Proper Parametrization of Surfaces with a Rational Pencil

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Abstract

We give an algorithm for the following problem: given a surface defined over $\mathbb{Q}$, and a rational pencil on it, compute a proper parametrization with coefficients in $\mathbb{Q}$.

Introduction

A surface with a rational pencil is an algebraic surface S equipped with a rational map $f : S \rightarrow \mathbb{P}^1$, such that the generic fiber is an irreducible curve of genus zero. The class of these surfaces includes many classical surfaces such as the conical surfaces, the cyclides of Dupin, or the surfaces of Darboux. They also arise naturally in CAD/CAM as canal surfaces [24, 23], and in kinematics [11, 12].

In this paper, we address the following problem: given a surface S with a rational pencil, find a proper parametrization of S by rational functions. A solution for this problem has applications in CAD/CAM, where one wants to represent curves and surfaces via rational parametrizations, at least piecewise (see [9, 25]). It is obvious that parametric varieties are easier to visualize than implicitly given varieties.

Over the complex numbers, a classical result [22] says that a surface with a rational pencil is indeed a rational surface. An efficient algorithm for the construction of a parametrization can be found in [27].

Over the real numbers, a theorem of Comesatti (see [14]) tells that a surface with a rational pencil is rational iff it is connected. Efficient algorithms for the construction have been given in [26, 28].

In the algorithms for the complex and for the real case, rounding errors are unavoidable. In situations where exact computation is essential (e.g. because the result is re-used for other symbolic computations), it is preferable to compute parametrizations with coefficients in $\mathbb{Q}$. That is the goal in this paper.

In theory, the question of rationality of surfaces with a rational pencil is well understood. Any such surface is birationally equivalent to a minimal one. For the minimal ones, Iskovskih proved in a series of papers [14, 15, 16]. that rationality is equivalent to ind ≤ 3 and the existence of a point defined over $\mathbb{Q}$. Moreover, the Minkowski-Hasse principle (see [19]) holds in the case ind ≤ 3, which makes the existence of points decidable.

From the computational point of view, the question has not been solved up to now. The main computational problem is to actually compute the reduction to a minimal one.

In order to perform this reduction concretely and algorithmically, we have to measure the distance to the minimalization somehow. For this purpose, we introduce the notion of *degree defect* (see section 2). In contrast to Iskovskih's numerical characters, it is defined by elementary and symbolic means.

We then give a fast algorithm for computing this reduction (algorithm *DegreeMinimize* in section 2). Its complexity is comparable to the complexity of polynomial factorization (see theorem 7). The parametrization for the minimal surfaces turn out to be quite easy. Together, these two algorithms give an algorithm for computing a parametrization over $\mathbb{Q}$ or detecting its non-existence.

When looking to our computations with quadratic forms, it stands out that we keep the mixed terms most of the time, instead of eliminating them by the usual Gaussian elimination. Example 5 shows that this is indeed necessary.

The algorithmic solution has interesting implications for the general proper parametrization problem, which are discussed in the last section.

We have formulated our algorithms for $\mathbb{Q}$, because this was the main motivation. Most of them work without any changes in any computable field with factorization, provided that the characteristic is not equal to 2.

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1 Reduction to Standard Forms

We consider algebraic surfaces defined over $\mathbb{Q}$, i.e. complex zero sets S of two-dimensional ideals in $\mathbb{Q}[x_1, \ldots, x_n]$. The problem addressed here is to find a proper parametrization: an n-tuple $(X_1, \ldots, X_n)$ of rational functions in $\mathbb{Q}[s, t]$ satisfying the equations of the surface, such that the parameters s, t can be expressed as rational functions in $\mathbb{Q}(S)$. For instance, the parametrization

$$(X_1, X_2, X_3) = \left(\frac{2s}{s^2+t^2+1}, \frac{2t}{s^2+t^2+1}, \frac{s^2+t^2-1}{s^2+t^2+1}\right)$$

is a parametrization of the unit sphere $x_1^2 + x_2^2 + x_3^2 - 1 = 0$, because we can obtain s, t by the stereographic projection $s = \frac{x}{1-z}, t = \frac{y}{1-z}$. If a surface has a proper parametrization, then we say that it is $\mathbb{Q}$-rational or (shortly) rational.

