The Parametrization Problem for Algebraic Surfaces

Josef Schicho

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Abstract

The parameterization problem asks for a parameterization of an implicitly given surface, in terms of rational functions in two variables. We give an overview on the history of this problem, including recent algorithmic solutions and open questions.

The problem is related to the parametrization problem for curves and to the problem of resolution of singularities.

Introduction

The parameterization problem is the following. Given is the implicit representation of a surface S, in terms of an equation in three variables over a field k of characteristic zero. Wanted is a parametrization of S, in terms of two rational functions of two independant parameters s, t. Such a parametrization may not exist; so, a first step in an algorithmic solution is to decide the existence. In the affirmative case, we say that S is *unirational*. A parametrization which is birational (as a rational map from the st-plane to S) is called *proper*. A surface is called *rational* iff it has a proper parametrization.

The reverse problem, implicitization, has been solved in [7, 15, 21, 9, 16]. The parametrization problem for curves has been solved in [43, 1, 35, 29, 40, 23, 22, 36, 41]; actually, we will need the curve parametrization algorithm as a subalgorithm in surface parametrization.

For complex surfaces, it is well-known that rationality and unirationality are equivalent, and Castelnuovo's criterion $p_a = P_2 = 0$ (see [8, 44, 37]) is necessary and sufficient. We give an algorithm that computes the arithmetic genus p_a and the second plurigenus P_2, and which can be used to decide (uni)rationality in the complex case.

In the general case, Castelnuovo's criterion is still necessary. It is interesting to ask how much the surface S can be birationally simplified, provided that Castelnuovo's criterion is fulfilled. This question is answered by the theorem of Enriques and Manin [14, 24]. A surface with $p_a = P_2 = 0$ has a pencil of rational curves or is birationally equivalent to a del Pezzo surface.

Computationally, a pencil of rational curves is already a big step towards the final result, because it gives one parameter t in terms of the coordinate functions x, y, z. It remains to find a parametrization of the rational curves $t =$ constant, in terms of the second parameter s. This leads to the problem of parametrizing a curve over the function field $k(t)$. It is

well-known (see [36]) that this problem can be reduced to the problem of parametrizing a conic over $k(t)$, or a surface with an equation of the type $A(t)x^2+B(t)y^2+C(t)=0$, where $A, B, C \in k[t]$ (such a surface is also called a *tubular*.)

The class of tubular surfaces is also interesting because many surfaces arising in CAD/CAM and other geometric applications, such as conical surfaces [11, 12], canal surfaces [28, 27], or generalized offset surfaces of curves (see [3]) can be birationally reduced to it. The theory is closely related to the theory of quadratic forms. The main problem is to find a cross section to the pencil, i.e. a rational curve on S with parametrization $(X(t), Y(t), t)$, where $X, Y \in k(t)$.

In the complex case, the cross section problem was algorithmically solved by [26]. This algorithm was improved in [32]. In the real case, there are tubular surface without cross section. A necessary and sufficient condition for the existence was given by [10]. In the affirmative case, we have algorithms to compute cross sections [27, 31]. The algorithm given in [19] works for arbitrary fields, however it is very inefficient. An efficient algorithm for the general case will be contained in a joint paper with E. Hillgarter, G. Landsmann, and F. Winkler.

The non-existence of a cross section does not imply the non-existence of a parametrization. For instance, in the real case we know that all tubular surfaces are unirational (see [20, 31]). In the real case, we can also decide rationality and compute proper parametrizations (see [34]).

The second case in the reduction of Enriques and Manin is the case of del Pezzo surfaces. In the complex case, del Pezzo surfaces have several pencils of conics, and can therefore be reduced to the tubular case above (in several ways). Computationally, this requires to solve a univariate equation of quite large degree. Over the fields $\mathbf{Q}$ and $\mathbf{R}$, the parametrization problem for del Pezzo surfaces is still open.

We turn to the question: how can we effectively compute the reduction asserted by the theorem of Enriques and Manin. The answer is: by the method of adjoints. A polynomial is an adjoint if it vanishes with a certain prescribed order at the singularities of the surface. The vector spaces of all adjoints of some particular degree define the rational maps which achieve the Enriques/Manin reduction.

Computing the adjoints is the most expensive part of the parametrization algorithm [32], because it requires a *resolution of the singularities*, i.e. a birational proper map $f : \tilde{S} \to S$ such that $\tilde{S}$ is nonsingular. It suffices to know $\tilde{S}$, but already the local problem is complicated enough.

We do have an algorithm that computes a resolution of singularities by blowing up [42, 6]. The algorithm returns a tree of charts, each containing a small portion of the blown up variety. Unfortunately, this algorithm is far too expensive to be useful for the computation of adjoints.

