all $x \in \mathrm{Sing}(J, b)$. One can easily show that the transform of a simple object is again simple (a stronger statement will also be shown in lemma 3.3).

A *tagged object* is a simple object (W, J, b, E) together with a subset E_- of E, satisfying an additional "condition T", which will be explained in subsection 3.3. For the moment, it suffices to know two things about condition T:

1 If (W, J, b, E) is simple, then $(W, J, b, E, E_- = E)$ is tagged.
2 The transform of a tagged object is tagged.

Here, the transform of the component E_- is done in the obvious way: E'_- is defined as the set of proper transforms of the hypersurfaces in E_-.

For any weighted object (W, J, b, E, a) with nonzero proper part, the *equimultiple* is defined as the tagged object constructed by the following algorithm.

Algorithm *Equimultiple*(W, J, b, E, a);
  $M, \overline{J} :=$ the monomial part and the proper part;
  $\overline{b} :=$ the largest order of $\overline{J}$ at points in the singular locus;
  if $\overline{b} \geq b$ then
    $I := \overline{J}$;
    $w := \overline{b}$;
  else
    $I := \overline{J}^{b - \overline{b}} + M^{\overline{b}}$;
    $w := (b - \overline{b})\overline{b}$;
  $E_- := E$;
  return (W, I, w, E, E_-);

The equimultiple enjoys the following properties.

LEMMA 3.2. *Let $\mathcal{W}$ be a weighted object with positive multiplicity, and let $\mathcal{T}$ be its equimultiple. Then the following hold.*

1 The singular locus of $\mathcal{T}$ is equal to the subset of the singular locus of $\mathcal{W}$ where the proper part has maximal order.
2 Let Y be permissible with respect to $\mathcal{T}$. Then Y is also permissible with respect to $\mathcal{W}$.

3 Let $\mathcal{W}'$ and $\mathcal{T}'$ be the transform of $\mathcal{W}$ and $\mathcal{T}$, respectively, along the blowing up of Y. If $\mathcal{T}'$ is not resolved, then multiplicity($\mathcal{W}'$) = multiplicity($\mathcal{W}$), and $\mathcal{T}'$ is the equimultiple of $\mathcal{W}'$.

4 If $\mathcal{T}'$ is resolved, then either multiplicity($\mathcal{W}'$) < multiplicity($\mathcal{W}$), or $\mathcal{W}'$ is resolved.

Proof. We use the same notation as in the description of the algorithm *Equimultiple.*

(1): We distinguish two cases.

If $\overline{b}{\geq}b$, then every point in the singular locus of $(\overline{J},\overline{b})$ is also a singular point of $\mathcal{W}$, and it consists precisely of those points where the order is maximal (namely $\overline{b}$).

Now, suppose $\overline{b} < b$. By lemma 3.1, the singular locus of $\mathcal{T}$ is the intersection of the singular locus of $(\overline{J},\overline{b})$ and the singular locus of $(M, b-\overline{b})$. But

$$\mathrm{ord}_x(\overline{J}){\geq}\overline{b} \wedge \mathrm{ord}_x(M){\geq}b-\overline{b}$$

is equivalent to

$$\mathrm{ord}_x(\overline{J}){\geq}\overline{b} \wedge \mathrm{ord}_x(J){\geq}b,$$

hence the singular locus of $\mathcal{T}$ consists of all the points in Sing($\mathcal{W}$) where $\overline{J}$ has maximal order.

(2): follows immediately from (1).

(3) and (4): we distinguish two cases again.

If $\overline{b}{\geq}b$, then $(\overline{J},\overline{b})$ is simple. The transform is $(\overline{J}',\overline{b})$, where $\overline{J}'$ is the proper part of $\mathcal{W}'$. This is again a simple object. If it is not resolved, then the multiplicity of $\mathcal{W}'$ is $\overline{b}$, and $\mathcal{T}'$ is the equimultiple of $\mathcal{W}'$. If it is resolved, then the multiplicity is smaller than $\overline{b}$. Thus the multiplicity has dropped—unless $\mathcal{W}'$ is resolved, in which case the multiplicity is not defined.

Now, suppose $\overline{b} < b$. By lemma 3.1, $\mathcal{T}'$ is equal to $((\overline{J}')^{b-\overline{b}} + (M')^{\overline{b}}, (b-\overline{b})\overline{b})$, where M' is the monomial part of $\mathcal{W}'$. The object $(\overline{J},\overline{b})$ is not necessarily simple, it could have points of higher order outside the singular locus of $\mathcal{W}$. But it behaves like a simple object because the blowing up center consists of points of order $\overline{b}$. If $\mathcal{T}'$ is not resolved, then the maximal order of $(\overline{J},\overline{b})$ within the singular locus of $(M', b-\overline{b})$ is $\overline{b}$, and $\mathcal{T}'$ is the equimultiple of $\mathcal{W}'$. Otherwise, the multiplicity has dropped —unless $\mathcal{W}'$ is resolved. □

Weighted objects with multiplicity zero are also called *monomial objects.* One can easily show that the transform of a monomial object is again monomial. Monomial objects can be resolved by the algorithm below. We say that the hypersurfaces $H_1,\ldots,H_r$ have *common singularities* iff the intersection of $H_1,\ldots,H_r$ and the singular locus is non-empty.

Algorithm *ResolveMonomial*($\mathcal{T}$);
  while not resolved do
    r := the smallest number such that r hypersurfaces in E have common singularities;
    $\overline{b}$:= the largest possible value of $a(H_1)+\ldots+a(H_r)$, where $H_1,\ldots,H_r{\in}E$ have common singularities;
    $H_1,\ldots,H_r$:= the first r hypersurfaces for which the maximum $\overline{b}$ is achieved (with respect to lexicographical ordering);
    Y := the intersection of $H_1,\ldots,H_r$;
    BlowUp(Y);

THEOREM 3.1. *The algorithm* ResolveMonomial *is correct.*

PROOF. (This proof was communicated to us by S. Encinas.) It is obvious that the center is permissible and that the algorithm will not stop before the basic object is resolved. The difficult part is termination.

For any monomial object, we define the number m as the set of all r-subsets $\{H_1, \ldots, H_r\}$ of E such that $a(H_1) + \ldots + a(H_r) = \overline{b}$, where r and $\overline{b}$ are defined as in the description of the algorithm *ResolveMonomial*. We claim that the triple $(\dim W - r, \overline{b}, m)$ drops in every blowing up step in the algorithm. Here, triples are compared by lexicographical order. Obviously, this claim then implies termination.

To prove the claim, we assume indirectly that the claim does not hold. Then we have either $r' < r$, or ($r' = r$ and $\overline{b}' > \overline{b}$), or ($r' = r$ and $\overline{b}' = \overline{b}$ and $m' \geq m$), where the postfix $'$ denotes the corresponding value of the transform.

Assume that $r' < r$. By definition of r', there are elements $G'_1, \ldots, G'_{r'} \in E'$ such that $a(G'_1) + \ldots + a(G'_{r'}) \geq b$. The new divisor F must be among the G_i, say $F = G'_1$. The other divisors are proper transforms of old divisors $G_2, \ldots, G_{r'}$. Among $H_1, \ldots, H_r$, there must be a divisor, say H_1, which is not in $\{G_2, \ldots, G_{r'}\}$. Then

$$b \leq a(G'_1) + \ldots + a(G'_{r'}) = a(H_1) + \ldots + a(H_r) - b + a(G_2) + \ldots + a(G_{r'}) =$$

$$= (a(H_1) + a(G_2) + \ldots + a(G_{r'})) + (a(H_2) + \ldots + a(H_r)) - b < b + b - b,$$

a contradiction.

