LNCA: A Lazy Narrowing Calculus for Applicative Term Rewriting Systems

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Abstract. The integration of higher-order functions into functional logic programming is widely seen as a powerful and desirable feature. The natural way to deal with higher-order functions in the well-studied framework of first-order term rewriting is through so-called applicative term rewriting systems ($\mathcal{A}$TRSs). We propose a new calculus, called LNCA, to deal efficiently with confluent $\mathcal{A}$TRSs and prove its soundness and completeness.

1 Introduction

Research on integrating functional and logic programming ([1]) has established narrowing as a suitable computational model. Some powerful features like higher-order functions, which are common and useful in functional languages, are not yet consolidated in narrowing calculi. The following program illustrates the expressiveness of higher-order functional logic programming:

```
plus 0 y     = y                 map f []      = []
plus (S x) y = S (plus x y)      map f [x | y] = [f x | map f y]
double x     = plus x x          compose f g x = f (g x)
```

The functions `map` and `compose` are higher-order. For instance, solving the goal

```
map f [S 0, 0, S 0]=[S (S (S 0)), S 0, S (S (S 0))]
```

means finding substitutions like $\{\mathtt{f} \mapsto \mathtt{compose\ S\ double}\}$ which can not be done by first order narrowing. This difficulty can be overcome by the use of applicative term rewriting systems ($\mathcal{A}$TRSs for short). In an $\mathcal{A}$TRS terms are built from variables, constants and a special binary function symbol `ap` which is denoted by juxtaposition of its two arguments. For example, the term

```
map f [S 0, 0, S 0]
```

will be represented as:

```
((map f) ((cons (S 0)) ((cons 0) ((cons (S 0)) []))))
```

where `map`, `cons`[1], `S`, `0` and `[]` are regarded as constants.

The narrowing calculus LNC ([4]) can now be used to solve such a goal but it is quite inefficient: while just one inference step extracts all arguments of a first-order term, for $\mathcal{A}$TRSs we need one inference step for each argument. In other words, we have to extract all arguments step by step before we even know whether, e.g., we are using the appropriate rewrite rule, even in the case that the head-symbol is a known function. Starting from this observation we define a new calculus which is a specialization of LNC for applicative TRSs. We call the new calculus LNCA (**L**azy **N**arrowing **C**alculus for **A**pplicative Term Rewriting Systems) and prove that it is sound and complete for confluent $\mathcal{A}$TRSs with respect to normalized substitutions.

The extension of LNC to LNCA is similar to that from OINC to NCA ([2, 5]). NCA is an earlier attempt to define a narrowing calculus which deals efficiently with $\mathcal{A}$TRSs. However, its completeness is proven to hold only if we restrict our attention to orthogonal $\mathcal{A}$TRSs, right-normal goals, and normalizable solutions. The advantages of LNCA over NCA are:

- completeness holds in the general case of confluent $\mathcal{A}$TRSs with respect to normalized substitutions,
- there are no restrictions on the shape of the goals.

The completeness proof of LNCA is very similar to that of LNC. We briefly mention the main ideas of the proof:

Fact 1 For every normalized solution θ of a goal G there exists a substitution $\theta' \leq \theta\ [\mathcal{V}ar(G)]$ and a normal NC-refutation $\Pi : G \rightsquigarrow^*_{\theta'} \top$,

[1] `[s | t]` is syntactic sugar of `cons(s,t)`

Fact 2 There exists a subclass $\mathcal{WF}$ of LNC-refutations, the well-formed LNC-refutations, such that for every normal NC-refutation $\Pi : G \rightsquigarrow_\theta^* \top$ there exists a substitution $\theta' \leq \theta\ [\mathcal{V}ar(G)]$ and $\Psi \in \mathcal{WF}$ such that $\Psi : G \Rightarrow_{\theta'}^* \Box$,

Fact 3 For every well-formed LNC-refutation $\Psi : G \Rightarrow_\theta^* \Box$ with respect to an $\mathcal{A}$TRS there exists a substitution $\theta' \leq \theta\ [\mathcal{V}ar(G)]$ and an LNCA-refutation $A : G \Rrightarrow_{\theta'}^* \Box$.

Fact 1 is proven in [4]. Fact 2 is a refinement of Lemma 36 in [4], which describes how a non-empty normal NC-refutation $\Pi : G \rightsquigarrow_\theta^+ \top$ can be lifted to an LNC-refutation $\Psi : G \Rightarrow_{\theta'}^+ \Box$ with $\theta' \leq \theta\ [\mathcal{V}ar(G)]$. We noticed that the LNC-refutations generated by lifting normal NC-refutations as described in [4] have some interesting properties, which were used to define the class of well-formed LNC-refutations. The proof of Fact 3 involves a deep analysis of the structure of well-formed LNC-refutations for $\mathcal{A}$TRSs.

We believe that LNCA can be later used as a basis for some deterministic calculi to further improve efficiency, like LNC_d ([**?**]). LNC_d is a deterministic extension of LNC that is complete for left-linear, confluent, constructor-based term rewriting systems, and goals with strict equality.

The rest of this paper is organized as follows. Section 2 reviews some basic definitions and notations used in term rewriting and narrowing. In Section 3 we define LNCA and prove its soundness. The proof of completeness of LNCA is given in Section 4.

2 Preliminaries

A *signature* is a set $\mathcal{F}$ of function symbols which is the disjoint union of two sets: $\mathcal{F}_\mathcal{D}$, the set of *defined symbols* and $\mathcal{F}_\mathcal{C}$, the set of *constructors*. Associated with every function symbol $f \in \mathcal{F}$ is a natural number $\mathrm{arity}(f)$ denoting its *arity*. Function symbols of arity 0 are called *constants*. The set $\mathcal{T}(\mathcal{F}, \mathcal{V})$ of *terms* built from a signature $\mathcal{F}$ and a countably infinite set of variables $\mathcal{V}$ is the smallest set containing $\mathcal{V}$ such that $f(t_1, \ldots, t_n) \in \mathcal{T}(\mathcal{F}, \mathcal{V})$ whenever $f \in \mathcal{F}$ with $\mathrm{arity}(f) = n$ and $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$. We write c instead of $c()$ whenever c is a constant. The set of variables occurring in a syntactical object S is denoted by $\mathcal{V}ar(S)$. We use the symbol $\equiv$ to denote the identity of two syntactic entities.

A *position* is a sequence of natural numbers identifying a subterm in a term. The set $\mathcal{P}os(t)$ of positions of a term t is inductively defined as follows: $\mathcal{P}os(t) = \{\varepsilon\}$ if t is a variable or a constant, and $\mathcal{P}os(t) = \{\varepsilon\} \cup \bigcup_{i=1}^n \{i \cdot p \mid p \in \mathcal{P}os(t_i)\}$ if $t \equiv f(t_1, \ldots, t_n)$. Here ε, the empty sequence, denotes the root position. If $p \in \mathcal{P}os(t)$ then $t_{|p}$ denotes the *subterm* of t at position p, and $t[s]_p$ denotes the term that is obtained from t by replacing the subterm at position p by s.

A *substitution* is a map θ from $\mathcal{V}$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$ with the property that the set $\mathcal{D}(\theta) = \{x \in \mathcal{V} \mid \theta(x) \neq x\}$ is finite. This set is called the *domain* of θ.

We denote by ε the substitution with empty domain. Substitutions are extended to homomorphisms from $\mathcal{T}(\mathcal{F}, \mathcal{V})$ to $\mathcal{T}(\mathcal{F}, \mathcal{V})$. In the following we write $t\theta$ instead of $\theta(t)$. We denote the set $\bigcup_{x \in \mathcal{D}(\theta)} \mathcal{V}ar(x\theta)$ of variables *introduced* by θ by $\mathcal{I}m(\theta)$. The *composition* $\theta_1\theta_2$ of two substitutions θ_1 and θ_2 is defined by $x\theta_1\theta_2 = (x\theta_1)\theta_2$ for all $x \in \mathcal{V}$. θ_1 is *at least as general as* θ_2, denoted by $\theta_1 \leq \theta_2$, if there exists a substitution θ such that $\theta_1\theta = \theta_2$. The *restriction* $\theta\restriction_V$ of a substitution θ to a set $V (\subseteq \mathcal{V})$ is defined by $\theta\restriction_V(x) = \theta(x)$ if $x \in V$ and $\theta\restriction_V(x) = x$ if $x \in \mathcal{V} - V$. A *variable substitution* maps variables to variables. A *variable renaming* is a bijective variable substitution. We write $\theta_1 = \theta_2\ [V]$ if $\theta_1\restriction_V = \theta_2\restriction_V$, and $\theta_1 \leq \theta_2\ [V]$ if there exists a substitution θ such that $\theta_1\theta = \theta_2\ [V]$. Two terms s and t are called *unifiable* if there exists a substitution θ such that $s\theta = t\theta$. A *most general unifier* of s and t is a unifier θ such that $\theta \leq \theta'$ for any other unifier θ' of s and t. Most general unifiers of unifiable terms always exist and are unique up to variable renaming.

A *rewrite rule* is a pair of terms of the form $l \to r$ such that $l, r \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, $\mathcal{V}ar(r) \subseteq \mathcal{V}ar(l)$, $l \equiv f(t_1, \ldots, t_n)$ for some terms $t_1, \ldots, t_n \in \mathcal{T}(\mathcal{F}, \mathcal{V})$, and $f \in \mathcal{F}_\mathcal{D}$. l is the *left-hand* side (LHS for short) of the rule and r the right-hand side (RHS for short). A *term rewriting system* (TRS for short) is a finite set $\mathcal{R}$ of rewrite rules. The *rewrite relation* $\to_\mathcal{R}$ associated with a TRS $\mathcal{R}$ is a binary relation on $\mathcal{T}(\mathcal{F}, \mathcal{V})$ defined as follows: $s \to_\mathcal{R} t$ if there exists a rewrite rule $l \to r \in \mathcal{R}$, a substitution θ and a position $p \in \mathcal{P}os(s)$ such that $s_{|p} = l\theta$ and $t = s[r\theta]_p$. The transitive and reflexive closure of $\to_\mathcal{R}$ is denoted by $\to^*_\mathcal{R}$, and the symmetric closure by $\leftrightarrow_\mathcal{R}$. We usually omit the subscript $\mathcal{R}$. A term t is a *normal form* if there exists no term s such that $t \to s$. A substitution θ is *normalized* if all instances $x\theta$ of variables $x \in \mathcal{D}(\theta)$ are normal forms. A TRS $\mathcal{R}$ is *terminating* if there are no infinite rewrite sequences $t_0 \to t_1 \to \cdots$, and *confluent* if for all terms t_1, t_2, t_3 with $t_1 \to^* t_2$ and $t_1 \to^* t_3$ there exists a term t_4 such that $t_2 \to^* t_4$ and $t_3 \to^* t_4$. Two terms t_1 and t_n are *convertible* if there exists a sequence $t_1 \leftrightarrow \cdots \leftrightarrow t_n$ $(n \geq 1)$ written as $t_1 \leftrightarrow^* t_n$. If θ is a variable renaming then the rewrite rule $l\theta \to r\theta$ is called a *variant* of $l \to r$. A variant is *fresh* if all variables occurring in it have never been used before. An *applicative term* is a term built from variables, constants and a special binary function symbol `ap` which is denoted by juxtaposition of its two arguments. Parentheses are omitted under the convention of association to the left, so `(f (S 0)) 0` and `f (S 0) 0` denote the same term. The *head symbol* of an applicative term is the symbol that occurs at the leftmost-innermost position. In the sequel we denote head symbols which are either function symbols or variables by the letters a, b, c, $\ldots$, variables by x, y, z, function symbols by f, g, h, arbitrary terms by $s, t, u, \ldots$, and integer numbers by $i, j, k, \ldots$ An *applicative rule* is a rewrite rule $l \to r$ between two applicative terms such that l has the form $f\ l_1\ \ldots\ l_n$ where f is a function symbol with arity n. We say that such a rule *defines* the symbol f. An applicative term rewriting system ($\mathcal{A}$TRS) is a finite set of applicative rewrite rules. We abbreviate an applicative term $a\ t_1\ \ldots\ t_n$ to $a\ \mathbf{t}_n$. If $n = 0$ then $a\ \mathbf{t}_n$ denotes a. By the same convention $b\ \mathbf{s}_m\ \mathbf{t}_n$ stands for $b\ s_1\ \ldots s_m\ t_1\ \ldots\ t_n$ and $c\ \mathbf{t}_{i,j}$ for $c\ t_i\ \ldots\ t_j$.

We distinguish a binary function symbol $\approx$ written in infix notation. A term of the form $s \approx t$ where s and t do not contain occurrences of $\approx$ is called *equation* with the LHS s and the RHS t. The symbol $\simeq$ denotes the symmetric closure of $\approx$. A *goal* is a finite sequence of equations. Given an equation $e = s \approx t$, we say that a substitution θ is a *solution* of e if $s\theta$ and $t\theta$ are convertible w.r.t. the given TRS $\mathcal{R}$. Given a goal G consisting of equations $e_1, \ldots, e_n$, we say that a substitution θ is a solution of G if θ is a solution of every e_i $(1 \leq i \leq n)$. We denote an empty goal by $\Box$.

2.1 Normal NC-refutations

In this subsection we assume that $\mathcal{R}$ is a confluent TRS. We distinguish a special constant `true` and allow it as an equation. We use $\top$ as generic notation for goals consisting only of `true` constants. In this setting it can be shown that θ is a solution of a goal G iff $G\theta \to^*_{\mathcal{R}_+} \top$ where $\mathcal{R}_+ = \mathcal{R} \cup \{x \approx x \to \texttt{true}\}$. An *NC-step* is a relation on goals defined by

$$E_1, \underline{s} \simeq t, E_2 \rightsquigarrow_{\theta, l \to r} (E_1, s[r]_p \simeq t, E_2)\theta \tag{1}$$

where $p \in \mathcal{P}os(s)$ such that $s_{|p} \notin \mathcal{V}$, $l \to r$ is a fresh variant of a rewrite rule from $\mathcal{R}$ and θ is a most general unifier of l and $s_{|p}$. When confusions may occur, we underline the chosen subterm.

An *NC-derivation* is a finite sequence of inference steps abbreviated by $G \rightsquigarrow_\theta^n G'$ where n is the number of inference steps and θ the composition of all substitutions involved in the inference steps. We may write $\rightsquigarrow^*$ instead of $\rightsquigarrow^n$ if $n \geq 0$ and $\rightsquigarrow^+$ instead of $\rightsquigarrow^n$ if $n > 0$. An NC-derivation that ends in $\top$ is called *NC-refutation.* In the sequel we will denote NC-derivations by the symbol Π, sometimes subscripted. Given an NC-refutation Π and $m \in \mathbb{N}$, we define the *length* $|\Pi|$ of Π as the the number of NC-steps of Π, and the *sub-refutation* $\Pi_{>m}$ of Π as the NC-refutation obtained by omitting the first m goals and inference steps in Π.

Definition 1 An NC-refutation $\Pi : G \rightsquigarrow_\theta^* \top$ is called *normal* if it consists of NC-steps of form (1) with $E_1 = \top$, and for every representation of Π in the form:

$$\Pi : G \equiv E_1', s \simeq t, E_2' \rightsquigarrow_{\theta_1}^* \top, (\underline{s} \simeq t, E_2')\theta_1 \rightsquigarrow_{\theta_2}^* \top,$$

the substitution $\theta_2\restriction_{\mathcal{V}ar(s\theta_1)}$ is normalized.

In the sequel we denote by $\mathcal{NC}$ the class of normal NC-refutations. The following property of normal NC-refutations is of interest to our paper.

Theorem 1 For every normalized solution θ of a goal G there exists $\Pi \in \mathcal{NC}$ such that $\Pi : G \rightsquigarrow_{\theta'}^* \top$ and $\theta' \leq \theta\ [\mathcal{V}ar(G)]$.

2.2 The LNC Calculus

Definition 2 LNC consists of the following inference rules:

[o] *outermost narrowing*

$$\frac{f(s_1, \ldots, s_n) \simeq t, E}{s_1 \approx l_1, \ldots, s_n \approx l_n, r \approx t, E}$$

if $f(l_1, \ldots, l_n) \to r$ is a fresh variant of some rule of $\mathcal{R}$.

[i] *imitation*

$$\frac{f(s_1, \ldots, s_n) \simeq x, E}{(s_1 \approx x_1, \ldots, s_n \approx x_n, E)\theta}$$

if $x \in \mathcal{V}$, where $\theta = \{x \mapsto f(x_1, \ldots, x_n)\}$ with $x_1, \ldots, x_n$ fresh variables and $n > 0$.