A more efficient way is to resolve the discriminant by blowing up, thereby reducing to the quasi-ordinary case. In the quasi-ordinary case, the singularities are resolved using two-dimensional Puiseux expansions [2]. This approach is taken in the forthcoming paper (joint work with W. Stöcher).

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1 The Problem

Let S be a surface given by an equation $F(x,y,z)=0$, where k is a field of characteristic zero, and $F \in k[x,y,z]$ is irreducible. Algebraically, a parametrization is a triple $(X,Y,Z) \in k(s,t)$, such that $F(X,Y,Z)=0$, and the Jacobian matrix

$$\begin{pmatrix} X & Y & Z \\ \partial_s X & \partial_s Y & \partial_s Z \\ \partial_t X & \partial_t Y & \partial_t Z \end{pmatrix}$$

has maximal rank 2 (both conditions are supposed to hold in the field $k(s,t)$). A parametrization is called proper iff there exist rational functions $S,T \in k(x,y,z)$, such that the composites $S(X,Y,Z)$ and $T(X,Y,Z)$ are defined in $k(s,t)$, and are equal to s and t. (The existence of such rational functions imply the Jacobian condition.)

Example 1.1. An easy example is the parametrization of the unit sphere $x^2+y^2+z^2-1=0$ by

$$(X,Y,Z) = \left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1}, \frac{s^2+t^2-1}{s^2+t^2+1}\right).$$

This is indeed a proper parametrization, its inverse is the stereographic projection:

$$(S,T) = \left(\frac{x}{1-z}, \frac{y}{1-z}\right)$$

The *parametrization problem* is the following:

Given: F, the implicit equation of a surface S;

Find: A parametrization (X,Y,Z) of S, if exists;
`NotExist` otherwise.

The *proper parametrization problem* is the following:

Given: F, the implicit equation of a surface S;

Find: If exists, a parametrization (X,Y,Z) of S, and its inverse (S,T);
`NotExist` otherwise.

A surface is called *unirational* iff it has a parametrization, and *rational* iff it has a proper parametrization.

2 Two theoretical results

For the definition of the arithmetic genus p_a and the plurigeni P_n of an algebraic variety, we refer to [37]. Here, we will only need the facts that these numbers are birational invariants, and that their definition does not depend on the choice of the field.

The following theorem is very helpful for deciding existence.

Theorem 2.1 (Castelnuovo). *Let S be a surface. Over the complex numbers, the following are equivalent.*

a) S is rational.

b) S is unirational.

c) $p_a(S) = P_2(S) = 0$.

For the computation of p_a and P_2, we refer to section 5.

The next theorem is even more relevant, because it gives a strategy for computing a parametrization. It holds for arbitrary fields.

Theorem 2.2 (Enriques,Manin). *A surface with $p_a = P_2 = 0$ has a pencil of rational curves or is birationally equivalent to a del Pezzo surface.*

In the following sections, we discuss these two cases separately.

3 Surfaces with a Pencil of Rational Curves

A pencil is a family of curves arising as the fibers of a rational map $f : S \to bfP^1$. Except in a finite number of base points which are common to all, the curves in a pencil are pairwise disjoint, and their union is equal to S. In this section, we assume that S has such a pencil, and the generic curve of the pencil is an irreducible rational curve.

Let $k(t)$ be a simple transcendental extension of k. Then $(t : 1)$ is a generic point of bfP^1_k. The generic fiber $f^{-1}((t : 1))$ is a curve defined over the field $k(t)$. We know that this curve is irreducible and rational over the algebraic closure $\mathbf{R}(t)$.

Typical examples are surfaces with an equation $A(t)x^2 + B(t)y^2 + C(t) = 0$, where $A, B, C \in k[t]$. Such surfaces are called *tubular surfaces*. The pencil is formed by the horizontal sections $t =$ constant. The generic horizontal section is an irreducible conic. For special values of t, the conic may split into two lines.

From a birational point of view, any surface with a pencil of rational curves may be reduced to a tubular surface. To obtain this result, we recall a well-known theorem from the theory of algebraic curves.

Theorem 3.1 (Noether). *Let K be an arbitrary field. Let C be a curve which is irreducible and rational over the algebraic closure of K. Then C is rational over K, or there is a birational equivalence, defined over K, transforming C to a conic.*

For a modern treatment and an algorithmic version of this theorem, see [36].

Applying theorem 3.1 to the field $K = k(t)$, we get the desired result.

