An Implementation for the Rational Parametrization of Surfaces

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Abstract

The parametrization problem for rational algebraic surfaces had been solved algorithmically only for a very limited class of surfaces until the first complete algorithm was formulated and proven in 1995. This algorithm parametrizes an implicitly given surface by first computing a birational map to an easier-to-parametrize surface of one of five different types. Then this surface is parametrized and the parametrization is composed with the inverse of the birational map to obtain a parametrization of the original surface.
The method used by the parametrization algorithm is based on the concept of adjoints. In the first step the parametrization algorithm computes adjoint vectorspaces of a given surface to construct a birational map to a simpler surface. The second part of the parametrization then consists in the parametrization of the birationally equivalent surface.
The topic of this thesis was the implementation of this second part of the entire algorithm. The goal was reached, i.e. we now have an implementation computing the parametrization of a given rational surface, using an interface to an algorithm that computes the adjoint vectorspaces of an implicitly given surface.
We will not go into technical details for the first part of the algorithm, adjoints computation. We will just briefly sketch an algorithm and use adjoints as a black box for the rest of the paper.
During the implementation several theoretical and practical changes and refinements turned out to be useful. We describe here this second part of the algorithm in detail, discuss the underlying theory and practical aspects of the implementation. Detailed proofs for the various subalgorithms we use are given.

Eidesstattliche Erklärung

Hiermit erkläre ich an Eides statt, daß ich die vorliegende Diplomarbeit selbständig verfaßt habe. Ich habe keine anderen als die angegebenen Quellen und Hilfsmittel verwendet und mich auch sonst keiner unerlaubten Hilfsmittel bedient.

Innsbruck, Juni 1999

Werner Heiß

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Chapter 0

Introduction

The usual way algebraic surfaces in three dimensional space are given is by specifying one polynomial equation in three variables. The surface consists of all points that fulfill the equation. This is called the *implicit representation* of an algebraic surface. A triple of rational functions in two variables is called a *parametrization* of this surface if the image of the rational map defined by these functions is a two-dimensional subset of the surface.

As an example consider the general solution of the equation of the unit sphere $x^2 + y^2 + z^2 - 1 = 0$. It can be written as

$$(x, y, z) = (\frac{2s}{s^2 + t^2 + 1}, \frac{2t}{s^2 + t^2 + 1}, \frac{s^2 + t^2 - 1}{s^2 + t^2 + 1}).$$

For any two (real) values of s, t, one gets a point on the sphere. Vice versa, any point on the sphere except the north pole $(0, 0, 1)$ can be obtained this way. The variables s, t are called the *parameters* of the parametrization. Given an algebraic surface in implicit form the parametrization problem asks for a parametrization of the surface.

The implicit representation of a surface is convenient whenever we want to check whether a given point is lying on the surface. The main reason why the parametric representation is important is that an arbitrary number of points on the surface can be produced in little time by simply plugging in different values for the parameters. It is used for many tasks in computational geometry, applications are reliable surface plotting, motion display, computing cutter offset surfaces and many other.

First of all not any implicitly given surface allows a parametrization. If

the surface has a parametrization then it is called *unirational.* If there exists a parametrization in terms of a rational map such that this map possesses a rational inverse, i.e. is birational, then the surface is called *rational* and the birational map is a *proper* parametrization of the surface. The image of a proper parametrization is almost all of the surface, at most a one-dimensional subset of the surface is missing.
For instance, the above example is a proper parametrization. It has the stereographic projection

$$(s,t) = (\frac{x}{1-z}, \frac{y}{1-z})$$

as inverse.
A theorem of Castelnuovo (1939) gives a necessary and sufficient condition for the existence of a parametrization over algebraically ground fields of characteristic zero. It says that a surface has a parametrization iff it has a proper parametrization iff both the arithmetical genus p_a and the second plurigenus P_2 vanish. A definition and discussion of these notions can be found in [Sh]. For us it is only important to know that these numerical invariants can be computed algorithmically. An algorithm can be found in A.3.

Whereas in the curve-case the parametrization-problem has been solved algorithmically some time ago ([Wa], [SeWi],[Sch4]), for the surface-case only algorithms for limited classes of surfaces were known until [Sch1], where the first complete algorithm was formulated. This algorithm uses the concept of adjoint functions as the main tool. Adjoints were introduced by Clebsch and Noether in the last century and have played a fundamental role in the surface theory of the Italian school.
Adjoints computation is a difficult problem and the only algorithm known is the one in [Sch1]. This algorithm is based on an algorithm for the resolution of singularities of Hironaka ([Hi]).

In [Sch1] an algorithm was formulated in pseudo-code and has been refined in [Sch2]. There was no implementation for it up to now. It is the purpose of an ongoing project at RISC-Linz to provide such an implementation in the CASA-package, which is part of the computer algebra program Maple.
The "adjoints-project" consist of three main parts: the implementation of an algorithm for the resolution of singularities, the implementation of the algo-

rithm that does the adjoints computation and finally the implementation of the algorithm that actually computes a parametrization of a given surface.
It was my part to do the implementation of the rational parametrization algorithm and this diploma thesis can be seen as a kind of report on the theoretical and practical work that had to be done for it. Several proofs had to be worked out in more detail and some of the algorithms had to be changed slightly to be possible to implement them.
While the implementation was written in Maple-code I decided to describe the algorithms in this paper in pseudo-code; on the one side all of them are fully detailed and it should be able to translate them without problems into any code of a suitable computer algebra package that is able to perform Gröbner basis computation. On the other hand if the reader is interested in using the Maple-code please contact Josef Schicho (jschicho@risc.uni-linz.ac.at) or me.
Throughout the work I used the adjoints-algorithm as a black box, therefore it will not be described in more detail in this paper. The necessary information about adjoints can be found in chapter 1, a description of the algorithm can be found in A.3.

Problem Specification and Sketch of Solution Method The parametrization algorithm described in this paper works in the projective setting, i.e. the surface equation is a homogeneous polynomial in 4 variables over some ground field k. Homogenization and dehomgenization are simple computations and do not lead into trouble. Generally the given surface has to be absolutely irreducible (that is it remains irreducible over the algebraic closure of the ground field k).
We also require a non-singular point lying on the surface as input. In some cases this facilitates the computation notably. Algorithms for computing such a point are known but usually these algorithms compute points that are not rational. Since the performance of the parametrization algorithm described here depends strongly on the field containing the coordinates of the point we decided to let the user have the control on the non-singular point used in the algorithm.
We assume here that the given surface has a parametrization. Here are the input and output specifications for the algorithm $ParametrizeWithAdjoints$:

Input: A homogeneous polynomial $F \in k[x, y, z, w]$,

representing a surface.
p a smooth point on the surface.
Output: A quadruple $(X : Y : Z : W) \in k[s,t]^4$
of bivariate polynomials,
representing a rational parametrization of the surface
or
A set G containing the equations of a Del Pezzo surface of degree ≤ 3.

ParametrizeWithAdjoints uses the adjoints algorithm to construct a birational map to one of the following types:

1. The projective plane.
2. A quadric surface in $\mathbf{P}^3$.
3. A surface with a pencil of lines.
4. A surface with a pencil of conics.
5. A Del Pezzo surface.

For each of these surfaces computing a parametrization is easier than for the original surface. We compute a parametrization of the birational equivalent surface above and then compose the maps to obtain a parametrization of the original surface.

Guideline for the Reading I tried to submit the most important information as soon as possible. Therefore the main algorithm is described in the first chapter. For the cases 3, 4 and 5 *ParametrizeWithAdjoints* forks to a subalgorithm that treats the special case. These algorithms are described in chapter 2. For case 2 an extra-algorithm was used in the original version in [Sch2]. During the work on this thesis it turned out that it can be reduced to case 1 directly. How this works can also be found in 2.
In case 1 we only have to invert the birational map to get a parametrization. Also in most of the other cases the corresponding subalgorithms construct a birational map from the surface to $\mathbf{P}^2$ and the parametrization is found by inverting this map. Therefore we also need an algorithm that computes the inverse of a birational map. This algorithm is described in chapter 3.
For cases 3 and 4 the parametrization is computed sequentially. To find the first parameter we use an algorithm *RationalNormalCurve* to parametrize a

rational normal curve. This algorithm can also be found in chapter 3.
The fifth case is an exception. Here *ParametrizeWithAdjoints* returns either a parametrization of the input surface or the equations of the Del Pezzo surface for certain types. For those types where only the equations are returned it is up to the user how to further proceed with this information.
These Del Pezzo surfaces can be parametrized and algorithms are known, see e.g. [Sch1]. But during the implementation work these algorithms turned out to be of minor practical use. For a general input they would either require too much memory or the running time would not be reasonable. Therefore we let the user decide which method is suitable for parametrizing a given surface in each of these cases.
The diagram on the next page is meant to give a simplifying overview on the information flow in the main algorithm.

I suppose that the potential reader has some knowledge about the concepts and notions used in this paper or is only interested in the algorithms and how they work. Therefore I postponed information on the underlying theory and concepts that are used in the first three chapters to several sections of the appendix. Only necessary information and essential proofs of the algorithms can be found in the main part.

Concerning notation there is not much to say. I tried to stick to the notation that is used in standard books on commutative algebra and algebraic geometry such as [Sh], [Har], [Ei] and others. Other notations that are used will be described at the appropriate place in the text.

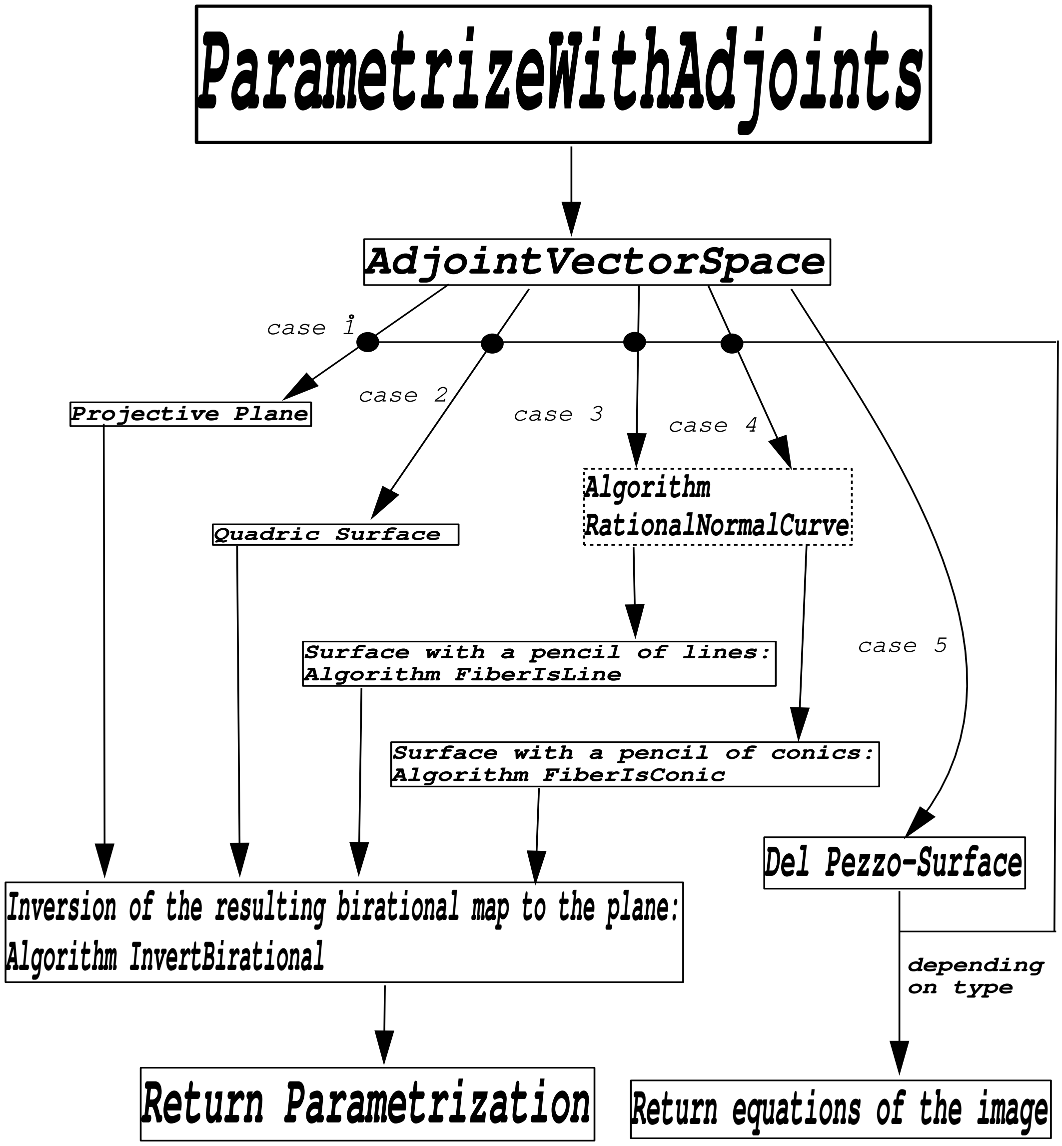

Figure 0.1: Scheme of the Algorithm

Chapter 1

The Parametrization Algorithm

In this chapter we describe the main algorithm *ParametrizeWithAdjoints*. This algorithm uses an algorithm *AdjointVectorSpace* to construct a birational map to a surface of one of the five types:

1. The projective plane.
2. A quadric surface in $\mathbf{P}^3$.
3. A surface with a pencil of lines.
4. A surface with a pencil of conics.
5. A Del Pezzo surface.

We briefly describe *AdjointVectorSpace* and give lemmas that tell us which of the five cases applies.

The Algorithm AdjointVectorSpace Throughout the work on this thesis this algorithm was used as a "black-box" and therefore will not be described in more detail at this point. More information about this and the concept of adjoints in general can be found in A.3 or in [Sch1].
The only thing we should know here is what we need to pass to this algorithm as an input and what we get from it as output. Here are the specifications of *AdjointVectorSpace* :

Input: A homogeneous polynomial $F \in k[x, y, z, w]$
of degree d, representing a surface.
Two integers n, m.
Output: A list of forms of degree $n + m(d - 4)$
(spanning a vector-space $V_{n,m}$)

Adjoint-Vector-spaces and Rational Maps Let F be an irreducible polynomial of degree d defining a surface X. A *form* on the surface is an equivalence class of polynomials modulo multiples of F, which we represent by homogeneous polynomials that are reduced modulo F (with respect to the lexicographical term order).

For an integer $m \geq 0$, a form G is an m-adjoint iff it vanishes with order $\geq m(r-1)$ at each r-fold curve singularity of F and with order $\geq m(r-2)$ at each r-fold point singularity of F. It is obvious that any form is a 0-adjoint.

Now m-adjoints of degree $n + m(d-4)$ form a finite dimensional vector-space $V_{n,m}$ of dimension $v_{n,m}$. We can represent $V_{n,m}$ by choosing a basis for it. Such a basis is the output of *AdjointVectorSpace* and for any m, n defines a rational map in the following way:
Suppose the dimension of $V_{n,m}$ is $k+1$. Then a basis consists of $k+1$ elements of $V_{n,m}$, say $\{v_0, \ldots, v_k\}$. We define the *associated map* $i_{V_{n,m}} : X \to \mathbf{P}^k$ by

$$i_{V_{n,m}} := (v_0 : \ldots : v_k).$$

The map $i_{V_{n,m}}$ is well defined. If we choose another basis for $V_{n,m}$ then this corresponds to making a projective change of coordinates for the image. For more information about the relation between vector-spaces and rational maps we again refer to the appendix (A.1).

1.1 The Five Cases

In this section we describe by what means we can decide which of the five cases listed in the preceding section applies.

We globally assume that the surface has positively passed a test-algorithm *DecideRationality*- an algorithm that checks whether or not a given surface X is rational, i.e. can be parametrized at all. If so this algorithm returns *true* otherwise we get *false* (and of course do not have to continue with the parametrization algorithm). *DecideRationality* is an application of *AdjointVectorSpace* and therefore described in the corresponding section A.3.

Lemma 1.1.1 *There is an integer m such that $v_{1,m} = 0$.*

We will make use of this fact in the main-algorithm *ParametrizeWithAdjoints* where we will compute all adjoint-vector-spaces $V_{1,0}, V_{1,1}, V_{1,2}, \ldots$ in a while-loop until we find $v_{1,m} = 0$ for a certain m- lemma 1.1.1 ensures termination of this process.
Throughout, let μ be the smallest integer with $v_{1,\mu+1} = 0$. We sometimes might call μ the **adjoint depth** and refer to the maximal sequence of nonzero vector-spaces

$$V_{1,0}, V_{1,1}, V_{1,2}, \ldots, V_{1,\mu}$$

as the **adjoint series** of the surface X. Note that μ cannot be -1, because $v_{1,0} = 4$.

Applying the following lemmas we are able to construct a birational map to a surface of one of the five types listed above.

Lemma 1.1.2 (case 1) *Suppose that $v_{1,\mu} = 3$ and $v_{2,2\mu} = 6$. Then $V_{1,\mu}$ defines a birational map to $\mathbf{P}^2$.*

Lemma 1.1.3 (case 2) *Suppose that $v_{2,2\mu+1} = 1$. Then $v_{1,\mu} = 4$, and $V_{1,\mu}$ defines a birational map to a quadric surface.*

Lemma 1.1.4 (case 3 and case 1) *Suppose that $v_{2,2\mu+1} \geq 2$. Let ϕ' be the map defined by $V_{2,2\mu+1}$. Then one of the following two cases holds.*

a) The image of the map ϕ' is a rational normal curve. The map defined by $V_{1,\mu}$ is birational and maps the generic fiber of ϕ' to a line.

b) We have $v_{2,2\mu+1} = 3$, and ϕ' is birational onto $\mathbf{P}^2$.