[d] *decomposition*

$$\frac{f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n), E}{s_1 \approx t_1, \ldots, s_n \approx t_n, E}$$

[v] *variable elimination*

$$\frac{x \approx t, E}{E\{x \mapsto t\}} \quad x \notin \mathcal{V}ar(t)$$

[t] *removal of trivial equations*

$$\frac{x \approx x, E}{E} \; x \in \mathcal{V}$$

In LNC, the equations $s_1 \approx l_1, \ldots, s_n \approx l_n$ which are created by the outermost narrowing step are called *parameter-passing* equations. If G and G' are the upper and lower goal in the inference rule $[\alpha]$ ($\alpha \in$ {o,i,d,v,t}) then we write $G \Rightarrow_{[\alpha]} G'$ and call it $[\alpha]$-*step*. An *LNC-step* is an $[\alpha]$-step where $\alpha \in$ {o,i,d,v,t}. The additional rewrite rule or substitution may be supplied as a subscript, that is, we may write things like $G \Rightarrow_{[\mathrm{o}],l \to r} G'$ and $G \Rightarrow_{[\mathrm{i}],\theta} G'$.

For the [i] rule, if the variable x appears to the RHS (LHS) of the selected equation then we say that [i] is applied to the LHS (RHS). For the [o] rule, if the term $f(s_1, \ldots, s_n)$ that is being narrowed is to the LHS (RHS) of the selected equation then we say that [o] is applied to the LHS (RHS). To avoid confusions concerning the side on which an [o] or [i]-step is performed, we either underline it or provide an additional subscript to identify it. For example, $e, E \Rightarrow_{[\mathrm{i}],k,\sigma} G'$ denotes an [i]-step applied to the LHS if $k = 1$ and to the RHS if $k = 2$.

LNC-derivations, sub-derivations and the corresponding notations are defined as in the case of NC. We denote LNC derivations by the symbol Ψ and its derivatives, sometimes subscripted. For the particular case of applicative term rewriting systems we use the symbol A and its derivatives to denote LNC-derivations. An *LNC-refutation* is an LNC-derivation of the form $\Psi : G \Rightarrow^*_\theta \Box$, where $\Box$ is the empty goal. We denote by $\mathcal{LNC}$ the class of LNC-refutations.

The selected equation $f(s_1, \ldots, s_n) \simeq t$ in an [o]-step has the equation $r \approx t$ as the only one-step descendant. In the imitation rule [i], the one-step descendants of the selected equation $f(s_1, \ldots, s_n) \simeq x$ are the equations $s_i\theta \approx x_i$ ($1 \le i \le n$). The selected equation $f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n)$ in the decomposition rule [d] has the equations $s_i \approx t_i$ ($1 \le i \le n$) as one-step descendants. The selected equations of [v] and [t] have no one-step descendants. The one-step descendant of a non-selected equation $s \approx t$ is the equation $(s \approx t)\theta$ where θ is the substitution created in that inference step. The *descendant* relation is the reflexive-transitive closure of the one-step descendant relation.

LNC is a calculus which is complete with respect to normalized solutions. The following lemma is extracted from the completeness proof of LNC given in [4]. We adopt the following convention for the formulas appearing in the lemma: the symbols k, i, n denote non-negative integers such that $1 \le k, i \le 2$; j and p denote term positions; if $n > 0$ then $j \in \{1, \ldots, n\}$, otherwise $j = \varepsilon$.

Lemma 1 There exists a well-founded order $\ll \; \subseteq \mathcal{NC} \times \mathcal{NC}$ such that:

$$\begin{array}{l} \forall \Pi : G = e, E \rightsquigarrow^+_\theta \top \in \mathcal{NC}. \exists \langle \Psi_1, \Pi_1 \rangle. \\ \quad \Psi_1 : G \Rightarrow_\sigma G_1 \wedge \Pi_1 : G_1 \rightsquigarrow^*_{\theta'} \top \in \mathcal{NC} \wedge \\ \quad \Pi_1 \ll \Pi \wedge \mathcal{R}el(\Pi, \Psi_1, \Pi_1) \wedge \sigma\theta' \le \theta \; [\mathcal{V}ar(G)] \end{array}$$

where $\mathcal{R}el(\Pi, \Psi_1, \Pi_1)$ is defined as follows:

1. The descendants of E are narrowed in Π at the same positions and in the same order as the descendants of $E\sigma$ in Π_1.
2. If Ψ_1 is a [d]-step:

$$\Psi_1 : G \equiv f(s_1, \ldots, s_n) \approx f(t_1, \ldots, t_n), E \Rightarrow_{[\mathrm{d}]} s_1 \approx t_1, \ldots, s_n \approx t_n, E$$

then $i \cdot j \cdot p$ is a narrowing position to a descendant of e in Π iff $i \cdot p$ is a narrowing position to a descendant of $s_j \approx t_j$ in Π_1.

3. If Ψ_1 is an [o]-step:

$$G \equiv \underline{f(s_1, \ldots, s_n)} \simeq t, E \Rightarrow_{[\mathrm{o}],k,f(l_1,\ldots,l_n)\to r} s_1 \approx l_1, \ldots, s_n \approx l_n, r \approx t, E$$

then:
(a) Π narrows a descendant of e at position k.
(b) Π_1 does not narrow descendants of $s_i \approx l_i$ at positions in the RHS.
(c) $1 \cdot p$ is a narrowing position to a descendant of $s_j \approx l_j$ in Π_1 iff $k \cdot j \cdot p$ is a narrowing position to a descendant of $f(s_1, \ldots, s_n) \simeq t$ in Π.
(d) $2 \cdot p$ is a narrowing position to a descendant of $r \approx t$ in Π iff $(3-k) \cdot p$ is a narrowing position to a descendant of $f(s_1, \ldots, s_n) \simeq t$ in Π.

4. If Ψ_1 is an [i]-step:

$$G \equiv f(s_1, \ldots, s_n) \simeq x, E \Rightarrow_{[\mathrm{i}],k,\sigma=\{x \mapsto f(x_1,\ldots,x_n)\}} s_1\sigma \approx x_1, \ldots, s_n\sigma \approx x_n, E\sigma$$

then:
(a) Π starts with an NC-step at a position of the form $k \cdot j \cdot p$.
(b) $i \cdot j \cdot p$ is a narrowing position to a descendant of e in Π iff $i' \cdot p$ is a narrowing position to a descendant of $s_j\sigma \approx x_j$ in Π_1 where $i' = i$ if $k = 1$ and $i' = 3 - i$ if $k = 2$.

5. If Ψ_1 is a [v]-step then Π starts with an NC-step at root position.

Corollary 1 Let $\Pi : G \rightsquigarrow^+_\theta \top \in \mathcal{NC}$. Then the successive applications of Lemma 1, starting from Π, yields an LNC-refutation $\Psi : G \Rightarrow^+_{\theta'} \square \in \mathcal{LNC}$ such that $\theta' \leq \theta$ $[\mathcal{V}ar(G)]$.

Proof. The result of successive applications of Lemma 1, starting from Π, is depicted in the figure below:

$$\begin{array}{ll}
\Pi = \Pi_0 : & G_0 = G \rightsquigarrow^+_{\theta_0=\theta} \top \\
& \Downarrow_{\sigma_0} \\
\Pi_1 : & G_1 \rightsquigarrow^+_{\theta_1} \top \\
& \vdots \\
\Pi_i : & G_i \rightsquigarrow^+_{\theta_i} \top \\
& \Downarrow_{\sigma_i} \\
\Pi_{i+1} : & G_{i+1} \rightsquigarrow^+_{\theta_{i+1}} \top \\
& \Downarrow_{\sigma_{i+1}} \\
& \vdots
\end{array}$$

Since $\forall i. \Pi_{i+1} \ll \Pi_i$, this process will eventually terminate with an NC-refutation $\Pi_{n+1} : G_{n+1} = \square \rightsquigarrow^0_{\theta_{n+1}=\varepsilon} \top$. The LNC-refutation generated in this way is:

$$\Psi : G_0 \Rightarrow_{\sigma_0} G_1 \Rightarrow_{\sigma_1} \cdots \Rightarrow_{\sigma_n} G_{n+1} = \square$$

According to Lemma 1, $\forall i \in \{1, \ldots, n\}. \sigma_i\theta_{i+1} \leq \theta_i$ $[\mathcal{V}ar(G_i)]$. Then $\theta' = \sigma_0\sigma_1 \ldots \sigma_n$ $= \sigma_0\sigma_1 \ldots \sigma_n\theta_{n+1} \leq \sigma_0\sigma_1 \ldots \sigma_{n-1}\theta_n \leq \ldots \leq \sigma_0\theta_1 \leq \theta_0 = \theta$ $[\mathcal{V}ar(G)]$. ■

3 The LNCA Calculus

LNC does not handle applicative terms efficiently because the applicable inference rules in the case of [o]-steps are determined by the outermost symbol of the term. This symbol is almost always the binary function symbol `ap` which does not impose any restriction on the choice of rewrite rules. We overcome this problem by specializing the inference rules of LNC to look at the head symbol rather than at the outermost symbol of the term under consideration and, if that symbol is an operator, to choose only the rewrite rules which define it.

Definition 3 Let $\mathcal{R}$ be an $\mathcal{A}$TRS. LNCA consists of the following inference rules:

[of] *outermost narrowing for head-function terms*

$$\frac{f\ \mathbf{s}_m\ \mathbf{t}_n \simeq t, E}{s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{t}_n \approx t, E}$$

if there exists a fresh variant $f\ \mathbf{u}_m \to r$ of a rewrite rule in $\mathcal{R}$.

[ov] *outermost narrowing for head-variable terms*

$$\frac{x\ \mathbf{s}_m\ \mathbf{t}_n \simeq t, E}{(s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{t}_n \approx t, E)\theta}$$

if there exists a fresh variant $f\ \mathbf{u}_k\ \mathbf{v}_m \to r$ of a rewrite rule in $\mathcal{R}$, $m > 0$, and $\theta = \{x \mapsto f\ \mathbf{u}_k\}$.

[if] *imitation for head-function terms*

$$\frac{f\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n, E}{(s_1 \approx x_1, \ldots, s_m \approx x_m, t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\theta}$$

if $m > 0$, $\theta = \{x \mapsto f\ \mathbf{x}_m\}$ with $\mathbf{x}_m$ fresh variables.

[iv] *imitation for head-variable terms*

$$\frac{y\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n, E}{(s_1 \approx x_1, \ldots, s_m \approx x_m, t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\theta}$$

if $m > 0$, $x \neq y$ and $\theta = \{x \mapsto y\ \mathbf{x}_m\}$ with $\mathbf{x}_m$ fresh variables.

[df] *decomposition for head-function terms*

$$\frac{f\ \mathbf{s}_n \approx f\ \mathbf{t}_n, E}{s_1 \approx t_1, \ldots, s_n \approx t_n, E}$$

[dv] *decomposition for head-variable terms*

$$\frac{x\ \mathbf{s}_n \approx x\ \mathbf{t}_n, E}{s_1 \approx t_1, \ldots, s_n \approx t_n, E}$$

[vf] *variable-elimination for head-function terms*

$$\frac{f\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n, E}{(t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\theta}$$

if $x \in \mathcal{V} - \mathcal{V}ar(f\ \mathbf{s}_m)$ and $\theta = \{x \mapsto f\ \mathbf{s}_m\}$.

[vv] *variable-elimination for head-variable terms*

$$\frac{y\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n, E}{(t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\theta}$$

if $x \in \mathcal{V} - \mathcal{V}ar(y\ \mathbf{s}_m)$, $y \in \mathcal{V}$ and $\theta = \{x \mapsto y\ \mathbf{s}_m\}$.

We write $G \Rrightarrow_{[\alpha],\theta} G'$ to denote an LNCA-step corresponding to an inference rule $[\alpha]$ with $\alpha \in \{\text{of, ov, if, iv, df, dv, vf, vv}\}$, upper goal G, lower goal G', and involved substitution θ. We assume $\theta = \varepsilon$ when $\alpha \in \{\text{of, df, dv}\}$.

The notions of derivation, refutation, and length of a derivation for LNCA are similar to those for LNC. We denote by $\mathcal{LNCA}$ the class of LNCA-refutations. A [V]-*step* is either a [fv]-step or a [vv]-step of LNCA. An [I]-*step* is either an [if]-step or an [iv]-step of LNCA.

The soundness of LNCA is stated in the following theorem.

Theorem 2 Let $\mathcal{R}$ be a confluent $\mathcal{A}$TRS and G a goal. If there exists an LNCA-refutation $A : G \Rrightarrow_\theta^* \Box$ then θ is a solution of G.

Proof. The proof is performed in two steps. We first prove that for every LNCA-step of the form $G \Rrightarrow_\theta G'$ the following property holds: if θ' is a solution of G' then $\theta\theta'$ is a solution of G. Next, we prove by induction on n that if $G \Rrightarrow_\theta^n \Box$ then θ is a solution of G.

Let $G \Rrightarrow_\theta G'$ be an arbitrary LNCA-step and θ' a solution of G'. We prove that $\theta\theta'$ is a solution of G. The proof is by case distinction on the nature of the LNCA-step from G to G'.

- Assume $G \equiv f\ \mathbf{s}_m\ \mathbf{t}_n \simeq t, E \Rrightarrow_{[\text{of}]} G' \equiv s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{t}_n \approx t, E$ where $f\ \mathbf{u}_m \to r$ is a fresh variant of some rule in $\mathcal{R}$. Then $\theta = \varepsilon$ and $\theta\theta' = \theta'$. We have to prove that θ' is a solution of G. Since θ' is a solution of G', the following conditions hold: (1) $\forall i \in \{1, \ldots, m\}.s_i\theta' \leftrightarrow_{\mathcal{R}}^* u_i\theta'$, (2) $(r\ \mathbf{t}_n)\theta' \leftrightarrow_{\mathcal{R}}^* t\theta'$, (3) θ' is a solution of E. Because of (3), we only have to prove that θ' is a solution of the equation $f\ \mathbf{s}_m\ \mathbf{t}_n \simeq t$. We note that:
$$(f\ \mathbf{s}_m\ \mathbf{t}_n)\theta' \overset{\text{by (1)}}{\longleftrightarrow}{}_{\mathcal{R}}^* (f\ \mathbf{u}_m\ \mathbf{t}_n)\theta' \to_{\mathcal{R}} (r\ \mathbf{t}_n)\theta' \overset{\text{by (2)}}{\longleftrightarrow}{}_{\mathcal{R}}^* t\theta'$$
and hence θ' is solution of the equation $f\ \mathbf{s}_m\ \mathbf{t}_n \simeq t$.
- Assume $G \equiv x\ \mathbf{s}_m\ \mathbf{t}_n \simeq t, E \Rrightarrow_{[\text{ov}],\theta} G' \equiv (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{t}_n \approx t, E)\theta$ where $f\ \mathbf{u}_k\ \mathbf{v}_m \to r$ is a fresh variant of a rewrite rule in $\mathcal{R}$, $m > 0$ and $\theta = \{x \mapsto f\ \mathbf{u}_k\}$. Then (1) $\forall i \in \{1, \ldots, m\}.s_i\theta\theta' \leftrightarrow_{\mathcal{R}}^* v_i\theta\theta'$, (2) $(r\ \mathbf{t}_n)\theta\theta' \leftrightarrow_{\mathcal{R}}^* t\theta\theta'$, (3) $\theta\theta'$ is a solution of E. Because of (3), we only have to prove that $\theta\theta'$ is a solution of the equation $x\ \mathbf{s}_m\ \mathbf{t}_n \simeq t$. We note that:
$$(x\ \mathbf{s}_m\ \mathbf{t}_n)\theta\theta' = (f\ \mathbf{u}_k\ \mathbf{s}_m\ \mathbf{t}_n)\theta\theta' \overset{\text{by (1)}}{\longleftrightarrow}{}_{\mathcal{R}}^* (f\ \mathbf{u}_k\ \mathbf{v}_m\ \mathbf{t}_n)\theta\theta' \to_{\mathcal{R}} (r\ \mathbf{t}_n)\theta\theta' \overset{\text{by (2)}}{\longleftrightarrow}{}_{\mathcal{R}}^* t\theta\theta'$$
and hence $\theta\theta'$ is solution of the equation $x\ \mathbf{s}_m\ \mathbf{t}_n \simeq t$.
- Assume $G \equiv a\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n, E \Rrightarrow_{[\alpha],\theta} G' \equiv (s_1 \approx x_1, \ldots, s_m \approx x_m, t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\theta$, where $a \in \mathcal{V} \cup \mathcal{F}$, $\alpha \in \{\text{if, iv}\}$, $\theta = \{x \mapsto a\ \mathbf{x}_m\}$, with $\mathbf{x}_m$ fresh variables. Then we have: (1) $\forall i \in \{1, \ldots, m\}.s_i\theta\theta' \leftrightarrow_{\mathcal{R}}^* x_i\theta\theta'$, (2) $\forall j \in \{1, \ldots, n\}.t_j\theta\theta' \leftrightarrow_{\mathcal{R}}^* u_j\theta\theta'$, (3) $\theta\theta'$ is a solution of E. Because of (3), we only have to prove that $\theta\theta'$ is solution of $a\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n$. Since:
$$(x\ \mathbf{u}_n)\theta\theta' = (a\ \mathbf{x}_m\ \mathbf{u}_n)\theta\theta' \overset{\text{by (1)}}{\longleftrightarrow}{}_{\mathcal{R}}^* (a\ \mathbf{s}_m\ \mathbf{u}_n)\theta\theta' \overset{\text{by (2)}}{\longleftrightarrow}{}_{\mathcal{R}}^* (a\ \mathbf{s}_m\ \mathbf{t}_n)\theta\theta'$$
the substitution $\theta\theta'$ is a solution of the equation $a\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n$.