Example 3.2. Let S be the surface with equation $x^4 + z^4 + (xy + z^2)^3 = 0$. The sections of the planes through the triple line $x = z = 0$ cut out cubic plane curves. The structure map is

$$f : S \to \mathbf{P}^1, (x, y, z) \mapsto (x : z).$$

The generic fiber has equations

$$x - tz = (t^4 + 1)z + (ty + z)^3 = 0.$$

This curve has a double point at infinity. By the genus formula (see [43]) it is rational. A parametrization is

$$(X, Y, Z) = (\frac{-s^3 t}{t^4 + 1}, \frac{s^3 + st^4 + s}{t^5 + t}, \frac{-s^3}{t^4 + 1}).$$

The same functions yield a parametrization of the surface S.

Example 3.3. Let S be the surface with equation

$$x^2 + 2xz^2 + 2xyz + 2y^3 + 2z^3y + x^4 + y^4 + z^4 = 0.$$

A rational pencil (found by some means that will be explained later) is given by the structure map

$$f : S \to \mathbf{P}^1, (x, y, z) \mapsto (x : y).$$

The generic fiber, with equations

$$x - ty = x^2 + 2xz^2 + 2xyz + 2y^3 + 2z^3y + x^4 + y^4 + z^4 = 0$$

is a quadric plane curve with an double point at the origin and two infinitely near double points. It is therefore a rational curve, but it is not easy to construct a parametrization.

We can, however, compute a birational map to the conic over $k(t)$ with equation

$$-2u^2 + tv^2 + 2tuv + 2v + (t^5 + t) = 0$$

(which is also the equation of a tubular surface T). It is given by

$$(x, y, z) \mapsto (U, V) = (\frac{z}{y}, \frac{ty + z^2}{y^2}),$$

$$(u, v) \mapsto (X, Y, Z) = (\frac{t^2}{v - u^2}, \frac{t}{v - u^2}, \frac{tu}{v - u^2}).$$

The same functions induce a birational map between S and T, keeping t fixed.

3.1 Tubular Surfaces

In order to parametrize a given tubular surface $S : A(t)x^2 + B(t)y^2 + C(t) = 0$, the main problem is to find a *cross section*, i.e. a curve on S with a parametrization $(X(t), Y(t), t)$. (A cross section intersects the generic horizontal section in precisely one point.) Once we have a cross section, it is easy to parametrize: we then have a point, defined over $k(t)$, on the generic horizontal section (which is a conic). This conic can then be parametrized by the method of stereographic projection (see [43]). Interpreting t as a variable again, we have found a parametrization of S.

Example 3.4.

The existence of cross sections depends heavily on the choice of the ground field k.

Theorem 3.5 (Noether). *In the complex case, any tubular surface has a cross section.*

A proof can be found in [26] or [17] (as a consequence of the more general theorem of Tsen-Lang). Nöther's proof is algorithmic, but the proposed algorithm is practically useless. It requires to solve a system of n algebraic equations in n unknowns, where n is of the order of $3\deg(S)$ (so we already have problems in degree 3).

A better algorithm was given in [32]. It requires to factor the coefficient polynomials A, B, C into linear factors, and to solve a system of n linear equations in n unknowns, where n is the same as in Nöther's method. For a good performance, one has to compute with floating point numbers (symbolic factorization into linear factors is too expensive if the degree is larger than 4).

Theorem 3.6 (Comesatti). *In the real case, a tubular surface has a cross section if and only if every horizontal cross section is non-empty.*

A proof can be found in [10] or [20]. Algorithms for computing cross sections can be found in [27, 32].

For the general case, we reduce to the case where A, B, C are squarefree and pairwise relatively prime (it is easy to compute such a reduction, see [32]).

Theorem 3.7. *Let A, B, C squarefree and relatively prime. Then the tubular surface $A(t)x^2 + B(t)y^2 + C(t) = 0$ has a cross section if and only if the following conditions are fulfilled.*

a) $-AB$ is a square in the field $k[t]/\langle C\rangle$.

b) $-BC$ is a square in the field $k[t]/\langle A\rangle$.

c) $-AB$ is a square in the field $k[t]/\langle B\rangle$.

d) If the parities of $\deg(A), \deg(B), \deg C$ are not all equal, then the quadratic form $\mathrm{lc}(A)x^2 + \mathrm{lc}(B)y^2 + \mathrm{lc}(C)z^2$ has a nontrivial solution.

For the proof, we refer to a forthcoming joint paper with Hillgarter/Landsmann/Winkler. An algorithm which computes a cross section can be found in [19].

Even if there exists no cross section, it is sometimes possible to construct a parametrization.