Now, assume that $r' = r$ and $\overline{b}' > \overline{b}$. Then it follows that $a(G'_1) + \ldots + a(G'_r) > \overline{b}$ for some $G'_1, \ldots, G'_{r'} \in E'$. Then the new divisor F must be among the G_i, say $F = G_1$. Among $H_1, \ldots, H_r$, there is again a divisor, say H_1, which is not in $\{G_2, \ldots, G_r\}$. Then

$$\overline{b} < a(G'_1) + \ldots + a(G'_r) = a(H_1) + \ldots + a(H_r) - b + a(G_2) + \ldots + a(G_r) =$$

$$= (a(H_1) + a(G_2) + \ldots + a(G_r)) + (a(H_2) + \ldots + a(H_r)) - b < \overline{b} + b - b,$$

again a contradiction.

Finally, assume that $r' = r$ and $\overline{b}' = \overline{b}$ and $m' \geq m$. By the inequality above, any r-subset containing F does not achieve the maximum value $\overline{b}$. All r-subsets achieving the maximum are old, i.e. consist of proper transforms. But the proper transforms of $H_1, \ldots, H_r$ do not have common singularities, because their intersection was blown up. Hence $m' < m$, which is again a contradiction. □

REMARK 3.2. The statement *BlowUp*(Y) implies that we pass from the given basic object to its transform. Moreover, we will have in general a whole stack of "current objects" of all three types, due to the recursive calls. In this case, Y will be permissible with respect to all of them, and the statement *BlowUp*(Y) means that we pass to the proper transform for all basic objects in the stack.

Here is the algorithm for resolution of weighted objects (it calls the resolution algorithm for tagged objects).

Algorithm *ResolveWeighted*($\mathcal{W}$);
while not resolved and multiplicity>0 do
$\mathcal{T}$:= *Equimultiple*(current object);
ResolveTagged($\mathcal{T}$);
if not resolved then
ResolveMonomial(current object);

THEOREM 3.2. *The algorithm* ResolveWeighted *is correct (i.e. relative correct with respect to the subalgorithm* ResolveTagged*).*

PROOF. By repeated application of lemma 3.2, we see that resolving the equimultiple either resolves the weighted object or it makes the multiplicity drop. Repeating, we obtain a resolution or multiplicity zero. In the latter case, *ResolveMonomial* resolves. □

3.2. BALANCED OBJECTS

Let (W, J, b, E) be a simple object. The *delta-ideal* $\Delta(J, b)$ is defined as the ideal generated by all partial derivatives of elements in J of order less than b. Its zero set contains the singular locus. For every point $x \in \mathrm{Sing}(J, b)$, there exists a $z \in \Delta(J, b)$ which has order 1 at x (because for each point x in the singular locus, there is a partial derivative of order b not vanishing at x).

The simple object (W, J, b, E) is called *balanced* iff the following condition holds. For every point $x \in \mathrm{Sing}(J, b)$, there exists a $z \in \Delta(J, b)$ which has order 1 at x and $\mathbb{V}(z)$ has normal crossing with E in a neighborhood of x.

Let (W, J, b, E) be a balanced simple object, and let $z \in \Delta(J, b)$ be an element which has order 1 and normal crossing with E at any point of its zero set. Then we say that this zero set (called V below) is a *hypersurface of maximal contact.*

Suppose that $W = \mathbb{A}^n$, and z is one of the coordinate functions. Then we define the *coefficient ideal* $C(J, b, V)$ as follows. Let

$$f_{0,1} + \ldots + f_{d_1,1} z^{d_1}, \ldots, f_{0,r} + \ldots + f_{d_r,r} z^{d_r}$$

be generators of J, where $f_{i,j}$ are polynomials in the remaining $n-1$ variables. The ideal C is generated by

$$f_{0,1}^{\frac{b!}{b}}, \ldots, f_{b-1,1}^{\frac{b!}{1}}, \ldots, f_{0,r}^{\frac{b!}{b}}, \ldots, f_{b-1,r}^{\frac{b!}{1}}.$$

In the general case, one can construct the coefficient ideal by partial differentiation.

The coefficient ideal is zero iff V is equal to the singular locus. In this case, V is permissible for blowing up. Since V is a hypersurface, the blowing up map is an isomorphism on W. It is easy to check that the transform of (W, J, b, E) is then resolved.

REMARK 3.3. In general, V may be reducible, and there might be components of V where the coefficient ideal is zero, and other components where it is not. In this section, we assume for simplicity that V is irreducible. For the general case, refer to section 5.

If the coefficient ideal is not zero, then the *coefficient object* is defined as

$$(V, C(J, b, V), b!, E \cap V, 0),$$

where 0 is the constant zero function. It enjoys the following properties.

LEMMA 3.3. *Let $\mathcal{B}$ be a balanced object. Let V be a hypersurface of maximal contact. Suppose that the coefficient ideal of $\mathcal{B}$ in V is not zero. Let $\mathcal{W}$ be the coefficient object. Then the following hold.*

1 *The singular loci of $\mathcal{B}$ and $\mathcal{W}$ coincide.*
2 *Let $Y \subset V$ be permissible with respect to $\mathcal{W}$. Then Y is also permissible with respect to $\mathcal{B}$.*
3 *Let $\mathcal{B}'$ be the transform of $\mathcal{B}$ along the blowing up of Y, and let V' be the proper transform of V. Then V' has maximal contact to $\mathcal{B}'$. Especially, $\mathcal{B}'$ is again balanced.*
4 *The transform of $\mathcal{W}$ along the blowing up of Y is equal to the coefficient object of $\mathcal{B}'$ with respect to V'.*

PROOF. We assume that $W = \mathbb{A}^n$, and z is one of the coordinate functions. We use the same notation as in the definition of the coefficient object.

(1): since V is a hypersurface of maximal contact, the singular locus of $\mathcal{B}$ is contained in V. Moreover, it is the zero set of all partial derivatives of $f_{m,n}$ of order less than $b-m$ with respect to the variables different from z. But this is precisely the singular locus of $\mathcal{W}$.

(2): follows immediately from (1).

(3): for any ideal I, let ∂I be the ideal generated by I and by all partial derivatives of elements of I (it suffices to take the partial derivatives of a generating set).

Let Y be permissible with respect to (I, m), $m \geq 2$, and let (I', m) be the transform. Obviously, Y is also permissible with respect to $(\partial I, m-1)$. Let $((\partial I)', m-1)$ be the transform. A straightforward calculation using the coordinates introduced in 2.1 shows that $(\partial I)' \subset \partial(I')$. Applying this relation $b-1$ times to (J, b), we get that the transform of $\Delta(J, b)$—transformed with integer component 1—is contained in $\Delta(J', b)$. Especially, the local equation of V' is in $\Delta(J', b)$. Clearly V' has normal crossing with E', hence V' has maximal contact.