We assume that we have given a rational map $f : S \rightarrow \mathbb{P}^1$, defined over $\mathbb{Q}$, with generic fiber being an irreducible curve of genus zero. Such a pencil is called a *rational pencil*.

For instance, S may be the offset surface of a rational curve in space. Then there is a rational map from S to the generating curve with circles as fibers. Below is another example. For parametrizations in the general case, we refer to section 4.

Example 1. Let S be the surface with equation $A(z)x^2 + B(z)y^2 + C(z) = 0$, where $A, B, C \in Q[z]$, without common factor. The generic fiber of the projection to the z-axis is an irreducible conic, hence it has genus zero.

In some sense, the example above is generic: any surface with a rational pencil is birationally equivalent to an instance of the example. This follows from the following classical theorem.

Theorem 1. *Let K be an arbitrary field. Let C be a curve of genus zero defined over K. Then there is a birational map, defined over K, transforming C to a conic.*

For a modern proof and an algorithm producing the birational map, see [29].

Let $\mathbb{Q}(t)$ be a simple transcendental extension of $\mathbb{Q}$. Then $(t:1)$ is a generic point of $\mathbb{P}^1$. The generic fiber $f^{-1}((t:1))$ is defined over the field $\mathbb{Q}(t)$. By assumption, it is an irreducible curve of genus zero. Hence we can apply theorem 1 for $K = \mathbb{Q}(t)$, to obtain a birational map to a conic defined over $\mathbb{Q}(t)$. Interpreting t as a variable again, we get a birational map from S to a surface in $\mathbb{P}^2 \times \mathbb{Q}$ with an irreducible equation

$$\sum_{0 \leq i \leq j \leq 2} A_{ij}(t) x_i x_j = 0, \tag{1}$$

where $A_{ij} \in \mathbb{Q}[t]$. We will call an equation of type (1) a *standard form.*

Example 2. Let S be the surface with equation

$$x^2 + 2xz^2 + 2xyz + 2y^3 + 2z^3 y + x^4 + y^4 + z^4 = 0.$$

A pencil is given by the structure map

$$f : S \to \mathbf{P}^1, (x, y, z) \mapsto (x : y).$$

The generic fiber, with equations

$$t^2y^2 + 2tyz^2 + 2ty^2z + 2y^3 + 2z^3y + (t^4+1)y^4 + z^4 = x - ty = 0$$

is a quadric plane curve with an double point at the origin and two infinitely near double points. By the genus formula (see [30]), it has genus zero.

We give a birational map to the conic over $\mathbb{Q}(t)$ with equation

$$-2x_1^2 + tx_2^2 + 2tx_1x_2 + 2x_2x_0 + (t^5 + t)x_0^2 = 0, \tag{2}$$

which is also the equation of a surface T over $\mathbb{Q}$. Here it is:

$$(x, y, z) \mapsto (X_0 : X_1 : X_2) = (y^2 : yz : ty + z^2),$$

$$(x_0 : x_1 : x_2) \mapsto (X, Y, Z) = \left(\frac{t^2 x_0^2}{x_2 x_0 - x_1^2}, \frac{t x_0^2}{x_2 x_0 - x_1^2}, \frac{t x_1 x_0}{x_2 x_0 - x_1^2} \right).$$

Using a computer algebra system, it is easy to check that the image of the map has indeed the above equation and that the composition of the two given maps is the identity.

The same functions induce a birational map between S and T, keeping t fixed.

For the parametrization problem of a surface with a rational pencil, we may always assume that we are in a standard form, by the algorithm [29].