Theorem 3.8. *In the real case, any tubular surface is unirational.*

For the proof, we introduce the concept of the *spine* of a tubular surface, which is the image of the projection to the t-axis (disregarding isolated points).

Proof. Let S be a tubular surface. Let $[a,b]$ be an interval which is contained in the spine. By substituting $t = \frac{as^2+b}{s^2+1}$ in the equation of S and clearing denominators, we get a tubular surface S' and a rational map

$$f : S' \to S, (x,y,s) \mapsto (x,y,\frac{as^2+b}{s^2+1}).$$

The tubular surface S' has full spine, therefore it has a cross section by theorem 3.6. Thus, S' has a parametrization $p : \mathbf{R}^2 \to S'$. Then the composition $f \circ p : \mathbf{R}^2 \to S$ is a parametrization of S. □

Example 3.9.

The parametrizations constructed by theorem 3.8 are always improper. Here is a theorem which allows to decide the existence of a proper parametrization.

For any real algebraic variety V, we define the number of dp-connected components as the number of connected components of the desingularization of the projectivization of V.

Theorem 3.10. *A tubular surface S is rational if and only if its spine is connected as a subset of $\mathbf{P}^1$, if and only if S is dp-connected.*

An algorithmic proof can be found in [34].

4 Del Pezzo Surfaces

A surface is called del Pezzo surfaces iff its generic plane section is elliptic and it has irregularity zero (see [37] for a definition of irregularity). Typical examples are the nonsingular cubic and the nonsingular intersections of two quadrics in $\mathbf{P}^4$. The degree of a del Pezzo surfaces is very small (less than or equal to 9).

In the complex case, a del Pezzo surface has several pencils of conics. For example, a nonsingular cubic has 27 lines; for each line L, the sections with planes through L cut out a pencil of conics. Therefore we can reduce to the previous case of tubular surfaces.

Computationally, it is quite complicated to find such a pencil. In the case of the nonsingular cubic, this requires to solve a univariate equation of degree 27 (see [4] for details).

The real classification is as follows.

Theorem 4.1. *A del Pezzo surface has at most five dp-connected components.*

A dp-connected del Pezzo surface is rational.

A del Pezzo surface with two or three components is not rational, but it has a pencil of conics.

A del Pezzo surface with four components does not have a pencil of rational curves (and is therefore not rational), but it is unirational.

A del Pezzo surface with five components does not have a pencil of rational curves.

Proof. See [24, 39]. □

As a consequence, we obtain the following result.

Theorem 4.2. *A real algebraic surface is rational if and only if it satisfies Castelnuovo's criterion $p_a = P_2 = 0$ and it is dp-connected.*

Proof. Necessity follows from the birational invariance of the number dp-connected components (see [5]). Sufficiency follows from theorem 3.10, the above classification, and theorem 2.2. □

The question of unirationality is still open (the problem consists of the del Pezzo surfaces with five components).

For other ground fields (e.g. the rational numbers), neither the parametrization problem nor the proper parametrization problem could be solved completely. An overview on the existing results can be found in [25].

5 Adjoints

The construction of a pencil of rational curves, the construction of a birational reduction to a del Pezzo surface, and the computation of the numbers p_a and P_2 can be accomplished with one single tool: adjoints.

Definition 5.1. A polynomial is an m-adjoint iff it vanishes with order at least $m(r-1)$ at each r-fold singular curve, and it vanishes with order at least $m(r-2)$ at each r-fold singular point.

Infinitely near singularities have to be taken into account. We consider only polynomials that are reduced with respect to the equation of S, i.e. we compute modulo the ideal of the surface.

For any two natural numbers n, m, we define the vector space $V_{n,m}$ as the space of all m-adjoints of degree at most $n + m(d-4)$, where d is the degree of the given surface S.

For any m, the dimension of the vector-space $V_{0,m}$ is a birational invariant, the m-th *plurigenus* P_m. Another birational invariant, the arithmetic genus p_a, can be computed by the formula

$$p_a = d + 2\dim(V_{1,1}) - \dim(V_{2,1}) - 1,$$

where d is the degree of the surface S (see [32]).

For each n, m, such that $\dim(V_{n,m}) > 0$, we define a rational map

$$f_{n,m} : S \to \mathbf{P}^{\dim V_{n,m}-1},$$

$$(x, y, z) \mapsto (p_1(x, y, z) : \ldots : p_{\dim V_{n,m}}(x, y, z)),$$

where $p_1, \ldots, p_{\dim V_{n,m}}$ is a basis for $V_{n,m}$. The choice of the basis is not important.