Lemma 1.1.5 (case 4) *Suppose that $v_{1,\mu} \geq 2$, and $v_{2,2\mu+1} = 0$, and either $v_{1,\mu} \neq 3$ or $v_{2,2\mu} \neq 6$. Let ϕ' be the map defined by $V_{1,\mu}$. Then the map defined by $V_{2,2\mu-1}$ is birational and maps the generic fiber of ϕ' to a conic.*

Lemma 1.1.6 (case 5a) *Suppose that $\mu \geq 2$ and $v_{1,\mu} = 1$.*
Then $2 \leq v_{1,\mu-1} \leq 10$, and one of the following holds.

a) *If* $v_{1,\mu-1}\geq 4$*, then* $V_{1,\mu-1}$ *defines a birational map to a Del Pezzo surface of degree* $v_{1,\mu-1}-1$.

b) *If* $v_{1,\mu-1}=3$*, then* $(V_{1,\mu-1};V_{2,2\mu-2})$ *defines a birational map to a Del Pezzo surface of degree 2.*

c) *If* $v_{1,\mu-1}=2$*, then* $(V_{1,\mu-1};V_{2,2\mu-2};V_{3,3\mu-3})$ *defines a birational map to a Del Pezzo surface of degree 1.*

Lemma 1.1.7 (case 5b) *Suppose that* $\mu\leq 1$ *and* $v_{1,\mu}=1$*. Then* $\mu=1$*, and* $4\geq v_{2,1}\geq 10$*, and* $V_{2,1}$ *defines a birational map to a Del Pezzo surface of degree* $v_{2,1}-1$.

In lemma 1.1.6 case *b)* and *c)* we work in weighted projective spaces (see section 2.3 for further information). In case *b)* $(V_{1,\mu-1};V_{2,2\mu-2})$ denotes a basis for the 4-dimensional vector-space spanned by the forms in $V_{1,\mu-1}$ and $V_{2,2\mu-2}$. To find it we take the forms in $V_{1,\mu-1}$ and look for a form in $V_{2,2\mu-2}$ not contained in $(V_{1,\mu-1})^2$ (the multiplication has to be understood as the multiplication of forms). The forms on the basis fulfill an algebraic relation of weighted degree 4, where the weights are $(w_0,w_1,w_2,w_3)=(1,1,1,2)$.
For case *c*) we proceed in an analogous way. We first search an element X_2 of $V_{2,2\mu-2}$ not contained in $(V_{1,\mu-1})^2$. Then we check for the fourth element in $V_{3,3\mu-3}$ linearly independent of $(V_{1,\mu-1})^3$ and $V_{1,\mu-1}\cdot X_2$. The resulting forms fulfill an algebraic relation of weighted degree 6 with weights $(w_0,w_1,w_2,w_3)=(1,1,2,3)$.

As one can easily check, the above lemmas cover all possibilities. For the proofs of the lemmas we refer to [Sch2].

1.2 The Main Algorithm

Here is an algorithm that tries to avoid unnecessary calls of the sub-algorithm *AdjointVectorSpace*.

Input: A homogeneous polynomial $F\in k[x,y,z,w]$,
representing a surface.
p a smooth point on the surface.
Output: A quadruple $(X:Y:Z:W)\in k[s,t]^4$

of bivariate polynomials,
representing a rational parametrization of the surface
or
A set G containing the equations of a Del Pezzo surface of degree ≤ 3.

Algorithm *ParametrizeWithAdjoints*(F):
$m := 0$;
while *AdjointVectorSpace*($F, 1, m+1$) is not zero do
$m := m + 1$;
$V :=$ *AdjointVectorSpace*($F, 1, m$); $r := \# \ V - 1$;
if $r = 0$ (**case 5**) then
$V' :=$ *AdjointVectorSpace*($F, 1, m-1$); $r' := \# \ V' - 1$;
if $m = 1$ (**case 5b**) then
$V' :=$ *AdjointVectorSpace*($F, 2, 1$); $r' := \# \ V' - 1$;
if $r' = 2$ then
$X_3 :=$ an element in *AdjointVectorSpace*($F, 2, 2m-2$),
linearly independent of V'^2;
append X_3 to V';
if $r' = 1$ then
$X_2 :=$ an element in *AdjointVectorSpace*($F, 2, 2m-2$),
linearly independent of V'^2;
$X_3 :=$ an element in *AdjointVectorSpace*($F, 3, 3m-3$),
linearly independent of V'^3, $V' \cdot X_2$;
append X_2, X_3 to V';
if $r' \leq 3$ then
G:=*DelPezzo*(F, V', r', p);
else
$(X : Y : Z : W) :=$ *DelPezzo*(F, V', r', p);
else if $r = 2$ and # *AdjointVectorSpace* ($F, 2, 2m$) $= 6$ (**case 1**) then
$(X : Y : Z : W) :=$ *InvertBirational*(F, V);
else if *AdjointVectorSpace*($F, 2, 2m+1$) is not zero do
$V' :=$ *AdjointVectorSpace*($F, 2, 2m+1$); $r' := \# \ V' - 1$;
if $r' = 0$ then (**case 2**)
V':= a basis of $V{\downarrow}p$;
$(X : Y : Z : W) :=$ *InvertBirational*(F, V');
else
$\Gamma(x_0, \ldots, x_{r'})$:= find the relations in V' (degree 2);
if $r' = 2$ and Γ is empty then (**case 1**)

$(X:Y:Z:W) :=$ *InvertBirational*(F, V');
else (**case 3**)
$(T:U) :=$ *RationalNormalCurve*(F,p,V');
$(X:Y:Z:W) :=$ *FiberIsLine*$(F,(T:U),V)$;
else (**case 4**)
$(T:U) :=$ *RationalNormalCurve*(F,p,V);
$V' :=$ *AdjointVectorSpace*$(F,2,2m-1)$;
$(X:Y:Z:W) :=$ *FiberIsConic*$(F,(T:U),V')$;
if G is not empty then
return G;
else
return $(X:Y:Z:W)$;

The algorithm is nothing more than an algorithmic formulation of the result of the lemmas given above. The only thing that might need some explanation is how to perform the computation of the image of a rational map. There are several ways how to do this. A description of the methods that can be used can be found in section A.1.
For case 2 we compute a subvectorspace $V{\downarrow}p$ of V that maps the surface to the projective plane. Here the image of the map V is a quadric surface in $\mathbf{P}^3$. By computing $V{\downarrow}p$ we basically obtain the composition of V and a projection map.
For an explanation of the notion $V{\downarrow}p$ and further information about this concept of *assigning base points* please look at section A.1.

Example 1.2.1 Let $F = x^2w^3 + y^3w^2 + z^5$ and $p = (0:1:1:0)$.

After the while loop, we have $m = 3$, because $V_{1,1}$, $V_{1,2}$, $V_{1,3}$ are not zero and $V_{1,4}$ is zero. Since $v_{1,3} = 1$, we have $r = 0$, and we are in the case (5). We have

$$V' = V_{1,2} = \langle zw^2, w^3\rangle$$

and $r = 1$, so we have a Del Pezzo surface of degree 1.

An element in $V_{2,4}$ which is not in $V_{1,2}^2$ is $X_2 = yw^5$. An element in $V_{3,6}$ which is not in $V_{2,4}{\cdot}V_{1,3}$ is $X_3 = xw^8$. We call *DelPezzo* with the input $(F,(zw^2 : w^3 : yw^5 : xw^8), 1, p)$. It computes the equation of the Del Pezzo surface - the weighted homogeneous algebraic relation between $X_0 = zw^2$,$X_1 = w^3$,X_2,X_3 :

$$G := \{x_0^5x_1 + x_2^3 + x_3^2\}.$$

Therefore the output of the algorithm is

$$\{x_0^5 x_1 + x_2^3 + x_3^2\}$$

Example 1.2.2 Let $F = (x^4 + z^4)w^2 + (xy + z^2)^3$ and $p = (-1 : 1 : 0 : 1)$.

After the while loop, we have $m = 1$, because $V_{1,1}$ is not zero and $V_{1,2}$ is zero. Then,

$$V = V_{1,1} = \langle z^2 w, xzw, x^2 w, xyz + z^3, x^2 y + x^2 z\rangle$$

and $r = 4$.

Next, we find that $V_{2,3}$ is not zero. We have

$$V' = V_{2,3} = \langle xz^5 w^2, z^6 w^2\rangle$$

and $r' = 1$. Therefore, we are in the case (3): The image of the map $\phi' = (xz^5 w^2 : z^6 w^2)$ is a rational normal curve C, and its generic fiber is mapped to a line by the birational map

$$\phi = (z^2 w : xzw : x^2 w : xyz + z^3 : x^2 y + x^2 z).$$

In this case, the rational normal curve is already $\mathbf{P}^1$, so the algorithm *RationalNormalCurve* just returns its input. We apply the algorithm *FiberIsLine* to the input

$$F, (xz^5 w^2 : z^6 w^2) = (x : z), \phi$$

and get

$$(X : Y : Z : W) = (-s^3 t^2 : t^3 + s^4 t + t : -st^3 : s^5 + s).$$

Chapter 2

Algorithms for the Distinct Cases

2.1 Quadric Surfaces

In this section we describe a general method to compute the birational inverse of a parametrization of a quadric surface. An algorithm is formulated in 2.1.1. This algorithm was used in the original version of *ParametrizeWithAdjoints*. In section 2.1.2 we describe the actual method used. It is based on the same but constructs the inverse map of a parametrization of the input surface of *ParametrizeWithAdjoints* directly. For this reason there is no need of an algorithm that parametrizes quadric surfaces separately any more.

2.1.1 Parametrization of Quadric Surfaces

A quadric surface in projective space is defined by a homogeneous polynomial of degree 2. Here we are dealing with quadric surfaces in $\mathbf{P}^3$, i.e. surfaces given by a homogeneous polynomial in four variables of degree 2. A good introduction to quadric surfaces can be found in e.g [Har].

In order to parametrize a quadric surface $F \in \mathbf{P}^3$ we compute a birational map from the surface to $\mathbf{P}^2$. The idea is to find a point p lying on the surface and then to project from p onto $\mathbf{P}^2$ (see A.4 for more information on projections). If p is a smooth point of F then the projection map will be birational.
Here is an algorithm for computing the inverse of a parametrization.

Input: A homogeneous polynomial $F \in k[x, y, z, w]$ of degree 2,
describing a quadric surface in $\mathbf{P}^3$.
Output: Three linear forms $(S : T : U)$ in x, y, z, w,
representing a birational map
from the surface to $\mathbf{P}^2$.

Algorithm *QuadricSurface*(F):
$p :=$ a smooth point on F;
$\{S, T, U\} :=$ 3 linearly independent linear forms vanishing at p;
return $(S : T : U)$

Example 2.1.1 Let $F = x^2 + y^2 + z^2 - w^2$, the equation of the unit sphere. For p, we choose the north pole $(0 : 0 : 1 : 1)$. The resulting map is the stereographic projection $(x : y : w - z)$.

In order to define the projection map from a point p on the surface to the projective plane, we need to find three linearly independent linear forms that vanish at p. It is easy to see why this is true if one views the projective plane as the set of all lines through the origin in affine 3-space. So by finding linearly independent linear forms vanishing at p what we do is nothing else but shifting p into the origin and mapping all points q on the surface to the line through p and q.

2.1.2 Direct Parametrization

Let F be the surface given as an input to *ParametrizeWithAdjoints* and let p be a non-singular point on it.
Suppose the birational map i_V obtained from *AdjointVectorSpace* maps F to a quadric surface. The following lemma gives us the key for constructing a parametrization of F without computing the equation of the image of i_V.

Lemma 2.1.1 *In the above situation the point $i_V(p)$ is a non-singular point on $i_V(F)$.*

We do not give a proof of this lemma here since this would require the introduction of lots of theory only for this purpose. However the lemma is a consequence of lemma 3.6 in [Sch1] where the reader can also find more information on the underlying theory.

Now we can make use of the fact that the sub-vector-space $V{\downarrow}p$ of V corresponds to the composition of i_V and the projection $\pi_{i_V(p)}(i_V(F))$ (see A.1). Since $i_V(p)$ is a non-singular point on $i_V(F)$ the map $\pi_{i_V(p)}$ is birational onto $\mathbf{P}^2$. Hence $i_{V\downarrow p}$ maps F birationally to the projective plane and we only need to invert $i_{V\downarrow p}$ to get a parametrization of F.

Remark 2.1.1 *In the early versions of* ParametrizeWithAdjoints *there was no non-singular point required as an input to the algorithm. This decision was taken later during the implementation.*
The improvement of avoiding the computation of the image of i_V and computing the inverse of a parametrization directly was possible only because of this change of the input specifications of the main algorithm.

2.2 Parametrization of a Generic Fiber

As we have seen in chapter 1 in two cases we construct a parametrization of a surface F sequentially.
To find the first parameter t, we will have a rational map

$$\pi_1 : F \rightarrow \mathbf{P}^1, (x : y : z : w) \mapsto (T : U)$$

with two polynomials T, U of the same degree, such that the generic fiber $F_t := {\pi_1}^{-1}(t : 1)$ is a rational curve. Since F_t is a projective space curve over the ground field $k(t)$, a parametrization of F_t would be of the form $(X : Y : Z : W)$, where X, Y, Z, W are polynomials in $k(t)[s]$. Since we are in the projective case, we can clear denominators. By changing the semantics of t from a generic constant to a variable, we obtain a parametrization of the surface F.
Maybe the following diagrams help for a better understanding of the situation:

$$\begin{tikzcd}
 & F \arrow[r, "\phi\ (birat.)"] \arrow[dl, dotted, "\pi_1'\ (rat.)"'] \arrow[ddl, dotted, "\pi_1\ (rat.)"] & \phi(F) \subset \mathbf{P}^r \arrow[d, dotted, "\pi_2 (rat.)"] \\
C \subset \mathbf{P}^k \arrow[d, "\pi_1''\ (birat.)"'] & & \mathbf{P}^1 \\
\mathbf{P}^1 & &
\end{tikzcd}$$

$$\begin{tikzcd}
F_t = \pi_1^{-1}(t : 1) \arrow[r, mapsto, "\phi"] \arrow[d, mapsto, "\pi_1"'] & C_t = \phi(F_t) \arrow[d, mapsto, "\pi_2"] \\
(t : 1) & \mathbf{P}^1
\end{tikzcd}$$

We start from a birational map ϕ of F to a surface in $\mathbf{P}^r$. The map ϕ is determined by a basis of a suitable adjoint-vector-space of F.
Then we will construct the map π_1 such that ϕ maps the generic fiber of π_1 to a projective curve $C_t \in \mathbf{P}^r$ in the following way:
Using a basis of a certain adjoint vector-space of F we will have a rational map $\pi_1' : F \rightarrow C$, where C is a rational normal curve. Applying the algorithm *RationalNormalCurve*(see section 3.2) we can find the inverse π_1'' of a parametrization of C and the map $\pi_1 := \pi_1'' \circ \pi_1'$.
It remains to find a parametrization of C_t. We know in advance that C_t is

either a line or a conic. To decide in which case we are we can use the lemmas in section 1.

In this section we describe the two tools for finding the second parameter s. This is the task of the algorithms *FiberIsLine* and *FiberIsConic*.

2.2.1 The Algorithm *FiberIsLine*

The case that C_t is a line is rather simple. Here we only need to project the line C_t in $\mathbf{P}^r$ birationally to $\mathbf{P}^1$. Then by inverting the composed map of F_t to $\mathbf{P}^1$ we will find the second parameter. Here is the algorithm:

Input: A homogeneous polynomial $F \in k[x, y, z, w]$,
representing the surface.
A pair of homogeneous polynomials $(T : U) \in k[x, y, z, w]^2$
of the same degree, representing a rational map
from the surface to $\mathbf{P}^1$.
An $r + 1$-tuple of homogeneous polynomials
$(P_0 : \ldots : P_r) \in k[x, y, z, w]^{r+1}$ of
the same degree, representing a birational map which
maps the generic fiber of $(T : U)$ to a line in $\mathbf{P}^r$.
Output: A quadruple $(X : Y : Z : W) \in k[s, t]^4$ of bivariate polynomials,
representing a parametrization of F.

Algorithm *FiberIsLine*$(F, (T : U), (P_0, \ldots, P_r))$:
$i := 0$;
$G' := \{1\}$;
while $G' = \{1\}$ do
$i := i + 1$;
$G := \{F, P_iU - s, P_0T - t, P_0U - 1\}$;
G':=the reduced Gröbner basis of G over $k(s, t)[x, y, z, w]$
with total degree order;
$(S, T', U') := (P_iU, P_0T, P_0U)$;
return *InvertBirational*$(F, (S : T' : U'))$;

In order to find two suitable coordinates for projecting the algorithm proceeds in the following way:
For the first coordinate P_0 is chosen. Then for finding the second coordinate P_i the algorithm computes the Gröbner basis G'. The set G is defined in the same manner as in the algorithm *InvertBirational* (see section 3.1.1). Here

$(P_iU : P_0T : P_0U)$ is the "candidate" for a birational map to the projective plane. This map comes from clearing denominator in $(\frac{P_i}{P_0} : \frac{T}{U} : 1)$ were $\frac{P_i}{P_0}$ is the second parameter we look for. If P_0 and P_i were not independent, then the image of the projection of C_t were a single point, and therefore $(P_iU : P_0T : P_0U)$ did not define a birational map from the surface F to $\mathbf{P}^2$. Hence there were no solutions for the equations in G. This means that the reduced Gröbner basis of G contained 1.

The algorithm *InvertBirational* computes the inverse map of $(P_iU : P_0T : P_0U)$. For more information about this algorithm see 3.1.1.

Example 2.2.1 *Let*

$$F = (x^4 + z^4)w^2 + (xy + z^2)^3,$$
$$(T : U) = (x : z),$$
$$(P_0 : P_1 : P_2 : P_3 : P_4) = (z^2w : xzw : x^2w : xyz + z^3 : x^2y + x^2z).$$

For finding two suitable components P_i, P_j we successively test all components starting at P_1. We set

$$G = \{F, xz^2w - s, xz^2w - t, z^3w - 1\}.$$

It is clear that for generic s and t there is no common solution of the polynomials in G, hence computation of the reduced Gröbner basis gives us

$$G' = \{1\}.$$

We go on and try P_2 in the same way, but again we get $G' = \{1\}$.
We have more luck with P_3. The reduced Gröbner basis of the ideal generated by

$$\{F, (xyz + z^3)z - s, xz^2w - t, z^3w - 1\}$$

is

$$\{z + t^4z + s^3w, yt + yt^5 - ws - t^4ws - s^3w,$$
$$x + xt^4 + ts^3w, 1 + 3t^4 + 3t^8 + t^12 + s^9w^4\}$$

Hence we have found the two projection components P_0 and P_3 and apply InvertBirational to the input

$$F = (x^4 + z^4)w^2 + (xy + z^2)^3,$$
$$((xyz + z^3)z : xz^2w : z^3w).$$

The output

$$(t^2s^3 : -s(1 + t^4 + s^2) : ts^3, -t(1 + t^4))$$

is a parametrization of F.