(4): assume that Y is the zero set of $(z, x_1, \ldots, x_{m-1})$. The chart corresponding to z is disjoint from V'. In the chart corresponding to x_1, $\mathcal{B}'$ is generated by the polynomials

$$f_{0,i}/x_1^b + f_{1,i}/x_1^{b-1}\tilde{z} + \ldots + f_{d_i,i}/x_1^{b-d_i}\tilde{z}^{d_i}, \; i = 1, \ldots, r.$$

The coefficient ideal is generated by

$$(f_{j,i}/x_1^b)^{b!/b-j}, \; i = 1, \ldots, r, \; j = 0, \ldots, b-1.$$

By lemma 3.1, this is precisely the transform of the coefficient ideal of $\mathcal{B}$. □

Here is the algorithm for resolution of balanced objects. It calls a resolution algorithm for weighted objects of smaller dimension.

Algorithm *ResolveBalanced*($\mathcal{B}$);
  $V :=$ a hypersurface of maximal contact;
  $C :=$ the coefficient ideal of $\mathcal{B}$ with respect to V;
  if C is the zero ideal then
    BlowUp(V);
  else

$\mathcal{W}$:= the coefficient of object of $\mathcal{B}$ with respect to V;
ResolveWeighted($\mathcal{W}$);

THEOREM 3.3. *The algorithm* ResolveBalanced *is correct.*

PROOF. If the coefficient ideal is zero, we already mentioned that V is permissible and the transform is resolved.

Otherwise, repeated application of lemma 3.3 tells that by resolving $\mathcal{W}$, we also resolve $\mathcal{B}$. □

An obstacle is that the ideal $\Delta(J,b)$ may have no global element of order 1, even though it has elements of order 1 at any point by definition of being "simple". But then we may replace W by open subsets which cover, and in each of the subsets we have a global element of order 1.

3.3. TAGGED OBJECTS

A tagged object is a simple object (W,J,b,E) together with a subset E_- of E, such that the simple object $(W,J,b,E-E_-)$—which we call the *shadow*— is a balanced object. (This condition was referred to as "property T" in subsection 3.1.)

Let $\mathcal{T}$ be a tagged basic object. Then the transform of $\mathcal{T}$ along some permissible blowing up is again tagged, because its shadow is the transform of the shadow of $\mathcal{T}$, which is balanced by lemma 3.3. This shows a fact that we have used in section 3.1: the transform of a tagged object is again tagged. The second statement that we have used in section 3.1 is that $E=E_-$ and being simple implies being tagged; but this is obvious.

Let r be the largest number such that r hypersurfaces in E_- have common singularities. If $r>0$, let m be the number of such r-subsets of E_-. The pair (r,m) (or $(0,$ "undefined"$)$) is called the *balance* of (W,J,b,E,E_-). It is a measure of the severeness of the tagged object. Balances are compared by the lexicographic order.

The *support* is defined as the basic object constructed as in the following algorithm.

Algorithm *Support*(W,J,b,E,E_-);
 r := the largest number such that r hypersurfaces in E_- have common singularities;
 $H_1,\ldots,H_r$:= the first r hypersurfaces in E_- having common singularities
  (with respect to lexicographical ordering);
 $I := J+\mathbf{I}(H_1)^b+\ldots+\mathbf{I}(H_r)^b$;
 $E_+ := E-E_-$;
 return (W,I,b,E_+);

By enlarging the ideal, one does not destroy the property of being simple. Hence the support is a simple object. If $r>0$, then the hypersurface H_1 has maximal contact and normal crossing with E_+, hence the support is balanced. If $r=0$, then the support is balanced by property T.

We have the following lemma.

LEMMA 3.4. *Let $\mathcal{T}$ be a tagged object. Let $\mathcal{B}$ be its support.*

1 The singular locus of $\mathcal{B}$ is equal to the set of common singularities of a maximal number r of hypersurfaces in E_-.
2 Let Y be permissible with respect to $\mathcal{B}$. Then Y is also permissible with respect to $\mathcal{T}$.
3 Let $\mathcal{T}'$ and $\mathcal{B}'$ be the transforms of $\mathcal{T}$ and $\mathcal{B}$, respectively. If $\mathcal{B}'$ is not resolved, then $\mathcal{B}'$ is the support of $\mathcal{T}'$, and balance$(\mathcal{T}')$ = balance$(\mathcal{T})$.
4 If $\mathcal{B}'$ is resolved, then balance$(\mathcal{T}')$ < balance$(\mathcal{T})$ or $\mathcal{T}'$ is resolved.

Proof. We use the same notation as in the description of the algorithm *Support*.

(1): follows immediately from lemma 3.1.

(2): by (1), Y is contained in the singular locus of $\mathcal{T}$. Let X be the intersection of $H_1,\ldots,H_r$. Since Y is a nonsingular subvariety of X that has normal crossing with $E_-\cap X$, we can extend the system of parameters defined by $E_-\cap X$ by $p_1,\ldots,p_m$, such that Y is defined by a subset of the extended system. Let $P_1,\ldots,P_m$ be elements in $k[W]$ that restrict to $p_1,\ldots,p_m$ on X. Then for any $x\in Y$, there is a neighborhood such that the elements $P_1,\ldots,P_m$ together with E_- and $H_1,\ldots,H_r$ form a system of parameters. Thus Y has normal crossing with E.

(3) and (4): Let $H_1',\ldots,H_r'$ be the proper transforms of $H_1,\ldots,H_r$. By lemma 3.1, $\mathcal{B}'$ is $(W',J'+\mathbf{I}(H_1')^b+\ldots+\mathbf{I}(H_r')^b,b,E'-E_-')$.

If $\mathcal{B}'$ is not resolved, then we still have the r hypersurfaces $H_1',\ldots,H_r'\in E_-'$ having common singularities. Note that Y is disjoint from all hypersurfaces in $E_- - \{H_1,\ldots,H_r\}$, so that the intersection relations between the other hypersurfaces in E_- is not changed. Hence balance$(\mathcal{T})$ = balance$(\mathcal{T}')$, and $\mathcal{B}'$ is the support of $\mathcal{T}'$.

If $\mathcal{B}'$ is resolved, then the number of r-subsets of E_- with common singularities has dropped, which implies that the balance has dropped, or the singular locus of $\mathcal{T}$ is empty. □

Here is the algorithm for the resolution of tagged objects. It calls the algorithm for resolution of balanced objects.

Algorithm *ResolveTagged*$(\mathcal{T})$;
    while not resolved do
        $\mathcal{B}$:= *Support*(current object);
        ResolveBalanced$(\mathcal{B})$;

Theorem 3.4. *The algorithm* ResolveTagged *is correct.*

Proof. By repeated application of lemma 3.4, we see that resolving the equimultiple either resolves the weighted object or it makes the multiplicity drop. Since the balance can only drop finitely many times, we obtain a resolution. □

3.4. Desingularization

The problem of embedded desingularization of hypersurfaces can be reduced to the resolution of basic objects in the following way. Given a nonsingular variety U and a hypersurface X, we just have to resolve the weighted basic object $(U,\mathbf{I}(X),1,\emptyset,$empty function$)$.

Then the final transform has ideal (1) and therefore the total transform is a product of ideals of exceptional divisors (i.e. elements in E), which always have normal crossing.

A desingularization (not necessarily embedded) is already obtained as soon as the maximal order of the proper part has become less than 2. This follows from the fact that the proper part is nothing but the equation of the proper transform.