Theorem 5.2 (Schicho). *If $p_a = P_2 = 0$, then one can find n, m such that one of the following holds.*

1. *$f_{n,m}$ is a map to a rational curve with rational curves as fibers.*
2. *$f_{n,m}$ is a birational map to a del Pezzo surface.*
3. *$f_{n,m}$ is a birational map to a projective plane.*
4. *$f_{n,m}$ is a birational map to a quadric surface in $\mathbf{P}^3$.*

More precisely, let μ be the smalles integer such that $V_{1,\mu+1}$ is the zero space. (One can show that such a μ always exists.) Then (n, m) is one of $(1, \mu)$, $(1, \mu - 1)$, or $(2, 2\mu + 1)$.

We discuss the four cases separately.

1. Here we can construct a parametrization in two steps, each one requiring a curve parametrization. First, we have to parametrize the image of $f_{n,m}$, in order to construct the structure map $f : S \to \mathbf{P}^1$. Second, we parametrize the generic fiber $f^{-1}(t : 1)$, as described in section 3.
2. Any parametrization of the equivalent del Pezzo surface can be propagated to a parametrization of the input surface. The degree of a del Pezzo surface is small, however they are still sometimes hard to parametrize (see section 4).
3. Here, we just have to invert the birational map $f_{n,m}$ (see [38, 30] for algorithms that invert birational maps).
4. Here, we observe that the stereographic projection from a nonsingular point on a quadric surface is birational onto the plane. By composing $f_{n,m}$ with this projection, we reduce to the previous case.

All subalgorithms that are needed in the four cases above have been implemented in Maple [18].

The full proof of theorem 5.2 is too long to be reproduced here, so we refer to [32]. Instead, we give the "secret definition" of adjoints, which gives an idea why the birational behavior of adjoints is so good. For a general introduction to this theory, see [45].

Definition 5.3. A rational differential form is said to be *of the first kind* iff its integral over any compact set is finite.

Differential forms of the first kind are regular at all regular points, but they may have poles of restricted order at the singularities.

Example 5.4. Let S be the quadric cone $xy - z^2 = 0$. Then

$$\frac{dx \wedge dy}{z} = x^{-1/2} y^{-1/2} dx \wedge dy$$

is a differential form of the first kind. Since

$$\frac{dx\wedge dy}{z} = \frac{2dx\wedge dz}{x} = \frac{2dz\wedge dy}{y},$$

we see that the singularity has an isolated pole at the vertex $(x, y, z) = (0, 0, 0)$.

Theorem 5.5. *Let $F(x, y, z)$ be the equation of the surface S. Then the polynomial $G(x, y, z)$ is a 1-adjoint iff*

$$\frac{Gdx\wedge dy}{\partial F/\partial z}$$

is a differential form of the first kind.

In the above example, we verified that $\frac{dx\wedge dy}{\partial F/\partial z}$ is a differential form of the first kind. Because a polynomial multiple of a differential form of the first kind is again of the first kind, we see that any polynomial is a 1-adjoint of the quadric cone. This agrees with the definition, because the only singularity of the quadric cone is a double point, which yields no restriction in the condition of adjunction.

Differential forms of the first kind have very good birational behavior.

Theorem 5.6. *Any proper birational map induces an isomorphism of the vector-spaces of rational differential forms of the first kind.*

6 Resolution of the Singularities

The computation of adjoints is the most expensive part of the parametrization algorithm. The problem is that one has to resolve the singularities, in order to detect the infinitely near singularities.

Here is a recursive algorithm that computes adjoints.

Input: A surface F in 3-space, given implicitly.
A basis for a vectorspace V of polynomials.
An integer $m > 0$.
Output: A basis for the subvectorspace of m-adjoints.

Algorithm *AdjointSubspace*(F, V, m):
    if F is smooth then
        return V;
    else
        choose a singularity S;
        $r :=$ the multiplicity of F at S;
        if S is a curve then
            $p := m(r-1)$;
        else (S is a point)

$p := m(r-2)$;
$V_1 :=$ the subvectorspace of the elements that vanish with order $\geq p$ at S;
$(\tilde{F}, e) := (BlowUp(F,S)$, equation of the exce ptional divisor);
$f : V_1 \to \{$forms on $\tilde{F}\} :=$ division by $e^p \circ$ pullback;
$V_2 := AdjointSubspace(\tilde{F}, f(V_1), m)$;
return $f^{-1}(V_2)$;

The problem of choosing a singularity to be blown up in such a way that the above algorithm terminates is called the problem of *resolution of singularities*. It is in itself a famous problem.