- Assume $G \equiv a\ \mathbf{s}_n \approx a\ \mathbf{t}_n, E \Rrightarrow_{[\alpha],\varepsilon} G' \equiv (s_1 \approx t_1, \ldots, s_n \approx t_n, E$ where $a \in \mathcal{V} \cup \mathcal{F}$ and $\alpha \in \{\text{df, dv}\}$. In this case $\theta = \varepsilon$, $\theta\theta' = \theta'$, and we have: (1) $\forall i \in \{1, \ldots, n\}. s_i\theta' \leftrightarrow^*_{\mathcal{R}} t_i\theta'$, (2) θ' is a solution of E. Because of (2), we only have to prove that θ' is a solution of $a\ \mathbf{s}_n \approx a\ \mathbf{t}_n$, which is obvious because of property (1).
- Assume $G \equiv a\ \mathbf{s}_m\ \mathbf{t}_n \approx x\ \mathbf{u}_n, E \Rrightarrow_{[\alpha],\theta} G' \equiv (t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\theta$ where $a \in \mathcal{V} \cup \mathcal{F}$, $\alpha \in \{\text{vf, vv}\}$, $x \notin \mathcal{V}ar(f\ \mathbf{s}_m)$ and $\theta = \{x \mapsto a\ \mathbf{s}_m\}$. Then: (i) $\forall i \in \{1, \ldots, n\}. t_i\theta\theta' \leftrightarrow^*_{\mathcal{R}} u_i\theta\theta'$, (ii) $\theta\theta'$ is solution of G. Because of (2), we only have to prove that $\theta\theta'$ is solution of the equation $f\ \mathbf{s}_m\ \mathbf{t}_n \approx x\ \mathbf{u}_n$. We have:

$$(x\ \mathbf{u}_n)\theta\theta' = (a\ \mathbf{s}_m\ \mathbf{u}_n)\theta\theta' \overset{\text{by (1)}}{\longleftrightarrow^*_{\mathcal{R}}} (a\ \mathbf{s}_m\ \mathbf{t}_n)\theta\theta'$$

We prove now that if $G \Rrightarrow^n_\theta \Box$ then θ is a solution of G. The only possible LNCA-refutations of length $n = 1$ are:

$$\begin{aligned}
&f \approx f \Rrightarrow_{[\text{df}],\varepsilon} \Box, \\
&x \approx x \Rrightarrow_{[\text{dv}],\varepsilon} \Box, \\
&f\ \mathbf{s}_m \simeq x \Rrightarrow_{[\text{vf}],\theta} \Box, \text{ where } x \notin \mathcal{V}ar(f\ \mathbf{s}_m) \text{ and } \theta = \{x \mapsto f\ \mathbf{s}_m\} \\
&y\ \mathbf{s}_m \simeq x \Rrightarrow_{[\text{vv}],\theta} \Box, \text{ where } x \notin \mathcal{V}ar(y\ \mathbf{s}_m) \text{ and } \theta = \{x \mapsto y\ \mathbf{s}_m\}
\end{aligned}$$

Obviously, all these LNCA-refutations are sound. Assume now $A \in \mathcal{LNCA}$ with $|A| = n > 1$. Then we can write $A : G \Rrightarrow_\theta G' \Rrightarrow^+_{\theta'} \Box$ where $\theta\theta' = \sigma$. We want to prove that σ is a solution of G. By the induction hypothesis for $A_{>1}$, θ' is a solution of G'. According to our first proof step, this implies that $\theta\theta'$ is a solution of G. Thus, $\sigma = \theta\theta'$ is a solution of G. ■

4 Completeness

In this section we prove the completeness of LNCA for confluent $\mathcal{A}$TRSs with respect to normalized substitutions. Subsection 4.1 contains an analysis of the structure of LNC-refutations generated from normal NC-refutations as shown in Corollary 1. Based on this analysis, we introduce the class of well-formed LNC-refutations and prove the completeness of LNC with respect to this class. Subsection 4.2 is concerned with the study of LNC-refutations for $\mathcal{A}$TRSs. In Subsection 4.3 we state some properties of well-formed LNC-refutations for $\mathcal{A}$TRSs. Based on these preliminary results, we prove the completeness of LNCA in Subsection 4.4.

4.1 Well-formed LNC-refutations

We first introduce some useful notations. Let: $\Pi : G \rightsquigarrow^* \top \in \mathcal{NC}$ and $e \in G$. We define:

- $\mathcal{P}(e, \Pi)$ the property that narrowing is never applied at positions of the RHS of a descendant of e in Π.
- $E_p(\Pi)$ the longest prefix of G such that $\forall e \in E_p(\Pi).\mathcal{P}(e, \Pi)$.
- $\psi(\Pi)$ the LNC-step constructed from Π as shown in Lemma 1.
- $\pi(\Pi)$ the NC-refutation constructed from Π as shown in Lemma 1.
- $\Psi(\Pi)$ the LNC-refutation constructed from Π as described in Corollary 1.

First we prove the following lemma:

Lemma 2 Let $\Pi : G = e, E \rightsquigarrow_{\theta}^{+} \top \in \mathcal{NC}$ and assume $\pi(\Pi) : G_1 \rightsquigarrow_{\theta'}^{*} \top$. Then the following conditions hold:

1. If $\psi(\Pi)$ is an [i]-step:

$$G \equiv f(s_1, \dots, s_n) \simeq x, E \Rightarrow_{\sigma=\{x \mapsto f(x_1,\dots,x_n)\}} (s_1 \approx x_1, \dots, s_n \approx x_n, E)\sigma$$

   then in $\pi(\Pi)$ narrowing is applied to at least one of the descendants of the equations $s_1\sigma \approx x_1, \dots, s_n\sigma \approx x_n$ at a position of the LHS.
2. If $E_p(\Pi) \neq \Box$ and $\psi(\Pi) : E_p(\Pi), E_i \Rightarrow_\sigma (E', E_i)\sigma$ then:
   (a) If $\psi(\Pi)$ is an [i]-step then it is applied to the LHS of e.
   (b) $E'\sigma = E_p(\pi(\Pi))$.
   (c) if $\psi(\Pi)$ is an [o]-step then it is applied to the LHS of e.

Proof.

1. If $\psi(\Pi)$ is an [i]-step:

$$\begin{aligned}\psi(\Pi) : f(s_1, \dots, s_n) \simeq x, E &\Rightarrow_{[\mathrm{i}],k,\sigma=\{x \mapsto f(x_1,\dots,x_n)\}} \\ s_1\sigma \approx x_1, \dots, s_n\sigma \approx x_n, E\sigma&\end{aligned}$$

   then $\pi(\Pi) : s_1\sigma \approx x_1, \dots, s_n\sigma \approx x_n, E\sigma \rightsquigarrow^{*} \top$. According to Lemma 1, 4.(a), the first NC-step of Π is applied at a position of the form $k{\cdot}j{\cdot}p$ where $1 \leq j \leq n$. Then, according to Lemma 1, 4.(c), $\pi(\Pi)$ narrows a descendant of $s_j\sigma \approx x_j$ at position $1{\cdot}p$, which is a position of the LHS.
2. Since $E_p(\Pi) \neq \Box$ there exists $e \in E_p(\Pi)$.
   (a) Assume $\psi(\Pi)$ is an [i]-step. Then e is of the form $x \simeq f(s_1, \dots, s_n)$ with $x \in \mathcal{V}$ and $n > 0$. We want to prove that the RHS of e is x. If this is not the case then $e \equiv x \approx f(s_1, \dots, s_n)$. By Lemma 1, 4.(a), Π starts with an NC-step at a position of the form $2{\cdot}j{\cdot}p$ in e where $1 \leq j \leq n$. Since $e \in E_p(\Pi)$, this case is impossible and therefore we must have $e \equiv f(s_1, \dots, s_n) \approx x$.
   (b) Let $e' \in E'\sigma$. We have to prove that $e' \in E_p(\pi(\Pi))$, i.e. that the property $\mathcal{P}(e', \pi(\Pi))$ holds. We distinguish two cases:
   (b1) e' is a descendant of e in $\psi(\Pi)$. Then $\psi(\Pi)$ is an [o]-, [d]- or [i]-step. If $\psi(\Pi)$ is an [o]-step then it is applied to the LHS since, by Lemma 1, 3.(a), an [o]-step to the RHS would imply $e \notin E_p(\Pi)$. Therefore, we can write:

$$\psi(\Pi) : \underline{f(s_1, \dots, s_n)} \approx t, E \Rightarrow_{[\mathrm{o}],f(l_1,\dots,l_n)\to r} s_1 \approx l_1, \dots, s_n \approx l_n, r \approx t, E$$

   such that $e' \equiv r \approx t$. If property $\mathcal{P}(r \approx t, \pi(\Pi))$ does not hold then there is a narrowing position to a descendant of $r \approx t$ in $\pi(\Pi))$ of the form $2{\cdot}p$. From Lemma 1, 3.(d) results the existence of a narrowing position of the form $2{\cdot}p$ to a descendant of e in Π. Since this contradicts the condition $e \in E_p(\Pi)$, we deduce that property $\mathcal{P}(e', \pi(\Pi))$ holds.
   If $\psi(\Pi)$ is a [d]-step:

$$f(s_1, \dots, s_n) \approx f(t_1, \dots, t_n), E \Rightarrow_{[\mathrm{d}]} s_1 \approx t_1, \dots, s_n \approx t_n, E$$

   then $e' \equiv s_j \approx t_j$ for some $j \in \{1, \dots, n\}$. Because $e \in E_p(\Pi)$, Π does not perform narrowing at positions of the form $2{\cdot}j{\cdot}p$ to descendants of e. From Lemma 1, 3.(b) we deduce that $\pi(\Pi)$ does not narrow descendants of $s_j \approx t_j$ at positions of the RHS. Thus, $\mathcal{P}(s_i \approx t_i, \pi(\Pi))$ holds.

If $\psi(\Pi)$ is an [i]-step then according to 2.(a) of this lemma e is of the form $f(s_1, \ldots, s_n) \approx x$ with $x \in \mathcal{V}$ and $n > 0$. In this case we can write:

$$f(s_1, \ldots, s_n) \approx x, E \Rightarrow_{[\mathrm{i}], \sigma = \{x \mapsto f(x_1, \ldots, x_n)\}} s_1\sigma \approx x_1, \ldots, s_n\sigma \approx x_n, E\sigma$$

and assume $e' \equiv s_j\sigma \approx x_j$ for some $j \in \{1, \ldots, n\}$. We want to prove that $\pi(\Pi)$ does not narrow descendants of e' at positions of the RHS. If narrowing is applied to a descendant of e' at a position of the RHS then from Lemma 1, 4.(b) we deduce that narrowing is applied to a descendant of e at a position of the RHS. This contradicts our assumption that $e \in E_p(\Pi)$. Therefore, $\mathcal{P}(e', \pi(\Pi))$ must hold.

(b2) e' is not an LNC-descendant of e in $\psi(\Pi)$. Then e' is either a parameter passing equation of e or an LNC-descendant of some $\overline{e} \in E_p(\Pi) \cap E$. The case when e' is a parameter-passing equation of e is covered by Lemma 1, 3.(b). The other case is an immediate consequence of Lemma 1, 1.

Hence $e' \in E'$ implies $e' \in E_p(\sigma(\Pi))$.

If $e' \notin E'$ then e' is a one-step descendant of an equation $\overline{e} \notin E_p(\Pi)$. Then narrowing is applied to the RHS of a descendant of e' in Π and by Lemma 1 1. narrowing is applied to the RHS of e' in $\pi(\Pi)$. Thus, $e' \notin E_p(\sigma(\Pi))$.

(c) Assume $\psi(\Pi)$ is an [o]-step. From $e \in E_p(\Pi)$ and Lemma 1 3.(a) we deduce that $\psi(\Pi)$ is applied to the LHS of e. ■

Lemma 3 Let $\Pi : G \rightsquigarrow_\theta^+ \top \in \mathcal{NC}$.

1. If an [i]-step is applied to a descendant e' of an equation $e \in E_p(\Pi)$ in $\Psi(\Pi)$ then it is applied to the LHS of e'.
2. If an [o]-step is applied to a descendant e' of an equation $e \in E_p(\Pi)$ in $\Psi(\Pi)$ then it is applied to the LHS of e'.

Proof. During the proof we will make use of the following notations:

$\mathcal{P}_{E_p,[\mathrm{i}]}(\Pi)$: If an [i]-step is applied to a descendant e' of an equation $e \in E_p(\Pi)$ in $\Psi(\Pi)$ then it is applied to the LHS of e'.

$\mathcal{P}_{E_p,[\mathrm{o}]}(\Pi)$: If an [o]-step is applied to a descendant e' of an equation $e \in E_p(\Pi)$ in $\Psi(\Pi)$ then it is applied to the LHS of e'.

We prove by induction with respect to the order $\ll$ on $\mathcal{NC}$ that the properties $\mathcal{P}_{E_p,[\mathrm{o}]}(\Pi)$ and $\mathcal{P}_{E_p,[\mathrm{i}]}(\Pi)$ hold.

Let $\Pi_1 = \pi(\Pi)$. Because $\Pi_1 \ll \Pi$, from the induction hypothesis we get that $\mathcal{P}_{E_p,[\mathrm{i}]}(\Pi_1)$ and $\mathcal{P}_{E_p,[\mathrm{o}]}(\Pi_1)$ hold. According to Lemma 2, 2.(b), all one-step descendants of equations of $E_p(\Pi)$ are in $E_p(\Pi_1)$. Then, by the induction hypothesis for Π_1, all [o]-steps to descendants of equations of $E_p(\Pi)$ in $\Psi(\Pi_1) = \Psi(\Pi)_{>1}$ are applied to the LHS. Moreover, if $\psi(\Pi)$ is an [o]-step then, by Lemma 2, 2.(c), $\psi(\Pi)$ is applied to the LHS of e. We conclude that $\mathcal{P}_{E_p,[\mathrm{o}]}(\Pi)$ holds.

It remains to prove that $\mathcal{P}_{E_p,[\mathrm{i}]}(\Pi)$ holds. Assume $e \in E_p(\Pi)$ such that an [i]-step is applied to a descendant of e in $\Psi(\Pi)$. We distinguish two cases:
(i) $\Psi(\Pi)$ starts with an [i]-step to e. Then, by Lemma 2, 2.(a), [i] is applied to the LHS of e.
(ii) [i] is applied in $\Psi(\Pi_1)$ to a descendant of an immediate descendant e' of e in Π. According to Lemma 2, 2.(b), we have $e' \in E_p(\Pi_1)$ and the result follows from the induction hypothesis applied to Π_1. ■

Definition 4 Let $\Psi \in \mathcal{LNC}$. We define:

$\mathcal{P}_{[o]}(\Psi)$: if an [o]-step is applied to a descendant e of a parameter-passing equation then it is applied to the LHS of e.

$\mathcal{P}_{[i]}(\Psi)$: if an [i]-step is applied to a descendant e of a parameter-passing equation then it is applied to the LHS of e.

The following theorem summarizes the main properties of LNC-refutations obtained by lifting normal NC-refutations.

Theorem 3 Let $\Pi : G \rightsquigarrow_\theta^* \top \in \mathcal{NC}$ and $\Psi_0 = \Psi(\Pi)$. Then Ψ_0 satisfies the following properties:

1. If Ψ_0 contains a sub-refutation Ψ' that starts with an [i]-step:

$$\begin{aligned}\Psi' : & f(s_1,\dots,s_n) \simeq x, E \\ & \Rightarrow_{[i],\sigma=\{x\mapsto f(x_1,\dots,x_n)\}} s_1\sigma \approx x_1,\dots,s_n\sigma \approx x_n, E\sigma \Rightarrow_{\theta'}^* \square\end{aligned}$$

   then:
   (a) The first step is of Ψ' not directly followed by n [v]-steps.
   (b) If $x \in \mathcal{V}ar(s_1,\dots,s_n)$ then $x\sigma\theta'$ is normalized.
2. The properties $\mathcal{P}_{[i]}(\Psi_0)$ and $\mathcal{P}_{[o]}(\Psi_0)$ hold.

Proof. If $|\Pi| = 0$ then there is nothing left to prove. Otherwise, we can write $\Pi : G \Rightarrow_\theta^+ \square$. By Corollary 1 we have $\Psi_0 : G \Rightarrow_{\theta'}^* \square$ where $\theta' \le \theta\ [\mathcal{V}ar(G)]$.