4. Charts and Blowing Ups

The varieties generated during the resolution are represented via open coverings, i.e. sets of affine *charts*. In this section we present the basic data structures of charts and basic objects, and describe the transformations of charts under the fundamental operations of the resolution process.

An affine chart U of dimension n is described by the following data:

- A list of affine variables VAR $= (x_1, \dots, x_m)$, for $m \geq n$, generating the polynomial ring $k[x_1, \dots, x_m]$.
- A list DEP $= (D_1, \dots, D_s) \subset k[\mathrm{VAR}]$. As a subset of the ambient affine space $\mathbb{A}^m$ the chart is $\mathbf{V}(\mathrm{DEP})$.
- A list IND $= (P_1, \dots, P_n) \subset k[U]$.
- An $m \times n$ matrix PDER $= (p_{ij})$, with elements from $k[U]$.

We postulate that:

1 U is a nonsingular algebraic set in $\mathbb{A}^m$ of dimension n.
2 The elements of IND give rise to a regular system of parameters in every point of U.
3 The matrix PDER is

$$p_{ij} = \partial_j x_i = \frac{\partial x_i}{\partial P_j} \qquad i = 1, \dots, m, \ j = 1, \dots, n.$$

We represent elements of $k[U]$ (i.e. $k[\mathrm{VAR}]/\langle\mathrm{DEP}\rangle$) with polynomials from $k[\mathrm{VAR}]$ and compute modulo $\langle DEP\rangle$ with them. The partial derivative of $f \in k[U]$ w.r.t. the j-th parameter can be computed as $\partial_j f = \sum_{i=1}^m p_{ij} \cdot \partial f / \partial x_i$.

Remark 4.1. A necessary condition for a chart to satisfy the postulates, which is also easy to check, is $\partial_i D_k = 0$ and $\partial_i P_j = \delta_i^j$ in $k[U]$, for all possible i, j, k.

In an initially setup VAR $= (x_1, \dots, x_n)$, DEP is empty, IND=VAR, PDER $= \mathrm{id}_{n\times n}$. This chart obviously fulfills the requirements of the previous postulates.

Basic objects are associated with charts. Their definition and the computations with them assume an underlying chart in which they exist. Basic objects are described by the following data:

- A list N of different elements from IND; generators for the ideal of W.
- A finite list G of elements from $k[W]$, generators for J.
- A positive integer b.
- A list E of different elements from IND; the “history” of exceptional divisors.

- A list a of natural numbers representing the function $a : E \to \mathbb{N}$; only in case of a weighted object.
- A list E_- of different elements from IND; only in case of a tagged object.

The fundamental operations on charts are called *Cover*, *Exchange* and *BlowUp*. In the descriptions of the following algorithms we use the notation for chart data as above.

4.1. Cover

The *Cover* operation takes as input a chart and a finite list $F \subset k[U]$, with the property $1 \in \langle F \rangle$. It returns a list of charts with the properties:

1 Each new chart is isomorphic to an affine open subset of the input chart.
2 The new charts cover the input chart.
3 The postulates are fulfilled by the new charts.

Algorithm *Cover*((VAR, DEP, IND, PDER), F);
    L := ();
    for each $f \in F$ do
        VAR$'$:= append x_{m+1} to VAR;
        DEP$'$:= append $x_{m+1}f - 1$ to DEP;
        R := $(-x_{m+1}^2 \partial_k f \mid k = 1, \ldots, n)$;
        PDER$'$:= append R as the $(m+1)$-st row to PDER;
        L := attach a new chart defined by (VAR$'$, DEP$'$, IND, PDER$'$) to L;
    return L;

Lemma 4.1. *The algorithm* Cover *is correct.*

Proof. The statements can be easily checked, provided that they hold for the input chart. □

4.2. Exchange

The *Exchange* operation takes as input a chart, an element $q \in k[U]$, an index i to IND such that $\partial_i q$ is invertible in $k[U]$, and the inverse $a = (\partial_i q)^{-1}$. It returns a new chart being isomorphic to the input chart, having P_i replaced with q in IND, and fulfilling the requirements of the postulates.

Algorithm *Exchange*((VAR, DEP, IND, PDER), q, i, a);
    IND$'$:= replace P_i by q in IND;
    for j from 1 to m do
        PDER$'[j,i]$:= ap_{ji};
        for k from 1 to m except i do
            PDER$'[j,k]$:= $p_{jk} - ap_{ji}\partial_k q$;
    return (VAR, DEP, IND$'$, PDER$'$);

Lemma 4.2. *The algorithm* Exchange *is correct.*

PROOF. We assume that the input chart fulfills the requirements.

(1): Obvious.

(2): Since $\Omega(U)$ is freely generated by the dP_k's, $dP_1, \ldots, dP_{i-1}, dq, dP_{i+1}, \ldots, dP_n$ also form a free basis because

$$dq = \sum_{j=1, j\neq i}^{n} \frac{\partial q}{\partial P_j} dP_j + \frac{\partial q}{\partial P_i} dP_i$$

and $\frac{\partial q}{\partial P_i}$ is invertible over U with inverse a, therefore $dP_i = a\big(dq - \sum_{j=1,j\neq i}^{n} (\partial q/\partial P_j) dP_j\big)$.

(3): Holds, because of the construction of the entries of $\mathrm{PDER}_{\mathrm{new}}$. For any $f \in k[x_1, \ldots, x_m]$:

$$\frac{\partial f}{\partial P_i} = \sum_{j=1,j\neq i}^{n} \frac{\partial_{\mathrm{new}} f}{\partial P_j} \cdot \frac{\partial P_j}{\partial P_i} + \frac{\partial_{\mathrm{new}} f}{\partial q} \cdot \frac{\partial q}{\partial P_i}, \quad \text{so}$$

$$\frac{\partial_{\mathrm{new}} f}{\partial q} = \frac{\partial f}{\partial P_i} \left(\frac{\partial q}{\partial P_i}\right)^{-1} = \frac{\partial f}{\partial P_i} \cdot a,$$

and

$$\frac{\partial f}{\partial P_k} = \sum_{j=1,j\neq i}^{n} \frac{\partial_{\mathrm{new}} f}{\partial P_j} \cdot \frac{\partial P_j}{\partial P_k} + \frac{\partial_{\mathrm{new}} f}{\partial q} \cdot \frac{\partial q}{\partial P_k} = \frac{\partial_{\mathrm{new}} f}{\partial P_k} + \frac{\partial_{\mathrm{new}} f}{\partial q} \cdot \frac{\partial q}{\partial P_k}, \quad \text{so}$$

$$\frac{\partial_{\mathrm{new}} f}{\partial P_k} = \frac{\partial f}{\partial P_k} - \frac{\partial f}{\partial P_i} \cdot a \cdot \frac{\partial q}{\partial P_k},$$

where $k = 1, \ldots, n,\ k \neq i$, and $\frac{\partial_{\mathrm{new}}}{\partial}$ denotes partial differentiation in the new differential frame. □

4.3. BlowUp

The most important operation performed on charts in the resolution is the blowing up along closed nonsingular centers. In our algorithm the permissible centers are determined by the intersections of hypersurfaces given by elements from IND. (for more properties see 3):

The *BlowUp* operation takes as input a chart and a list of different indices to IND. For the sake of better readability we assume that the blowing up center is given by $(P_1, \ldots, P_r)$. The operation returns a list of charts with the following properties:

1 Each new chart is isomorphic to an affine open subset of the blown up variety.
2 The new charts cover the blown up variety.
3 The postulates are fulfilled by the new charts.

