Definite Summation and Hypergeometric Identities

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Abstract

This is a concise survey about recent developments in the mentioned field, which is to appear as a section of the *Computer Algebra Handbook*, E. Kaltofen, J. Grabmeier, V. Weispfenning (eds.), Springer-Verlag, 2000.

1 The definite breakthrough

Until quite recently the algorithmic task of *summation*, compared to its continuous analogue *integration* (see the section on "Symbolic Integration") has received relatively little attention within the computer algebra community. The survey article [21] by J. Lafon characterizes the situation in the eighties: in textbooks and monographs the problem was hardly mentioned; only few methods had been implemented, at least in "commercial" systems. Indeed, down to date: B. Gosper's method [12] for indefinite hypergeometric summation served as the work horse most of the time; sometimes variations of R. Moenck's approach [22] for rational summation were used, however the work of S. Abramov [1, 2] and of M. Karr [13, 14] remained widely unnoticed.

The current intensified interest in summation was mainly initiated by two achievements of D. Zeilberger: his successful application ("creative telescoping") of Gosper's algorithm for *indefinite* hypergeometric summation to problems of *definite* hypergeometric summation [40, 42] and, based on I. Bernstein's theory of holonomic systems, his "holonomic systems approach to special function identities" [41]. We want to emphasize that algorithmic proofs of definite hypergeometric summation identities had sometimes even been considered as unfeasible. After this breakthrough in the single sum case, Wilf and Zeilberger further extended this algorithmic proof theory to closed form evaluation ("WZ-method") and to multisums and integrals ranging over hypergeometric and hyperexponential terms [38, 39]. The approach is based on the fact that these sums and integrals are holonomic and that annihilating linear difference and differential operators can be constructed effectively. This constitutes the realization of an old idea, known as *Sister Celine's technique* to researchers in the area of special functions for quite some time. One characteristic of the method is that, in addition to the operators, "proof certificates" in the form of multivariate rational functions will be constructed, whose verification by itself requires only exact rational function arithmetic.

A comprehensive presentation of this development and the algorithms around is now available in book form [31, 17]. These monographs describe carefully the five basic algorithmic methods, their applications and extensions, in the field: Sister Celine's technique, Gosper's algorithm, creative telescoping, WZ-method, and Petkovšek's algorithm `hyper`. The latter [29] finds closed form solutions of linear higher order polynomial recurrences, and hence provides a decision procedure for closed form evaluation of definite hypergeometric summations.

For further references to original papers and other relevant literature we refer to the extensive bibliographies of [31] and [17], see also the section on "Orthogonal Polynomials'.

2 Further developments

A concise survey of some recent developments in symbolic summation is given in [28]. It is shown that normal form aspects such as the Gosper-Petkovšek form for rational functions or Paule's "greatest factorial factorization" for polynomials, a shift-analogue of square-free factorization, lead to effective algorithms (e.g., the optimal solution by R. Pirastu und V. Strehl [32] in the case of rational summation) and to better understanding and extensions (e.g., Paule's algebraic approach to Gosper's algorithm [25] and to its q- and multibasic analogues [26, 33, 3]).

As to the holonomic aspects of the field, N. Takayama developed non-commutative Groebner bases methods for holonomic elimination problems (e.g., recurrence and contiguous relations for hypergeometric functions, zero-recognition, etc.); see [35, 36]. F. Chyzak and B. Salvy formulated holonomic elimination in the generic setting of Ore algebras, which provides additional flexibility and tools in automatically proving special function identities. This context allows, e.g., to treat systems of mixed differential-difference type including ordinary as well as q-hypergeometric recurrences; see [8]. In [7] F. Chyzak underlines the central role of multivariate D-finiteness and exploits it algorithmically.

Proving binomial identities and finding recurrences satisfied by such sums, a task notoriously difficult for the human mathematician, has become a routine business for computers, even in the q-case and also for ordinary hypergeometric multisums. The literature abounds with successful applications of this "technology"; for sake of brevity we only mention a few instances:

— The paper by I. Nemes et al. [23] describes how the typical Monthly

problem (on summation) can be treated automatically with the computer.
— In [11], I. Gessel shows the effectiveness of the WZ-method if applied systematically.
— The celebrated Rogers-Ramanujan identities in additive numbertheory find "one-line" computer proofs as explained by P. Paule in [24].
— Spectacular determinant evaluations with essential help of creative telescoping have been achieved by C. Krattenthaler et al. [9, 20] and by M. Petkovšek and H. Wilf [30].

3 Implementations

There are now various implementations available of the work mentioned above. In particular:
— The algorithms discussed in "A=B" [31] are contained on a diskette which comes with the book or can else be taken in Maple or Mathematica versions from the web site `http://www.cis.upenn.edu/~wilf/AeqB.html` (from which even the entire book can be downloaded).
— Mathematica implementations of creative telescoping in the ordinary and q-case have been written at RISC-Linz [26, 27]; efficient multisum algorithms are available through a package developed by K. Wegschaider [37]. Karr's algorithms and extensions have been implemented by C. Schneider [34]. All can be taken from `http://www.risc.uni-linz.ac.at/research/combinat/risc/`. For a Macsyma implementation of Zeilberger's method see [6].
— Maple implementations in the ordinary and q-case have been written by T. Koornwinder [18], available from `http://turing.wins.uva.nl/~thk/recentpapers/zeilbalgo.html`, by W. Koepf [17, 5], available from `http://www.imn.htwk-leipzig.de/~koepf/research.html`, and by B. Gauthier [10], from `http://http://www-igm.univ-mlv.fr/~gauthier/HYPERG.html`
— The Maple library now contains the package `sumtools`, written by W. Koepf [15]; a Reduce version is also available, see [16, 4].
— For implementations of N. Takayama's work consult
`http://www.math.s.kobe-u.ac.jp/HOME/taka/index.html`
— The packages developed by F. Chyzak can be taken from
`http://www-rocq.inria.fr/algo/libraries/libraries.html`
— Finally we mention C. Krattenthaler's Mathematica packages HYP and HYPQ [19], though different in character, are invaluable tools for the investigation of hypergeometric transformation identities; they are to be found at `http://radon.mat.univie.ac.at/People/kratt/hyp_hypq/hyp.html`

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