Assume that Ψ_0 contains a sub-refutation Ψ' that starts with an [i]-step. Then we have the following situation:

$$\begin{array}{lll}\Pi_0 = \Pi : & G_0 = G & \rightsquigarrow^+ \top \\ & \Downarrow^* & \\ \Pi_k = \pi(\Pi_{k-1}) : & G_k = f(s_1,\dots,s_n) \simeq x, E & \rightsquigarrow^+ \top \\ & \Downarrow_{[i],\sigma} & \\ \Pi_{k+1} = \pi(\Pi_k) : & G_{k+1} = s_1\sigma \approx x_1,\dots,s_n\sigma \approx x_n, E\sigma & \rightsquigarrow^+ \top \\ & \Downarrow^* & \\ & \square & \rightsquigarrow_\varepsilon^0 \top\end{array}$$

where $\sigma = \{x \mapsto f(x_1,\dots,x_n)\}$. Since $\Pi_k \in \mathcal{NC}$, according to Lemma 2, 1., narrowing is applied in Π_{k+1} to at least one of the descendants of the equations $s_1\sigma \approx x_1,\dots,s_n\sigma \approx x_n$ at a position of the LHS. Suppose $s_i\sigma \approx x_i$ is narrowed at a position of the LHS.

Assume now that the first step of Ψ' is followed by n [v]-steps. Then the construction of $\Psi'_{>1}$ is as depicted in the figure below.

$$\begin{array}{lll}\Pi_{k+1} : & G_{k+1} = s_1\sigma \approx x_1,\dots,s_i\sigma \approx x_i,\dots,s_n\sigma \approx x_n, E\sigma & \rightsquigarrow^+ \top \\ & \Downarrow_{[v]}^{i-1} & \\ \Pi_{k+i} : & G_{k+i} = s_i\sigma_i \approx x_i,\dots,s_n\sigma_i \approx x_n, E\sigma_i & \rightsquigarrow^+ \top \\ & \Downarrow_{[v]}^{n-i+1} & \\ \Pi_{k+n+1} & G_{k+n+1} = E\sigma_n & \rightsquigarrow^+ \top \\ & \Downarrow^* & \\ & \square & \end{array}$$

where $\sigma_1 = \sigma\{x_1 \mapsto s_1\sigma\}$, ..., $\sigma_n = \sigma_{n-1}\{x_n \mapsto s_n\sigma_{n-1}\}$. According to Lemma 1, 1., the descendants of the equation $s_i\sigma \approx x_i$ are narrowed at a position of the LHS in Π_{k+1}, ..., Π_{k+i}. Since $\psi(\Pi_{k+i})$ is [v], from Lemma 1, 5. we deduce that Π_{k+i}

starts with a narrowing step at root position. This contradiction proves the validity of condition 1.(a).

We next prove condition 1.(b). Assume that $x \in \mathcal{V}ar(s_1, \ldots, s_n)$. We want to prove that $x\sigma\theta'_1$ is normalized. By Lemma 1, 4.(a), Π_k starts with a step at non-root position. Since Π_k is normal, $\theta\lceil_{\mathcal{V}ar(f(s_1,\ldots,s_n))}$ is normalized. In particular, $x\theta$ is a normal form. Since $\theta' \leq \theta$ $[\mathcal{V}ar(G)]$ and $x \in \mathcal{V}ar(G)$, we deduce that $x\theta$ is an instance of $x\theta'$, and therefore $x\theta'$ is normalized.

We prove now that $\mathcal{P}_{[o]}(\Psi(\Pi))$ and $\mathcal{P}_{[i]}(\Psi(\Pi))$ hold. Let e be a parameter-passing equation in $\Psi(\Pi)$. Then the construction of $\Psi(\Pi)$ from Π looks as follows:

$$
\begin{array}{lll}
\Pi_0 = \Pi : & G_0 = G & \rightsquigarrow^{+} \top \\
& \Downarrow^{*} & \\
\Pi_k = \pi(\Pi_{k-1}) : & G_k = f(l_1, \ldots, l_n) \simeq t, E & \rightsquigarrow^{+} \top \\
& \Downarrow_{[o], f(s_1,\ldots,s_n)\to r} & \\
\Pi_{k+1} = \pi(\Pi_k) : & G_{k+1} = s_1 \approx l_1, \ldots, \underbrace{s_i \approx l_i}_{e}, \ldots, s_n \approx l_n, r \approx t, E & \rightsquigarrow^{+} \top \\
& \Downarrow^{*} & \\
& \square & \rightsquigarrow^{0}_{\varepsilon} \top
\end{array}
$$

By Lemma 1, 3.(b) we have that $s_1 \approx l_1, \ldots, s_n \approx l_n \in E_p(\Pi_{k+1})$. In particular, $e \in E_p(\Pi_{k+1})$. From Lemma 3, 1. for $\Pi_{k+1} \in \mathcal{NC}$ we know that if [i] is applied to a descendant e' of e in $\Psi(\Pi)_{>k+1} = \Psi(\Pi_{k+1})$ then it is applied to the LHS. Hence $\mathcal{P}_{[i]}(\Psi(\Pi))$ holds. Also, from Lemma 3, 2. for $\Pi_{k+1} \in \mathcal{NC}$ we know that if [o] is applied to a descendant e' of e in $\Psi_{>k+1} = \Psi(\Pi_{k+1})$ then it is applied to the LHS of e'. Hence $\mathcal{P}_{[o]}(\Psi(\Pi))$ holds. ■

It is now appropriate to characterize the LNC-refutations generated by Ψ from normal NC-refutations.

Definition 5 (Well-formed LNC-refutation) $\Psi \in \mathcal{LNC}$ is *well-formed* if it satisfies the following properties:

1. If Ψ contains a sub-refutation that starts with an [i]-step

$$
\begin{array}{l}
\Psi' : f(s_1, \ldots, s_n) \simeq x, E \\
\quad \Rightarrow_{[i], \sigma = \{x \mapsto f(x_1,\ldots,x_n)\}} s_1\sigma \approx x_1, \ldots, s_n\sigma \approx x_n, E\sigma \Rightarrow^{*}_{\theta'} \square
\end{array}
$$

   then:
   (a) the first step of Ψ' is not directly followed by n [v]-steps.
   (b) if $x \in \mathcal{V}ar(s_1, \ldots, s_n)$ then $x\sigma\theta'$ is normalized.
2. Properties $\mathcal{P}_{[i]}(\Psi)$ and $\mathcal{P}_{[o]}(\Psi)$ hold.

We denote by $\mathcal{WF}$ the class of well-formed LNC-refutations. An immediate consequence of Theorem 1 and Theorem 3 is:

Corollary 2 For every normalized solution θ of G there exists $\Psi : G \Rightarrow^{*}_{\theta'} \square \in \mathcal{WF}$ with $\theta' \leq \theta$ $[\mathcal{V}ar(G)]$.

At the end of this subsection we state some useful properties of well-formed LNC-refutations.

Lemma 4 Every sub-refutation $\Psi_{>i}$ of $\Psi \in \mathcal{WF}$ is well-formed.

Lemma 5 Let $\Psi \in \mathcal{WF}$ such that $\Psi_{>k} : s_1 \approx u_1, \ldots, s_n \approx u_n, E \Rightarrow^*_\theta \Box$, where $s_1 \approx u_1, \ldots, s_n \approx u_n$ are descendants of parameter-passing equations. Then

$$\forall 1 \leq i \leq n.(s_i\theta \to^* u_i\theta) \tag{2}$$

Proof. Let $\Phi = \Psi_{>k}$. The proof is by induction on $|\Phi|$. If $|\Phi| = 1$ then $n = 1$ and Ψ consists of a [d]-, [v]- or [t]-step. In each of these cases, property (2) holds. Assume now $|\Phi| > 1$. We distinguish the following cases:

- Φ starts with a [v]- or a [t]-step. Then $s_1\theta = u_1\theta$. From the induction hypothesis for $\Phi_{>1}$ we have $s_i\theta \to^* u_i\theta$ if $2 \leq i \leq n$.
- Φ starts with an [o]-step to the LHS. Then $s = f(s'_1, \ldots, s'_k)$ and:

$$\begin{aligned}\Phi_{>1} :\ & s_1 \approx u_1, \ldots, s_n \approx u_n, E \\ & \Rightarrow_{[\mathrm{o}], f(u'_1,\ldots,u'_k)\to r}\ s'_1 \approx u'_1, \ldots, s'_k \approx u'_k, r \approx u_1, s_2 \approx u_2, \ldots, s_n \approx u_n, E \\ & \Rightarrow^*_\theta \Box\end{aligned}$$

  From the induction hypothesis we have $s'_i\theta \to^* u'_i\theta$ $(1 \leq i \leq k)$, $r\theta \to^*_\theta u_1\theta$, $s_j\theta \to^*_\theta u_j\theta$ $(2 \leq j \leq n)$. It remains to prove that $s_1\theta \to^*_\theta u_1\theta$, which is obvious because $s_1\theta = f(s'_1\theta, \ldots, s'_k\theta) \to^* f(u'_1, \ldots, u'_k)\theta \to r\theta \to^* u_1\theta$.
- Φ starts with a [d]-step. Then $s_1 = f(s'_1, \ldots, s'_\ell), u_1 = f(u'_1, \ldots, u'_\ell)$, and:

$$\begin{aligned}\Phi :\ & f(s'_1, \ldots, s'_\ell) \approx f(u'_1, \ldots, u'_\ell), s_2 \approx u_2, \ldots, s_n \approx u_n, E \\ & \Rightarrow_{[\mathrm{d}]}\ s'_1 \approx u'_1, \ldots, s'_\ell \approx u'_\ell, \ldots, s_n \approx u_n, E \Rightarrow^*_\theta \Box\end{aligned}$$

  From the induction hypothesis we have $s'_i\theta \to^* u'_i\theta$ $(1 \leq i \leq \ell)$ and $s_j\theta \to^* u_j\theta$ $(2 \leq j \leq n)$. It remains to prove that $s_1\theta \to^*_\theta u_1\theta$, which is obvious because $s_1\theta = f(s'_1\theta, \ldots, s'_\ell\theta) \to^* f(u'_1, \ldots, u'_\ell)\theta = u_1\theta$.
- Φ starts with an [i]-step. By Lemma 2, 2.(a), Φ is of the form

$$\begin{aligned}\Phi :\ & f(s'_1, \ldots, s'_\ell) \approx u_1, s_2 \approx u_2, \ldots, s_n \approx u_n, E \\ & \Rightarrow_{[\mathrm{i}], \sigma_1 = \{u_1 \mapsto f(x_1,\ldots,x_n)\}}\ s'_1\sigma_1 \approx x_1, \ldots, s'_\ell\sigma_1 \approx x_\ell, \ldots, s_n\sigma_1 \approx u_n\sigma_1, E\sigma_1 \\ & \Rightarrow^*_{\theta'} \Box\end{aligned}$$

  From the induction hypothesis we have $s'_i\theta = s'_i\sigma_1\theta' \to^* x_i\theta'$ $(1 \leq i \leq \ell)$ and $s_j\theta = s'_j\sigma_1\theta' \to^* u_j\sigma_1\theta = u_j\theta$ $(2 \leq j \leq n)$. It remains to prove that $s_1\theta \to^*_\theta u_1\theta$, which is obvious because $s_1\theta = f(s'_1\theta, \ldots, s'_\ell\theta) \to^* f(x_1\theta', \ldots, x_\ell\theta') = f(x_1, \ldots, x_\ell)\theta' = u_1\sigma_1\theta' = u_1\theta$. ■

Note that property 1. of well-formedness is not necessary to prove Lemma 5.

Corollary 3 If $\Psi : \underline{s} \simeq t, E \Rightarrow_{[\mathrm{o}], l\to r} \Rightarrow^*_\theta \Box \in \mathcal{WF}$ then $s\theta$ is not a normal form.

Proof. Ψ can be written as:

$$f(s_1, \ldots, s_n) \simeq t, E \Rightarrow_{[\mathrm{o}], f(l_1,\ldots,l_n)\to r}, s_1 \approx l_1, \ldots, s_n \approx l_n, r \approx t, E.$$

By Lemma 5 we have $\forall i \in \{1, \ldots, n\}.s_i\theta \to^* l_i\theta$. This implies:

$$s\theta = f(s_1, \ldots, s_n)\theta \to^* f(l_1, \ldots, l_n)\theta \to r\theta$$

and hence $s\theta$ is reducible. ■

Lemma 6 Let $\Psi : E_1, s \approx t, E_2 \Rightarrow_\theta^* \Box \in \mathcal{WF}$ such that $s \approx t$ is the n-th equation in the initial goal of Ψ. We denote by $\phi_{swap}(\Psi, n) : E_1, t \approx s, E_2 \Rightarrow_\theta^* \Box$ the LNC-refutation obtained from Ψ by performing the same inference steps in the same order at corresponding positions. Then $\phi_{swap}(\Psi, n)$ is well-formed.

Proof. From the construction of $\phi_{swap}(\Psi, n)$ we see that $\phi_{swap}(\Psi, n)$ verifies condition 1. of well-formedness. The validity of condition 2. of well-formedness for $\phi_{swap}(\Psi, n)$ follows from its validity for Ψ and the observation that, due to the asymmetry of the [o]-rule, the descendants of parameter-passing equations are identical in Ψ and $\phi_{swap}(\Psi, n)$. ∎

In the sequel we confine our attention to the case of applicative term rewriting systems.

4.2 The Structure of LNC-refutations for $\mathcal{A}$TRSs

In this subsection we analyze the structure of LNC-refutations for the particular case of $\mathcal{A}$TRSs. We first introduce the notions of immediate a-descendant and a-descendant of an equation.

Definition 6 (immediate a-descendant) Let $A : G = e, E \Rightarrow G'$ be an LNC inference step.

- If $G \equiv s_1\ s_2 \simeq t, E \Rightarrow_{[\mathrm{o}], l_1\ l_2 \to r} G' \equiv s_1 \approx l_1, s_2 \approx l_2, r \approx t, E$ then $s_1 \approx l_1$ in G' is the only immediate a-descendant of e.
- If $G \equiv f \simeq t, E \Rightarrow_{[\mathrm{o}], f \to r} G' \equiv r \approx t, E$ then there is no immediate a-descendant of e.
- If $G \equiv s_1\ s_2 \simeq x, E \Rightarrow_{[\mathrm{i}], \sigma = \{x \mapsto x_1\ x_2\}} G' \equiv s_1\sigma \approx x_1, s_2\sigma \approx x_2, E\sigma$ then $s_1\sigma \approx x_1$ in G' is the only immediate a-descendant of e.
- If $G \equiv s_1\ s_2 \approx t_1\ t_2, E \Rightarrow_{[\mathrm{d}]} s_1 \approx t_1, s_2 \approx t_2, E$ then $s_1 \approx t_1$ in G' is the only immediate a-descendant of e.
- If A is a [v]- or a [t]-step then e has no immediate a-descendants.

Definition 7 (a-descendant) The relation of *a-descendant* is the reflexive-transitive closure of the relation of immediate a-descendant.

Note the difference between the notions of a-descendant and descendant.

Lemma 7 Let $\Psi : G = s \approx t, E \Rightarrow_\theta^* \Box$. If the first [o]-step of Ψ is applied to an a-descendant of $s \approx t$ then there exists $\Psi' \in \{\Psi, \phi_{swap}(\Psi, 1)\}$ such that:

(i) all [i]-steps before the first [o]-step in Ψ' are applied to the LHS,
(ii) the first [o]-step of Ψ' is applied to the LHS of an a-descendant of $s \approx t$.

Proof. A simple case analysis reveals that if an [o]-step is applied to an a-descendant of $s \approx t$ then A starts with $m \geq 0$ [d]-steps, followed by $p \geq 0$ [i]-steps, followed by an [o]-step.

If $p = 0$ then we can write Ψ in the form:

$$\begin{aligned}\Psi : G &\equiv a\ \mathbf{u}_n\ \mathbf{s}_m \simeq x\ \mathbf{t}_m, E \\ &\Rightarrow_{[\mathrm{d}]}^m \underline{a\ \mathbf{u}_n} \simeq x, s_1 \simeq t_1, \ldots, s_m \simeq t_m, E \Rightarrow_{[\mathrm{o}], k, l \to r} \Rightarrow_{\theta_2}^* \Box\end{aligned}$$

Then $\Psi' = \Psi$ if $k = 1$ and $\phi_{swap}(\Psi, 1)$ if $k = 2$ obviously satisfies conditions (i)-(ii).