2.2.2 The Algorithm *FiberIsConic*

The second case in which we compute the parameters sequentially is the more difficult one. The strategy basically is the same as in *FiberIsLine* . Here we know that the image C_t of the generic fiber F_t is a conic in $\mathbf{P}^r$. Again we project the conic birationally to the plane $\mathbf{P}^2$. Then we find a parametrization of the projected conic in a similar way as we did it in section 2.1 for quadric surfaces. We compute a point on the conic and project birationally from this point to $\mathbf{P}^1$.
Composing the projection maps and the (restriction of) the birational map ϕ from F_t to C_t we get a birational map from the generic fiber F_t to $\mathbf{P}^1$ and by inverting the composed map we have found the second parameter s of a parametrization of F.
For being able to parametrize a conic we need to compute a rational point on it. This is the task of the algorithm *FindPoint*.

The Algorithm

Input: A homogeneous polynomial $F \in k[x, y, z, w]$,
representing the surface.
A pair of homogeneous polynomials $(T : U) \in k[x, y, z, w]^2$
of the same degree,
representing a rational map from the surface to $\mathbf{P}^1$.
An $r + 1$-tuple of homogeneous polynomials
$(P_0 : \ldots : P_r) \in k[x, y, z, w]^{r+1}$ of the same degree,
representing a birational map which maps the generic
fiber of $(T : U)$ to a conic in $\mathbf{P}^r$.
Output: A quadruple $(X : Y : Z : W) \in k[s, t]^4$ of bivariate polynomials,
representing a parametrization of F.

Algorithm *FiberIsConic*$(F, (T : U), (P_0, \ldots, P_r))$:
$i := 0$;
$G' := \{1\}$;
while $G' = \{1\}$ do
$i := i + 1$;
$G := \{F, P_i U - s, P_0 T - t, P_0 U - 1\}$;
G':=the reduced Gröbner basis of G over $k(s, t)[x, y, z, w]$
with total degree order;
$j := i + 1$;

$C := 0$
while $C = 0$ do
  $G := \{F, P_0 - p_0, P_i - p_i, P_j - p_j, T - t, U - 1\}$;
  I:=the ideal generated by G in $k(t)[x, y, z, w, p_0, p_i, p_j]$;
  C:=a homogeneous element of degree 2 in $I \cap k(t)[p_0, p_i, p_j]$
    if exists;
  $j := j + 1$;
(now C is a conic equation in the variables p_0, p_i, p_j
  with coefficients in $k(t)$)
$(Q_0 : Q_i : Q_j) := FindPoint(C)$;
  (Note that $Q_0, Q_i, Q_j \in k(t)$.)
if $Q_0 = 0$ then
  $S := \frac{P_0}{Q_i(\frac{T}{U})P_j - Q_j(\frac{T}{U})P_i}$;
else
  $S := \frac{Q_0(\frac{T}{U})P_i - Q_i(\frac{T}{U})P_0}{Q_0(\frac{T}{U})P_j - Q_j(\frac{T}{U})P_0}$;
(S', T', U') := clear denominators in $(SU : T : U)$;
return $InvertBirational(F, (S' : T' : U'))$;

Example 2.2.2 *Let*

$$F = x^2y^2 + 8x^3y + 4x^4 + xyz^2 - x^2z^2 - y^2w^2 - 7xyw^2 + 8x^2w^2,$$

$$(T : U) = (y : x),$$

$$(P_0 : P_1 : P_2 : P_3) = (x : y : z : w).$$

We find the second component in the same way we did it in the algorithm FiberIsLine. After the first while-loop we have $i = 2$. To find the third suitable component for projecting we compute a basis of

$$G := \{F, P_0 - p_0, P_2 - p_2, P_j - p_j, T - t, U - 1\}.$$

starting with $j = 3$ with respect to a term order that eliminates x, y, z, w. The basis is given by

$$\{p_0 - 1, -p_3^2t^2 + tp_2^2 - p_2^2 - 7tp_3^2 + 8p_3^2 + t^2 + 8t + 4, w - p_3, z - p_2, y - t, x - 1\}.$$

Eliminating x, y, z, w we get

$$\{p_0 - 1, -p_3^2t^2 + tp_2^2 - p_2^2 - 7tp_3^2 + 8p_3^2 + t^2 + 8t + 4\}.$$

Now the basis consists of 2 inhomogeneous elements, one of degree 1 and the second of degree 2. Composing these two elements we find the quadratic relation between P_0, P_2, P_3 .

$$C = (t^2 + 8t + 4)p_0^2 + (t - 1)p_2^2 + (-t^2 - 7t + 8)p_3^2 = 0.$$

We do not have the sub-algorithm FindPoint *yet, but it is easily checked that*

$$(Q_0 : Q_2 : Q_3) = (-t + 1 : 5t + 34 : t + 12)$$

is a rational point on C. Since $Q_3 \neq 0$ we define the second parameter

$$S = \frac{(-\frac{y}{x} + 1)z - (5\frac{y}{x} + 34)x}{(-\frac{y}{x} + 1)w - (\frac{y}{x} + 12)x} =$$

$$= \frac{-yz + xz - 5xy - 34x^2}{-yw + xw - xy - 12x^2}.$$

By clearing denominators, we find

$$(S' : T' : U') =$$

$$(-xyz + x^2z - 5x^2y - 34x^3 : -y^2w + xyw - xy^2 - 12x^2y :$$
$$: -xyw + x^2w - x^2y - 12x^3).$$

We apply InvertBirational *and find*

$$(X : Y : Z : W) = (t^2 + 7t - s^2t + s^2 - 8 : t^3 + 7t^2 - s^2t^2 + s^2t - 8t :$$
$$: -5t^2 + 2st^2 - 74t - 5s^2t + 40st + 192s - 34s^2 - 272 :$$
$$: t^2 - 10st + 20t + s^2t + 12s^2 - 68s + 96).$$

Correctness of the Algorithm In the algorithm *FiberIsConic* we have to find 3 suitable components of the map $(P_0 : \ldots : P_r)$ in order to project the conic in $\mathbf{P}^r$ birationally to the plane.
Again we simply take P_0 for the first one and then search for the others. For finding the second component we can use the same arguments as in *FiberIs-Line* and proceed in the same way.
For finding a third independent component in the second while-loop we intersect the ideal $I \subset k(t)[x, y, z, w, p_0, p_i, p_j]$ generated by the set $G :=$

$\{P_0 - p_0, P_i - p_i, P_j - p_j, F, T - t, U - 1\}$ with $k(t)[p_0, p_i, p_j]$ and then try to find a homogeneous polynomial of degree two (the conic equation) in this intersection. If this is possible then we took the right third component P_j.
To see why we can proceed in this way consider the following diagram; we denote $X := ZS(< F, T - t, U - 1 >_{K(t)[x,y,z,w]}) \subset K(t)^4$ and $Y := ZS(I \cap k(t)[p_0, p_i, p_j]) \subset K(t)^2$.
By $\sim$ we denote the projective equivalence relation, i.e. $\sim$ is the identification of all points of an affine space that are on the same line through the origin; finally $F_t = \overline{X_{/\sim}} \subset \mathbf{P}^3{}_{K(t)}$ denotes the projective closure of the points identified that way and similarly we define $\tilde{C}_t := \overline{Y_{/\sim}} \subset \mathbf{P}^2{}_{K(t)}$.

We claim that the diagram

$$\begin{tikzcd}
F_t \subset \mathbf{P}^3{}_{K(t)} \arrow[r, "(P_0:P_i:P_j)"] & \tilde{C}_t \subset \mathbf{P}^2{}_{K(t)} \\
X \subset K(t)^4 \arrow[u, "\sim"] \arrow[r, "(P_0,P_i,P_j)"'] & Y \subset K(t)^3 \arrow[u, "\sim"']
\end{tikzcd}$$

commutes.
It is clear that Y is just the image of X under the map defined by (P_0, P_i, P_j) . On the other hand if we can show that F_t is the generic fiber of the map $(T : U)$ then the diagram obviously commutes.
To see that we indeed obtain F_t by computing $X/_\sim$, we need to prove that X is not empty for generic t, i.e. that for every projective point p in the inverse image F_t of the map $(T : U)$ of the generic point $(t : 1)$ of $\mathbf{P}^1$ we have a corresponding point in X.
The argument is the same as in the proof of *InvertBirational*, nevertheless we give the proof:
Let p denote a projective point in the inverse image F_t of the generic point $(t : 1)$ of $\mathbf{P}^1$. This means

$$\begin{aligned} F(p) &= 0 \\ (T(p) : U(p)) &= (t : 1) \end{aligned} \tag{2.1}$$

Can we find homogeneous coordinates $\overline{p}$ of p such that

$$(T(\overline{p}), U(\overline{p})) = (t, 1)$$

and $F(\overline{p}) = 0$?
Now suppose we have (arbitrary but) fixed homogeneous coordinates p' for

p. As a consequence of (2.1) we have a λ such that

$$\lambda T(p') = t$$
$$\lambda U(p') = 1$$
$$(\lambda := \frac{1}{U(p')})$$

Let n denote the common degree of T and U. Then we define

$$d := \sqrt[n]{U(p')}$$

and make the projective change of coordinates

$$\overline{p} = \frac{p'}{d}$$

The so defined homogeneous coordinates $\overline{p}$ satisfy

$$T(\overline{p}) = \frac{1}{\sqrt[n]{U(p')}^n} T(p') = \lambda T(p') = t$$
$$U(\overline{p}) = \frac{1}{\sqrt[n]{U(p')}^n} U(p') = \lambda U(p') = 1$$

Therefore for every point of F_t we have a corresponding point in X.

Now in checking that a given P_j is suitable for projecting, the final step we take is to look for a homogeneous polynomial of degree two for $\tilde{C}_t$ in the affine(!) ideal $J := (I \cap k(t)[p_0, p_i, p_j]) \subset k(t)[p_0, p_i, p_j]$. We can do this by the following proposition that guarantees that we can find the homogeneous equation for $\tilde{C}_t$ in J :

Proposition 2.2.1 *Let $Y \subset K^n$ be an affine variety, let $I := ideal(Y)$.*
Let $X \subset \mathbf{P}^{n-1}$ the projective closure of $Y_{/\sim}$.
Then the homogeneous elements of $ideal(X)$ are precisely the homogeneous elements of I.

Proof: $\subset$: Let F be homogeneous in $ideal(X)$. $F(x_1, ..., x_n) = 0$ for all $(x_1 : ... : x_n) \in X$. In particular $F(x_1, ..., x_n) = 0$ for all $(x_1, ..., x_n) \in Y$.
$\supset$: Let F be homogeneous of degree d in I. Then $F(x_1, ..., x_n) = 0$ for

all $(x_1, ..., x_n) \in Y$. Now since F is homogeneous for all $\lambda \in K$ we have $F(\lambda x_1, ..., \lambda x_n) = \lambda^d F(x_1, ..., x_n) = 0$ for all $(x_1, ..., x_n) \in Y$.

Now to actually find the conic equation C we do the following:
We compute the reduced Gröbner basis G' of I with respect to a term-order that eliminates x, y, z, w. Then $G'' := G' \cap k(t)[p_0, p_i, p_j]$ is a Gröbner basis of the intersection-ideal. If we have the right P_j then C is contained in the ideal generated by G'' (but C is not necessarily contained in G''). To decide whether or not we have the right P_j we have to distinguish the following cases:

1. G'' contains no element of degree 1 or 2 or
   G'' contains a homogeneous element of degree 1 or
   G'' contains more than one inhomogeneous element of degree 1 or
   G'' contains only one inhomogeneous element of degree 1 and no element of degree 2 or
   G'' contains no element of degree 1 and only one inhomogeneous polynomial of degree 2

2. If G'' already contains a homogeneous element of degree 2, there is nothing to do.

3. G'' contains one inhomogeneous polynomial of degree 1 and at least one inhomogeneous polynomial of degree 2.

4. G'' contains no polynomial of degree 1 and more than only one inhomogeneous polynomial of degree 2.

In case 1 we have the wrong P_j. Either we can construct a homogeneous polynomial of degree one or no homogeneous algebraic relation of degree less than 3 exists. By construction P_0, P_i and P_j do not fulfill a linear homogeneous relation.
In the cases 3 and 4 we have found the right P_j. What is left is to actually compute the conic equation C out of the elements of G''.
In case 3 we do the following:

Let

$$f_1 := l_1(p_0, p_i, p_j) + c_1$$

$$f_2 := q_2(p_0, p_i, p_j) + l_2(p_0, p_i, p_j) + c_2$$

be two elements of G'', where q_s, l_s, c_s denote the quadratic, linear and constant portion of f_1, f_2 respectively. Then we define

$$C := q_2(p_0, p_i, p_j) - \frac{l_1(p_0, p_i, p_j) l_2(p_0, p_i, p_j)}{c_1} + c_2 (\frac{l_1(p_0, p_i, p_j)}{c_1})^2.$$

Clearly C is a homogeneous polynomial of degree 2 contained in the ideal generated by G''.

In case 4 we construct C in a different way:
Let

$$f_1 := q_1 + l_1 + c_1$$

$$\dots$$

$$f_n := q_n + l_n + c_n$$

denote the inhomogeneous elements of degree 2 contained in G''. We know that a homogeneous quadratic equation exists. To find we homogenize all the f_s's introducing a new variable v

$$f_1' := q_1 + v l_1 + v^2 c_1$$

$$\dots$$

$$f_n' := q_n + v l_n + v^2 c_n$$

and compute the reduced Gröbner basis of the (homogeneous) ideal I generated by $\{f_1', \dots, f_n'\}$ with respect to a term order that eliminates v.
This basis consists necessarily of homogeneous polynomials that generate the polynomials contained in I. Therefore it also contains a homogeneous polynomial of degree 2 depending on p_0, p_i, p_j only (we already know about its existence).

Example 2.2.3 *Let*

$$G'' = \{p_0 - 1, -p_3^2t^2 + tp_2^2 - p_2^2 - 7tp_3^2 + 8p_3^2 + t^2 + 8t + 4\}$$

as in the previous example. It consists of 2 inhomogeneous elements, one of degree 1 and the second of degree 2, therefore we are in case 3. Here we have $l_1 = p_0$, $c_1 = -1$, $q_2 = -p_3^2t^2 + tp_2^2 - p_2^2 - 7tp_3^2 + 8p_3^2$, $l_2 = 0$, $c_2 = t^2 + 8t + 4$. *Now for all points in the zero set of* G'' *we have* $\frac{l_1}{c_1} = p_0$ *equals 1 and hence*

$$C = (t^2 + 8t + 4)p_0^2 + (t - 1)p_2^2 + (-t^2 - 7t + 8)p_3^2 = 0$$

is contained in the ideal of G''.

Now suppose that we have found C in the way described above - C is a quadratic equation in p_0, p_i, p_j with coefficients in $k(t)$. To find the second parameter of the parametrization of F we now need to parametrize C.

The usual way to parametrize a conic is to find a rational point on it and then project birationally from this point to the line. In our situation C is a conic over $k(t)$ and we need to compute a point whose coordinates are rational functions in t. But the field $k(t)$ is not algebraically closed, so the Nullstellensatz does not guarantee that there exists such a point.

To see that we indeed can find a rational point on C we give the following proposition(taken from [Sch1]).

Proposition 2.2.2 *Let* k *be algebraically closed. Then any conic over* $k(t)$ *has a* $k(t)$*-rational point.*

Proof: We can write the equation as a homogeneous polynomial of degree 2 in x, y, z with polynomial coefficients:

$$F(x, y, z) = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5xz + a_6yz = 0,$$

$$a_i \in k[t]$$

Let n be an integer such that $\deg_t a_i \leq n$ for all i. We make the ansatz

$$x = x_0 + \ldots + x_{n-1}t^{n-1},$$

$$y = y_0 + \ldots + y_{n-1}t^{n-1},$$

$$z = z_0 + \ldots + z_{n-1}t^{n-1},$$

and plug it into the equation. By comparing coefficients, we get at most $3n-1$ homogeneous equations (of degree 2) in $3n$ indeterminates. Hence there must be a solution, because k is algebraically closed.

The proof of the proposition suggests a method to actually compute a rational point on a conic defined over $K(t)$. But the resulting system has so many unknowns that the time needed to solve is rather long.
A better solution for this problem is provided by the algorithm *FindPoint* described in the next paragraph. It first appeared in [Sch2].
Now suppose we have found a $k(t)$-rational point $(Q_0(t) : Q_i(t) : Q_j(t))$ on $C(p_0, p_i, p_j)$. In order to define the birational projection from the point $(Q_0 : Q_i : Q_j)$ onto $\mathbf{P}^1$ we need to find two linear forms vanishing at this point. We distinguish the two cases:

1. If $Q_0 = 0$ then two such forms are
$$(p_0 : p_i : p_j) \mapsto (p_0 : Q_j(t)p_i - Q_i(t)p_j).$$
2. If $Q_0 \neq 0$ we define the projection by
$$(p_0 : p_i : p_j) \mapsto (Q_0(t)p_i - Q_i(t)p_0 : Q_i(t)p_j - Q_j(t)p_i).$$

Now composing the maps we get a birational map from the generic fiber F_t to $\mathbf{P}^1$ defined by

$$(P_0 : Q_j(\frac{T}{U})P_i - Q_i(\frac{T}{U})P_j)$$

or

$$(Q_0(\frac{T}{U})P_i - Q_i(\frac{T}{U})P_0 : Q_0(\frac{T}{U})P_j - Q_j(\frac{T}{U})P_0)$$

respectively.
Hence for every t we have found a parametrization of F_t (namely the inverse of this map). We complete the parametrization of the surface by composing the two parameters and clearing denominator in the resulting map

$$(\frac{P_0U}{Q_j(\frac{T}{U})P_i - Q_i(\frac{T}{U})P_j} : T : U)$$

or

$$(\frac{(Q_0(\frac{T}{U})P_i - Q_i(\frac{T}{U})P_0)U}{Q_0(\frac{T}{U})P_j - Q_j(\frac{T}{U})P_0} : T : U)$$

The Algorithm FindPoint For the description of the algorithm *FindPoint* we follow [Sch2], where it first appeared . *FindPoint* uses the following idea : it finds a solution modulo $t = t_i$ for sufficiently many numbers t_i, at which the equation of the conic splits into linear factors. We have to assume *char* $k \neq 2$, to be able to eliminate the mixed terms by suitable linear transformations (see e.g. [CoLiSh]).

Input: A homogeneous polynomial $G \in k(t)[x, y, z]$ of degree 2.
Output: A non-trivial solution $(X : Y : Z) \in k[t]^3$.