Algorithm *BlowUp*((VAR, DEP, IND, PDER), $(P_1, \ldots, P_r)$);
  L := ();
  for i from 1 to r do
    VAR$'$:= append $(x_{m+j} \,|\, j = 1, \ldots, r,\ j \neq i)$ to VAR;
    DEP$'$:= append $(x_{m+j} P_i - P_j \,|\, j = 1, \ldots, r,\ j \neq i)$ to DEP;
    IND$'$:= replace P_j by x_{m+j} in IND for $j = 1, \ldots, r,\ j \neq i$;
    for j from 1 to m do
      PDER$'[j, i]$:= $p_{ji} + \sum_{k=1,k\neq i}^{r} x_{m+k} p_{jk}$;

```
            for k from 1 to r except i do
                  PDER'[j, k] := P_i p_jk;
            for k from r + 1 to n do
                  PDER'[j, k] := p_jk;
      for j from 1 to r except i do
            R := a list of zeros of length n with a 1 at the j-th position;
            PDER' := append R as a new row to PDER';
      L := append (VAR', DEP', IND', PDER') to L;
return L;
```

LEMMA 4.3. *The algorithm* BlowUp *is correct.*

PROOF. We assume that the input chart $U \subseteq \mathbb{A}^m$ whose coordinate ring is generated by $\overline{x}_1, \ldots, \overline{x}_m$, is a nonsingular variety.

(1): Let Y be the blown up of U along $\mathbf{V}(P_1, \ldots, P_r)$, with

$$\mathbf{I}(Y) = \{P \in k[x_1, \ldots, x_m, y_1, \ldots, y_r] \mid P(\overline{x}_1, \ldots, \overline{x}_m, tP_1, \ldots, tP_r) = 0 \quad \text{in } k[U][t]\}.$$

The dehomogenized ideal of $\mathbf{I}(Y)$ w.r.t. y_1 is

$$I_1 = \{P \mid \text{the homogenization } P' \text{ of } P \text{ w.r.t. } y_1 \text{ fulfills:} \\ P'(\overline{x}_1, \ldots, \overline{x}_m, tP_1, \ldots, tP_r) = 0 \quad \text{in } k[U][t]\}.$$

Here the homogenization of $P \in I_1$ with total degree d in the y's is

$$P(x_1, \ldots, x_m, y_2/y_1, \ldots, y_r/y_1)y_1^d.$$

Because of the definition of $\mathbf{I}(Y)$ this is equivalent to

$$P(\overline{x}_1, \ldots, \overline{x}_m, tP_2/tP_1, \ldots, tP_r/tP_1)P_1^d t^d = 0 \quad \text{in } k(U)[t].$$

This is equivalent to $P(\overline{x}_1, \ldots, \overline{x}_m, P_2/P_1, \ldots, P_r/P_1) = 0$ in $k(U)$ because $P_1, \ldots, P_r$ is a regular sequence for U (see (Eisenbud, 1994) Corollary 10.15, p 241). So I_1 is the kernel of the homomorphism sending $x_j \mapsto \overline{x}_j,\ j = 1, \ldots, m,\ y_i \mapsto P_i/P_1,\ i = 2, \ldots, r$. We also know that this kernel is generated by the polynomials $D_1, \ldots, D_s, y_iP_1 - P_i,\ i = 2, \ldots, r$ (see (Eisenbud, 1994) Exercise 17.14, p 441). Therefore the first chart is isomorphic to $Y \cap (\mathbb{A}^m \times \mathbb{A}_1^{r-1})$, where $\mathbb{A}^m \times \mathbb{P}^{r-1} = \bigcup_{i=1}^r (\mathbb{A}^m \times \mathbb{A}_i^{r-1})$ is the covering by principal open subsets of $\mathbb{P}^{r-1}$.

This proves the case of the chart with exceptional divisor P_1, and the proof is analogous for the rest of the charts.

(2): Holds, because the principal open subset decomposition covers $\mathbb{A}^m \times \mathbb{P}^{r-1}$, thus also the restrictions $Y \cap (\mathbb{A}^m \times \mathbb{A}_i^{r-1})$ cover Y.

(3): It has three subparts to show, each including the assumption that the requirements of the postulates are fulfilled by the input chart of the blowing up operation.

The first one holds because the blowing up of a nonsingular variety along a smooth center results a smooth blown up, therefore the charts are also nonsingular. The new charts are of dimension n, because the dimension is a birational invariant.

The second part requires to show that $dP_1, dx_{m+2}, \ldots, dx_{m+r}, dP_{r+1}, \ldots, dP_n$ generate $\Omega(Y \cap (\mathbb{A}^m \times \mathbb{A}_1^{r-1}))$ (taking only the first chart again). It suffices to show that $dP_2, \ldots, dP_r$ is generated, and this is so because $P_i = x_{m+i}P_1$, by the new algebraic relations, thus $dP_i = dx_{m+i}P_1 + dP_1x_{m+i},\ i = 2, \ldots, r$ everywhere. Since the cardinality of the IND list has not changed, $\Omega(Y \cap (\mathbb{A}^m \times \mathbb{A}_1^{r-1}))$ is generated freely in the new chart.

The third part can be checked by computation similar to the case of the exchange operation. Let f depend only on the variables $x_1, \ldots, x_m$, then for $k = 1, \ldots, r,\ k \neq i$

$$\frac{\partial f}{\partial P_k} = \sum_{j=1, j\neq i}^{r} \frac{\partial x_{m+j}}{\partial P_k} \cdot \frac{\partial_{\text{new}} f}{\partial x_{m+j}} + \frac{\partial P_i}{\partial P_k} \cdot \frac{\partial_{\text{new}} f}{\partial P_i} + \sum_{j=r+1}^{n} \frac{\partial P_j}{\partial P_k} \cdot \frac{\partial_{\text{new}} f}{\partial P_j} =$$

the surviving summand is $j = k$ from the first sum

$$= \frac{\partial (P_k/P_i)}{\partial P_k} \cdot \frac{\partial_{\text{new}} f}{\partial x_{m+k}} = \left(\frac{\partial P_k}{\partial P_k} P_i - \frac{\partial P_i}{\partial P_k} P_k \right) \frac{1}{P_i^2} \cdot \frac{\partial_{\text{new}} f}{\partial x_{m+k}}, \quad \text{so}$$

$$\frac{\partial_{\text{new}} f}{\partial x_{m+k}} = P_i \cdot \frac{\partial f}{\partial P_k}.$$

For the case of the exceptional divisor we use the same decomposition, and the surviving parts are:

$$\frac{\partial f}{\partial P_i} = \sum_{j=1, j\neq i}^{r} \frac{\partial P_j/P_i}{\partial P_i} \cdot \frac{\partial_{\text{new}} f}{\partial x_{m+j}} + \frac{\partial_{\text{new}} f}{\partial P_i} = \sum_{j=1, j\neq i}^{r} -x_{m+j} \cdot \frac{\partial f}{\partial P_j} + \frac{\partial_{\text{new}} f}{\partial P_i}, \quad \text{so}$$

$$\frac{\partial_{\text{new}} f}{\partial P_i} = \frac{\partial f}{\partial P_i} + \sum_{j=1, j\neq i}^{r} x_{m+j} \cdot \frac{\partial f}{\partial P_j}$$