If $p > 0$ then we can write:

$$\begin{aligned}\Psi : G \equiv a\ \mathbf{u}_n\ \mathbf{s}_m \simeq x\ \mathbf{t}_m, E \Rightarrow^m_{[\mathrm{d}]} \underline{a\ \mathbf{u}_n} \simeq x, s_1 \simeq t_1, \dots, s_m \simeq t_m, E \\ \Rightarrow^p_{[\mathrm{i}],k,\theta_1} (a\ \mathbf{u}_{n-p} \approx x_{n-p}, u_{n-p+1} \approx x_{n-p+1}, \dots, u_n \approx x_n, \\ s_1 \approx t_1, \dots, s_m \approx t_m, E)\theta_1 \Rightarrow_{[\mathrm{o}],k,l\to r} \Rightarrow^*_{\theta_2} \Box\end{aligned}$$

If the first [i]-step is to the LHS (i.e., $k = 1$) then we can take $\Psi' = \Psi$, otherwise we can take $\Psi' = \phi_{swap}(\Psi, 1)$. ■

Lemma 8 Let $G = f\ \mathbf{s}_m \approx g\ \mathbf{t}_n, E$ such that $f \neq g$ or $m \neq n$. Then for every $A : G \Rightarrow^* \Box \in \mathcal{LNC}$ there exists an application of an [o]-step to an a-descendant of $f\ \mathbf{s}_m \approx g\ \mathbf{t}_n$.

Proof. By induction on $n + m$. Obviously, A starts with an [o]-step or with a [d]-step. If A starts with an [o]-step then there is nothing more to prove. Assume now that A starts with a [d]-step. If $m = 0$ then the only possibility is $n = 0$ and $g = f$. Since this contradicts our hypothesis, we must have $m > 0$. By a similar argument we infer that $n > 0$ and therefore A can be written as:

$$A : G \Rightarrow_{[\mathrm{d}]} f\ \mathbf{s}_{m-1} \approx g\ \mathbf{t}_{n-1}, s_m \approx t_n, E.$$

We can now apply the induction hypothesis to $A_{>1}$ and get the desired result. ■

Lemma 9 Let A be an LNC-refutation $f\ \mathbf{s}_n \approx f\ \mathbf{t}_n, E \Rightarrow^*_\theta \Box$. If there are no [o]-steps applied to a-descendants of $f\ \mathbf{s}_n \approx f\ \mathbf{t}_n$ in A, then A is of the form:

$$A : f\ \mathbf{s}_n \approx f\ \mathbf{t}_n, E \Rightarrow^{n+1}_{[\mathrm{d}]} s_1 \approx t_1, \dots, s_n \approx t_n, E \Rightarrow \Box.$$

Proof. By induction on n. If $n = 0$ then the first step must be [d] and we are done. Suppose $n > 0$. A starts with a [d]-step. Therefore A can be written as $A : f\ \mathbf{s}_n \approx f\ \mathbf{t}_n, E \Rightarrow_{[\mathrm{d}]} f\ \mathbf{s}_{n-1} \approx f\ \mathbf{t}_{n-1}, s_n \approx t_n, E \Rightarrow^*_\theta \Box$ and the conclusion follows from the induction hypothesis for $A_{>1}$. ■

Lemma 10 Let $G = x\ \mathbf{s}_n \simeq g\ \mathbf{t}_m, E$ such that $m < n$. Then for every LNC-refutation $A : G \Rightarrow^* \Box$ there exists an application of an [o]-step to an a-descendant of $x\ \mathbf{s}_n \simeq g\ \mathbf{t}_m$.

Proof. By induction on $n + m > 0$. Since $0 \leq m < n$, A starts either with an [o]-step or with a [d]-step. If A starts with an [o]-step then we are done. If not, then A starts with a [d]-step:

$$A : G \Rightarrow_{[\mathrm{d}]} x\ \mathbf{s}_{n-1} \simeq g\ \mathbf{t}_{m-1}, s_n \simeq t_m, E \Rightarrow^* \Box.$$

From the induction hypothesis for $A_{>1}$ we infer the existence of an [o]-step which is applied to an a-descendant of the immediate a-descendant of $x\ \mathbf{s}_n \simeq g\ \mathbf{t}_m$ in $A_{>1}$, and therefore to an a-descendant of $x\ \mathbf{s}_n \simeq g\ \mathbf{t}_m$ in A. ■

Lemma 11 Let $A : G = f\ \mathbf{s}_m \approx g\ \mathbf{t}_n, E \Rightarrow^* \Box$ such that $m < \mathrm{arity}(f)$ and $n < \mathrm{arity}(g)$. Then $m = n$, $f = g$, and $G \Rightarrow^{m+1}_{[\mathrm{d}]} s_1 \approx t_1, \dots, s_m \approx t_m, E \Rightarrow^* \Box$.

Proof. By induction on $|A|$. If $|A| = 1$ then A must consist of only a [d]-step.This implies $f = g$ and $m = n = 0$. Assume now that $|A| > 1$. We distinguish three cases for the first step in A:

1. $G \Rightarrow_{[\mathrm{d}]} f\ \mathbf{s}_{m-1} \approx g\ \mathbf{t}_{n-1}, s_m \approx t_n, E \Rightarrow^* \Box$.
   From the induction hypothesis for $A_{>1}$ we get $f = g$ and $m-1 = n-1$, and we are done.
2. A starts with an [o]-step. Note that we can not have $m = 0$ in this case because there are no rewrite rules in $\mathcal{R}$ with LHS f. Hence $m > 0$ and we can assume that the first step of A is:
   $\underline{f\ \mathbf{s}_m} \approx g\ \mathbf{t}_n, E \Rightarrow_{[\mathrm{o}],h\ \mathbf{l}_k \to r} f\ \mathbf{s}_{m-1} \approx h\ \mathbf{l}_{k-1}, s_m \approx l_k, r \approx g\ \mathbf{t}_n, E \Rightarrow^* \Box$
   where $k = \mathrm{arity}(h)$. From the induction hypothesis for $A_{>1}$ we get $f = h$ and $m-1 = k-1$. This implies $\mathrm{arity}(f) = \mathrm{arity}(h) = k = m$. This case is impossible because we assume that $m < \mathrm{arity}(f)$.
3. $f\ \mathbf{s}_m \approx \underline{g\ \mathbf{t}_n}, E \Rightarrow_{[\mathrm{o}],h\ \mathbf{l}_k \to r} g\ \mathbf{t}_{n-1} \approx h\ \mathbf{l}_{k-1}, t_n \approx l_k, r \approx f\ \mathbf{s}_m, E \Rightarrow^* \Box$.
   This case is also impossible and the proof similar to the previous one. ■

Lemma 12 Let $A : G = f\ \mathbf{s}_m \approx t, E \Rightarrow^* \Box$ such that it contains an [o]-step which is applied to an a-descendant of $f\ \mathbf{s}_m \approx t$. If the first [o]-step to an a-descendant of $f\ \mathbf{s}_m \approx t$ is applied to the LHS then $m \geq \mathrm{arity}(f)$.

Proof. By induction on $|A|$. If $|A| = 0$ there is nothing more to prove. If $|A| = 1$ then A consists of a [d]- or a [v]-step and the lemma trivially holds. Otherwise we distinguish the following cases for the first step in A:

1. A starts with a [v]-step. Then there are no more a-descendants left.
2. A starts with an [o]-step to the LHS. If $m = 0$ then A this case is possible only if $\mathrm{arity}(f) = 0$ and then we are done. Otherwise $m > 0$ and we have:
   $$A : f\ \mathbf{s}_m \approx t, E \Rightarrow_{[\mathrm{o}],h\ \mathbf{l}_k \to r} f\ \mathbf{s}_{m-1} \approx h\ \mathbf{l}_{k-1}, s_m \approx l_k, r \approx t, E \Rightarrow^* \Box$$
   where $\mathrm{arity}(h) = k > 0$. If $m < \mathrm{arity}(f)$, then by Lemma 11 we must have $h = f$ and $m-1 = k-1$. But this implies $m = k = \mathrm{arity}(h) = \mathrm{arity}(f)$, which contradicts the assumption that $m < \mathrm{arity}(f)$. Thus, $m \geq \mathrm{arity}(f)$.
3. A starts with a [d]-step. If $m = 0$ then $t \equiv f$ and there are no a-descendants left. If $m > 0$ then A can be written as:
   $$A : f\ \mathbf{s}_m \approx g\ \mathbf{t}_n, E \Rightarrow_{[\mathrm{d}]} f\ \mathbf{s}_{m-1} \approx g\ \mathbf{t}_{n-1}, s_m \approx t_n, E \Rightarrow^* \Box.$$
   From the induction hypothesis for $A_{>1}$ we deduce $m-1 \geq \mathrm{arity}(f)$, and hence $m \geq \mathrm{arity}(f)$.
4. A starts with an [i]-step. Then $m > 0$ and A has the form:
   $$A : f\ \mathbf{s}_m \approx x, E \Rightarrow_{[\mathrm{i}],\sigma=\{x \mapsto x_1\ x_2\}} (f\ \mathbf{s}_{m-1} \approx x_1, s_m \approx x_2, E)\sigma \Rightarrow^* \Box$$
   The induction hypothesis for $A_{>1}$ yields immediately the desired result. ■

Lemma 13 Let $A : \underline{f\ \mathbf{s}_n} \approx t, E \Rightarrow_{[\mathrm{o}],l \to r} \Rightarrow^* \Box$ where $n = \mathrm{arity}(f)$. Then l has the form $f\ \mathbf{l}_n$.

Proof. If $n = 0$ then $l \equiv f$. Otherwise l is of the form $h\ \mathbf{l}_k$ with $\mathrm{arity}(h) = k > 0$ and A is of the form $A : f\ \mathbf{s}_n \approx t, E \Rightarrow_{[\mathrm{o}],h\ \mathbf{l}_k \to r} f\ \mathbf{s}_{n-1} \approx h\ \mathbf{l}_{k-1}, s_n \approx l_k, r \approx t, E \Rightarrow^* \Box$. From Lemma 11 for $A_{>1}$ we have $f = h$ and $n = k$. ■

Lemma 14 Let $A : \underline{x\ \mathbf{s}_n} \approx t, E \Rightarrow_{[\mathrm{o}],f\ \mathbf{l}_k \to r} \Rightarrow^* \Box$ such that $n > 0$ and [o] is never applied to an a-descendant of $x\ \mathbf{s}_n \approx t$ in $A_{>1}$ .Then $k \geq n$.

Proof. Without loss of generality, A can be written as:

$$\begin{aligned}\underline{x\ \mathbf{s}_n} \approx t, E &\Rightarrow_{[\mathrm{o}], f\ \mathbf{l}_k \to r} x\ \mathbf{s}_{n-1} \approx f\ \mathbf{l}_{k-1}, s_n \approx l_k, r \approx t, E \\ &\Rightarrow^{n-1}_{[\mathrm{d}]} x \approx f\ \mathbf{l}_j, s_1 \approx l_{j+1}, \ldots, s_n \approx l_k, r \approx t, E \Rightarrow^* \Box\end{aligned}$$

such that $j \geq 0$ and $k = j + n$. Then $k \geq n$. ■

Lemma 15 If $A \in \mathcal{LNC}$ is of the form

$$A : G = a\ \mathbf{t}_n \approx x, E \Rightarrow_{[\mathrm{v}], \sigma = \{x \mapsto a\ \mathbf{t}_n\}} E\sigma \Rightarrow^*_\theta \Box$$

where $n > 0$ then there exists a refutation:

$$\begin{aligned}A' : G = a\ \mathbf{t}_n \approx x, E &\Rightarrow_{[\mathrm{i}], \sigma_1 = \{x \mapsto x_1\ x_2\}} (a\ \mathbf{t}_{n-1} \approx x_1, t_n \approx x_2, E)\sigma_1 \\ &\Rightarrow_{[\mathrm{v}], \sigma_2 = \{x_1 \mapsto a\ \mathbf{t}_{n-1}\}} (t_n \approx x_2, E)\sigma_1\sigma_2 \\ &\Rrightarrow_{[\mathrm{V}], \sigma_3 = \{x_2 \mapsto t_n\}} E\sigma_1\sigma_2\sigma_3 \Rightarrow^*_\theta \Box\end{aligned}$$

such that $A_{>1} = A'_{>3}$ and $\sigma_1\sigma_2\sigma_3 \restriction_{\mathcal{V}ar(G)} = \sigma$.

Proof. From the applicability of [v] to the equation $a\ \mathbf{t}_n \approx x$ we infer that $x \notin \mathcal{V}ar(a\ \mathbf{t}_n)$. Let x_1, x_2 be fresh variables and $\sigma_3 = \{x_2 \mapsto t_n\}$. Because x, x_1, $x_2 \notin \mathcal{V}ar(a\ \mathbf{t}_n)$ we can construct an LNC-derivation:

$$\overline{A} : G \Rightarrow_{[\mathrm{i}], \sigma_1} a\ \mathbf{t}_{n-1} \approx x_1, t_n \approx x_2, E\sigma_1 \Rightarrow_{[\mathrm{v}], \sigma_2} t_n \approx x_2, E\sigma_1\sigma_2 \Rightarrow_{[\mathrm{v}], \sigma_3} E\sigma_1\sigma_2\sigma_3$$

Let $G_1 = t_n \approx x_2, E\sigma_1\sigma_2$. Note that we can apply a [V]-step (with $n = 0$) to G_1:

$$G_1 \Rrightarrow_{[\mathrm{V}], \sigma_3} E\sigma_1\sigma_2\sigma_3$$

We have $\sigma_1\sigma_2\sigma_3 = \{x \mapsto a\ \mathbf{t}_n, x_1 \mapsto a\ \mathbf{t}_{n-1}, x_2 \mapsto t_n\}$. Then $\sigma_1\sigma_2\sigma_3 \restriction_{\mathcal{V}ar(G)} = \sigma$ because $x_1, x_2 \notin \mathcal{V}ar(G)$. Since $\mathcal{V}ar(E) \subseteq \mathcal{V}ar(G)$, we have $E\sigma_1\sigma_2\sigma_3 = E\sigma$. Therefore, we can replace the second [v]-step of $\overline{A}$ with a [V]-step and obtain the (mixed) refutation A'. ■

4.3 The Structure of Well-formed LNC-refutations for $\mathcal{A}$TRSs

Lemma 16 Let $A : G = a\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n, E \Rightarrow^*_\theta \Box \in \mathcal{WF}$. If [o] is never applied to an a-descendant of $a\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n$, then there exists an LNCA-derivation $B : G \Rrightarrow^*_\sigma G_1$ and $A' : G_1 \Rightarrow^*_{\theta'} \Box \in \mathcal{WF}$ such that $|A'| < |A|$ and $\theta = \sigma\theta'$ $[\mathcal{V}ar(G)]$.

Proof. Because [o] is never applied to a-descendants of $a\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n$ of G, the first n LNC-steps of A must be [d]-steps. Hence, A can be written as:

$$A : G = a\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n, E \Rightarrow^n_{[\mathrm{d}]} a\ \mathbf{s}_m \simeq x, t_1 \simeq u_1, \ldots, t_n \simeq u_n, E \Rightarrow^*_\theta \Box$$

We distinguish the following situations:

(1) $\underline{a = x.}$ We prove that in this case we must have $m = 0$. Assume $m \neq 0$. Then the only possibility is to start $A_{>n}$ with an [i]-step. We note that in this case all the subsequent LNC-steps are [i]-steps and A is non-terminating. Therefore $m = 0$ and the first step of $A_{>n}$ is a [t]-step:

$$A_{>n} : x \approx x, t_1 \simeq u_1, \ldots, t_n \simeq u_n, E \Rightarrow_{[t]} t_1 \simeq u_1, \ldots, t_n \simeq u_n, E \Rightarrow^*_\theta \Box$$

We note that we can replace the first $n+1$ LNC-steps of A with a [dv]-step:

$$B : G \Rrightarrow_{[\mathrm{dv}]} t_1 \simeq u_1, \ldots, t_n \simeq u_n, E \Rightarrow^*_{\theta} \Box$$

We can take $A' = A_{>n+1}$ with $\theta' = \theta$, $\sigma = \varepsilon$.

(2) $\underline{a \neq x}$. Then the only possibility is to start $A_{>n}$ with a sequence of i [i]-steps, where $0 \leq i \leq m$, followed by a [v]-step. There are two possibilities:

(2a) $\underline{i = 0}$. In this case $A_{>n}$ can be written as:

$$a\ \mathbf{s}_m \simeq x, t_1 \simeq u_1, \ldots, t_n \simeq u_n, E \Rightarrow_{[\mathrm{v}],\sigma} (t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\sigma \Rightarrow_{\theta'} \Box$$

with $\sigma = \{x \mapsto a\ \mathbf{s}_m\}$. This implies that $x \notin \mathcal{V}ar(a\ \mathbf{s}_m)$ and therefore we can perform the [V]-step:

$$B : G \Rrightarrow_{[\mathrm{V}],\sigma} (t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\sigma$$

In this case we can choose $G_1 = (t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\sigma$ and $A' = A_{>n+1}$.