Algorithm *FindPoint*:
  eliminate the mixed terms by suitable linear transformations;
  clear denominators;
  eliminate multiple factors of the coefficients
    by suitable linear transformations;
  eliminate common factors of the coefficients:
  if two coefficients have a common factor P then
      multiply the equation with P;
      eliminate the double factors;
  (Now, $G = Ax^2 + By^2 + Cz^2$ and ABC is square-free.)
  if two polynomials of A, B, C have degree 0 then
    $(X : Y : Z) :=$ two suitable constants and 0;
  else
    Make an ansatz with undetermined coefficients:
      $b_x := [\frac{\deg B + \deg C - 1}{2}]$;
      $X := x_0 + \ldots + x_{b_x} t^{b_x}$;
      $b_y := [\frac{\deg A + \deg C - 1}{2}]$;
      $Y := y_0 + \ldots + y_{b_y} t^{b_y}$;
      $b_z := [\frac{\deg A + \deg B - 1}{2}]$;
      $Z := z_0 + \ldots + z_{b_z} t^{b_z}$;
    $n := b_x + b_y + b_z + 2$;
    $(t_1, \ldots, t_n) := n$ different zeroes of ABC;
    for i from 1 to n do
      $L(x, y, z) :=$ a linear factor of $G(t_i, x, y, z)$;
      $E_i := L(X(t_i), Y(t_i), Z(t_i))$;
      (E_i is linear in $x_0, \ldots, z_{b_z}$.)
    $x_0, \ldots, z_{b_z} :=$ a nontrivial solution of $(E_1, \ldots, E_n)$;
  transform the solution back;

return $(X : Y : Z)$

Example 2.2.4 Let

$$G = (t^2 + 8t + 4)x^2 + (t-1)y^2 + (-t^2 - 7t + 8)z^2$$

as in the previous example. There are no mixed terms and no multiple zeroes, but $t = 1$ is a common zero of two of the coefficients. We eliminate the common zero and get

$$G' = (t^2 + 8t + 4)(t-1)x^2 + y'^2 - (t+8)z'^2$$

with $y' = (t-1)y$, $z' = (t-1)z$. We find $b_x = 0$, $b_y = b_z = 1$ and make the ansatz

$$x = x_0, y' = y_0 + y_1 t, z' = z_0 + z_1 t.$$

We compute linear equations using the zeroes

$$t_1 = 2\sqrt{3} - 4, t_2 = -2\sqrt{3} - 4, t_3 = 1, t_4 = -8$$

of the coefficients.

To find an equation using t_1, we substitute $t = t_1$ in G' and factor:

$$G'(t_1) = y'^2 - (2\sqrt{3} + 4)z'^2 = (y' - (1-\sqrt{3})z')(y' + (1-\sqrt{3})z').$$

We substitute the ansatz into the first factor and get the equation

$$y_0 + (2\sqrt{3} - 4)y_1 - (1 - \sqrt{3})(z_0 + (2 + \sqrt{3})z_1) = 0.$$

The other equations are

$$\begin{gathered} y_0 + (-2\sqrt{3} - 4)y_1 - (1 + \sqrt{3})(z_0 + (2 - \sqrt{3})z_1) = 0, \\ y_0 + y_1 - 3(z_0 + z_1) = 0, \\ 6x_0 - y_0 + 8y_1 = 0. \end{gathered}$$

We find the nontrivial solution

$$x_0 = -1, y_0 = -34, y_1 = 5, z_0 = 12, z_1 = 1,$$

which yields the point

$$(X : Y' : Z') = (-1 : 5t + 34 : t + 12).$$

We transform back and get

$$(X : Y : Z) = (-1 : \frac{5t+34}{t-1} : \frac{t+12}{t-1}) = (-t + 1 : 5t + 34 : t + 12).$$

Theorem 2.2.1 *The algorithm FindPoint is correct.*

Proof: The case that two of A, B, C have degree 0 is trivial.

Assume that at most one of A, B, C has degree 0. We have either $n = \deg A + \deg B + \deg C$ or $n = \deg A + \deg B + \deg C - 1$, depending on the parities of the degrees. Therefore, we can find n different zeroes of ABC.

Because of the inequality

$$\max(\deg A + 2b_x, \deg B + 2b_y, \deg C + 2b_z) \leq n - 1,$$

the polynomial $AX^2 + BY^2 + CZ^2$ has degree $\leq n - 1$. Since it vanishes at $t_1, \ldots, t_n$ it vanishes identically.

2.3 Del Pezzo Surfaces

In this section we describe the algorithm that treats case 5 in the parametrization algorithm. In this case the output of *AdjointVectorSpace* provides a birational map from F to a Del Pezzo surface. We begin with a short description of these surfaces for which we follow [Sch2].

2.3.1 General Information on Del-Pezzo Surfaces

Del Pezzo surfaces are surfaces of degree n in $\mathbf{P}^n$ (not in some linear subspace) for some n, which satisfy the following conditions.

- The generic hyperplane section is an elliptic curve.
- $p_a = P_2 = 0$.

For instance, a smooth cubic surface in $\mathbf{P}^3$ is a Del Pezzo surface.

Theorem 2.3.1 *The largest possible degree of a Del Pezzo surface is 9.*

Proof: See [Sch2].

For $n = 1$ or 2, Del Pezzo surfaces of degree n are constructed in a non-standard way, adapting the definition appropriately. We start with a generalization of the notion of projective space.

Definition 2.3.1 Let k be a field. Let $w_0, \ldots, w_n$ be positive integers. The *weighted projective space* over k with weights $w_0, \ldots, w_n$ is the set of all elements in $k^{n+1} - \{0\}$ modulo the following equivalence relation: Two elements $(p_0, \ldots, p_n)$ and $(q_0, \ldots, q_n)$ are equivalent iff there exists an element λ in the algebraic closure of k, such that

$$(p_0, \ldots, p_n) = (\lambda^{w_0} q_0, \ldots, \lambda^{w_n} q_n).$$

If all weights are equal, then the weighted projective space is just the old $\mathbf{P}^n$. In general, the weighted projective space is a projective variety of dimension n, which may have singularities.

Example 2.3.1 Let $(w_0, w_1, w_2) = (1, 1, 2)$. This weighted projective space is isomorphic to a quadratic cone in $\mathbf{P}^3$. An isomorphism to the cone with equation $xz - y^2 = 0$ is given by

$$(x_0 : x_1; x_2) \mapsto (x : y : z : w) = (x_0^2 : x_0 x_1 : x_1^2 : x_2),$$

$$(x : y : z : w) \mapsto (x_0 : x_1; x_2) = (x : y; xw) = (y : z; zw).$$

The weighted projective space has a singular point $(0 : 0; 1)$, which is the vertex of the cone.

A hypersurface in weighted projective space can be described by a weighted homogeneous polynomial, with weights $w_1, \ldots, w_n$. The ‘hyperplanes’ are the zero sets of weighted homogeneous polynomials of weighted degree 1, i.e. the linear forms in the coordinates with weight 1.

Del Pezzo surfaces of degree 2. We consider the weighted projective space with weights $(w_0, w_1, w_2, w_3) = (1, 1, 1, 2)$. Let F be a hypersurface (i.e. a surface) given by a weighted homogeneous polynomial of weighted degree 4. For suitable choices of F, the following are true (see [Ma]).

- There is a natural projection of F to $\mathbf{P}^2$, which is a map of degree 2.
- The generic ‘hyperplane section’ is an elliptic curve.
- $p_a = P_2 = 0$.

In this situation, we say that F is a Del Pezzo surface of degree 2.

Example 2.3.2 Let F be given by the equation

$$x_0^4 + x_1^4 - x_2^4 - x_3^2 = 0.$$

The surface F is a 2-sheeted branched covering of $\mathbf{P}^2$, and the branch curve is a smooth quartic with equation $x_0^4 + x_1^4 - x_2^4 = 0$.

Let C be the intersection with a generic 'hyperplane' $x_0 + sx_1 + tx_2 = 0$. We have a degree 2 map from C to the projective line with 4 branch points (the intersections with the branch curves). By the Hurwitz genus formula (see [Hart]), we see that C is elliptic.

Del Pezzo surfaces of degree 1. We consider the weighted projective space with weights $(w_0, w_1, w_2, w_3) = (1, 1, 2, 3)$. Let F be a surface given by a weighted homogeneous polynomial of weighted degree 6. For suitable choices of F, the following are true (see [Ma]).

- There is a natural projection of F to $\mathbf{P}^1$.
- The generic 'hyperplane section' is an elliptic curve.
- The number of intersections of two generic 'hyperplane sections' is 1.
- $p_a = P_2 = 0$.

In this situation, we say that F is a Del Pezzo surface of degree 1.

Example 2.3.3 Let F be given by the equation

$$x_0^5 x_1 + x_2^3 + x_3^2 = 0.$$

The surface F is a 2-sheeted branched covering of the weighted homogeneous space with weights $(w_0, w_1, w_2) = (1, 1, 2)$. The branch locus consists of the the curve with equation $x_0^5 x_1 + x_2^3 = 0$ and an isolated branch point $(0 : 0; 1)$ (which is the singular point on the weighted projective space, see example ?).

The generic 'hyperplane section' C has equations

$$x_0^5 x_1 + x_2^3 + x_3^2 = x_0 + tx_1 = 0.$$

We have a degree 2 map from C to the projective line with 4 branch points, namely the isolated branch point and three intersection points with the branch curve. By the Hurwitz genus formula, we see that C is elliptic.

2.3.2 The Algorithm

For Del Pezzo surfaces of degree $4 \leq d \leq 9$ a parametrization is computed using knowledge on the standard way these surfaces are constructed.
Let V be the map defined by the vector-space of all homogeneous cubic forms in three variables. The dimension of V is 10. A basis for V is given by

$$\{x^3, x^2y, x^2z, xy^2, xz^2, xyz, y^3, y^2z, yz^2, z^3\}.$$

V determines a map

$$i_V : \mathbf{P}^2 \to \mathbf{P}^9$$

the image of which is a Del Pezzo surface of degree 9. If we consider sub-vector-spaces of V consisting of all forms passing through a collection of points $p_1, \ldots, p_r \in \mathbf{P}^2$ (in general position) the images of the corresponding maps are again Del Pezzo surfaces of degree $9 - r$.
There are some non-standard constructions for Del Pezzo surfaces of degree 8 (using e.g. the concept of bihomogeneous polynomials), that shall not be described here (see [Ma]).
Now by assigning an m-fold base point q (other than the p_i, see A.1 for this notion) to these rational maps we can construct birational maps to $\mathbf{P}^2$. If the rational map is defined by a vector-space V of forms then we write $i_{V\downarrow q^m}$ for this assignment- for further information on how to perform this please look at section A.1. We do not have a strict proof of the following proposition in which we comprise the parametrization method used in the algorithm *DelPezzo*; therefore it might as well be seen as a conjecture since we were also not able to find any reference for it. Considerations like the above strongly suggest correctness- however a proof for this might get cumbersome (considering all special cases that can occur) and would require more sophisticated methods similar to the ones used for the proof of the lemmas in section 1.1 - methods that are beyond the scope of this paper.

For Del Pezzo surfaces of degree $1 \leq d \leq 3$ we do not compute a parametrization. Here algorithms are known and can be found in e.g. [Sch1] or [Ma], but since these algorithms turned out to be of minor practical relevance because of enormous memory requirement and not reasonable running time we decided to just compute the equations of the Del Pezzo surface and let the user decide how to further proceed.

Here is the proposition for the cases $4 \leq d \leq 9$:

Proposition 2.3.1 *Let F be the surface, p be a smooth point on it and let V be the birational map from F to a Del Pezzo surface (as determined in the algorithm* ParametrizeWithAdjoints*). Let d be the degree of the Del Pezzo surface. Then the following holds:*
If $d = 4$ then $i_{V\downarrow p}$ maps F birationally to a surface with a pencil of conics. The conics are the images (under $i_{V\downarrow p}$) of the fibers of the map $i_{V\downarrow p^2}$ the image of which is the projective line.
If $d = 5$ then $i_{V\downarrow p^2}$ maps F birationally to the projective plane.
If $d = 6$ then $i_{V\downarrow p^2}$ maps F birationally to a quadric surface in $\mathbf{P}^3$.
If $d = 7$ then $i_{V\downarrow p^2}$ maps F birationally to a surface with a pencil of lines. These lines are the images (under $i_{V\downarrow p^2}$) of the fibers of the map $i_{V\downarrow p^3}$ the image of which is the projective line.
If $d = 8$ then one of the following holds:

- *F is mapped birationally to the plane by the map $i_{V\downarrow p^3}$.*

- *$i_{V\downarrow p^2}$ maps F birationally to a surface with a pencil of lines. These lines are the images (under $i_{V\downarrow p^2}$) of the fibers of the map $i_{V\downarrow p^3}$ the image of which is a rational normal curve.*

If $d = 9$ then $i_{V\downarrow p^2}$ maps F birationally to a surface with a pencil of lines. These lines are the images (under $i_{V\downarrow p^2}$) of the fibers of the map $i_{V\downarrow p^3}$ the image of which is a rational normal curve.

Now using this proposition we can easily reduce the Del Pezzo case to one of the cases already described in this paper. The algorithm follows immediately. For convenience we denote $V\downarrow p^m$ for both the vectorspace and a basis for it.

Input: A homogeneous polynomial $F \in k[x, y, z, w]$,
representing a surface.
An $r + 1$-tuple of homogeneous polynomials
$V = (V_0 : \ldots : V_r) \in k[x, y, z, w]^{r+1}$ representing a birational from the surface to a Del Pezzo surface.
d - the degree of the Del Pezzo surface

p a smooth point on the surface.

Output: A quadruple $(X : Y : Z : W) \in k[s,t]^4$
of bivariate polynomials, representing a rational parameterization of the surface
or
A set G containing the equations of the image of V (for $d = 3, 2, 1$).

Algorithm *DelPezzo*(F, V, d, p):
  if $d \in \{1, 2, 3\}$ then
    G:=Compute the equations of the image of V;
    return(G);
  else
    $\{u_1, u_2\}$:=Compute a regular system of parameters for F at p;
    if $d = 4$ then
      $(X : Y : Z : W)$:= *FiberIsConic*$(F, V{\downarrow}p^2, V{\downarrow}p)$;
    if $d = 5$ then
      $(X : Y : Z : W)$:= *InvertBirational*$(F, V{\downarrow}p^2)$;
    if $d = 6$ then
      V':= a basis of the vectorspace of forms $v \in V{\downarrow}p^2$
        such that $\partial^2_{u_{1,2}}(v)(p) = 0$ for **one**
        of the second partial derivatives of v
      $(X : Y : Z : W)$:= *InvertBirational*(F, V');
    if $d = 7$ then
      $(X : Y : Z : W)$:=*FiberIsLine*$(F, V{\downarrow}p^3, V{\downarrow}p^2)$;
    if $d = 8$ then
      $(X : Y : Z : W)$:=*InvertBirational* $(F, V{\downarrow}p^3)$;
      if $(X : Y : Z : W)$ =*failed* then
        $(T : U)$:=*RationalNormalCurve*$(F, p, V{\downarrow}p^3)$;
        $(X : Y : Z : W)$:=*FiberIsLine*$(F, (T : U), V{\downarrow}p^2)$;
    if $d = 9$ then
      $(T : U)$:=*RationalNormalCurve*$(F, p, V{\downarrow}p^3)$;
      $(X : Y : Z : W)$:=*FiberIsLine*$(F, (T : U), V{\downarrow}p^2)$;
  return $(X : Y : Z : W)$;

For the computation of a regular system of parameters at a point p we need to dehomogenize the equations of F and V. A detailed description of how to find the local parameters and how to compute the partial derivatives is given in A.6.

For the case of a Del-Pezzo-surface of degree six ($d = 6$) we again use the same method described in section 2.1 for computing a birational map to the plane . For further information please look at the section on quadric surfaces in this chapter.

Chapter 3

Additional Tools

In this section we present two algorithms that do not fit into the preceding chapter but are indispensable for the algorithm *ParametrizeWithAdjoints* .

3.1 Inversion of Birational Maps

As we have seen before, it is the general strategy of the algorithm *Parametrize-WithAdjoints* to first compute a birational map from the surface to the projective plane and then find the parametrization of the surface by inverting this map. So in most of the cases discussed above for completing the parametrization we need an algorithm to do this task.
The inversion of birational maps can be done algorithmically. We discuss here two different algorithms for this task. The first one works entirely in the projective setting and is discussed in more detail here. The second one is an affine version for inverting birational maps. It is described in detail in [Sch3], where it first appeared. Here we will only sketch this algorithm in order to be able to compare the two versions and to point out certain advantages and disadvantages of *InvertBirational*.

3.1.1 The Projective Algorithm

Input: A homogeneous polynomial $F \in k[x, y, z, w]$,
representing the surface.
A triple of homogeneous polynomials $(S : T : U) \in k[x, y, z, w]^3$
of the same degree, representing a birational map

from the surface to $\mathbf{P}^2$.

Output: A quadruple $(X : Y : Z : W) \in k[s,t]^4$ of bivariate polynomials, representing the inverse of $(S : T : U)$.

The implementation I have written in Maple is a straight-forward implementation of the following algorithm.

Algorithm *InvertBirational*(F,(S:T:U)):
Cancel common factors of S, T, U by finding and taking away
the greatest common divisor of the three polynomials;
$G := \{S - s, T - t, U - 1, F\}$
(s and t are regarded as constants here);
$G' :=$ the reduced Gröbner basis of G over $k(s,t)[x,y,z,w]$
with total degree order;
$G'' :=$ the set of all elements of G' of degree 1;
if $\#G'' < 3$ then
return *failed*;
(*the input was not a birational map*)
$(X, Y, Z, W) :=$ a nontrivial solution of G'
(considered as a linear equation system in (x, y, z, w));
return $(X : Y : Z : W)$;

Example 3.1.1 Let $F = x^2 + y^2 + z^2 - w^2$, the equation of the unit sphere. We will invert the stereographic projection $(x : y : w - z)$.

We compute a Gröbner base of

$$G = \{x - s, y - t, z - w - 1, x^2 + y^2 + z^2 - w^2\}$$

(total degree order, s and t are constants). It consists entirely of linear elements:

$$G' = G'' = \{x - s, y - t, 2z + 1 - s^2 - t^2, 2w - 1 - s^2 - t^2\}.$$

We solve for x, y, z, w and obtain

$$(X : Y : Z : W) = (s : t : \frac{s^2 + t^2 - 1}{2} : \frac{s^2 + t^2 + 1}{2}).$$

Example 3.1.2 Let $F = (x^4 + z^4)w^2 + (xy + z^2)^3$ and let the birational map be defined by $(xyz^2 + z^4 : xwz^2 : z^3 w)$.