The existence was proven by Walker and Zariski in the 30's. Hironaka gave an existence proof for arbitrary dimension. From a constructive point of view, the breakthrough was by Villamayor and Encinas, who gave explicit strategies [42, 13]. Bodnar and the author filled in the necessary details [6] and implemented the algorithm in Maple, downloadable at the web site `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup`.

The input of this algorithm is a polynomial in n variables, representing a singular hypersurface. The output is a tree of charts, where

- the root node is $\mathbf{A}^n$, the ambient space of the given hypersurface,
- the children of each node are obtained from the parent node by blowing up, or by replacement by an open covering, or by a change of coordinates,
- the proper transform of the given hypersurface in the leaf nodes is nonsingular, and the total transform is a normal crossing divisor.

Unfortunately, a closer look at the performance of the algorithm shows that the algorithm is only of theoretical interest. No bound for the asymptotical complexity is known. Even for small examples, the computing time and the size of the output are enormous.

We have designed a faster algorithm to compute adjoints (joint work with W. Stöcher). We proceed in two steps: first, we resolve the discriminant by blowing ups, thereby reducing to the case where all singularities are quasi-ordinary. Second, we analyze the quasi-ordinary case by computing the two-dimensional Puiseux expansions [2].

The fact that we compute with infinite series also requires more sophisticated data types. For instance, power series have to be represented as black box procedures, producing the n-th coefficient when fed with the integer n.

To show how to compute with this data structure, we present the Maple code of an algorithm that factors an element in $F \in k[x][[t]]$ by lifting a factorization of $F(x,0) \in k[x]$. The program is actually a functor which takes an input procedure and produces an output procedure. The code was written by W. Stöcher.

```
# Lift the factor f=p^k               (repr. as [p,k]) of P(0)
# to a factor F of the power series P (repr. as a proc. n->coeff(P,x^n)),
# and return factor together with rest R=P/F
#                      (repr. as a proc. n->[coeff(F,x^n),coeff(P,x^n)]).
```

```
#
# precondition: f divides P(0), f and P(0)/f are rel. prime and
#               p and the resulting factor must be monic in x.
#               The resulting factor will always be monic in x, if P is.
#
# examples:
#   > P := n -> coeff(y^2-1-x,x,n):
#   > F := Lift([y-1,1],P,y):
#   > seq(F(i),i=0..2);
#        [y-1,y+1],[-1/2,1/2],[1/8,-1/8]
#
Lift := proc(f::[polynom,posint], P::procedure, x::name)
  local F0, R0, coF0, coR0;

  # consider only one root for each irreducible factor of P(0)
  # F0 corresponds to the root, R0 is the cofactor.
  # Now find coF0 and coR0 such that F0*coF0+R0*coR0=1

  F0   := (x-RootOf(f[1],x))^(f[2]);
  R0   := simplify(P(0)/F0);
  gcdex(F0,R0,1,x,'coF0','coR0');    # we know: gcd(F0,R0)=1

  # Construct the two Weierstrass polynomials F and R such that F*R=P
  # as a procedure giving the coefficients in pairs.
  # "procname" is a maple keyword, that allows recursive calls for
  # anonymous procedures.

  proc(n::nonnegint)
    local i,G,Fn,Rn;
    option remember;  # remember previously computed coefficients
    if n = 0 then
      [F0,R0]
    else
      G  := simplify(P(n) - `+`(seq(procname(i)[1] * procname(n-i)[2],
                                i=1..n-1)));
      Fn := simplify(rem(coR0*G,F0,y));
      Rn := simplify((G-R0*Fn)/F0);
      [Fn,Rn]
    fi
  end
end;
```

7 Degree Bounds

We define the degree of a parametrization $(\frac{X_n(s,t)}{D(s,t)}, \frac{Y_n(s,t)}{D(s,t)}, \frac{Z_n(s,t)}{D(s,t)})$ as the maximum of the total degrees of the four polynomials X_n, Y_n, Z_n, D.

In the curve case, it is easy to show that the degree of a proper parametrization is equal to the degree of the equation of the curve, and that the degree of an improper parametrization is larger. In the surface case, the degree of a parametrization can be lower, as shown by the following example.

Example 7.1. The surface of degree n^2 with equation

$$(x^n - y)^n - z = 0$$

has a parametrization of degree n, namely

$$(X, Y, Z) = (t, s + t^n, s^n).$$

The parametrization in this example reaches a lower bound.

Theorem 7.2. *If P is a parametrization of the surface S, then* $\deg P \geq \sqrt{\deg S}$.