(2b) $\underline{i > 0}$. In this case $A_{>n}$ can be written as:

$$\begin{aligned} A_{>n} : &\ a\ \mathbf{s}_m \simeq x, t_1 \simeq u_1, \ldots, t_n \simeq u_n, E \\ &\Rightarrow^i_{[\mathrm{i}],\sigma_1\ldots\sigma_i} (a\ \mathbf{s}_{m-i} \approx x'_{m-i+1}, s_{m-i+1} \approx x_{m-i+1}, \ldots, s_m \approx x_m, \\ &\qquad t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\sigma_1 \ldots \sigma_i \\ &\Rightarrow_{[\mathrm{v}],\sigma'_i} G_1 = (s_{m-i+1} \approx x_{m-i+1}, \ldots, s_m \approx x_m, \\ &\qquad\qquad t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\sigma_1 \ldots \sigma_i \sigma'_i \Rightarrow^*_{\theta'} \Box \end{aligned}$$

where $\sigma_1 = \{x \mapsto x'_m\ x_m\}, \ldots, \sigma_i = \{x'_{m-i+2} \mapsto x'_{m-i+1}\ x_{m-i+1}\}$ and $\sigma'_i = \{x'_{m-i+1} \mapsto a\ \mathbf{s}_{m-i}\}$ with $x_{m-i+1}, x'_{m-i+1}, \ldots, x_m, x'_m \in \mathcal{V} - \mathcal{V}ar(G_1)$ fresh variables. By applying Lemma 15 $m-i$ times to the first [v]-step of A, we obtain:

$$\begin{aligned} A'' :\ G &\Rightarrow^n_{[\mathrm{d}]} a\ \mathbf{s}_m \simeq x, t_1 \simeq u_1, \ldots, t_n \simeq u_n, E \\ &\Rightarrow^{i+(m-i)}_{[\mathrm{i}],\sigma_1\ldots\sigma_m} (a \approx x'_1, s_1 \approx x_1, \ldots, s_m \approx x_m, \\ &\qquad t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\sigma_1 \ldots \sigma_m \\ &\Rightarrow_{[\mathrm{v}],\sigma'_m} (s_1 \approx x_1, \ldots, s_m \approx x_m, \\ &\qquad t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\sigma_1 \ldots \sigma_m \sigma'_m \\ &\Rrightarrow^{m-i}_{[\mathrm{V}],\sigma} G_1 = (s_{m-i+1} \approx x_{m-i+1}, \ldots, s_m \approx x_m, \\ &\qquad t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\sigma_1 \ldots \sigma_i \sigma'_i \\ &\Rightarrow_{\theta'} \Box \end{aligned}$$

where $\sigma' = \{x_1 \mapsto s_1, \ldots, x_{m-i} \mapsto s_{m-i}\}$, $\sigma_1 \ldots \sigma_m \sigma'_m \sigma\restriction_{\mathcal{V}ar(G)} = \sigma_1 \ldots \sigma_i \sigma'_i$, and $A''_{>n+m+1+m-i} = A_{>n+i+1}$. We have $\sigma_1 \ldots \sigma_m \sigma'_m \restriction_{\mathcal{V}ar(G)} = \{x \mapsto a\ \mathbf{x}_n\}$. We let $\rho = \{x \mapsto a\ \mathbf{x}_n\}$ and the LNCA step:

$$a\ \mathbf{s}_m\ \mathbf{t}_n \simeq x\ \mathbf{u}_n, E \Rrightarrow_{[\mathrm{I}],\rho} (s_1 \approx x_1, \ldots, s_m \approx x_m, t_1 \simeq u_1, \ldots, t_n \simeq u_n, E)\rho$$

replace the first $n+m+1$ LNC-steps of A''. Now we can choose:

$$B : G \Rrightarrow_{[\mathrm{I}],\rho} \Rrightarrow^{m-i}_{[\mathrm{V}],\sigma'} G_1$$

and $A' = A_{>n+i+1}$. By 2., we have $\theta = \rho\sigma'\theta'\ [\mathcal{V}ar(G)]$. ■

The following lemma is of importance when lifting a well-formed LNC-refutation to an LNCA-refutation requires the introduction of an [i]-step.

Lemma 17 Let $A \in \mathcal{WF}$ be of the form $A : G = (E_1, r \approx x_1, s \approx x_2, E_2)\sigma \Rightarrow^*_\theta \Box$, where $\sigma = \{x \mapsto x_1\ x_2\}$, such that:

(i) $x_1, x_2 \in \mathcal{V} - \mathcal{V}ar(r, s, x, E_1, E_2)$,
(ii) if $x \in \mathcal{V}ar(E_1, r, s)$ then $x\sigma\theta$ is normalized.

Then there exists $A' \in \mathcal{WF}$ of the form: $A' : G' = (E_1, r\ s \approx x, E_2) \Rightarrow^*_{\theta'} \Box$ such that $\sigma\theta = \theta'$ and $|A'| \leq |A| + 1$.

Proof. We distinguish two cases:

1. A has a sub-refutation $(x_1\ x_2 \simeq t, E_1', r \approx x_1, s \approx x_2, E_2)\sigma\theta_1 \Rightarrow^* \Box$ which does not start with a [v]- or an [o]-step applied to t,
2. A does not have such a sub-refutation.

First we prove case 1. Let

$$A'' = A_{>i_1} : G_{i_1} = (x_1\ x_2 \simeq t, E_1', r \approx x_1, s \approx x_2, E_2)\sigma\theta_1 \Rightarrow^* \Box$$

be the longest sub-refutation of A which does not start with an [o]- or a [v]-step. Then obviously $x \in \mathcal{V}ar(E_1)$ and, according to our hypothesis, $x\sigma\theta$ is normalized. This implies that $(x_1\ x_2)\theta$ is a normal form.

We note that the only way a term of the form $x_1\ x_2$ is decomposed in the sub-derivation $B : G \Rightarrow^{i_1}_{\theta_1} G_{i_1}$ of A is by applying a [d]-, [i]- or [o]-step to an equation of the form $x_1\ x_2 \simeq w$ where w is any term. From the definition of A'' we deduce that such steps do not appear in B and therefore the following conditions hold:

- $x_1, x_2 \notin \mathcal{D}(\theta_1)$,
- if x_1 and x_2 appear in $\mathcal{I}m(\theta_1)$ then they appear in subterms of the form $x_1\ x_2$.

As a consequence $(x_1\ x_2)\sigma\theta_1 = x_1\ x_2$. Because $A \in \mathcal{WF}$, from Corollary 3 we obtain by contraposition that A'' does not start with an [o]-step applied to $x_1\ x_2$. If A'' starts with an [i]-step then we must have $t\sigma\theta_1 = z \in \mathcal{V}$ and A'' is of the form:

$$A'' : x_1\ x_2 \simeq z, E' \Rightarrow_{[\mathrm{i}],\sigma'} x_1 \approx y_1, x_2 \approx y_2, E'\sigma' \Rightarrow^2_{[\mathrm{v}],\sigma''} G''\sigma'\sigma'' \Rightarrow^* \Box$$

with $G'' = (E_1', r \approx x_1, s \approx x_2, E_2)\sigma\theta_1$ and $\sigma' = \{z \mapsto y_1\ y_2\}$. Since this contradicts the assumption that A'' is well-formed, we have that A'' does not start with an [i]-step. Hence the next step of A must be a [d]-step. In this case $t\sigma\theta_1 = v_1\ v_2$ for some terms v_1, v_2 an E_1 is of the form $E_0', x \simeq t, E_1'$ such that we can write:

$$\begin{aligned}
A : G &\equiv (E_0', x \simeq t, E_1', r \approx x_1, s \approx x_2, E_2)\sigma \\
&\Rightarrow^{i_1}_{\theta_1} G_{i_1} = x_1\ x_2 \simeq v_1\ v_2, (E_1', r \approx x_1, s \approx x_2, E_2)\sigma\theta_1 \\
&\Rightarrow_{[\mathrm{d}]} x_1\ \simeq v_1, x_2 \simeq v_2, (E_1', r \approx x_1, s \approx x_2, E_2)\sigma\theta_1 \\
&\Rightarrow^{i_2}_{\theta_2} (E_1', r \approx x_1, s \approx x_2, E_2)\sigma\theta_1\theta_2 \\
&\Rightarrow^{i_3}_{\theta_3} G_{i_1+i_2+i_3+1} = (r \approx x_1, s \approx x_2, E_2)\sigma\theta_1\theta_2\theta_3 \\
&\Rightarrow^{i_4}_{\theta_4} \Box
\end{aligned}$$

Starting from A, we construct A' as follows. Let $B_4 \in \mathcal{LNC}$ be of the form:

$$B_4 : (r\ s \approx x_1\ x_2, E_2)\sigma\theta_1\theta_2\theta_3 \Rightarrow_{[\mathrm{d}]} G_{i_1+i_2+i_3+1} \Rightarrow^{i_4}_{\theta_4} \Box$$

such that $(B_4)_{>1} = A_{>(i_1+i_2+i_3+1)}$. Then $B_4 \in \mathcal{WF}$ because $A_{>(i_1+i_2+i_3+1)} \in \mathcal{WF}$. Let $B_3' \in \mathcal{LNC}$ be of the form:

$$B_3' : x_1 \simeq v_1, x_2 \simeq v_2, (E_1', r\ s \approx x_1\ x_2, E_2)\sigma\theta_1 \Rightarrow^{i_2+i_3+i_4+1}_{\theta_2\theta_3\theta_4} \Box$$

such that:

– The first $i_2 + i_3$ steps of B'_3 coincide with the first $i_2 + i_3$ steps of $A_{>i_1+1}$,
– $(B'_3)_{>(i_2+i_3)} = B_4$.

Then $B'_3 \in \mathcal{WF}$. From B'_3 we construct

$$B_3 : v_1 \approx x_1, v_2 \approx x_2, (E'_1, r\ s \approx x_1\ x_2, E_2)\sigma\theta_1 \Rightarrow^{i_2+i_3+i_4+1}_{\theta_2\theta_3\theta_4} \square$$

by permuting, if necessary, the sides of the first two equations and applying the same inference steps in the same order at corresponding positions. Since $B'_3 \in \mathcal{WF}$, from Lemma 6 we obtain that $B_3 \in \mathcal{WF}$.

Since $x_1, x_2 \notin E_1$ we have that $x_1, x_2 \notin E'_1$. We already noticed that $x_1, x_2 \notin \mathcal{D}(\theta_1)$ and if x_1 and x_2 appear in $\mathcal{I}m(\theta_1)$ then they appear in subterms of the form $x_1\ x_2$. Therefore we can remove all occurrences of x_1 and x_2 from $\mathcal{I}m(\theta_1)$ by replacing all the occurrences of $x_1\ x_2$ by x. Assume that by this transformation we obtain δ_1 from θ_1. Because $\sigma\theta_1 = \delta_1\sigma$ we can consider the LNC refutation

$$\begin{aligned} B'_2 : G'' &= (x \approx t, E'_1, r\ s \approx x, E_2)\delta_1 \\ &\Rightarrow_{[\mathrm{i}],\sigma} v_1 \approx x_1, v_2 \approx x_2, (E'_1, r\ s \approx x_1\ x_2, E_2)\sigma\theta_1 \\ &\Rightarrow^{i_2+i_3+i_4+1}_{\theta_2\theta_3\theta_4} \square \end{aligned}$$

where $(B'_2)_{>1} = B_3$. Then $x\delta_1\sigma\theta_2\theta_3\theta_4 = x\sigma\theta_1\theta_2\theta_3\theta_4 = x\sigma\theta$ is normalized. The only case when $B'_2 \notin \mathcal{WF}$ is where the first [i]-step is followed by two [v]-steps. In this case we define B_2 as the LNC-refutation obtained from B'_2 by replacing the first three steps by a [v]-step. Otherwise we assume $B_2 = B'_2$. Then $B_2 \in \mathcal{WF}$ and $|B_2| \leq i_2 + i_3 + i_4 + 2$. We finally define:

$$\begin{aligned} A' : G' &= (E_1, r\ s \approx x, E_2) \\ &\Rightarrow^{i_1}_{\delta_1} G'' = (x \approx v_1\ v_2, E''_1, r\ s \approx x, E_2)\delta_1 \\ &\Rightarrow^{\leq i_2+i_3+i_4+2}_{\sigma\theta_2\theta_3\theta_4} \square \end{aligned}$$

where the first i_1 steps coincide with those of A and are applied in the same order at the same positions, and $A'_{>i_1} = B_2$. Then $A' \in \mathcal{WF}$ and $|A'| = i_1 + |B_2| \leq i_1 + i_2 + i_3 + i_4 + 2 = |A| + 1$.

We now prove case 2. In this case A can be written as:

$$A : G \equiv (E_1, r \approx x_1, s \approx x_2, E_2)\sigma \Rightarrow^{i_1}_{\theta_1} G_{i_1} = (r \approx x_1, s \approx x_2, E_2)\sigma\theta_1 \Rightarrow^{i_2}_{\theta_2} \square$$

such that $x_1, x_2 \notin \mathcal{D}(\sigma\theta_1)$. From A we construct the LNC-refutation B as follows:

$$\begin{array}{l} A : G \Rightarrow^{i_1}_{\theta_1} (r \approx x_1, s \approx x_2, E_2)\sigma\theta_1 \Rightarrow^{i_2}_{\theta_2} \square \\ \qquad\qquad\qquad\qquad \Downarrow_{[\mathrm{i}],\sigma} \\ B : G \Rightarrow^{i_1}_{\delta_1} G'_{i_1} = (E_1, r\ s \approx x)\delta_1 \end{array}$$

where δ_1 is defined like in case 1, the first i_1 steps of B coincide with the first i_1 steps of A and are applied in the same order and at the same positions, and $B_{>i_1+1} = A_{>i_1}$. If $A_{>i_1}$ starts with two [v]-steps then we define A' to be B in which the sequence $\Rightarrow_{[\mathrm{i}]}\Rightarrow_{[\mathrm{v}]}\Rightarrow_{[v]}$ of steps to G'_{i_1} is replaced by a [v]-step. Otherwise $A' = B$. Then $A' \in \mathcal{WF}$ and $|A'| \leq i_1 + i_2 + 1 = |A| + 1$. ■

Lemma 18 Let $A : f\ \mathbf{s}_n \approx t, E \Rightarrow^*_\theta \square \in \mathcal{WF}$ such that:

(i) There exists a first [o]-step of A which is applied to the LHS of an a-descendant of $f\ \mathbf{s}_n \approx t$,

(ii) All [i]-steps before the first [o]-step are applied to the LHS.

Then there exists a fresh variant $f\ \mathbf{u}_m \to r$ of a rewrite rule such that:

(a) The last [o]-step to an a-descendant of $f\ \mathbf{s}_n \approx t$ is of the form

$$A_1 : (f\ \mathbf{s}_m \approx t')\sigma, E' \Rightarrow_{f\ \mathbf{u}_m \to r} (f\ \mathbf{s}_{m-1} \approx f\ \mathbf{u}_{m-1}, s_m \approx u_m, r\ \mathbf{s}_{m+1,n} \approx t', E')\sigma$$

where $m \leq n$,

(b) There exists $A' \in \mathcal{WF}$ of the form

$$A' : s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n} \approx t, E \Rightarrow^*_\theta \Box$$

such that $|A'| < |A|$.

Proof. We first prove (a). Because of (ii), the a-descendant e' of $f\ \mathbf{s}_m \approx t$ to which the first [o]-step is applied is of the form $e' = (f\ \mathbf{s}_p \approx t'')\sigma'$ where $p \leq n$. Since the a-descendants of e' are descendants of parameter-passing equations and the conditions $\mathcal{P}_{[o]}(A)$ and $\mathcal{P}_{[i]}(A)$ of well-formedness hold, we deduce that the a-descendant of e to which the last [o]-step is applied is of the form $(f\ \mathbf{s}_m \approx t')\sigma$ where $m \leq p \leq n$. It remains to prove that the rewrite rule variant employed in this last [o]-step is of the form $f\ \mathbf{u}_m \to r$. By Lemma 13, it suffices to prove that $m = \text{arity}(f)$. By Lemma 12 for the well-formed sub-refutation A'' of A starting with A_1 we have $m \geq \text{arity}(f)$. If $m = 0$ then also $\text{arity}(f) = 0$ and there is nothing more to prove. If $m > 0$ then we can write A'' as follows:

$$(f\ \mathbf{s}_m \approx t')\sigma, E' \Rightarrow_{h\ \mathbf{u}_k \to r} (f\ \mathbf{s}_{m-1} \approx h\ \mathbf{u}_{k-1}, s_m \approx u_k, r\ \mathbf{s}_{m+1,n} \approx t', E')\sigma \Rightarrow^* \Box$$

where $k = \text{arity}(h) > 0$. Since $A''_{>1}$ does not contain [o]-steps applied to a-descendants of $f\ \mathbf{s}_{m-1} \approx h\ \mathbf{u}_{k-1}$, we can apply Lemma 8 applied to $A''_{>1}$ and obtain by contraposition that $f = h$ and $m - 1 = k - 1$. Hence $\text{arity}(f) = \text{arity}(h) = k = m$.