One gets analogously for $k = r+1, \ldots, n, \quad \partial_{\text{new}} f/\partial P_k = \partial f/\partial P_k$. The values for the new variables come from the axiom $\partial^j_{\text{new}} x_{m+k} = \delta^j_k$, where $k = 1, \ldots, r,\ k \neq i,\ j = 1, \ldots, n$. Again, $\frac{\partial_{\text{new}}}{\partial}$ denoted partial differentiation in the new differential frame. $\square$

Example 4.1. We use the initial variables x, y, z, and v_1, v_2 as the introduced ones. Suppose we want to resolve the surface S in three-space with equation

$$z^2 + (x^2 + y^2 - 1)^2 y = 0.$$

The circle $x^2 + y^2 - 1 = z = 0$ is a double curve of S which has to be blown up. Actually, the algorithm 3 tells to first blow up the two points $(\pm 1, 0, 0)$. But for our purposes it suffices to consider what happens in the open chart $y{\neq}0$. In this chart, the algorithm will blow up the circle.

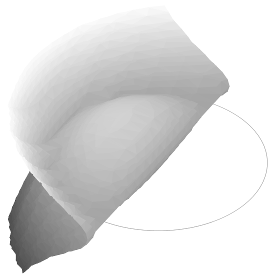

Figure 2. Outline of S in real space

So, we first do a cover operation, producing a chart C_1 isomorphic to the open set $y{\neq}0$:

1 VAR $= [x, y, z, v_1]$,
2 DEP $= [yv_1 - 1]$,
3 IND $= [x, y, z]$,
4 PDER $= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -v_1^2 & 0 \end{pmatrix}$.

The entry of the last row of the matrix PDER is introduced by the cover operation and comes from the formula $\partial_y v_1 = \partial_y \frac{1}{y} = -\frac{1}{y^2} = -v_1^2$. The cover operation also produces another chart C_2, isomorphic to the open sets $y^2 - 1 \neq 0$, but here we only treat C_1.

Next, we exchange y by $x^2 + y^2 - 1$ in IND. This is possible because $\partial_y(x^2 + y^2 - 1)$ is invertible in C_1. We obtain the chart E:

1 VAR $= [x, y, z, v_1]$,
2 DEP $= [v_1 y - 1]$,
3 IND $= [x, x^2 + y^2 - 1, z]$,
4 PDER $= \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & \frac{1}{2}v_1 & 0 \\ 0 & 0 & 1 \\ v_1^3 x & -\frac{1}{2}v_1^3 & 0 \end{pmatrix}$.

Now, we are ready for blowing up, and in this step we obtain two charts. The first, say, B_1 is given by

1 VAR $= [x, y, v_1, v_2]$,
2 DEP $= [v_1 y - 1]$,
3 IND $= [x, x^2 + y^2 - 1, v_2]$,
4 PDER $= \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & \frac{1}{2}v_1 & 0 \\ v_1^3 x & -\frac{1}{2}v_1^3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

The variable z has been eliminated. The proper transform of S has equation $v_2^2 + y$, hence it is nonsingular. The exceptional divisor is $x^2 + y^2 - 1$, and one can show easily that the total transform is a normal crossing divisor.

The second chart B_2 is given by

1 VAR $= [x, y, z, v_1, v_2]$,
2 DEP $= [v_1 y - 1, v_2 z - x^2 - y^2 + 1]$,
3 IND $= [x, v_2, z]$,
4 PDER $= \begin{pmatrix} 1 & 0 & 0 \\ -v_1 x & -\frac{1}{2}v_1 z & -\frac{1}{2}v_2 v_1 \\ 0 & 0 & 1 \\ v_1^3 x & -\frac{1}{2}v_1^3 z & -\frac{1}{2}v_2 v_1^3 \\ 0 & 1 & 0 \end{pmatrix}$.

Again the proper transform $v_2^2 y + 1$ is nonsingular and has normal crossing with the exceptional divisor z.

5. How the Computation is Done

In this section, we describe the computational subtasks of the desingularization algorithm. It turns out that they can be solved using Gröbner bases.

The computational subtasks in the desingularization algorithm are the following:

1 polynomial arithmetic and partial differentiation;
2 testing whether an ideal contains 1 (to find the maximal order, for instance);
3 computing the proper transform of a polynomial;
4 finding an element of order 1 in the delta-ideal;
5 separating the components of V where the coefficient ideal is zero from the others.

Task 1 is easy.

To perform task 2, we compute a Gröbner base and check whether it contains a constant.

To perform task 3, we need to divide an element in $k[U]$ by some power of an element in IND. If $U = \mathbb{A}^n$, then we can simply use polynomial division. In general, we have a set $\{R_1, \ldots, R_n\}$ (arising by DEP and N) of polynomials, and two polynomials F, G, and we want to compute a Q such that $F - QG$ is in the ideal $\langle R_1, \ldots, R_n \rangle$. We do this by solving the linear equation

$$X_1 R_1 + \ldots + X_n R_n + QG = F$$

by the method of extended Gröbner bases (see (Becker and Weispfenning, 1993), section 6.1).

The remaining tasks are now being discussed in some detail.

The following tasks seem to be necessary, but the are not because they are automatically provided by our way of representing charts/basic objects.

1 Computing equations for the blowing up center. Note that the blowing up center arises as an intersection of hypersurfaces of maximal contact and exceptional divisors. We always keep these hypersurfaces in IND. Thus, we already have the center generated by elements in IND when it comes to blowing up. (Especially, no radical computation and no primary decomposition is necessary.)
2 Computing the proper part of a weighted object. Checking carefully the algorithms *ResolveWeighted* and *Equimultiple*, we see that the proper part is equal to the ideal component of the just resolved tagged object, except in the beginning, where it is equal to J.

5.1. Finding an element of order 1

We have given an ideal I—the delta-ideal of a balanced object—and a set E of hypersurfaces with normal crossing (i.e. each hypersurface is the zero set of a function in IND). We know that the ideal ∂I generated by I and the partial derivatives of elements in E with respect to the parameters in IND $-$ E does not have any zeroes (otherwise the basic object would not be balanced), i.e. $\partial I = (1)$

Using the charts as described in section 4, we do not just want to compute an element of order 1, but we also want to be able to perform an *Exchange* operation that brings this element into IND. In general, there is no global solution, so the task has to be prepared by a *Cover* operation.

In each output charts, we want to find an element $g \in I$ such that $\frac{\partial g}{\partial P}$ is invertible (actually we need to know the inverse). Incidentally, the algorithm also produces charts which are disjoint from the zero set of I; these charts will be reported as being outside the singular locus.

Input:
a chart $U = (VAR, DEP, IND, PDER)$,
a subset E of IND;
a set G of elements in elements in $k[U]$, such that the ideal generated by G and its partial derivatives with respect to $\mathrm{IND} - \mathrm{E}$ is (1).

Output:
a set C of elements in $k[U]$, generating the ideal (1) (as required for an input set to *Cover*),
a subset $C_1 \subset C$ such that for each $f \in C_1$, the open set of $f \neq 0$ and the zero set of I are disjoint ;
for any element $f \in C - C_1$, a triple (g, P, a) such that
$g \in I$,
$P \in \mathrm{IND} - \mathrm{E}$,
$a \in k[U][1/f]$ (the coordinate ring of the subset corresponding to f)
$a \cdot \frac{\partial g}{\partial P} = 1$ (as required for an input to *Exchange*).