Computation of the reduced Gröbner base of

$$G = \{xyz^2 + z^4 - s, xwz^2 - t, z^3w - 1, (x^4 + z^4)w^2 + (xy + z^2)^3\}$$

yields

$$G' = \{z + zs^4 + wt^3, ys + ys^5 - wt - ws^4t - wt^3, x + xs^4 + wst^3, 1 + s^4 + w^2t^3\}.$$

The linear elements of G' are

$$G'' = \{z + zs^4 + wt^3, ys + ys^5 - wt - ws^4t - wt^3, x + xs^4 + wst^3\}.$$

We solve for x, y, z, w and obtain

$$(X : Y : Z : W) = (-s^2t^3 : t^3 + s^4t + t : -st^3 : s^5 + s).$$

Example 3.1.3 Let

$$F = x^2y^2 + 8x^3y + 4x^4 + xyz^2 - x^2z^2 - y^2w^2 - 7xyw^2 + 8x^2w^2$$

and let

$$(S : T : U) =$$
$$(-xyz + x^2z - 5x^2y - 34x^3 : -y^2w + xyw - xy^2 - 12x^2y :$$
$$: -xyw + x^2w - x^2y - 12x^3).$$

Computation of the reduced Gröbner base of

$$G = \{-xyz + x^2z - 5x^2y - 34x^3 - s, -y^2w + xyw - xy^2 - 12x^2y - t,$$
$$-xyw + x^2w - x^2y - 12x^3 - 1,$$
$$x^2y^2 + 8x^3y + 4x^4 + xyz^2 - x^2z^2 - y^2w^2 - 7xyw^2 + 8x^2w^2\}$$

gives us (after filtering the nonlinear elements)

$$G'' = \{tzs^2 + 5ts^2w + 34s^2w + 12zs^2 - 2wt^2s-$$
$$-40wts - 10tzs - 68zs - 192ws + zt^2 + 5t^2w + 20zt + 74tw + 96z + 272w,$$
$$t^2s^2w - ts^2w + ts^2y + 12s^2y - 10tys - 68ys - t^3w+$$
$$+t^2y - 7t^2w + 20ty + 8tw + 96y,$$
$$ts^2w + s^2xt - s^2w + 12t^2x - 10xts - 68xs - t^2w + xt^2-$$
$$-7tw + 20xt + 8w + 96x\}.$$

Solving for x, y, z, w we obtain

$$(X : Y : Z : W) = (t^2 + 7t - s^2t + s^2 - 8 : t^3 + 7t^2 - s^2t^2 + s^2t - 8t :$$
$$: -5t^2 + 2st^2 - 74t - 5s^2t + 40st + 192s - 34s^2 - 272 :$$
$$: t^2 - 10st + 20t + s^2t + 12s^2 - 68s + 96).$$

Correctness of the Algorithm

Here is a proof of the algorithm *InvertBirational*. Basically this proof is the same as the one that can be found in [Sch2]. We discuss some left out points in more detail here.

Theorem 3.1.1 *The algorithm InvertBirational is correct.*

Proof: Let $(X' : Y' : Z' : W') \in k[s,t]^4$ be an inverse of $(S : T : U)$.
Let $q = (s : t : 1)$ be a generic point of $\mathbf{P}^2$. Since $(S : T : U)$ is birational q has a unique preimage which we denote by p. Let I be the ideal in $\overline{k(s,t)}[x,y,z,w]$ generated by $G := \{S - s, T - t, U - 1, F\}$. The zeroes of I are on the line corresponding to the projective point p.
To see that the set of zeros of G is not empty consider the following:
We know that

$$\begin{aligned} F(p) &= 0 \\ (S(p) : T(p) : U(p)) &= (s : t : 1) \end{aligned} \tag{3.1}$$

and want to find homogeneous coordinates $\overline{p}$ of p such that

$$\begin{aligned} F(\overline{p}) &= 0 \\ (S(\overline{p}), (T(\overline{p}), U(\overline{p})) &= (s, t, 1). \end{aligned}$$

Now suppose we have (arbitrary but) fixed homogeneous coordinates p' for p. Then $F(p') = 0$ and as a consequence of (3.1) we have a λ such that

$$\begin{aligned} \lambda S(p') &= s \\ \lambda T(p') &= t \\ \lambda U(p') &= 1 \\ (\lambda &:= \frac{1}{U(p')}) \end{aligned}$$

Let n denote the common degree of S, T and U. Then we define

$$d := \sqrt[n]{U(p')}$$

and make the projective change of coordinates

$$\overline{p} = \frac{p'}{d}$$

The so defined homogeneous coordinates $\overline{p}$ satisfy

$$\begin{aligned} F(\overline{p}) &= \frac{1}{d^{degr(F)}}F(p') = 0 \\ S(\overline{p}) &= \frac{1}{\sqrt[n]{U(p')}^n}S(p') = \lambda S(p') = s \\ T(\overline{p}) &= \frac{1}{\sqrt[n]{U(p')}^n}T(p') = \lambda T(p') = t \\ U(\overline{p}) &= \frac{1}{\sqrt[n]{U(p')}^n}U(p') = \lambda U(p') = 1 \end{aligned}$$

and therefore we have found a zero of G.

Now, all solutions of the linear equations in I correspond to the projective point p, hence are inverse to $(S : T : U)$. From theorem 3.1.2 (see below) we know that G generates a radical ideal $I < \overline{k(s,t)}[x,y,z,w]$, where $\overline{k(s,t)}$ denotes the algebraic closure of $k(s,t)$. This means all linear equations in I are generated by the linear equations in the reduced Gröbner basis G' of I. Hence we can find linear equations contained in G' that generate the linear equations in I.

To complete the proof of theorem 3.1.1 we need the following theorem.

Theorem 3.1.2 *In the above situation the set* $G := \{S - s, T - t, U - 1, F\}$ *generates a radical ideal in* $\overline{k(s,t)}[x,y,z,w]$, *where* $\overline{k(s,t)}$ *denotes the algebraic closure of* $k(s,t)$.

Before we prove 3.1.2 we give the following propositions we need.

Proposition 3.1.1 *Let A be a Noetherian integral domain, let I be a radical ideal, and let S be a multiplicative set disjoint from I. Then $S^{-1}I$ is radical in $S^{-1}A$.*

Proof: Let $\frac{a}{s} \in S^{-1}A$ such that $\frac{a^n}{s^n} \in S^{-1}I$ for some natural number n. We have to show that $\frac{a}{s} \in S^{-1}I$. Since $\frac{a^n}{s^n} \in S^{-1}I$ we can take $b \in I$ and $t \in S$ such that $\frac{a^n}{s^n} = \frac{b}{t}$. This means that $ta^n = s^n b$. Now since $b \in I$ also $s^n b \in I$ and therefore $ta^n \in I$. From here we get $t^{n-1}(ta^n) \in I$ and since I is radical $ta \in I$. Since S is a multiplicative set $ts \in S$ and the element $\frac{ta}{ts} = \frac{a}{s}$ is contained in $S^{-1}I$. This proves that $S^{-1}I$ is a radical ideal in $S^{-1}A$.

Proposition 3.1.2 *Given two rings A and B, a radical ideal $I \subset B$ and a ring-homomorphism $\phi : A \to B$. Then $\phi^{-1}(I)$ is a radical ideal in A.*

Proof: We need to prove the following:

$$\forall a \in A : \ \exists n \ a^n \in \phi^{-1}(I) \Longrightarrow a \in \phi^{-1}(I)$$

(due to the definition of a radical ideal).
Now let $a \in A$ and n be such that the condition $a^n \in \phi^{-1}(I)$ is fulfilled. Since $a^n \in \phi^{-1}(I)$ also $\phi(a^n) \in I$. Since ϕ is a ring-homomorphism we get $\phi(a)^n \in I$. We know I is radical so $\phi(a) \in I$. It is clear that $a \in \phi^{-1}(I)$ and hence the proposition is proven.

Proposition 3.1.3 *Let F, U be homogeneous polynomials in $K[x_1, x_2, x_3, x_4]$. The set $\{F, U-1\}$ generates a radical ideal in $I \subset K[x_1, x_2, x_3, x_4]$.*

Proof: We use the criterion of lemma A.7.1. Hence we need to show that the ideal I generated by $\{F, U-1\}$ is a complete intersection and has generically normal crossing.
To show complete intersection consider the following:
I is the sum of the two ideals generated by F and $U-1$, or equivalently X is the intersection of the two hypersurfaces defined by F and $U-1$. $\tilde{X} = ZS(F) \subset k^4$ is an irreducible variety (because F is irreducible), hence its dimension is 3. On the other hand $U-1$ defines a non-zero form on $\tilde{X}$, hence the dimension of the intersection of $\tilde{X}$ and the zero set of $U-1$ is $dim(\tilde{X}) - 1 = 2 = dim(X)$. (see e.g.[Sh]).
Now by the definition I is a complete intersection (see A.7) if I can be generated by $n - dim(I) = 2$ elements and this is obviously the case.
To see that X has normal crossing at its generic points we use lemma A.7.2. Let Y be set of all points of X such that all the determinants of the 2×2-submatrices of the jacobian-matrix J vanish simultaneously. Y is exactly the set of points for which X does not have normal crossing.If we can show that the dimension of Y is strictly smaller than the dimension of X then we have generically normal crossing for X. Let

$$Y' := ZS(< F, U-1, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}, \frac{\partial F}{\partial x_4} >).$$

We show that $Y \subset Y'$ and then show that the dimension of Y' is already smaller than 2=dim(X).

To see the implication $Y \subset Y'$ let $p \in Y$. Suppose that $p \notin Y'$. Then at least one of the partial derivatives of F does not vanish at p. Now Y can also be viewed as the set of all points of X for which the rank of the jacobian-matrix J evaluated at p is smaller or equal 1. This means that the rows of $J|_p$ are linearly dependent, i.e. for every $p \in Y$ we can find a λ_p such that

$$\lambda_p \frac{\partial F}{\partial x_i}(p) = \frac{\partial U - 1}{\partial x_i}(p) = \frac{\partial U}{\partial x_i}(p)$$

for $i = 1, \ldots, 4$.
Now let d_U and d_F be the (homogeneous) degree of U and F respectively. By *Euler's formula* for homogeneous polynomials (see e.g. [CoLiSh]) we have

$$\sum_i \frac{\partial F}{\partial x_i} x_i = d_F F$$

$$\sum_i \frac{\partial U}{\partial x_i} x_i = d_U U$$

and starting from this we can derive

$$d_U U(p) = \sum_i \frac{\partial U}{\partial x_i}(p) p_i = \lambda_p \sum_i \frac{\partial F}{\partial x_i}(p) p_i = \lambda_p d_F F(p) = 0$$

applying the formula two times and using the fact that $F(p) = 0$. But now we have derived a contradiction because the equation holds only if $U(p) = 0$ and since on the other hand $p \in Y$ we have $U(p) = 1$ and therefore the assumption $\frac{\partial F}{\partial x_i}(p) \neq 0$ for at least one i was wrong. This proves the implication $Y \subset Y'$.
Now what is left is to see that the dimension of Y' is strictly smaller than two.
The variety Σ defined by $\{F, \frac{\partial F}{\partial x_1}, \frac{\partial F}{\partial x_2}, \frac{\partial F}{\partial x_3}, \frac{\partial F}{\partial x_4}\}$ is the so called *singular locus* of the hypersurface F (dimension 3), that is the subvariety of F that consists of all singular points of F. It is well known that Σ is a variety of strictly smaller dimension than F and therefore the dimension of Σ is smaller or equal two (see [CoLiSh]).
On the other hand Y' is the intersection of the hypersurface defined by $U - 1$ and Σ. Now $U - 1$ defines a non-zero form on any irreducible component of Σ therefore dim(Y')=dim(Σ)-1≤ 1 and I has generically normal crossing. This completes the proof of the proposition .

Proof:(theorem 3.1.2) If I is a radical ideal in $\overline{k(s,t)}[x,y,z,w]$ then $I \cap k(s,t)[x,y,z,w]$ is also radical in $k(s,t)[x,y,z,w]$.
In the other direction if we have a radical ideal in $k(s,t)[x,y,z,w]$ then its extension in $\overline{k(s,t)}[x,y,z,w]$ is again radical.
To prove this theorem we first reduce our problem to the problem of showing the radicality of the ideal generated by the same set G but in the ring $k[s,t][x,y,z,w]$. We can do this by proposition 3.1.1 (if we set $A := k[s,t][x,y,z,w]$, S the set of all nonzero elements of $k[s,t]$ and then look at the corresponding ideals).
To reduce the problem one more time we define the ringhomomorphism

$$\begin{aligned} \phi : k[s,t][x,y,z,w] &\longrightarrow k[x,y,z,w] \\ s &\mapsto S(x,y,z,w) \\ t &\mapsto T(x,y,z,w) \\ \phi\mid_{k[x,y,z,w]} &= id \end{aligned}$$

Now let I be the ideal generated by $G := \{F, S-s, T-t, U-1\}$ in $K[s,t,x,y,z,w]$, denoted by $< G >_{K[s,t,x,y,z,w]}$. Let I' be the ideal generated by $G' := \{F, U-1\}$ in $K[x,y,z,w]$, denoted by $< G' >_{K[x,y,z,w]}$. Note that $G' \subset G$.
If we want to apply proposition 3.1.2 we need to prove that $\phi^{-1}(I') = I$.
The implication $I \subset \phi^{-1}(I')$ is trivial. To see $\phi^{-1}(I') \subset I$ consider the following:
Let $u \in K[s,t,x,y,z,w]$ such that $\phi(u) \in I'$. The crucial point is to regard to I' as a subset of I:

$$\begin{aligned} I' =< G' >_{K[x,y,z,w]} &\subset < G' >_{K[s,t,x,y,z,w]} \\ &\subset < G >_{K[s,t,x,y,z,w]} = I \end{aligned}$$

On the other hand it is clear that the kernel of ϕ is the ideal $I'' :=< S - s, T - t >_{K[s,t,x,y,z,w]}$ and we have the implication

$$I'' =< S-s, T-t >_{K[s,t,x,y,z,w]} \subset < G >_{K[s,t,x,y,z,w]} = I$$

Since $\phi(u) \in I'$ we know also $\phi(u) \in I$. On the other hand from $u - \phi(u) \in I''$ we get $u - \phi(u) \in I$ and by adding $\phi(u) + (u - \phi(u)) = u$ we see that u is contained in I and the implication $\phi^{-1}(I') \subset I$ holds.

By proposition 3.1.2 we now are left with the problem of proving that the set $\{F, U-1\}$ generates a radical ideal in $k[x,y,z,w]$. This is true by proposition 3.1.3 and therefore the theorem is proven.

3.1.2 Comparison with an Affine Algorithm

The affine algorithm *InvertaffBirational* presented here works in a similar way to *InvertBirational* described above, but has one mentionable advantage: the Gröbner bases computed consists entirely of linear elements. We just give a sketch of this algorithm here to be able to discuss further advantages and disadvantages.
We have the following input specifications:

Input: An affine surface $V \subset \mathbf{A}^3$,
A birational map $f : V \to \mathbf{A}^2$.
Output: The inverse $f^{-1} : \mathbf{A}^2 \to V$, the parametrization.

By introducing a new variable x_4 this problem can be reduced to the following:

Reduced Problem :
Input: An affine variety $V \subset \mathbf{A}^4$.
A birational *polynomial* map $f : V \to \mathbf{A}^2$.
Output: The inverse $f^{-1} : \mathbf{A}^2 \to V$.

The Reduction is simple and works as follows:

Reduction :
Reduce the rational functions defining f
to the greatest common denominator D;
[Now, $f = (F_1/D, F_2/D)$.]
$S' := \{$g,$x_4D - 1$ $\}$;
($g \in K[x_1, x_2, x_3]$ is the defining function of the surface)
$f' := (F_1x_4, F_2x_4)$;
$(G_1, \ldots, G_4)$:= solution of the reduced problem for S', f';
return (G_1, G_2, G_3);

Now we can formulate the affine algorithm for this simpler input:

Algorithm *InvertaffBirational*:
Input: A set $S \subset k[x_1, x_2, x_3, x_4]$ of the form {g,wD-1}
An 2-tuple $(F_1, F_2) \in k[x_1, x_2, x_3, x_4]^2$,
representing a birational polynomial map f to $\mathbf{A}^2$.

Output: An 3-tuple $(G_1, \ldots, G_3) \in k(y_1, y_2)^3$,
representing the inverse f^{-1}.

$S' := S \cup \{y_1 - F_1, y_2 - F_2\}$;
$S'' :=$ a Gröbner basis of S' over $k(y_1, y_2)$;
$(G_1, \ldots, G_4) :=$ solution of S'' (trivial to compute);
return $(G_1, \ldots, G_3)$

Example 3.1.4 *Let V be the surface in $\mathbf{A}^3$ with equation*

$$x_1(x_2^2 - x_1x_3)(x_3^2 - x_2) + (x_1 - x_2x_3)^2 = 0.$$

Let $f : V \to \mathbf{A}^2$ be the map

$$(x_1, x_2, x_3) \mapsto (y_1, y_2) = (\frac{x_2^2 - x_1x_3}{x_1 - x_2x_3}, \frac{x_3^2 - x_2}{x_1 - x_2x_3}).$$

Reduction of the Problem: V' is defined as the surface in $\mathbf{A}^4$ defined by the equations

$$x_1(x_2^2 - x_1x_3)(x_3^2 - x_2) + (x_1 - x_2x_3)^2 = x_4(x_1 - x_2x_3) - 1 = 0,$$

and $f' : V' \to \mathbf{A}^2$ is the polynomial map

$$(x_1, x_2, x_3, x_4) \mapsto (y_1, y_2) = ((x_2^2 - x_1x_3)x_4, (x_3^2 - x_2)x_4).$$

We compute a Gröbner basis of the set

$$S' := \{x_1(x_2^2 - x_1x_3)(x_3^2 - x_2) + (x_1 - x_2x_3)^2, x_4(x_1 - x_2x_3) - 1, (x_2^2 - x_1x_3)x_4 - y_1,$$
$$(x_3^2 - x_2)x_4 - y_2\}$$

with total degree order, where y_1 and y_2 are regarded as constants. The result is

$$S'' = \{(y_2y_1^2 - y_1)x_2 + y_1^3 + 1, y_1y_2x_1 + 1, (y_2y_1^2 - y_1)x_3 - y_2 - y_1^2,$$
$$(y_2y_1^5 - y_1 + 3y_2y_1^2 + y_2^2)x_4 - y_2^3y_1^4 + 2y_2^2y_1^3 - y_2y_1^2\}.$$

Therefore, the solution is

$$(G_1, G_2, G_3, G_4) = (\frac{-1}{y_1y_2}, \frac{-y_1^3 - 1}{y_2y_1^2 - y_1}, \frac{y_2 + y_1^2}{y_2y_1^2 - y_1}, \frac{y_2^3y_1^4 - 2y_2^2y_1^3 + y_2y_1^2}{y_2y_1^5 - y_1 + 3y_2y_1^2 + y_2^2}).$$

This is a representation of the inverse map $f^{-1} : \mathbf{A}^2 \to V$. *The first three components form a parameterization of the surface* V.