We prove now (b). We prove by induction on $n - m$ the existence a well-formed LNC-refutation

$$A' : s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n} \approx t, E \Rightarrow^*_\theta \Box$$

that in addition to $|A'| < |A|$ it also satisfies the condition:

$\mathcal{C}(A, A')$: If $A : e, E \Rightarrow^* \Box$ then for every $e' \in E$ the following implication holds: if [o] is never applied to the RHS of the descendants of e' in A then [o] is never applied to the RHS of the descendants of e' in A'.

This condition is used in the proof of case 2., where we construct an LNC-refutation with a new parameter-passing equation.

Case I. Assume $n = m$. Then we distinguish two subcases:

(a) $\underline{m = 0}$. Because property (a) holds, A is of the form $A : \underline{f} \approx t, E \Rightarrow_{[o], f \to r} r \approx t, E \Rightarrow^*_\theta \Box$ and we can take $A' = A_{>1}$. Obviously, $\mathcal{C}(A_{>1}, A')$ implies $\mathcal{C}(A, A')$.

(b) $\underline{m > 0}$. Because of property (a), the first LNC-step of A coincides with the last [o]-step to an a-descendant of $f\ \mathbf{s}_m \approx t$. Therefore, we can write:

$$A : \underline{f\ \mathbf{s}_m} \approx t, E \Rightarrow_{[o], f\ \mathbf{u}_m \to r} f\ \mathbf{s}_{m-1} \approx f\ \mathbf{u}_{m-1}, s_m \approx u_m, r \approx t, E \Rightarrow^*_\theta \Box$$

From Lemma 11 we know that $A_{>1}$ contains a sub-refutation:

$$A' : s_1 \approx u_1, \ldots, s_m \approx u_m, r \approx t, E \Rightarrow^*_\theta \Box$$

such that $|A'| \leq |A_{>1}| < |A|$ and $\mathcal{C}(A_{>1}, A')$. Also, $\mathcal{C}(A_{>1}, A')$ implies $\mathcal{C}(A, A')$.

Case II. Assume $n > m$. We distinguish the following situations:

1. A starts with a [d]-step. Then $t \equiv t_1\ t_2$ for some terms t_1, t_2 and we have:

$$A : f\ \mathbf{s}_n \approx t_1\ t_2, E \Rightarrow_{[\mathrm{d}]} f\ \mathbf{s}_{n-1} \approx t_1, s_n \approx t_2, E \Rightarrow^*_\theta \Box$$

Since $A_{>1}$ has properties (i) and (ii), from the induction hypothesis we infer the existence of $B \in \mathcal{WF}$ of the form:

$$B : G'_1 = (s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n-1} \approx t_1, s_n \approx t_2, E) \Rightarrow^*_\theta \Box$$

such that $|B| < |A_{>1}|$ and $\mathcal{C}(A_{>1}, B)$ holds. Also, $\mathcal{C}(A_{>1}, B)$ implies $\mathcal{C}(A, B)$. Let $B_{>i}$ be the sub-refutation of B such that

$$B_{>i} : G_2 = (r\ \mathbf{s}_{m+1,n-1} \approx t_1, s_n \approx t_2, E)\theta_1 \Rightarrow^j_{\theta_2} \Box$$

We construct the LNC-refutation

$$\begin{aligned} A' : G_1 &= s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n} \approx t_1\ t_2, E \\ &\Rightarrow^i_{\theta_1} G'_2 = (r\ \mathbf{s}_{m+1,n} \approx t_1\ t_2, E)\theta_1 \\ &\Rightarrow_{[\mathrm{d}]} G_2 = (r\ \mathbf{s}_{m+1,n-1} \approx t_1, s_n \approx t_2, E)\theta_1 \Rightarrow^j_{\theta_2} \Box \end{aligned}$$

where the first i steps of A' coincide with those of B and $A'_{>(i+1)} = B_{>i}$. Then $A' \in \mathcal{WF}$ and $|A'| = i + j + 1 = |B| + 1 < |A_{>1}| + 1 = |A|$. Since $B \in \mathcal{WF}$, we deduce that $A' \in \mathcal{WF}$. Moreover, $\mathcal{C}(A, B)$ implies $\mathcal{C}(A, A')$.
2. A starts with an [o]-step to the LHS. Then A is of the form:

$$A : f\ \mathbf{s}_n \approx t, E \Rightarrow_{[\mathrm{o}], h\ \mathbf{v}_k \to r'} f\ \mathbf{s}_{n-1} \approx h\ \mathbf{v}_{k-1}, s_n \approx v_k, r' \approx t, E \Rightarrow^*_\theta \Box$$

where $k = \mathrm{arity}(h) > 0$. From the induction hypothesis for $A_{>1}$, there exists $B \in \mathcal{WF}$ of the form

$$B : G'_1 = s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n-1} \approx h\ \mathbf{v}_{k-1}, s_n \approx v_k, r' \approx t, E \Rightarrow^*_\theta \Box$$

such that $|B| < |A_{>1}|$ and $\mathcal{C}(A_{>1}, B)$. From the validity of property $\mathcal{P}_{[\mathrm{o}]}(A_{>1})$ we deduce that [o] is never applied to the RHS of descendants of the equation $t_n \approx v_k$ in $A_{>1}$. From $\mathcal{C}(A_{>1}, B)$ we have that [o] is never applied to the RHS of the descendants of $t_n \approx v_k$ in B. Let $B_{>i}$ be the sub-refutation of B such that $B_{>i} : G_2 = (r\ \mathbf{s}_{n-1} \approx h\ \mathbf{v}_{k-1}, s_n \approx v_k, r' \approx t, E)\theta_1 \Rightarrow^j_{\theta_2} \Box$. We construct the LNC-refutation:

$$\begin{aligned} A' : G_1 &= s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n} \approx t, E \\ &\Rightarrow^i_{\theta_1} G'_2 = (r\ \mathbf{s}_{m+1,n} \approx t, E)\theta_1 \\ &\Rightarrow_{[\mathrm{o}], h\ \mathbf{v}_k \to r'} G_2 = (r\ \mathbf{s}_{m+1,n-1} \approx h\ \mathbf{v}_{k-1}, s_n \approx v_k, r' \approx t, E)\theta_1 \Rightarrow^j_{\theta_2} \Box \end{aligned}$$

where the first i steps of A' coincide with those of B and $A'_{>(i+1)} = B_{>i}$. Then $|A'| = |B| + 1 < |A_{>1}| + 1 = |A|$. From our previous remark that no [o]-steps are applied to the RHS of descendants of $t_n \approx v_k$ we deduce $A' \in \mathcal{WF}$. From the construction of A' and the fact that $\mathcal{C}(A_{>1}, B)$ holds we infer that $\mathcal{C}(A, A')$ holds too.
3. A starts with an [i]-step to the LHS. Then:

$$A : f\ \mathbf{s}_n \approx x, E \Rightarrow_{[\mathrm{i}], \sigma = \{x \mapsto x_1\ x_2\}} (f\ \mathbf{s}_{n-1} \approx x_1, s_n \approx x_2, E)\sigma \Rightarrow^*_{\theta'} \Box$$

An application of the induction hypothesis to $A_{>1}$ reveals the existence of a $B \in \mathcal{WF}$ of the form:

$$B : (s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n-1} \approx x_1, s_n \approx x_2, E)\sigma \Rightarrow^*_{\theta'} \Box$$

such that $|B| < |A_{>1}|$ and $\mathcal{C}(A_{>1}, B)$. We distinguish two subcases:

(a) $x \in \mathcal{V}ar(f\ \mathbf{s}_n)$. Then $x\sigma\theta'$ is normalized because $A \in \mathcal{WF}$. We can now apply Lemma 17 to B and obtain $A' \in \mathcal{WF}$ of the form:

$$A' : s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n} \approx x, E \Rightarrow_{\theta''} \Box$$

such that $\sigma\theta' = \theta''$ and $|A'| \leq |B| + 1$. But $\sigma\theta' = \theta$ and hence $\theta'' = \theta$. Also, $|A'| \leq |B| + 1 < |A_{>1}| + 1 = |A|$. From the construction of A' from B given in the proof of Lemma 17 results that if [o] is never applied to the RHS of descendants of $e \in E$ in B then [o] is never applied to the RHS of descendants of e in A'. This observation together with $\mathcal{C}(A_{>1}, B)$ implies $\mathcal{C}(A, A')$.

(b) $x \notin \mathcal{V}ar(f\ \mathbf{s}_n)$. Then $x \notin \mathcal{V}ar(s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n-1}, s_n)$ and we can again apply Lemma 17 to construct from B the desired $A' \in \mathcal{WF}$ with property $\mathcal{C}(A, A')$. ■

Lemma 19 Let $A \in \mathcal{WF}$ be of the form

$$A : x\ \mathbf{s}_n \approx t, E \Rightarrow^*_{\theta} \Box \tag{3}$$

such that there exists a first [o]-step of A which is applied to the LHS of an a-descendant of $x\ \mathbf{s}_n \approx t$ and all the [i]-steps which precede it are applied to the LHS. Then there exists a fresh variant $f\ \mathbf{u}_k\ \mathbf{v}_m \to r$ of a rewrite rule such that:

(a) $0 < m \leq n$,

(b) The last [o]-step to an a-descendant of $x\ \mathbf{s}_n \approx t$ is of the form

$$\begin{aligned} A_1 : &(x\ \mathbf{s}_m \approx t')\theta_1, E' \\ &\Rightarrow_{[\text{o}], f\ \mathbf{u}_k\ \mathbf{v}_m \to r} (f\ \mathbf{s}_{m-1} \approx f\ \mathbf{u}_k\ \mathbf{v}_{m-1}, s_m \approx v_m, r\ \mathbf{s}_{m+1,n} \approx t', E')\theta_1 \end{aligned}$$

(c) There exists a $A' \in \mathcal{WF}$ of the form:

$$A' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n} \approx t, E)\sigma \Rightarrow^*_{\theta'} \Box \tag{4}$$

with $\sigma = \{x \mapsto f\ \mathbf{u}_k\}$ such that $\sigma\theta' = \theta$ and $|A'| < |A|$.

Proof. We note that the a-descendants of parameter-passing equations are parameter-passing equations. Since the first [o]-step to an a-descendant of $x\ \mathbf{s}_n \approx t$ is applied to the LHS then, because of property $\mathcal{P}_{[\text{o}]}(A)$, all the [o]-steps to a-descendants of $x\ \mathbf{s}_n \approx t$ are applied to the LHS. Also, because of property $\mathcal{P}_{[\text{i}]}(A)$, all [i] steps between the first and the last [o]-step to an a-descendant of $x\ \mathbf{s}_n \approx t$ are applied to the LHS. Therefore, the last a-descendant of $x\ \mathbf{s}_n \approx t$ to which an [o]-step is applied is of the form $\underline{(x\ \mathbf{s}_m)\theta_1} \approx t'\theta_1$ with $m \leq n$.

We prove now that $m > 0$. Assume that $m = 0$. Then from the applicability of an [o]-step to the LHS of $(x\ \mathbf{s}_m)\theta_1 \approx t'\theta_1$ we deduce that $x \in \mathcal{D}(\theta_1)$. Also, by Corollary 3, the term $x\theta_1$ is reducible. Since this contradicts property 1.(b) of well-formedness for A, we must have $m > 0$. From Lemma 14 we deduce that the variant of the rewrite rule employed in the [o]-step to the LHS of $(x\ \mathbf{s}_m)\theta_1 \approx t'\theta_1$ can be written as $f\ \mathbf{u}_k\ \mathbf{v}_m \to r$. Thus, conditions (a) and (b) hold.

We prove now that condition (c) also holds. Consider the sub-refutation $\overline{A}$ of A that starts with an [o]-step applied to the LHS of the a-descendant $(x\ \mathbf{s}_m \approx t')\theta_1$ of $x\ \mathbf{s}_n \approx t$. This refutation starts from a goal of the form $G_2 = (\underline{x\ \mathbf{s}_m} \approx t')\theta_1, E'$. Since $\overline{A} \in \mathcal{WF}$, it is of the form:

$$\overline{A} : G_2 \Rightarrow_{[\mathrm{o}], f\ \mathbf{u}_k\ \mathbf{v}_m \to r} (x\theta_1)\ \mathbf{s}'_{m-1} \approx f\ \mathbf{u}_k\ \mathbf{v}_{m-1}, s'_m \approx v_m, r \approx t'\theta_1, E' \Rightarrow^* \Box$$

where $s'_i = s_i\theta_1$ for $1 \le i \le m$. Since the first step of this sub-refutation is also the last one to an a-descendant of $x\ \mathbf{s}_n \approx t$, we deduce that the following $m-1$ steps must be [d]-steps. Therefore we can write:

$$\begin{array}{l} A : G \Rightarrow^*_{\theta_1} G_2 = (\underline{x\ \mathbf{s}_m} \approx t')\theta_1, E' \\ \Rightarrow_{[\mathrm{o}], f\ \mathbf{u}_k\ \mathbf{v}_m \to r} \underline{(x\theta_1)\ \mathbf{s}'_{m-1}} \approx f\ \mathbf{u}_k\ \mathbf{v}_{m-1}, s'_m \approx v_m, r \approx t'\theta_1, E' \\ \Rightarrow^{m-1}_{[\mathrm{d}]} x\theta_1 \approx f\ \mathbf{u}_k, \boxed{s'_1 \approx v_1}, \ldots, \boxed{s'_m \approx v_m}, r \approx t'\theta_1, E' \Rightarrow^* \Box \end{array} \tag{5}$$

The equations displayed within boxes are descendants of the parameter-passing equations generated from the equation $(x\ \mathbf{s}_m \approx t')\theta_1 \in G_2$. They are used in specifying the property $\mathcal{C}_2(A, A')$ defined below.

We now prove by induction on $|A|$ the existence of $A' \in \mathcal{WF}$. In the proof we make use of the property that the condition $\mathcal{C}(A, A') = \mathcal{C}_1(A, A') \wedge \mathcal{C}_2(A, A')$ holds in each induction step. Here, $\mathcal{C}_1(A, A')$ and $\mathcal{C}_2(A, A')$ are defined as follows:

Let A and A' be the LNC-refutations under consideration of the forms given in (3) and (4). Then:

$\mathcal{C}_1(A, A')$: for every $e' \in E$, if [o] is never applied to the RHS of descendants of e' in A then [o] is never applied to the RHS of descendants of e' in A'

$\mathcal{C}_2(A, A')$: assuming A is written in the form (5) described above then for every $i \in \{1, \ldots, m\}$ the following implication holds: if [v] is the only LNC-step applied to a descendant of $s'_i \approx v_i$ in A then [v] is the only LNC-step applied to a descendant of $s_i \approx v_i$ in A'.