Here is a surprisingly simple algorithm. The only real computation is to differentiate the elements in G.

Algorithm *MaximalContact*(U, E, G):
$C := G$;
$C_1 := G$;
for each $g \in G$ and $P \in \mathrm{IND} - \mathrm{E}$ do
$f := \frac{\partial g}{\partial P}$;
$a := 1/f$;
attach (g, P, a) to f;
add f to C;
return C and C_1;

It is easy to see that this algorithm would work in principle. However, it pays off to spend some more effort in order to reduce subsequent computations in the charts produced by subsequent *Cover* and *Exchange* operations. In our implementation, we first compute a reduced Gröbner basis for G with respect to a total degree order and then apply the algorithm above. Since elements of the ideal with minimal degree appear in the Gröbner basis, this has the effect that the degree of the hypersurfaces of maximal contact is as small as possible.

Remark 5.1. We have a strategy for avoiding redundant charts that works as follows: at the time of generating, it is checked if the new chart is covered by previously generated charts. If yes, then the chart is supressed.

An important consequence is that the order of an input set to *Cover* matters. Experience has shown that it is better to put C_1 *after* the rest of C.

5.2. Separating components

We have given a regular algebraic set V (a hypersurface of maximal contact) and an ideal $I \subset k[V]$ (the coefficient ideal). Recall that V is not necessarily irreducible, it may be a disjoint union of nonsingular varieties (the union is disjoint because intersections would be singular points). We want to cover V by open subsets of two types:

1 either I restricts to the zero ideal (in the sense of sheaf theory). Within such a chart, we have to blow up the hypersurface;

2 or I restricts to a nonzero ideal on all connected components. Within such a chart, we have to construct the coefficient object.

Let V_1 be a component of V of type 1. Then the restriction of the annihilator $\mathrm{Ann}(I)$ is (1), because 1 annihilates I inside V_1. On the other hand, the ideal $\partial^\infty I$ generated by I and its partial derivatives of any order is zero.

Now, let V_2 be a component of V of type 2. Then the restriction of the annihilator is zero, because $k[V_2]$ does not have zero divisors. On the other hand, one can show that $\partial^\infty I = (1)$, because the maximal order of I is finite.

Combining, we see that

$$\mathrm{Ann}(I) + \partial^\infty I = (1).$$

As an input to *Cover*, we take a finite generating set A for $\mathrm{Ann}(I)$ together with a finite generating set B for $\partial^\infty I$. For any $f \in A$, the open subset $f \neq 0$ contains only components of type 1, whereas for any $g \in B$, the open subset $g \neq 0$ contains only components of type 2.

In terms of polynomials, the annihilator is just the quotient ideal of I and $\mathbf{I}(V)$, and this can be computed by Gröbner bases (see (Becker and Weispfenning, 1993)).

Here is an algorithm that computes the multiplicity and the delta-ideal of the coefficient object at no extra cost.

Algorithm *Separate*(I):
  $b := 1; D := I$;
  $A :=$ a generating set for $\mathrm{Ann}(I)$;
  $B := D \cup$ the partial derivatives of D of order 1;
  while $\langle A, B \rangle$ does not contain 1 do
    $D := B$;
    $B := D \cup$ the partial derivatives of D of order 1;
    $b := b + 1$;
  return A (for creating components of type 1);
  return B (for creating components of type 2);
  return b (multiplicity of coefficient object);
  return D (delta-ideal of coefficient object);

6. An Extended Example

We take the singular hypersurface defined by $x_1^2 + x_2x_3^2$ in three-space (also known as the Whitney umbrella). We follow one branch of its resolution along steps of the algorithm described in this paper, trying to give an insight to the working program. The upper indices count charts, and the lower ones basic objects of the same chart.

The initial setup consists of the chart: $\mathrm{VAR}^1 = (x_1, x_2, x_3)$, $\mathrm{DEP}^1 = ()$, $\mathrm{IND}^1 = (x_1, x_2, x_3)$, $\mathrm{PDER}^1 = \mathrm{id}_{3\times 3}$, and the weighted object $N_1^1 = ()$, $G_1^1 = (x_1^2 + x_2x_3^2)$, $b_1^1 = 1$, $E_1^1 = ()$, $a_1^1 = ()$. This object is not resolved and not monomial, therefore the algorithm sets up a tagged object $N_2^1 = ()$, $G_2^1 = (x_1^2 + x_2x_3^2)$, $b_2^1 = 2$, $E_2^1 = ()$, $E_{2-}^1 = ()$, and continues the resolution of this. The next step is to create a balanced object, which is: $N_3^1 = ()$, $G_3^1 = (x_1^2 + x_2x_3^2)$, $b_3^1 = 2$, $E_3^1 = ()$, and to find a hypersurface of maximal contact. The algorithm finds $\mathbf{V}(x_1)$ to be appropriate and since the coefficient ideal of J_3^1 of the balanced object w.r.t. x_1 is not zero, it creates a new weighted object of dimension 2: $N_4^1 = (x_1)$, $G_4^1 = (x_2x_3^2)$, $b_4^1 = 2$, $E_4^1 = ()$, $a_4^1 = ()$.

For this weighted object the procedure happens to be quite the same: tagged object: $N_5^1 = (x_1)$, $G_5^1 = (x_2x_3^2)$, $b_5^1 = 3$, $E_5^1 = ()$, $E_{5-}^1 = ()$, balanced object: $N_6^1 = (x_1)$, $G_6^1 = (x_2x_3^2)$, $b_6^1 = 3$, $E_6^1 = ()$, hypersurface of maximal contact: $\mathbf{V}(x_3)$, coefficient ideal of J_6^1 w.r.t. x_3 is generated by x_2. The new weighted object is: $N_7^1 = (x_1, x_3)$, $G_7^1 = (x_2^6)$, $b_7^1 = 3$, $E_7^1 = ()$, $a_7^1 = ()$.

The weighted object is again not resolved and not monomial, so its tagged object is: $N_8^1 = (x_1, x_3)$, $G_8^1 = (x_2^6)$, $b_8^1 = 6$, $E_8^1 = ()$, $E_{8-}^1 = ()$, and the next balanced object is: $N_9^1 = (x_1, x_3)$, $G_9^1 = (x_2^6)$, $b_9^1 = 6$, $E_9 = ()$. The hypersurface of maximal contact is $\mathbf{V}(x_2)$, and the coefficient ideal is now zero, therefore a *BlowUp* operation is performed on the chart, with all the basic objects assigned to it.

The blowing up center is $\mathbf{V}(x_1, x_3, x_2)$, and we follow the resolution along the new chart with exceptional divisor given by x_3: $\mathrm{VAR}^2 = (x_3, x_4, x_5)$, $\mathrm{DEP}^2 = ()$, $\mathrm{IND}^2 = (x_4, x_5, x_3)$,

$$\mathrm{PDER}^2 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix},$$

where the affine variables x_1, x_2 were eliminated by the equations $x_4x_3 = x_1$, $x_5x_3 = x_2$.