To compare the result with InvertBirational we homogenize the input surface and the birational map:

$$F := x_1x_2^2x_3^2 - x_1x_2^3x_4 - x_1^2x_3^3 + x_1^2x_3x_2x_4 + x_1^2x_4^3 - 2x_1x_3x_2x_4^2 + x_2^2x_3^2x_4$$

$$(S : T : U) := (x_2^2 - x_1x_3 : x_3^2 - x_2x_4 : x_1x_4 - x_2x_3)$$

Computing the Gröbner basis as usual leads to

$$\{-y_2x_4 + y_2x_3y_1^2 - x_3y_1 - y_1^2x_4, y_2x_2y_1^2 - x_2y_1 + x_4y_1^3 + x_4, y_2x_1y_1 + x_4,$$
$$-y_2^3y_1^4 + 2y_2^2y_1^3 + y_2y_1^5x_4^2 - y_2y_1^2 + y_2^2x_4^2 + 3x_4^2y_2y_1^2 - x_4^2y_1\}$$

This set contains three linear and one quadratic equation in x_1, x_2, x_3, x_4. *We have to remove the non-linear element and then solve the system. The result is*

$$(y_2y_1 - 1 : y_2(y_1^3 + 1) : -y_2(y_2 + y_1^2) : -(y_2y_1 - 1)y_1y_2)$$

a parametrization of the surface F *(the same as above if we clear denominators there).*

Comparison of the two algorithms Both algorithms basically use the same method. But as mentioned above there are certain differences.

- The affine algorithm computes a Gröbner basis that contains only linear elements. If we take a look at the previous example the basis of the affine algorithm was

  $$S'' = \{(y_2y_1^2 - y_1)x_2 + y_1^3 + 1, y_1y_2x_1 + 1, (y_2y_1^2 - y_1)x_3 - y_2 - y_1^2,$$
  $$(y_2y_1^5 - y_1 + 3y_2y_1^2 + y_2^2)x_4 - y_2^3y_1^4 + 2y_2^2y_1^3 - y_2y_1^2\}.$$

  Here all elements are linear in the x_i.
  The basis computed in the projective algorithm was

  $$\{-y_2x_4 + y_2x_3y_1^2 - x_3y_1 - y_1^2x_4, y_2x_2y_1^2 - x_2y_1 + x_4y_1^3 + x_4, y_2x_1y_1 + x_4,$$
  $$-y_2^3y_1^4 + 2y_2^2y_1^3 + y_2y_1^5x_4^2 - y_2y_1^2 + y_2^2x_4^2 + 3x_4^2y_2y_1^2 - x_4^2y_1\}.$$

Here we have the quadratic equation $-y_2^3y_1^4 + 2y_2^2y_1^3 + y_2y_1^5x_4^2 - y_2y_1^2 + y_2^2x_4^2 + 3x_4^2y_2y_1^2 - x_4^2y_1$. Of course the computation of extra-elements in the Gröbner basis is costful and might bring us into trouble if we try to compute bigger examples, even more since we do not have a bound for the number of extra-elements that can occur in the basis nor a degree bound for them. However in all the examples computed so far there was only one additional element in the basis, so maybe this is generally the case and can be proven. Comparing the time consumed for computing the basis in the two algorithms there was no significant difference in the examples computed so far. Also the degree of the non-linear element was not arbitrarily high but we always received quadratic or cubic equations.

- An argument for using the projective version is that in the affine algorithm we solve a linear equation in the last step , namely the equation for x_4, that we throw away afterwards. Of course solving linear equations is not a big deal in comparison to computing Gröbner basis.

- Finally there is another reason for using *InvertBirational* . Throughout the entire parametrization algorithm we work in the projective setting, so also the algorithm that does the inversion will be called with projective input. This means that if we want to use the affine implementation we needed to dehomogenize the input equation F and the equations defining the birational map first, then make the reduction step described above and in the end again convert the solution computed to the corresponding projective solution. Of course all those operations are not at all complicated.

Anyway since up to now there was no problem in computing the Gröbner basis in the projective algorithm, it is the one we actually use, because I think the last point mentioned is a good argument from the logic point of view. The implementation of the affine version is done in a way that allows us to just exchange the two modules without any necessity of changing other things in *ParametrizeWithAdjoints*, so if problems occur we can switch to the affine version immediately.

3.2 Rational Normal Curves

3.2.1 General Comments

In the two cases where a parametrization is computed sequentially we need an additional algorithm to find the first parameter.
For the cases where the surface F has a pencil of lines (see 2.2.1) or a pencil of conics (see 2.2.2) we can use a basis of a certain adjoint vector-space that occurs in the adjoint-series to get a rational map π' from F to a rational normal curve $C \subset \mathbf{P}^k$. It is known that such curves can be parametrized easily.
Both algorithms *FiberIsLine* and *FiberIsConic* require as an input a rational map $(T : U)$ from the surface to $\mathbf{P}^1$. This is the map we get composing the rational map π' and a birational map π'' from C to $\mathbf{P}^1$. Now π'' is just the inverse of a parametrization of C, so before *ParametrizeWithAdjoints* is able to call these algorithms we need an algorithm that parametrizes rational normal curves. In our special situation we don't even need to have the equations of C explicitly but use an algorithm *RationalNormalCurve* that computes $\pi'' \circ \pi' {=} (T : U)$ directly.
Basically *RationalNormalCurve* finds π'' by projecting C from the tangent space at a smooth point $q \in C$ onto $\mathbf{P}^1$.
For q we can simply take the image $\pi'(p)$ of the point p of F, which we gave as an input to the main algorithm. For computing the tangent space at q we use the idea of a regular system of parameters at a point (see A.6).

What are rational normal curves Rational normal curves can be defined using their degree (see A.5).

Definition 3.2.1 *An irreducible (non-degenerate) curve $C \subset \mathbf{P}^k$ is a rational normal curve iff the degree of C is k.*

From this and proposition A.5.1 it follows that rational normal curves are exactly the curves of minimal degree.
To see that rational normal curves are rational, i.e. have a parametrization, we do the following:
If C is a curve of minimal degree k in some $\mathbf{P}^k$, then unless $k = 2$ we can project C birationally from a smooth point lying on C to $\mathbf{P}^{k-1}$. The projective closure of the image of C under this projection is again a rational normal

curve (because the degree decreases by one- see page 71 in section A.5). Iterating this process (projection from points on the curve) we will finally arrive at the rational normal curve $\tilde{C} \subset \mathbf{P}^2$, which is a hypersurface. Now $\tilde{C}$ is a hypersurface of degree 2 and hence defined as the zero-set of one homogeneous polynomial of degree 2 (see A.5)-it is a plane conic. We know that a plane conic can be parametrized by (again) projecting from a smooth point on it (this is what we have already done in the algorithm *FiberIsConic* for finding the second parameter).

Composing all the birational maps we get the birational inverse of a parametrization of the original curve C.

3.2.2 The Algorithm and Some Remarks

As mentioned above the algorithm we use takes advantage of the special situation we are in, so that we will not have to compute the equations of the rational normal curve.

The Algorithm RationalNormalCurve

Input: A homogeneous polynomial $F \in k[x, y, z, w]$,
representing the surface.
A smooth point p lying on the surface.
A $(k+1)$-tuple $(V_0 : \ldots : V_k)$ of polynomials in $k[x, y, z, w]$
homogeneous of the same degree,
representing a rational map from the surface
to a rational normal curve $C \in \mathbf{P}^k$.

Output: A pair $(T : U) \in k[x, y, z, w]^2$ of polynomials homogeneous
of the same degree,
representing a rational map from the surface to $\mathbf{P}^1$.

Algorithm *RationalNormalCurve*$(F, p, (V_0 : \ldots : V_k))$:
  Compute a regular system of parameters for F at p;
  r:= a suitable integer such that $V{\downarrow}p^r$ is 2-dimensional ;
  $(T : U)$:= a basis of $V{\downarrow}p^r$;
  return $(T : U)$;

The implementation is straight forward: We first dehomogenize F and the map $(V_0 : \ldots : V_k)$ with respect to a suitable variable in order to be able to compute a regular system of parameters for F at p. A more detailed description of how to perform this can be found in A.6 in the appendix.
Let V be the vector-space spanned by the V_i. Then we sequentially compute the sub-vector-spaces $V{\downarrow}p^r$ of V (see section A.1 for an explanation of this notion) for $r = 1, 2, \ldots$ until we arrive at 2-dimensional subspace.
The corresponding map $i_{V{\downarrow}p^r}$ is the composition of i_V with the projection from a higher order tangent space of the image of the surface under i_V (the rational normal curve) at $i_V(p)$ and this projection is a birational map from the rational normal curve to the projective line.

Appendix A

Theoretical background

A.1 Vector-spaces and Rational Maps

We recall here the notion of a rational map between projective varieties and the algebraic description of such maps that is used in this paper. A rational map can be described by specifying a vector-space of forms in the graded function ring. For the description of this concept we follow [Sch1].

Rational maps between projective varieties Let us first recall some topological notions we are going to use. A set $X \subset \mathbf{P}^n$ is **closed** if X is the zero-set of a finite set of (homogeneous) polynomials in $(n+1)$-variables. Given a closed subset $X \subset \mathbf{P}^n$ a subset of X is **closed in** X if it is the intersection of a closed set of $\mathbf{P}^n$ and X. If $Y \subset X$ are two closed sets, then $X \setminus Y$ is an **open set in** X. Any open set $U \subset X$ containing a point p is called a **neighbourhood** of p. Given a subset $M \subset X$ the **projective closure** $\overline{M}$ **of** M is the intersection of all the closed subsets of X containing M. M is **dense in X** if $\overline{M} = X$.

We come to the definition of a rational map:
A rational map $f : X \subset \mathbf{P}^n \to \mathbf{P}^m$ is defined by giving $(m+1)$ forms $(F_0 : \ldots : F_m)$ of the same degree in $(n+1)$ variables. At least one of the forms must not vanish on X. Two maps $(F_0 : \ldots : F_m)$ and $(G_0 : \ldots : G_m)$ are equal if $F_iG_j = F_jG_i$ for all i, j.
A rational map is **regular** at a point $p \in X$ if it can be defined by functions $(f_0 : \ldots : f_m)$ such that all the f_i are regular at p and do not vanish simul-

taneously at p (a function f_i is regular at p if it can be written in the form F/G with F, G homogeneous of the same degree and $G(p) \neq 0$).
The set of points at which a rational map is regular is open. If $X \subset \mathbf{P}^n$ and $Y \subset \mathbf{P}^m$ are projective varieties and $f : X \to \mathbf{P}^m$ is a rational map , then we say that f maps X to Y if there exists an open set $U \subset X$ on which f is regular and $f(U) \subset Y$. The union $\tilde{U}$ of all such open sets is the **domain of definition of** f and $f(\tilde{U}) \subset Y$ is the **image of** X **in** Y.
If the image of rational map $f : X \to Y$ has an inverse rational map then f is **birational** and X and Y are birationally equivalent.

Sub-vector-spaces of the graded function ring and rational maps

Definition A.1.1 *Let G be a commutative group (written additively) and let Λ be a set. A* **grading on** G **of type** Λ *is a family $(G_\lambda)_{\lambda \in \Lambda}$ of subgroups of G, of which G is the direct sum.*

Definition A.1.2 *Given a ring R and a grading R_λ of type Λ on the additive group R, the grading is said to be compatible with the ring structure on R if*

$$R_\lambda R_\mu \subset R_{\lambda+\mu} \ \forall \ \lambda, \mu \in \Lambda.$$

The ring R with this grading is then called a **graded ring (of type** Λ**)** .

Let X be a surface in $\mathbf{P}^3$, given by a homogeneous equation $F(x, y, z, w)$ of degree d. The graded ring $S(X)$ of forms on X is the graded polynomial ring $k[x, y, z, w]$ modulo the homogeneous ideal generated by F. The grading is a decomposition

$$S = \bigoplus_{i \in \mathbf{N}} S_i, S_i S_j \subset S_{i+j}.$$

The homogeneous components S_i are finite dimensional vector-spaces over k. Let n be a natural number. Let $V \subset S_n$ be a subspace of positive dimension $r+1$. V has a basis $\{v_0, \ldots, v_r\}$ consisting of forms (homogeneous polynomials modulo F) of degree n.

V defines a rational map $i_V : X \to \mathbf{P}^r$ by

$$(x : y : z : w) \mapsto (v_0(x, y, z, w) : \ldots : v_r(x, y, z, w)).$$

Since the forms in the basis are linearly independent, the image $i_V(X)$ does not lie in a proper linear subspace. Choosing another basis of V corresponds to making a projective change of coordinates for $i_V(X)$. The map is defined on all points on X where at least one form in V does not vanish. The closure of the image in $\mathbf{P}^r$ is a projective variety. Its equations can be computed by eliminating x, y, z, w in the ideal generated by

$$F, x_0 - v_0(x, y, z, w), \ldots, x_r - v_r(x, y, z, w).$$

It helps if a good degree bound m for the equations is known. In this case, it suffices to compute the linear relations of the vector-space V^m (the multiplication has to be understood as multiplication of forms).

The domain of definition of i_V may actually also contain points where all forms in V do vanish. The forms may have a common factor that vanishes at p. Division by the common factor does not affect the ratio, and after division some form may have a nonzero value.

To see the common factor, it may be necessary to pass to the local ring at p. The common factor is not always a globally defined form. For a discussion on local and global equations we refer to chapter 4 in [Sch1].

Base Point Assignment Let $p \in X$ be such that i_V is defined at p. By $V{\downarrow}p \subset V$ we denote the sub-vector-space of all forms that vanish at p. Its dimension is $\dim V - 1$. Its associated map is the composite of i_V with the projection from $i_V(p)$. This is not hard to see:
We can choose a basis $\{V_0, \ldots, V_m\}$ for V such that $V(p) = (1 : 0 : \ldots : 0)$. Then $\{V_1, \ldots, V_m\}$)is a basis of $V{\downarrow}p$ and its associated map is exactly the projection $\pi_{V(p)}(V(X))$ (see also example A.4.1).
We say that $V{\downarrow}p$ is obtained by *assigning p as a base point.*

Example A.1.1 *Let X be the unit sphere, homogeneously given by*

$$F(x, y, z, w) = x^2 + y^2 + z^2 - w^2 = 0.$$

Let

$$V = S_1 = \langle x, y, z, w \rangle.$$

The associated map is the embedding $X \hookrightarrow \mathbf{P}^3$. Let $p := (0 : 0 : 1 : 1) \in X$. Then the vector-space with base point p is

$$V{\downarrow}p = \langle x, y, w - z \rangle.$$

Its associated map $\pi = i_{V\downarrow p} : X \to \mathbf{P}^2$

$$(x : y : z : w) \mapsto (s : t : 1) = (x : y : w - z)$$

is the projection from p. *Since the generic line through* p *meets* X *in exactly one other point, the map is birational.*

Now let $\{u_1, u_2\}$ be a local coordinate system of X around the point p. Then by $V\downarrow p^r \subset V$ we denote the sub-vector-space of all forms $v \in V$ such that

$$\begin{aligned} v(p) &= 0 \\ \partial_{u_1}(v)(p) = \partial_{u_2}(v)(p) &= 0 \\ &\vdots \\ \partial^{r-1}_{u_{1,2}}(v)(p) &= 0 \end{aligned}$$

where $\partial^{r-1}_{u_{1,2}}(v)(p) = 0$ shall mean that all possible $(r-1)-fold$ derivatives (see A.6) of v (mixed and unmixed) vanish at p. We say that we obtain $V\downarrow p^r$ by *assigning* p *as an r-fold base point.*
It can be shown that $V\downarrow p^r$ again corresponds to the projection of $V(X)$ from a (higher order) tangential space at $V(p)$. This was already known to the Italians but we do not give a proof here since unfortunately we could not come up with a simple proof that does not require concepts other than the ones used in this paper.

A.2 Singular Points

The general definition of a singular point of an irreducible variety X is the following(see e.g. [CoLiSh]):

Definition A.2.1 *A point* $p \in X$ *for which the tangent space of* X *at* p *has dimension other than* $dim(X)$ *is called singular.*

In the situations where we need to check singularity in the algorithm we have a hypersurface X given by an irreducible polynomial $F \in k[x, y, z, w]$. Then the definition A.2.1 is equivalent to this:

Definition A.2.2 *A point* $p \in X$ *is singular iff* p *satisfies the conditions* $F(p) = 0$, $\frac{\partial F}{\partial x}(p) = 0$, $\frac{\partial F}{\partial y}(p) = 0$, $\frac{\partial F}{\partial z}(p) = 0$, $\frac{\partial F}{\partial w}(p) = 0$.

A.3 Adjoints

In this section, we introduce the main tool for parametrization, namely adjoint functions. Using adjoints, one can reduce the parametrization problem to one of the problems in the previous sections. For this section we follow [Sch2].
The theory of adjoints is classical ([En], [Za]). The first adjoints computation algorithm has appeared in [Sch1]. Here, we explain the basic idea of this algorithm.

Finally, we show how to compute the arithmetical genus, the geometrical genus and the plurigeni, using adjoints (see [Sh] for a definition of these notions).

What are Adjoints Let $F \in k[x, y, z, w]$ be a homogeneous polynomial of degree d, representing a surface. (As always, we assume that F is absolutely irreducible.) A *form* on the surface is an equivalence class of polynomials modulo multiples of F. We will represent forms by homogeneous polynomials which are reduced modulo F, with respect to the lexicographical term order.