Proof. Suppose that P is an e : 1-parameterization of F, i.e. a generic point of F is reached by e points in the parameter space. We may assume that X_n, Y_n, Z_n, D do not have a common divisor (otherwise we may cancel by this divisor and obtain a parameterization of strictly smaller degree). Let E_1, E_2 be two generic planes in k^3. Then the line $E_1 \cap E_2$ intersects F in exactly d points. The preimage of the set of these points is a set Σ of $e \cdot \deg F$ points. On the other hand, Σ is the set of common zeroes of the two linear combinations Q_1, Q_2 of X_n, Y_n, Z_n, D corresponding to E_1 and E_2 minus the set of common zeros of X_n, Y_n, Z_n, D. Since X_n, Y_n, Z_n, D do not have a common factor, and Q_1, Q_2 are generic linear combinations, Q_1 and Q_2 do not have a common factor. By Bezout's theorem, the number of common zeroes is less or equal $(\deg P)^2$. Hence $\deg F \leq e \cdot \deg F = \mathrm{card}(\Sigma) \leq (\deg P)^2$. □

On the other hand, there are rational surfaces which do not have a small parametrization.

Example 7.3. The surface of degree $2n - 1$ with equation

$$x^{2n-1} - y^n - z^2 = 0$$

has a parametrization of degree n^2, namely

$$(X, Y, Z) = (\frac{1}{s^n + t^2}, \frac{s}{(s^n + t^2)^2}, \frac{t}{(s^n + t^2)^n}).$$

One can show that then $\deg P \geq \frac{n^2}{2}$ for any parametrization. So, the degree of the smallest parametrization grows quadratically.

By an analysis of the algorithm [32], one can obtain the following upper bound. It is valid for the complex case only.

Theorem 7.4. *Let S be a rational surface. Then there exists a parameterization P of S, such that*

$$\deg P \leq 2(d^2 - 4d + 1)^2 + 2d + 2,$$

where $d := \deg S$.

For the proof, we refer to [33].

There are no known series of rational surfaces where the degree of the smallest possible parametrization grows larger than quadratically. Hence, it is quite possible that the upper bound above can be improved a lot.

Another open question is to compute the smallest possible parametrization for a given rational surface. The algorithm [32] is guaranteed to produce a parametrization which is at most twice as big as the smallest possible parametrization (see [33]).

References

[1] Abhyankar, S. S., and Bajaj, B. Automatic parametrization of curves and surfaces III. *Computer Aided Geometric Design 5-4* (1987), 309–323.

[2] Alonso, M. E., Luengo, I., and Raimondo, M. An algorithm on quasi-ordinary polynomials. *Applicable Algebra in Engineering, Communication and Computing 6* (1988), 59–73.

[3] Arrondo, E., Sendra, J., and Sendra, J. R. Parametric generalized offsets to hypersurfaces. *Journal of Symbolic Computation 23* (1997), 267–285.

[4] Bajaj, C., Holt, R., and Netravali, A. Rational parametrizations of nonsingular cubic surfaces. Tech. rep., Computer Science Department, Purdue University, West Lafayette, IN 47907, 1997.

[5] Bochnak, J., Coste, M., and Roy, M.-F. *Géometrie Algébrique Réelle*. Springer, 1987.

[6] Bodnar, G., and Schicho, J. Automated resolution of singularities for hypersurfaces. Tech. Rep. 99-06, RISC-Linz, Univ. Linz, A-4040 Linz, 1999.

[7] Canny, J., and Manocha, D. Implicit representation of rational parametric surfaces. *Journal of Symbolic Computation 13*, 5 (1992).

[8] Castelnuovo, G. Sulle superficie di genere zero. In *Memoria scelte*. Zanichelli, 1939, pp. 307–334.

[9] Chionh, E.-W., and Goldman, R. Degree, multiplicity and inversion formulas for rational surfaces using u-resultants. *Comp. Aided Geom. Des. 9* (1992), 93–108.

[10] Comesatti, A. Fondamenti per la geometria sopra le superficie rationali del punto di vista reale. *Math. Ann. 73* (1912), 1–72.

[11] Degen, W. On the projective differential geometry of surfaces that are generated by a one parameter family of conics I (in german). *Math. Ann. 155* (1964), 1–34.

[12] Degen, W. On the projective differential geometry of surfaces that are generated by a one parameter family of conics II (in german). *Math. Ann. 170* (1967), 1–36.

[13] Encinas, S., and Villamayor, O. Good points and constructive resolution of singularities. *Acta Math. 181* (1998), 109–158.

[14] Enriques, F. Sulle irrazionalita da cui puo farsi dipendere la risoluzione d'un equazione f(xyz)=0 con funzioni razionali di due parametri. *Math. Ann.* (1895), 1–23.