The transforms of the basic objects are as follows:

$N_1^2 = ()$, $G_1^2 = (x_4^2x_3 + x_5x_3^2)$, $b_1^2 = 1$, $E_1^2 = (x_3)$, $a_1^2 = (1)$,
$N_2^2 = ()$, $G_2^2 = (x_4^2 + x_5x_3)$, $b_2^2 = 2$, $E_2^2 = (x_3)$, $E_{2-}^2 = ()$,
$N_3^2 = ()$, $G_3^2 = (x_4^2 + x_5x_3)$, $b_3^2 = 2$, $E_3^2 = (x_3)$,
$N_4^2 = (x_4)$, $G_4^2 = (x_5x_3)$, $b_4^2 = 2$, $E_4^2 = (x_3)$, $a_4^2 = (1)$,

The basic objects with $x_3 \in N$ and the tagged object of index 5 got resolved, and the recursion returned to this stage.

The weighted object of index 4 is not resolved and not monomial, so the new tagged object is $N_5^2 = (x_4)$, $G_5^2 = (x_5, x_3)$, $b_5^2 = 1$, $E_5^2 = (x_3)$, $E_{5-}^2 = (x_3)$. The new balanced object is $N_6^2 = (x_4)$, $G_6^2 = (x_5, x_3)$, $b_6^2 = 1$, $E_6^2 = ()$. The hypersurface of maximal contact is $\mathbf{V}(x_3)$, and the coefficient ideal of J_6^2 w.r.t. x_3 is generated by x_5, hence the next weighted object is: $N_7^2 = (x_4, x_3)$, $G_7^2 = (x_5)$, $b_4^2 = 1$, $E_4^2 = ()$, $a_4^2 = ()$.

The latter weighted object is neither resolved nor monomial, so a tagged object is created: $N_8^2 = (x_4, x_3)$, $G_8^2 = (x_5)$, $b_8^2 = 1$, $E_8^2 = ()$, $E_{8-}^2 = ()$. The next step creates a balanced object $N_9^2 = (x_4, x_3)$, $G_9^2 = (x_5)$, $b_9^2 = 1$, $E_9^2 = ()$, and in *ResolveBalanced* the hypersurface of maximal contact, $\mathbf{V}(x_5)$, leads to blowing up.

The blowing up center is $\mathbf{V}(x_4, x_3, x_5)$, and we look at the new chart with exceptional divisor given by x_4: $\mathrm{VAR}^3 = (x_4, x_6, x_7)$, $\mathrm{DEP}^3 = ()$, $\mathrm{IND}^3 = (x_4, x_7, x_6)$,

$$\mathrm{PDER}^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix},$$

where the affine variables x_3, x_5 were eliminated by the equations $x_6x_4 = x_3$, $x_7x_4 = x_5$.

The transform of the basic object of index 1 is $N_1^3 = ()$, $G_1^3 = (x_4^2x_7x_6^2 + x_4^2x_6)$, $b_1^3 = 1$, $E_1^3 = (x_6, x_4)$, $a_1^3 = (1, 2)$. The rest of the basic objects got resolved and the recursion returned to this level.

At this point, since the ideal $\overline{J}_1^3$ is of order 1, we already have a resolution, which in this case even happens to be embedded. If we didn't need embedded resolution, the termination criterion for the resolution of a chart could be: $\mathrm{ord}(\overline{J}_1) \leq 1$. On the other hand, the previous condition does not guarantee embeddedness, so the algorithm continues to complete the resolution.

For the weighted object above the next tagged object is: $N_2^3 = ()$, $G_2^3 = (x_7x_6+1)$, $b_2^3 = 1$, $E_2^3 = (x_6, x_4)$, $E_{2-}^3 = (x_6, x_4)$. The subalgorithm *ResolveTagged* creates a balanced object $N_3^3 = ()$, $G_3^3 = (x_7x_6 + 1, x_4)$, $b_3^3 = 1$, $E_3^3 = ()$. In the next step $\mathbf{V}(x_4)$ is found to be the hypersurface of maximal contact, and since the corresponding coefficient ideal is generated by $x_7x_6 + 1$, the new weighted object is: $N_4^3 = (x_4)$, $G_4^3 = (x_7x_6 + 1)$, $b_4^3 = 1$, $E_4^3 = ()$, $a_4^3 = ()$.

For this weighted object the resolution goes similarly: creating a tagged object $N_5^3 = (x_4)$, $G_5^3 = (x_7x_6 + 1)$, $b_5^3 = 1$, $E_5^3 = ()$, $E_{5-}^3 = ()$, and the next balanced object: $N_6^3 = (x_4)$, $G_6^3 = (x_7x_6 + 1)$, $b_6^3 = 1$, $E_6^3 = ()$.

In order to find an appropriate element generating a hypersurface of maximal contact, the program has to do some preparation. Since the partial derivatives of $x_7x_6 + 1$ w.r.t. x_4, x_7, x_6 are $0, x_6, x_7$ respectively, it must make either x_6 or x_7 invertible in order to put $x_7x_6 + 1$ into IND and perform the next blowing up.

To do this, first it covers the chart with its open subsets having $\mathbf{V}(x_6)$ and $\mathbf{V}(x_7x_6+1)$ excluded. In the chart having $x_7x_6 + 1$ invertible, the basic objects become nonsingular. We look at the other chart: $\mathrm{VAR}^4 = (x_4, x_6, x_7, x_8)$, $\mathrm{DEP}^4 = (x_8x_6 - 1)$, $\mathrm{IND}^4 = (x_4, x_7, x_6)$,

$$\mathrm{PDER}^4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -x_8^2 \end{pmatrix},$$

The basic objects remain the same (the upper indices increase).

Now, the exchange of $x_7x_6 + 1$ with x_7, having $(\partial_2(x_7x_6 + 1))^{-1} = x_8$ is allowed: $\mathrm{VAR}^5 = (x_4, x_6, x_7, x_8)$, $\mathrm{DEP}^5 = (x_8x_6 - 1)$, $\mathrm{IND}^5 = (x_4, x_7x_6 + 1, x_6)$,

$$\mathrm{PDER}^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & x_8 & -x_8x_7 \\ 0 & 0 & -x_8^2 \end{pmatrix},$$

The basic objects are again unchanged (the upper indices increase).

By continuing in *ResolveBalanced*, the hypersurface of maximal contact is just $\mathbf{V}(x_7x_6+1)$, and the coefficient ideal is zero.

The chart is blown up along the center $\mathbf{V}(x_4, x_7x_6+1)$. We choose to follow the chart with exceptional divisor given by x_7x_6+1: $\mathrm{VAR}^6 = (x_6, x_7, x_8, x_9)$, $\mathrm{DEP}^6 = (x_8x_6-1)$, $\mathrm{IND}^6 = (x_9, x_7x_6+1, x_6)$,

$$\mathrm{PDER}^6 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & x_8 & -x_8x_7 \\ 0 & 0 & -x_8^2 \\ 1 & 0 & 0 \end{pmatrix},$$

The transform of the basic object of index 1 is: $N_1^6 = ()$, $G_1^6 = (1)$, $b_1^6 = 1$, $E_1^6 = (x_9, x_7x_6+1)$, $a_1^6 = (2,2)$.

Now, it is the monomial case which takes four blowing ups to complete, but since they do not produce any significant change we omit them. Eventually, we arrived to a chart which has got a single resolved object. This chart is a member of the affine covering of the smooth variety of a desingularization of the Whitney umbrella.

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