For $m \geq 0$, a form G is an m-adjoint iff it vanishes with order $\geq m(r-1)$ at each r-fold curve singularity of F and with order $\geq m(r-2)$ at each r-fold point singularity of F. Obviously, any form is a 0-adjoint. Singularities must be computed over the algebraic closure of k. 'Infinitely near singularities' (see [Wa]) must also be taken into account.

Example A.3.1 Let $F := x^4 + y^4 - z^2w^2$, $m = 1$. The surface has two isolated double points $(0:0:0:1)$ and $(0:0:1:0)$, which impose no conditions for the adjoints. But each of the two double points has a double line in the infinitely near (this can be seen when the points are blown up). The 1-adjoints have to vanish at the these double lines, and therefore also on the isolated double points.

Using the algorithm in the next subsection, one can show vice versa that all forms vanishing at the two double points are 1-adjoints.

The m-adjoints of fixed degree form a finite-dimensional vector-space over k. The vector-space of all m-adjoints of dimension $n + m(d-4)$ is denoted by $V_{n,m}$, and its dimension is denoted by $v_{n,m}$.

Here are some basic properties of the adjoints.

Proposition A.3.1 *The m-adjoints have the following properties.*

- *If F is smooth, then every form is an m-adjoint.*
- *The question if G is an m-adjoint can be decided locally.*
- *We have the inclusion $V_{n,m} \cdot V_{n',m'} \subset V_{n+n',m+m'}$.*

Proof: Obvious.

Adjoints Computation The algorithm *AdjointVectorSpace* has the following specification.

Input: A homogeneous polynomial $F \in k[x, y, z, w]$
of degree d, representing a surface.
Two integers n, m.
Output: A basis for the vector-space $V_{n,m}$
of m-adjoints of degree $n + m(d-4)$.

To compute the 'infinitely near singularities' which occur in the definition of adjoints, we have to compute a resolution of the singularities by blowing ups (see [Hart]). Here, we use the resolution method described in [Hi].

For convenience of the recursive description of the algorithm, we change the specification slightly. It is obvious how to reduce the original problem to the one below.

Input: A surface F in 3-space, given implicitly.
A vector-space V of forms on X.
An integer $m > 0$.
Output: A basis for the sub-vector-space of m-adjoints.

Algorithm *AdjointSubVectorSpace*(F, V, m):
    if F is smooth then
        return V;
    else
        choose a singularity S;
        $r :=$ the multiplicity of F at S;
        if S is a curve then
            $p := m(r-1)$;
        else (S is a point)
            $p := m(r-2)$;

$V_1 :=$ the sub-vector-space of the elements
that vanish with order $\geq p$ at S;
$(\tilde{F}, e) := (BlowUp(F, S)$, equation of the exceptional divisor);
$f : V_1 \rightarrow \{$forms on $\tilde{F}\} :=$ division by $e^p \circ$ pullback;
$V_2 := AdjointSubVectorSpace(\tilde{F}, f(V_1), m)$;
return $f^{-1}(V_2)$;

The singularity S must be chosen according to the following rules.

1. S must be either a point or a smooth curve.
2. If S is a point, then it must not lie on a smooth curve with the same multiplicity.
3. If S is a curve, then it all its points must have the same multiplicity.

The rules ensure termination (see [Hi]).

Example A.3.2 Let $F := x^4 + y^4 - z^2w^2$, as in the previous example. We compute $V_{1,1}$, the vector-space of 1-adjoints of degree 1.

In [Sch1], the blowing up is realized as a union of affine charts. The decomposition business starts with the input surface, so we dehomogenize with respect to w and restrict our attention to this patch for the moment. The input to *AdjointSubVectorSpace* is

$$x^4 + y^4 - z^2, \langle x, y, z, 1\rangle, 1.$$

The only singularity is the origin. It is a double point, so $p = 0$, and the subspace V_1 is equal to the input vector-space.

In the blowing up patch with affine coordinates $x, \tilde{y} = \frac{y}{x}, \tilde{z} = \frac{z}{x}$, the blown up surface has equation $\tilde{F}_1 = x^2 + x^2\tilde{y}^4 - \tilde{z}^2$. The equation of the exceptional divisor is x. We find

$$f(V_1) = \langle x, x\tilde{y}, x\tilde{z}, 1\rangle.$$

There are also other patches of $\tilde{F}$, but we restrict our attention for the moment.

We call *AdjointSubVectorSpace* with input

$$x^2 + x^2\tilde{y}^4 + \tilde{z}^2, \langle x, x\tilde{y}, x\tilde{z}, 1\rangle, 1.$$

As the singularity, we choose the double line $x = \tilde{z} = 0$. Then $p = 1$, and $V_1 = \langle x, x\tilde{y}, x\tilde{z} \rangle$.

The next blowing up is smooth. Therefore, the next recursion does not make the vector-space any smaller, and we get the result

$$\langle x, x\tilde{y}, x\tilde{z} \rangle.$$

Now, we have to compute the inverse image (in the outer recursion). The result is

$$\langle x, y, z \rangle.$$

Checking also the other patches, one finds that the whole vector-space $\langle x, y, z \rangle$ consists of 1-adjoints of the affine surface $x^4 + y^4 - z^2 = 0$. A new restriction comes from the affine patch obtained by dehomogenization with respect to z ($x^4 + y^4 - w^2 = 0$). The final result is

$$V_{1,1} = \langle x, y \rangle.$$

For technical details, especially how to perform the blowing ups of curves, consult [Sch1]. Another blowing up method can be found in [Ku].

Obviously, one can always find a singularity S matching the rules, provided that F is not smooth. Computationally, this amounts to solving a system of algebraic equations over the algebraic closure of k. This can be done with Gröbner bases (see [Bu1],[Bu2],[BeWe]) (as suggested in [Sch1]) or with resultants ([Mi]) or with characteristic sets ([PfWa]). An equation solver specialized to 3 unknowns can be found in [Ka].

If S is not defined over the coefficient field k (but in some algebraic extension), then the sub-vector-spaces V_1 cannot be generated by elements with coefficient in k. However, singularities which are not defined over k come in conjugated sets. Therefore, it is always possible to eliminate the algebraic coefficients in basis elements at some later step.

The precise meaning of 'infinitely near points' is implicitly contained in the algorithm: a singularity in the infinitely near to a singularity S is a singularity on the blowing up along S. Therefore, we see that the algorithm "matches the definition". For a correctness proof, we refer to [Sch1].

Remark A.3.1 A weakness of the algorithm *AdjointSubVectorSpace* is that we cannot give a bound for the worst-case complexity. The reason is that nobody knows a bound for the number of blowing ups required by Hironaka's resolution algorithm.

A consolation is the observation that surfaces are typically far less singular than they could be. Generically, there will only be a few, rather simple, singularities (see also [Te]).

Simple Applications Here are formulas for the arithmetical genus p_a and for the plurigeni P_m, including the geometrical genus $p_g = P_1$ (see [Sh] for a discussion of these numbers).

Theorem A.3.1 *For any surface in* $\mathbf{P}^3$ *of degree* d*, the following hold.*

a) $p_a = d + 2v_{1,1} - v_{2,1} - 1$.

b) $P_m = v_{0,m}$.

Proof: (b): See [En].

(a): The function $n \mapsto v_{n,1}$ is a polynomial for sufficiently large n, whose constant term is $p_a + 1$ (see [BlLi]). The function coincides with the polynomial for $n \geq 1$, the degree of the polynomial is 2, and that the leading coefficient is $\frac{d}{2}$ (see [Sch2]). Therefore, the constant term is $d + 2v_{1,1} - v_{2,1}$.

Using these formulas and Castelnuovo's rationality criterion, it is easy to decide the existence of a parametrization (possibly over an algebraic extension of k).

Input: A homogeneous polynomial $F \in k[x, y, z, w]$ of degree d,
representing a surface.
Output: `Exist` if there is a parametrization of F
over the algebraic closure of k,
`NotExist` otherwise.

Algorithm *DecideRationality*(F):
$v_{1,1} := \#$ *AdjointVectorSpace*$(F, 1, 1)$;
$v_{2,1} := \#$ *AdjointVectorSpace*$(F, 2, 1)$;
$v_{0,2} := \#$ *AdjointVectorSpace*$(F, 0, 2)$;
if $d + 2v_{1,1} - v_{2,1} - 1 = v_{0,2} = 0$ then
return `Exist`;
else
return `NotExist`;

The correctness of the algorithm *DecideRationality* follows immediately by Castelnuovo's criterion $p_a = P_2 = 0$ for the existence of a parametrization, and from the correctness of the sub-algorithm *AdjointVectorSpace* (the dimension $v_{n,m}$ is the cardinality of the basis returned by *AdjointVectorSpace*).

A.4 Projections

In most of the cases occurring in the algorithm *ParametrizeWithAdjoints* we find a birational map to the plane by projecting either from a point or from a subspace (see e.g. the sections on *QuadricSurface*, *DelPezzo* and *RationalNormalCurve*). We give a short introduction to this concept in this section. For this introduction we follow [Har].

Projections from points Let $p \in \mathbf{P}^n$ be a point not contained in a hyperplane $\mathbf{P}^{n-1} \subset \mathbf{P}^n$. We define the *projection from p to the hyperplane* $\mathbf{P}^{n-1}$ in the following way:

$$\begin{aligned} \pi_p : \mathbf{P}^n - \{p\} &\to \mathbf{P}^{n-1} \\ q &\mapsto \overline{qp} \cap \mathbf{P}^{n-1} \end{aligned}$$

where $\overline{qp}$ denotes the line passing through q and p.

Example A.4.1 *We take coordinates $Z = (Z_0 : \ldots : Z_n)$ on $\mathbf{P}^n$. Let $\mathbf{P}^{n-1}$ be given by $Z_n = 0$ and let p be the point given through $(0 : \ldots : 0 : 1)$. Then π_p is given by*

$$\pi_p : (Z_0 : \ldots : Z_n) \mapsto (Z_0 : \ldots : Z_{n-1}).$$

Now we can extend this definition to arbitrary varieties $X \subset \mathbf{P}^n$ in the obvious way:
The *projection of X from p to* $\mathbf{P}^{n-1}$ is the restriction of π_p to X. We denote $\overline{X}$ for $\pi_p(X)$.

Example A.4.2 *Let $X \subset \mathbf{P}^3$ be the irreducible variety given by $F := x^2 + y^2 + z^2 - w^2$. X is the unit sphere. As in section 2.1 on quadric surfaces we project from $(0 : 0 : 1 : 1)$. The projection is given by $(x : y : w - z)$.*

In this example the point p is lying on the variety X. If $p \notin X$ then for any irreducible variety X we have the following theorem:

Theorem A.4.1 *The projection $\overline{X}$ of X from $p \notin X$ to $\mathbf{P}^{n-1}$ is a projective variety.*

The proof of this theorem works with elimination theory and the resultant of two polynomials in one variable. A detailed proof can be found in Harris([Har]).

Projections from subspaces This is a generalization of the notion of projections from points.
Let $\Lambda \cong \mathbf{P}^k$ be a subspace of $\mathbf{P}^n$ and let $\mathbf{P}^{n-k-1}$ be a subspace complementary to Λ. Then we define the *projection from Λ to $\mathbf{P}^{n-k-1}$* to be the map

$$\begin{aligned} \pi_\Lambda : \mathbf{P}^n - \Lambda &\to \mathbf{P}^{n-k-1} \\ q &\mapsto \overline{q, \Lambda} \cap \mathbf{P}^{n-k-1} \end{aligned}$$

where $\overline{q, \Lambda}$ denotes the $(k+1)$-plane spanned by q and Λ.

Example A.4.3 *This is the obvious generalization of example A.4.1. We take coordinates $Z = (Z_0 : \ldots : Z_n)$ on $\mathbf{P}^n$. Let $\mathbf{P}^{n-3}$ be the subspace given by $Z_{n-2} = Z_{n-1} = Z_n = 0$ and let Λ be the subspace spanned by the points $(0 : \ldots : 1 : 0 : 0)$, $(0 : \ldots : 1 : 0)$, $(0 : \ldots : 0 : 1)$. Then π_Λ is given by*

$$\pi_\Lambda : (Z_0 : \ldots : Z_n) \mapsto (Z_0 : \ldots : Z_{n-3}).$$

Again we define the *projection of X from Λ to $\mathbf{P}^{n-k-1}$* as the restriction of π_Λ to the variety X and denote $\overline{X}$ for $\pi_\Lambda(X)$.

Example A.4.4 *Let $X \subset \mathbf{P}^3$ be the irreducible variety defined by the set*

$$\{x_0x_1 + x_2x_3, x_1^2 + x_2^2 - x_1x_3 - x_0x_2, x_0^2 - x_3^2 + x_1x_3 - x_0x_2\}.$$

X is a rational normal curve. We project from the subspace Λ spanned by the points

$$\{(1 : 0 : 1 : 0), (0 : 1 : 0 : 1)\}.$$

A representation for the projection π_Λ *is given by*

$$(x_0 - x_2 : x_1 - x_3).$$

π_Λ *defines a birational map to* $\mathbf{P}^1$*.*

It is not hard to see that any projection from a k-dimensional subspace Λ can be realized as a composition of projections from points $p_0, \ldots, p_k$ spanning Λ.

Example A.4.5 *We get the projection* π_Λ *in example A.4.3 by projecting sequentially from the points* $p_0 := (0 : \ldots : 1 : 0 : 0)$*,* $p_1 := (0 : \ldots : 1 : 0)$*,* $p_2 := (0 : \ldots : 0 : 1)$*. We first project as in example A.4.1 from* p_2*. Then we are left with the points* $\pi_{p_2}(p_0) = (0 : \ldots : 1 : 0)$ *and* $\pi_{p_2}(p_1) = (0 : \ldots : 0 : 1)$ *in* $\mathbf{P}^{n-1}$ *. Now we go on in the same way - making the projections* $\pi_{\pi_{p_2}(p_1)}$ *and then* $\pi_{\pi_{\pi_{p_2}(p_1)}(\pi_{p_2}(p_0))}$*- and obviously have realized* π_Λ *as the composition of projections from points.*

Another important result about projections is the following:

Proposition A.4.1 *Let* $X \subset \mathbf{P}^n$ *be an irreducible variety. Unless* X *is not a hypersurface a general projection (projection from a generic point)* $\pi_p : X \to \mathbf{P}^{n-1}$ *gives a birational isomorphism from* X *with its image* $\overline{X}$*.*

Knowing this it is not hard to conclude that every irreducible variety X is birationally equivalent to a hypersurface. We only need to project X birationally from suitable points until we get a hypersurface. Equivalently we can project X in one step from a suitable subspace (corresponding to these points) birationally to a hypersurface.

A.5 Degree of Varieties

The degree of a variety is one important attribute for describing it (among others like smoothness and singularity, dimension or the Hilbert function and Hilbert polynomial). In chapter 3 we used the degree of a variety to define rational normal curves and to show that these curves are rational.
Here we shortly introduce the degree and certain properties of it (the behaviour under projection), following [Har]. To be able to define degree we need to know what the dimension of a variety is, therefore to recall this notion we start with a (intuitive) definition of dimension. Let us assume we

know that the dimension of the projective space $\mathbf{P}^n$ is n. We then can define the dimension of an irreducible projective variety in the following way:

Definition A.5.1 *Let X be an irreducible variety. The* **dimension of** *X is k if X admits a "finite-to-one" dominant rational map to $\mathbf{P}^k$.*

Of course from this definition it is not immediately clear either that such a map can always be found or that if it can the dimension k of the target space $\mathbf{P}^k$ is the same for all such maps. However I think one gets a good imagination of dimension from it - to see how these problems can be solved please look at the book of Harris([Har]).

Definition A.5.2 *Let $X \subset \mathbf{P}^n$ be an irreducible k-dimensional variety. Let Ω be a general $(n-k)$-plane. The* **degree of** *X is the number of points of intersection of Ω with X.*

Example A.5.1 *Let $X \subset \mathbf{P}^2$ be the parabola defined by $F = x^2 - yz$. Clearly X has dimension one. A general line in the projective plane meets X at exactly two points, therefore the degree of X is two.*

Degree of varieties and projections We have the following behaviour of the degree of a variety under a projection:

- Suppose that $X \subset \mathbf{P}^n$, $p \notin X$ and let $\overline{X} = \pi_p(X) \subset \mathbf{P}^{n-1}$ be the image of X under the projection from p. Then

$$deg(\overline{X}) = deg(X)$$

- Suppose $X \subset \mathbf{P}^n$, $p \in X$ a smooth point and let $\overline{X} \subset \mathbf{P}^{n-1}$ be the projective closure of $\pi_p(X)$, the image of X under the projection from p. Then

$$deg(\overline{X}) = deg(X) - 1$$

Example A.5.2 *Let* $X \subset \mathbf{P}^n$ *be a hypersurface. Then* X *is given as the zero locus of an irreducible polynomial* $F \in k[x_0, \ldots, x_n]$*. In this case the degree of* X *is just the degree of the polynomial* F*.*

Example A.5.3 *Let* $X \subset \mathbf{P}^n$ *be an arbitrary* k*-dimensional irreducible variety. As a consequence of proposition A.4.1 we can project* X *from a general* $(n-k-2)$*-plane birationally onto a variety* $\overline{X} \subset \mathbf{P}^{k+1}$*, which is a hypersurface, without changing the degree. Then the degree of* X *equals the degree of* $\overline{X}$*, which in turn is given by the degree of the defining polynomial of* $\overline{X}$*.*

We say that a variety $X \subset \mathbf{P}^n$ is *non-degenerate* if it does not lie in any hyperplane of $\mathbf{P}^n$.
Here is another result on the degree of a variety:

Proposition A.5.1 *Let* $X \subset \mathbf{P}^d$ *be an irreducible non-degenerate variety of dimension* k*.*
Then the degree of X *is* **at least** $d - k + 1$*.*

As an immediate consequence of this we have the following two corollaries:

Corollary A.5.1 *Let* $C \subset \mathbf{P}^d$ *be an irreducible non-degenerate curve. Then* $deg(C) \geq d$

Corollary A.5.2 *Let* $X \subset \mathbf{P}^d$ *be an irreducible non-degenerate hypersurface. Then* $deg(X) \geq d - 1$

A.6 Regular Systems of Parameters

In this section we introduce the concept of a *regular system of parameters* - a sort of local coordinate system for a variety X around a point $p \in X$.
We work in the *local ring* $\mathcal{O}_p$ of a variety X at a point p. This is the basic local invariant of a point p of a variety X. Since any point p of X has an affine neighbourhood, in the study of local properties we may restrict to irreducible affine varieties.
Most of the theory presented in this section can be found in [Sh] or in [Ei], where the material is treated in greater generality.