First we note that because of assumption (ii) A can not start with a [v]-step or a [t]-step. We distinguish three possibilities for the first LNC-step of A:

Case I. A starts with an [i]-step. Then $t \in \mathcal{V}$. Because of assumption (ii) we must have $n > 0$. In this case A can be written as:

$$A : G \Rightarrow_{[\mathrm{i}], \sigma_0} (t\sigma_0)\ \mathbf{s}''_{n-1} \approx x_1, s''_n \approx x_2, E\sigma_0 \Rightarrow^* \Box$$

where $\sigma_0 = \{t \mapsto x_1\ x_2\}$ with $x_1, x_2 \in \mathcal{V}$ fresh variables, $s''_i = s_i\sigma_0$ for $1 \le i \le n$. We further distinguish two subcases:

1. $t = x$. Then A is of the form

$$A : G \Rightarrow_{[\mathrm{i}], \sigma_0} x_1\ x_2\ \mathbf{s}''_{n-1} \approx x_1, s''_n \approx x_2, E\sigma_0 \Rightarrow^*_{\theta''} \Box$$

where $\sigma_0\theta'' = \theta$, and the sub-refutation $\overline{A}$ of A can be written as:

$$\begin{array}{l} \overline{A} : G_2 = (\underline{x_1\ x_2\ \mathbf{s}_m} \approx t')\theta_1, E' \\ \quad \Rightarrow_{[\mathrm{o}], f\ \mathbf{u}_k\ \mathbf{v}_m \to r} (x_1\ x_2\ \mathbf{s}_{m-1} \approx f\ \mathbf{u}_k\ \mathbf{v}_{m-1}, s_m \approx v_m, r \approx t')\theta_1, E' \Rightarrow^* \Box \end{array}$$

The equation $(x_1\ x_2\ \mathbf{s}_m \approx t')\theta_1$ is an a-descendant of $x_1\ x_2\ \mathbf{s}''_{n-1} \approx x_1$ obtained by applying a sequence of [d]-, [i]- and [o]-steps. Since [i]- and [o]-steps of this sequence are applied only to the LHS, we have $\mathcal{V}ar(x_1\ x_2\ s''_{n-1} \approx x_1) \cap \mathcal{D}(\theta_1) = \{x_1\}$ and $\mathcal{V}ar(x_1\ x_2\ s''_{n-1} \approx x_1) \cap \mathcal{I}m(\theta_1) = \emptyset$. Therefore $x_2 \notin \mathcal{D}(\theta_1) \cup \mathcal{V}ar(x_1\theta_1)$. From Lemma 14 we deduce that $k \geq 1$. Since $\overline{A} \in \mathcal{WF}$, we can write:

$$\begin{aligned}\overline{A} : G_2 = (&\underline{x_1\ x_2\ \mathbf{s}_m} \approx t')\theta_1, E' \\ &\Rightarrow_{[\mathrm{o}], f\ \mathbf{u}_k\ \mathbf{v}_m \to r}\ x_1\ x_2\ \mathbf{s}'_{m-1} \approx f\ \mathbf{u}_k\ \mathbf{v}_{m-1}, s'_m \approx v_m, r \approx t'\theta_1, E' \\ &\Rightarrow^m_{[\mathrm{d}]}\ x_1\theta_1 \approx f\ \mathbf{u}_{k-1}, \boxed{x_2 \approx u_k}, \boxed{s'_1 \approx v_1}, \ldots, \boxed{s'_m \approx v_m}, r \approx t'\theta_1, E' \Rightarrow^* \Box\end{aligned}$$

where the equations displayed within boxes are descendants of the parameter-passing equations generated from the equation $(x_1\ x_2\ \mathbf{s}_m \approx t')\theta_1 \in G_2$. Since $x_2 \notin \mathcal{V}ar(x_1\theta_1 \approx f\ \mathbf{u}_{k-1})$, we can further write

$$\begin{aligned}\overline{A}_{>m+1} : &\ x_1\theta_1 \approx f\ \mathbf{u}_{k-1}, (x_2 \approx u_k, s_1 \approx v_1, \ldots, s_m \approx v_m, r \approx t')\theta_1, E' \\ &\Rightarrow^*_{\theta_2}\ x_2 \approx u_k\theta_2, (s_1 \approx v_1, \ldots, s_m \approx v_m, r \approx t')\theta_1\theta_2, E'\theta_2 \\ &\Rightarrow_{[\mathrm{v}]} \Rightarrow^* \Box\end{aligned}$$

We can now apply the the induction hypothesis to $A_{>1} \in \mathcal{WF}$ and deduce the existence of a well-formed LNC-refutation A'' of the form:

$$(x_2 \approx u_k, s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1} \approx x_1, s_n \approx x_2, E)\sigma_0\sigma_1 \Rightarrow^*_{\theta'''} \Box$$

where $\sigma_1 = \{x_1 \mapsto f\ \mathbf{u}_{k-1}\}$, $|A''| < |A_{>1}|$, and $\mathcal{C}(A_{>1}, A'')$, and $\sigma_1\theta''' = \theta''$. From the assumption $\mathcal{C}_2(A_{>1}, A'')$ and the observation that [v] is the only step applied to an a-descendant of $x_2 \approx u_k$ in A we conclude that the first LNC-step to $(x_2 \approx u_k)\sigma_0\sigma_1 = x_2 \approx u_k$ must be a [v]-step. Hence:

$$A''_{>1} : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1} \approx x_1, s_n \approx x_2, E)\sigma_0\sigma_1\{x_2 \mapsto u_k\} \Rightarrow^* \Box$$

Note that $\sigma_0\sigma_1\{x_2 \mapsto u_k\} = \{x \mapsto f\ \mathbf{u}_k, x_1 \mapsto f\ \mathbf{u}_{k-1}, x_2 \mapsto u_k\}$. Because $\sigma = (\sigma_0\sigma_1\{x_2 \mapsto u_k\})\restriction_{\mathcal{V}ar(s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1}, E)}$, we can write:

$$\begin{aligned}A''_{>1} : &\ (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1} \approx f\ \mathbf{u}_{k-1}, s_n \approx u_k, E)\sigma \\ &\Rightarrow^i_{\tau_1} (r\ \mathbf{s}_{m+1,n-1} \approx f\ \mathbf{u}_{k-1}, s_n \approx u_k, E)\sigma\tau_1 \Rightarrow^*_{\tau_2} \Box\end{aligned}$$

We perform the following construction of A' from $A''_{>1}$:

$$\begin{aligned}A' : &\ (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n} \approx x, E)\sigma \Rightarrow^i_{\tau_1} (r\ t_n \approx f\ \mathbf{u}_k, E)\sigma\tau_1 \\ &\Rightarrow_{[\mathrm{d}]} (r\ \mathbf{s}_{m+1,n-1} \approx f\ \mathbf{u}_{k-1}, s_n \approx u_k, E)\sigma\tau_1 \Rightarrow^*_{\tau_2} \Box\end{aligned}$$

where the first i LNC-steps of A' coincide with the first i LNC-steps of $A''_{>1}$ and $A'_{>i+1} = A''_{>i+1}$. Then A' is well-formed and satisfies the requirements of our lemma. The validity of $\mathcal{C}(A, A')$ results from the way in which A' is constructed from A'' and from the property $\mathcal{C}(A_{>1}, A'')$.

2. $t \neq x$. Then A is of the form

$$A : G \Rightarrow_{[\mathrm{i}],\sigma_0} x\ \mathbf{s}''_{n-1} \approx x_1, s''_n \approx x_2, E\sigma_0 \Rightarrow^*_{\theta''} \Box$$

where $\sigma_0\theta'' = \theta$. By the induction hypothesis for $A_{>1}$ there exists a $A'' \in \mathcal{WF}$ of the form:

$$A'' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1} \approx x_1, s_n \approx x_2, E)\sigma_0\sigma \Rightarrow^*_{\theta_0} \Box$$

such that $\sigma\theta_0 = \theta''$, $|A''| < |A_{>1}|$ and $\mathcal{C}(A_{>1}, A'')$. Since $\sigma_0\sigma = \sigma\sigma_0$, we can write A'' in the form:

$$A'' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1} \approx x_1, s_n \approx x_2, E)\sigma\sigma_0 \Rightarrow^*_{\theta_0} \Box$$

An application of Lemma 17 to A'' yields the desired $A' \in \mathcal{WF}$.

Case II. A starts with an [o]-step. We distinguish two cases:

- $n = m$. Then this step is also the last [o]-step to an a-descendant of $x\ \mathbf{s}_m \approx t$. We have then:

$$\begin{aligned} &A : x\ \mathbf{s}_m \approx t, E \\ &\Rightarrow_{[\mathrm{o}], f\ \mathbf{u}_k\ \mathbf{v}_m \to r}\ x\ \mathbf{s}_{m-1} \approx f\ \mathbf{u}_k\ \mathbf{v}_{m-1}, s_m \approx v_m, r \approx t, E \\ &\Rightarrow^{m-1}_{[\mathrm{d}]}\ x \approx f\ \mathbf{u}_k, s_1 \approx v_1, \ldots, s_m \approx v_m, r \approx t, E \\ &\Rightarrow_{[\mathrm{v}], \sigma = \{x \mapsto f\ \mathbf{u}_k\}} (s_1 \approx v_1, \ldots, s_m \approx v_m, r \approx t, E)\sigma \Rightarrow^*_{\theta'} \Box \end{aligned}$$

  and we can choose $A' = A_{>m+1}$.
- $n > m$. In this case we have:

$$A : x\ \mathbf{s}_n \approx t, E \Rightarrow_{[\mathrm{o}], l_1\ l_2 \to r'} x\ \mathbf{s}_{n-1} \approx l_1, s_n \approx l_2, r' \approx t, E \Rightarrow^*_{\sigma'} \Box$$

  By the induction hypothesis for $A_{>1}$ we infer the existence of $A'' \in \mathcal{WF}$ of the form:

$$A'' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1} \approx l_1, s_n \approx l_2, r' \approx t, E)\sigma \Rightarrow^*_{\theta'} \Box$$

  such that $\sigma\theta' = \theta$, $|A''| < |A_{>1}|$ and $\mathcal{C}(A_{>1}, A'')$. Let $i_1 \geq 0$ such that:

$$\begin{aligned} &A'' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1} \approx l_1, s_n \approx l_2, r' \approx t, E)\sigma \\ &\Rightarrow^{i_1}_{\sigma'} (r\ \mathbf{s}_{m+1,n-1} \approx l_1, t_n \approx l_2, r' \approx t, E)\sigma\sigma' \Rightarrow^* \Box \end{aligned}$$

  We construct:

$$\begin{aligned} &A' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n} \approx t, E)\sigma \Rightarrow^{i_1}_{\sigma'} (r\ \mathbf{s}_{m+1,n} \approx t, E)\sigma\sigma' \\ &\Rightarrow_{[\mathrm{o}], l_1\ l_2 \to r'} (r\ \mathbf{s}_{m+1,n-1} \approx l_1, s_n \approx l_2, r' \approx t, E)\sigma\sigma' \Rightarrow^* \Box \end{aligned}$$

  such that the first i_1 steps of A' coincide with the first i_1 steps of A'' and $A'_{>i_1+1} = A''_{>i_1}$. We notice that $|A'| = |A''| + 1 < |A_{>1}| + 1 = |A|$. We must show that $A' \in \mathcal{WF}$. Because $A'' \in \mathcal{WF}$, we only have to show that in A' there are no [o]-steps applied to the RHS of descendants of the parameter-passing equation $(t_n \approx l_2)\sigma\sigma'$. We note that $(s_n \approx l_2)\sigma$ is a parameter-passing equation in $A_{>1}$ and therefore [o]-steps are never applied to the RHS of the descendants of $(t_n \approx l_2)\sigma$. From $\mathcal{C}(A_{>1}, A'')$ we infer that in A'' [o]-steps are never applied to the RHS of the descendants of $(s_n \approx l_2)\sigma$. From the construction of A' it is easily seen that also in A' [o] is never applied to the RHS of $(s_n \approx l_2)\sigma\sigma'$.
  The validity of $\mathcal{C}(A, A')$ results from $\mathcal{C}(A_{>1}, A'')$ and the construction of A' from A''.

Case III. A starts with a [d]-step. Then $n > 0$, $t \equiv t_1\ t_2$ for some terms t_1, t_2 and:

$$A : x\ \mathbf{s}_m \approx t_1\ t_2, E \Rightarrow_{[\mathrm{d}]} x\ \mathbf{s}_{n-1} \approx t_1, s_n \approx t_2, E \Rightarrow^*_{\theta} \Box$$

and we can apply the induction hypothesis to $A_{>1}$ and obtain a well-formed LNC-refutation:

$$A'' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1} \approx t_1, s_n \approx t_2, E)\sigma \Rightarrow_{\theta'} \Box$$

with $\sigma\theta' = \theta$, $|A'| < |A_{>1}|$ and $\mathcal{C}(A_{>1}, A'')$. Let $i_1 \geq 0$ such that:

$$A'' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n-1} \approx t_1, s_n \approx t_2, r' \approx t, E)\sigma$$
$$\Rightarrow^{i_1} (r\ \mathbf{s}_{n-1} \approx t_1, s_n \approx t_2, E)\sigma\sigma' \Rightarrow^* \square$$

Then we define A' as follows:

$$A' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n} \approx t, E)\sigma \Rightarrow^{i_1}_{\sigma'} (r\ \mathbf{s}_{m+1,n} \approx t_1\ t_2, E)\sigma\sigma'$$
$$\Rightarrow_{[\mathrm{d}]} (r\ \mathbf{s}_{m+1,n-1} \approx t_1, s_n \approx t_2, E)\sigma\sigma' \Rightarrow^* \square$$

where the first i_1 steps of A' coincide with the first i_1 steps of A'' and $A'_{>i_1+1} = A''_{>i_1}$. Then $A' \in \mathcal{WF}$ and $|A'| = |A''|+1 < |A_{>1}|+1 = |A|$. Also, property $\mathcal{C}(A, A')$ follows from $\mathcal{C}(A_{>1}, A'')$ and the construction of A' from A''. ■

4.4 The Completeness Theorem of LNCA

Lemma 20 Let $\mathcal{R}$ be a confluent $\mathcal{A}$TRS and G be a goal. For every well-formed LNC-refutation $A : G \Rightarrow^*_\theta \square$ there exists an LNCA-derivation $B : G \Rrightarrow^*_\sigma G_1$ and a well-formed LNC-refutation $A' : G_1 \Rightarrow^*_{\theta'} \square$ such that $\sigma\theta' = \theta\ [\mathcal{V}ar(G)]$ and $|A'| < |A|$.

Proof. Let $G \equiv s \approx t, E$. We distinguish the following cases:

(1) No [o]-steps are applied to a-descendants of $s \approx t$ in A. We have to consider the following cases:

(1a) $s \equiv f\ \mathbf{s}_m$ and $t \equiv g\ \mathbf{t}_n$. According to Lemma 8, we must have $f = g$ and $n = m$. According to Lemma 9, there exists a $A' \in \mathcal{WF}$ of the form $A' : s_1 \approx t_1, \ldots, s_n \approx t_n, E \Rightarrow^*_\theta \square$ such that $|A'| < |A|$. Then, for:

$B : G \equiv f\ \mathbf{s}_n \approx f\ \mathbf{t}_n, E \Rrightarrow_{[\mathrm{df}]} G_1$
$G_1 = s_1 \approx t_1, \ldots, s_n \approx t_n, E,$
$\theta' = \theta$, $\sigma = \varepsilon$

the conclusion of Lemma 20 holds.

(1b) $s \approx t$ is of the form $a\ \mathbf{s}_m \simeq x\ \mathbf{t}_n$ with $m \geq n$. This case is covered by Lemma 16.

(1c) Otherwise, $s \approx t$ must be of the form $f\ \mathbf{s}_m \simeq x\ \mathbf{u}_n$ with $m < n$. According to Lemma 10, there exists an [o]-step in A which is applied to an a-descendant of $s \approx t$. Since we assumed the contrary, this case is impossible.

(2) The first [o]-step is applied to the LHS of an a-descendant of $s \approx t$. By Lemma 7, there exists $A'' \in \{A, \phi_{swap}(A, 1)\}$ such that all [i]-steps before the first [o]-step are applied to the LHS and the first [o]-step is applied to the LHS. According to Lemma 6, $A'' \in \mathcal{WF}$. Assume:

$$A'' : \underline{a\ \mathbf{s}_n} \approx r', E \Rightarrow^*_\theta \square$$

where $a \in \mathcal{F} \cup \mathcal{V}$. We distinguish two cases:
(a) $a \equiv f \in \mathcal{F}$. Let $m = \text{arity}(f)$. From Lemma 18 we infer the existence of a fresh variant $f\ \mathbf{u}_m \to r$ of a rewrite rule with $m \leq n$ and of an $A' \in \mathcal{WF}$ of the form

$$A' : s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n} \approx t, E \Rightarrow^*_\theta \square$$

such that $|A'| < |A|$. We note that we can choose B to be:

$$B : f\ \mathbf{s}_n \simeq r', E \Rrightarrow_{[\mathrm{of}], f\ \mathbf{u}_m \to r} s_1 \approx u_1, \ldots, s_m \approx u_m, r\ \mathbf{s}_{m+1,n} \approx r', E$$

(b) $a \equiv x \in \mathcal{V}$. By Lemma 19 we can assume the existence of a fresh variant $f\ \mathbf{u}_k\ \mathbf{v}_m \to r$ of a rewrite rule such that $0 < m \leq n$ and of an $A' \in \mathcal{WF}$ of the form:

$$A' : (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n} \approx t, E)\sigma \Rightarrow^*_{\theta'} \Box$$

with $\sigma = \{x \mapsto f\ \mathbf{u}_k\}$ such that $\sigma\theta' = \theta$ and $|A'| < |A|$. We can now consider the [ov]-step of LNCA:

$$\begin{aligned} B : G &\equiv (x\ \mathbf{s}_n \simeq r', E) \\ &\Rrightarrow_{[\mathrm{ov}], \sigma, f\ \mathbf{u}_k\ \mathbf{v}_m \to r} G_1 = (s_1 \approx v_1, \ldots, s_m \approx v_m, r\ \mathbf{s}_{m+1,n} \approx r', E)\sigma. \end{aligned}$$

(3) The first [o]-step is applied to the RHS of an a-descendant of $s \approx t$. Then obviously the first [o]-step is not preceded by [i]-steps, and in $\phi_{swap}(A, 1)$ the first [o]-step is applied to the LHS. This case reduces to case (2). ■

Theorem 4 Let $\mathcal{R}$ be a confluent $\mathcal{A}$TRS and G a goal. For every normalized solution θ of G there exists a successful LNCA-derivation $A : G \Rrightarrow^*_{\theta'} \Box$ such that $\theta' \leq \theta\ [\mathcal{V}ar(G)]$.

Proof. By Corollary 2 and induction on $|A|$ using Lemma 20. ■

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