[15] Gao, X.-S., and Chou, S.-C. Implicitization of Rational Parametric Equations. *J. Symb. Comput. 14* (1992), pages 459–470.

[16] González-Vega, L., and Trujillo, G. Implicitization of parametric curves and surfaces by using symmetric functions. In *Proceedings of the ISSAC'95, Montreal* (1995), pp. 180–186.

[17] Greenberg, M. J. *Lectures on forms in many variables.* Benjamnin, 1969.

[18] Heiss, W. An implementation for the parametrization of algebraic surfaces. Master's thesis, RISC-Linz, University of Linz, 1999.

[19] Hillgarter, E. Rational points on conics. Master's thesis, RISC Linz, 1996.

[20] Iskovskih, V. Rational surfaces with a pencil of rational curves. *Math. USSR Sb. 3* (1967), 563–587.

[21] Kalkbrener, M. Implicitization of Rational Curves and Surfaces. In *Lect. Notes in Comp. Sci. 508, AAECC-8* (Tokyo, Japan, Aug. 1990), Sakata, Ed.

[22] Mñuk, M. *Algebraic parametrization of rational curves.* PhD thesis, RISC Linz, 1996.

[23] Mñuk, M., Sendra, J. R., and Winkler, F. On the complexity of parametrizing curves. *Beitr. Algebra und Geometrie 37/2* (1996), 309–328.

[24] Manin, Y. Rational surfaces over perfect fields. *Math. USSR Sb. 1* (1967), 141–168.

[25] Manin, Y. *Cubic Forms.* North-Holland, 1974.

[26] NÖETHER, M. Über Flächen, welche Scharen rationaler Kurven besitzen. *Math.Ann. 3* (1870), 161–227.

[27] PETERNELL, M. *Rational parametrizations for envelopes of quadric families.* PhD thesis, Techn. Univ. Vienna, 1997.

[28] PETERNELL, M., AND POTTMANN, H. Computing rational parametrizations of canal surfaces. *Journal of Symbolic Computation 23* (1997), 255–266.

[29] SCHICHO, J. On the choice of pencils in the parametrization of curves. *J. Symb. Comp. 14*, 6 (1992), 557–576.

[30] SCHICHO, J. Inversion of rational maps with gröbner bases. In *Gröbner bases and applications* (1998), B. Buchberger and F. Winkler, Eds., Cambridge Univ. Press.

[31] SCHICHO, J. Rational parameterization of real algebraic surfaces. In *ISSAC-98* (1998), ACM Press, pp. 302–308.

[32] SCHICHO, J. Rational parametrization of surfaces. *J. Symb. Comp. 26*, 1 (July 1998), 1–30.

[33] SCHICHO, J. A degree bound for the parameterization of a rational surface. *J. Pure Appl. Alg.* (1999). to appear.

[34] SCHICHO, J. Proper parametrization of real algebraic surfaces. Tech. Rep. 99-19, RISC-Linz, Univ. Linz, A-4040 Linz, 1999.

[35] SENDRA, J., AND WINKLER, F. Symbolic parametrization of curves. *Journal of Symbolic Computation 12*, 6 (1991), 607–632.

[36] SENDRA, J., AND WINKLER, F. Parametrization of algebraic curves over optimal field extensions. *Journal of Symbolic Computation 23*, 2/3 (1997), 191–208.

[37] SHAFAREVICH, I. R. *Algebraic geometry.* Springer, 1974.

[38] SHANNON, D., AND SWEEDLER, M. Using gröbner bases to determine algebra membership, split surjective algebra homomorphisms and determine birational equivalence. *Journal of Symbolic Computation 6*, 2/3 (1988), 267–273.

[39] SILHOL, R. *Real Algebraic Surfaces.* Springer, 1980.

[40] VAN HOEIJ, M. Computing parametrizations of rational algebraic curves. In *ISSAC-94* (1994), ACM Press, pp. 187–190.

[41] VAN HOEIJ, M. Rational parametrizations of algebraic curves using canonical divisors. *Journal of Symbolic Computation 23* (1997), 209–227.

[42] VILLAMAYOR, O. Introduction to the algorithm of resolution. In *Algebraic geometry and singularities, La Rabida 1991* (1996), Birkhäuser, pp. 123–154.

[43] WALKER, R. J. *Algebraic Curves*. Springer, 1978.

[44] ZARISKI, O. Castelnuovos criterion of rationality $p_a = P_2 = 0$ of an algebraic surface. *Ill. J. Math. 2* (1958), 303–315.

[45] ZARISKI, O. *Algebraic surfaces*, 2 ed. Springer, 1971.