Maximal ideals and localization We recall the definition of a maximal ideal I in a ring R and describe the process of localization of a commutative ring R at a prime ideal P.

Definition A.6.1 *An ideal $I \subset R$ is said to be* **maximal** *if $I \neq R$ and any ideal J containing I is such that either $J = I$ or $J = R$.*

Example A.6.1 *Let $R := k[x_1, \ldots, x_n]$ be the polynomial ring in n variables. If k is an algebraically closed field every ideal of the form $< x_1 - a_1, \ldots, x_n - a_n >_{k[x_1,\ldots,x_n]}$, where $a_1, \ldots, a_n \in k$, is maximal. The other way around every maximal ideal in R is of this form. Therefore we have a one-to-one correspondence between points of k^n and maximal ideals of $k[x_1, \ldots, x_n]$ by*

$$ideal(\{(a_1, \ldots, a_n)\}) =< x_1 - a_1, \ldots, x_n - a_n >_{k[x_1,\ldots,x_n]} .$$

Definition A.6.2 *Let R be a ring. A proper ideal $I \subset R$ is* **prime** *if whenever $f, g \in R$ and $fg \in I$, then either $f \in I$ or $g \in I$.*

If R is a commutative ring and $P \subset R$ a prime ideal we may construct the **local ring of** R **at** P or **localization of** R **at** P which we denote by R_P.
R_P is the set of pairs (f, g) with $f, g \in R$ and $g \notin P$ identified according to the rule

$$(f, g) = (f', g') \Longleftrightarrow \exists h \in R \setminus P \text{ such that } h(fg' - gf) = 0.$$

Addition and multiplication on R_P are defined as follows:

$$\begin{aligned} (f, g) + (f', g') &= (fg' + gf', gg') \\ (f, g)(f', g') &= (ff', gg'). \end{aligned}$$

R_P with this operations is again a ring. Any element $u \in R_P$ can be written in the form f/g with $f, g \in R$ and $g \notin P$. The elements $g/1$ with $g \notin P$ are invertible in R_P. Moreover the elements of the form f/g with $F \in P$ and $g \notin P$ form a maximal ideal m that contains every other ideal of R_P (every element $u \notin m$ has an inverse).

Definition A.6.3 *A ring R is a* **local ring** *iff R has only one maximal ideal m that contains every other ideal of R.*

As we have seen in the example above there is a one-to-one correspondence of points in k^n and maximal ideals in $k[x_1, \dots, x_n]$. This remains true if we work in the affine coordinate ring $k[X]$ of an affine variety X. Note that every maximal ideal is prime.
If $p \in X$ is a point of X we denote m_p for the maximal ideal corresponding to this point. We then define the **local ring of X at p**, which we denote by $\mathcal{O}_p$, as the localization of $k[X]$ at m_p:

$$\mathcal{O}_p := k[X]_{m_p}.$$

By definition $\mathcal{O}_p$ is a local ring.

Regular local rings and regular systems of parameters

Definition A.6.4 *A local ring R with maximal ideal m is called* **regular** *if m can be generated by exactly d elements, where d is the (Krull-)dimension of R.*

The Krull-dimension of R is the maximal length of a chain of prime ideals in R. However in our special situation of a local ring of a variety X at a point $p \in X$ it is equal to the dimension of the variety at this point.
It is a consequence of the *principal ideal theorem* that m cannot be generated by fewer than d elements.

Definition A.6.5 *If R is a regular local ring of dimension d, then a set $\{u_1, \dots, u_d\} \subset m$ is said to be a* **regular system of parameters** *for R if $\{u_1, \dots, u_d\}$ generates the maximal ideal m.*

By the above a regular system of parameters of R is a minimal generating system of the maximal ideal $m \subset R$.

Computation of a regular system of parameters If p is a nonsingular point of a variety X then then the local ring $\mathcal{O}_p$ is regular. We turn to the question of how we can actually compute a regular system of parameters for

$\mathcal{O}_p$.

From now on we only consider the special case of an affine hypersurface $X \subset \mathbf{A}^3$. X is given as the zero set of an irreducible polynomial $F \subset k[x_1, x_2, x_3]$. If $p = (p_1, p_2, p_3) \in X$ is nonsingular then by definition we know that for some variable $x_k \in \{x_1, x_2, x_3\}$ the partial derivative of F with respect to z does not vanish at p, i.e.

$$\frac{\partial F}{\partial x_k}(p) \neq 0.$$

Now let x_i, x_j be the other two variables and p_i, p_j the corresponding components of p.

Claim: The set $\{x_i - p_i, x_j - p_j\}$ is a regular system of parameters for $\mathcal{O}_p$.

To see why this is true consider the following:

The maximal ideal m_p is generated by $\{x_1 - p_1, x_2 - p_2, x_3 - p_3\}$. Since X is a hypersurface in 3-space, i.e. of dimension 2, we are looking for a set of only two generators. This means that one of the three polynomials can be expressed in terms of the others.

Now we make use of the following proposition that gives us a relation to the **differential** of a function:

Proposition A.6.1 *$\{u_1, \ldots, u_d\}$ is a regular system of parameters of the local ring of a variety X at a nonsingular point p iff the set $\{du_1, \ldots, du_d\}$ generates $\Omega[U]$ - the ($k[U]$-) module of differential forms on U, where U is a neighbourhood of p in X .*

We need the following properties of the differential:

- For any polynomial $f \in k[x_1, \ldots, x_n]$ the differential of f can be written in the form
$$df(x_1, \ldots, x_n) = \sum_i \frac{\partial f}{\partial x_i} dx_i.$$
- For any $f \in ideal(X)$ the differential df is zero:
$$\forall f \in X \ : \ df = 0$$
- $d(f + g) = df + dg$
- $d(fg) = fdg + gdf$

- $dc = 0$ for any constant c

Now to see that $\{x_i - p_i, x_j - p_j\}$ is a regular system of parameters we need to see that $\{d(x_i - p_i), d(x_j - p_j)\} = \{dx_i, dx_j\}$ generates $\Omega[U]$ for some open set U containing p.If this is the case then we can write $d(x_k - p_k)$ as a $k[U]$-linear combination of dx_i and dx_j.
Now since F generates X we know $dF = 0$ and therefore

$$dF = \frac{\partial F}{\partial x_i} dx_i + \frac{\partial F}{\partial x_j} dx_j + \frac{\partial F}{\partial x_k} dx_k = 0.$$

Since $\frac{\partial F}{\partial x_k}(p) \neq 0$ we can also find an open set $\tilde{U} \subset X$ such that $\frac{\partial F}{\partial x_k}(\tilde{p}) \neq 0$ for all points $\tilde{p} \in \tilde{U}$. Therefore in $\tilde{U}$ we can divide by $\frac{\partial F}{\partial x_k}$ and write $dx_k = d(x_k - p_k)$ as linear combination

$$dx_k = -\frac{\frac{\partial F}{\partial x_i}}{\frac{\partial F}{\partial x_k}} dx_i - \frac{\frac{\partial F}{\partial x_j}}{\frac{\partial F}{\partial x_k}} dx_j.$$

Now we have found an open set $\tilde{U}$ (which is a neighbourhood of p) where $\{d(x_i - p_i), d(x_j - p_j)\}$ generates $\Omega[\tilde{U}]$ and by proposition A.6.1 this means that indeed $\{x_i - p_i, x_j - p_j\}$ is a regular system of parameters for $\mathcal{O}_p$.

Example A.6.2 *Let $X \subset \mathbf{A}^3$ be the hypersurface defined by $F = (x^4 + z^4) + (xy + z^2)^3$. The point $p = (-1, 1, 0)$ is a nonsingular point of X because $\frac{\partial F}{\partial x}(p) = -1$. Therefore the set $\{y - 1, z\}$ forms a regular system of parameters.*

Partial Derivatives in Regular Systems of Parameters Once we have found a regular system of parameters at a point p we can compute partial derivatives of elements of $k[U]$ with respect to the local coordinates. Throughout let us denote $u_1 := x_1 - p_1$ and $u_2 := x_2 - p_2$ for the local coordinates and let x_3 be the variable not involved.u_1 and u_2 determine k-linear mappings

$$\partial_{u_1}, \partial_{u_2} : k[U] \to k[U],$$

that is for $g, h \in k[U]$ and constants $c, d \in k$ we have

$$\partial_{u_k}(cg + dh) = c\partial_{u_k}(g) + d\partial_{u_k}(h),$$

and $\partial_{u_1}, \partial_{u_2}$ fulfill the chain rule

$$\partial_{u_k}(gh) = \partial_{u_k}(g)h + g\partial_{u_k}(h).$$

Moreover we have that

$$\partial_{u_k}(c) = 0$$

for any constant c and

$$\partial_{u_k}(u_i) = \delta_{i,k}.$$

If $g \in k[U]$ is any polynomial function on U then we can express $\partial_{u_k}(g)$ in the form

$$\partial_{u_k}(g) \;=\; \sum_{i=1}^{3} \frac{\partial g}{\partial x_i}\partial_{u_k}(x_i) \tag{A.1}$$

and hence the mappings ∂_{u_k} are completely determined by the images $\partial_{u_k}(x_i)$, $i = 1 \ldots 3$. Now for $i = 1, 2$ we have $\partial_{u_k}(x_i) = \partial_{u_k}(x_i - p_i) = \partial_{u_k}(u_i) = \delta_{i,k}$ and the only thing left is to compute $\partial_{u_k}(x_3)$.
On the other hand $U \subset X$ and therefore the defining function F of X is also the zero-function in $k[U]$. So by formula A.1 we have

$$0 \;=\; \sum_{i=1}^{3} \frac{\partial F}{\partial x_i}\partial_{u_k}(x_i) \;=\; \frac{\partial F}{\partial x_k}\overbrace{\partial_{u_k}(x_k)}^{=1} \;+\; \frac{\partial F}{\partial x_3}\partial_{u_k}(x_3).$$

Since by assumption $\frac{\partial F}{\partial x_3}$ is a non-zero form on U we can divide and get

$$\partial_{u_k}(x_3) \;=\; -\frac{\frac{\partial F}{\partial x_k}}{\frac{\partial F}{\partial x_3}}.$$

The resulting formula for $\partial_{u_k}(g)$ is

$$\partial_{u_k}(g) \;=\; \frac{\partial g}{\partial x_k} \;-\; \frac{\partial g}{\partial x_3}\frac{\frac{\partial F}{\partial x_k}}{\frac{\partial F}{\partial x_3}}.$$

Example A.6.3 *Let F and p be be as in example A.6.2 above. Here $u_1 := y - 1$ and $u_2 := z$. Then*

$$\partial_{u_1}(x) = \frac{-3x(x^2y^2 + 2xyz^2 + z^4)}{4x^3 + 3x^2y^3 + 6xy^2z^2 + 3z^4y}$$

$$\partial_{u_2}(x) = \frac{-2z(2z^2 + 3x^2y^2 + 6xyz^2 + 3z^4))}{4x^3 + 3x^2y^3 + 6xy^2z^2 + 3z^4y}$$

A.7 A Criterion for Radicality

In this section we will proof a criterion to test the radicality of an ideal-based on the primary decomposition of a complete intersection. We used this criterion in the proof of proposition 3.1.3. This section was contributed by Santiago Encinas whom I want to thank cordially at this point. We start with the formulation of the theorem in the final version that we are actually use in the proof of proposition 3.1.3. We then will show how this theorem can be deduced from theorem A.7.2 below.

Let $F_1, \ldots, F_n \subset k[x_1, \ldots, x_m]$ (where $n < m$). Let I be the ideal generated by the polynomials F_i and let X be the zero set of I. Let Y be the intersection of X and the zero set of the polynomial system of equations given by the determinants of the $n \times n$ sub-matrices of the jacobian of the F_i, that is

$$Y := ZS(F_1, \ldots, F_n, det_{n\times n}\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}).$$

Theorem A.7.1 *If $dim(X) = m - n$ and $dim(Y) < m - n$ then I is a radical ideal.*

Before we start with the deduction of this criterion we first recall the definition of a radical ideal :

Definition A.7.1 *Let $I \subset A$ be an ideal. We say that I is* **radical** *iff whenever I contains the n-th power a^n of an element $a \in A$ then it follows that I also contains a.*

We also recall here the concept of a generic point of a variety. Let X be a (projective or affine) variety in (projective or affine) space over a ground-field k.

Definition A.7.2 *A point $p \in X$ is a* **generic point of X** *iff all polynomials with coefficients in k that vanish at p vanish at all points of X.*

Now before we give the criterion for radicality we need to define when we say that a notion for an ideal is generical, for instance the radicality.

Definition A.7.3 *Let k be an algebraically closed field and I be an ideal in the polynomial ring $k[X_1, \dots, X_n]$. We say that the ideal I is generically radical if the localization of I at the generic points of the closed set $V(I)$ is a radical ideal.*

More algebraically, we can define the same concept.

Definition A.7.4 *Given an ideal I in a noetherian ring A, we say that the ideal is generically radical if for every minimum prime p, associated to I, the localization of the ideal I_p is a radical ideal.*

It is an easy exercise on primary decomposition to show that both definitions coincide for the polynomial case.

The criterion for radicality is enclosed in theorem A.7.2.We first recall the definition of complete intersection for the case of an ideal $I \subset k[x_1, \dots, x_n]$. As usual the dimension $dim(I)$ of an ideal I is defined as the dimension of its zero-set $ZS(I)$.

Definition A.7.5 *Let A be a noetherian ring. An ideal $I \subset A$ is a* **complete intersection** *iff it can be generated by exactly $n - dim(I)$ elements.*

Theorem A.7.2 *Let A be a noetherian ring. Let I be an ideal of A such that I is a complete intersection and I is generically radical, then I is a radical ideal.*

Proof: Consider the primary decomposition of the ideal I, say $I = q_1 \cap \cdots \cap q_r$. The radical of I is the intersection $p_1 \cap \cdots \cap p_r$, where p_i is the radical of q_i.

It is well known that if I is a complete intersection, then the ring A/I is Cohen-Macaulay ([Ei]), and it follows from the unmixedness theorem [Ei, corollary 18.14] that every associated prime of a Cohen-Macaulay ring is minimal, so that every prime p_i is minimal. By hypothesis, all the localizations I_{p_i} are radical ideals. On the other hand, since there are no embedded components in I, we have that $I_{p_i} = (q_i)_{p_i}$.

Now let f be in the radical of I. For any i, f is in the radical of I_{p_i}, so $f \in I_{p_i} = (q_i)_{p_i}$. There exist $g_i \in q_i$ and $s_i \notin p_i$, such that $s_i f = g_i$. So it follows that $f \in q_i$ for any i, which means that $f \in I$.

For applying the above criterion of radicality, we need to show that our ideal I is a complete intersection- for this there are several tools and criteria

- and that I is generical radical. But in many cases for the generic points we have normal crossing, and this implies radicality. We will describe below the meaning of **normal crossing** and a way how to check whether or not a given ideal has this property. First we give the corollary we are actually going to use to prove proposition 3.1.3.

In analogy to the above definition A.7.3 we say that an ideal I has generically normal crossing if the localization of I at every minimal prime is a normal crossing.

Lemma A.7.1 *Let A be a noetherian ring and I be an ideal in A such that I is complete intersection. If I has generically normal crossing (with respect to a minimal set of generators $\{F_1, \dots, F_n\}$) then I is a radical ideal.*

The geometric meaning of **normal crossing** is the following: Let X be a smooth variety and let $X_1, \dots, X_n$ be subvarieties of X of codimension one. We say that $X_1, \dots, X_n$ have **normal crossing** at a point $p \in X$ if the following conditions holds:

- If $p \in X_i$ then p is a smooth point of X_i.
- If $p \in X_{i_j}$ and $p \in X_{i_k}$ then X_{i_j} and X_{i_k} do not meet tangential at p.
- The number r of subvarieties containing p is at most $dim(X)$.

This translates to the following algebraic definition:
Given a variety X and a smooth point $p \in X$ let $u_1, \dots, u_n$ be non-zero forms on X (defining subvarieties of codimension 1 in X). Let T_pX denote the tangent space to X at p and let $(T_pX)^*$ be its dual space. Then $u_1, \dots, u_n$ have normal crossing at the point p if for the collection $u_{i_1}, \dots, u_{i_r}$ of forms vanishing at p every u_{i_j} is smooth at p and the differentials $d_p u_{i_1}, \dots, d_p u_{i_r}$ are linearly independent in $(T_pX)^*$.

Now what is again the precise meaning of **generically normal crossing** in lemma A.7.1?
We are given an ideal $I \subset A$ and a set of generators $\{F_1, \dots, F_n\}$ of I. I is the ideal-sum of the ideals generated by the F_i or equivalently the zero-set $ZS(I)$ of I is the intersection of the hypersurfaces defined by the F_i. Then we say that I has generically normal crossing (with respect to $\{F_1, \dots, F_n\}$) if $F_1, \dots, F_n$ have normal crossing at the generic points of $ZS(I)$.

We now turn to question of how we actually can check whether an ideal $< F_1, \ldots, F_n > \subset k[x_1, \ldots, x_m]$ has generically normal crossing.Note that the variety X containing the hypersurfaces X_i is now the whole space k^m and the hypersurfaces are defined by the polynomials $F_1, \ldots, F_n$. For a point $p \in ZS(I)$ we want to check if the F_i have normal crossing at p. If so then the differentials d_pF_i must be linearly independent. It is well known that for a (local) coordinate system $x_1, \ldots, x_m$ of X at p we can express the differentials d_pF_i by

$$d_pF_i = \sum_{j=1}^{m} \frac{\partial F_i}{\partial x_j}(p) d_p x_j.$$

Then saying that the differentials d_pF_i are linearly independent is the same as saying that the jacobian-matrix

$$J := \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}$$

evaluated at p has maximal rank n or that the determinants of all $n \times n$ submatrices of J do not vanish at p.
This brings us to the following lemma:

Lemma A.7.2 *Let $I := < F_1, \ldots, F_n > \subset k[x_1, \ldots, x_m]$. $F_1, \ldots, F_n$ have normal crossing at $p \in ZS(I)$ iff the rank of the jacobian matrix J of the F_i-evaluated at p- is maximal.*

From this theorem A.7.1 at the beginning of this section follows immediatly: if the dimension of the set of points of X (the zero set of I) for which the jacobian does not have maximal rank is strictly smaller than the dimension of X then I has normal crossing at its generic points.

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