2 Degree Reduction

Let $Q(t, x_0, x_1, x_2) := \sum_{0 \leq i \leq j \leq 2} A_{ij}(t) x_i x_j$, where $A_{ij} \in \mathbb{Q}[t]$, be a standard form of a surface S with a rational pencil. Let $W := (w_0, w_1, w_2)$ be a triple of positive integers. The weighted degree of Q with respect to the weight vector W is defined as

$$\deg_W(Q) := \max_{i,j}(\deg(A_{i,j}) + w_i + w_j).$$

Recall that the Hessian matrix of Q is defined as

$$M(Q) := \left[\frac{\partial^2 Q}{\partial x_i \partial x_j} \right]_{ij} = [A'_{ij}]_{ij},$$

where $A'_{ij} := A_{ij}$ for $i < j$, $A'_{ii} := 2A_{ii}$, and $A'_{ij} := A_{ji}$ for $i > j$, and that the Hessian is defined as $H(Q) := \det(M(Q))$. The inequalities $\deg(A_{i,j}) \leq \deg_W(Q) - w_i - w_j$ imply

$$\deg(H(Q)) \leq 3\deg_W(Q) - 2w_0 - 2w_1 - 2w_2. \tag{3}$$

The difference between the right hand side and the left hand side of (3) is called the *degree defect* $\mathrm{def}_W(Q)$. It may be interpreted as follows. Let

$$Q_h(t_0, t_1, x_0, x_1, x_2) := t_0^{\deg_W(Q)} Q\left(\frac{t_1}{t_0}, \frac{x_0}{t_0^{w_0}}, \frac{x_1}{t_0^{w_1}}, \frac{x_2}{t_0^{w_2}}\right)$$

be the weighted homogenization of Q. Note that the choice of the exponent of t_0 ensures that Q_h is a polynomial. The Hessian $H(Q_h)$ is a homoneous polynomial in t_0, t_1 of degree $\deg_W(Q) - 2w_0 - 2w_1 - 2w_2$. Moreover, $H(Q_h)(1,t) = H(Q)$. Thus, the degree defect is the exponent of t_0 in the factorization of $H(Q_h)$.

Lemma 1. *Any standard form is birationally equivalent to a standard form with squarefree Hessian and with degree defect less than or equal to 1, with respect to a suitable weight vector.*

Proof. By Gauss elimination on $M(Q)$, one can find an invertible matrix $T \in \mathrm{GL}(3, \mathbb{Q}(t))$ such that $M' := T^* M(Q) T$ is a diagonal matrix (T^* denotes the transpose). Let Q' be the primitive part of the numerator of $\sum_{i=0}^{2} M'_{i,i}(t) y_i^2$. Then the surfaces defined by Q and Q' are birationally equivalent by the invertible map

$$x_i = \sum_{j=0}^{3} T_{ij}(t) y_j, i = 0, \ldots, 2,$$

which leaves t fixed. Thus, we have reduced to the case where $M(Q)$ is a diagonal matrix.

Assume that one of the diagonal elements, say A_{00}, has a quadratic factor F. By the birational transformation $x_0 = F(t)y_0$, we transform Q to the surface $(A_{00}/F^2)y_0^2 + A_{11}x_1^2 + A_{22}x_2^2 = 0$, i.e. we have eliminated the quadratic factor.

Assume that two of the diagonal elements, say A_{11} and A_{22}, have a common factor F. Then we multiply the equation with F and eliminate the quadratic coefficients of x_1

and x_2, as above. Thus, we have reduced to the case where A_{00}, A_{11}, A_{22} are relatively prime and squarefree. Since the Hessian is equal to $A_{00}A_{11}A_{22}$, we already achieved that the Hessian is squarefree.

It is not hard to find a weight vector such that the deficiency is less than or equal to 1. If the numbers $\deg(A_{00}), \deg(A_{11}), \deg(A_{22})$ are all even, then we set $w_i := n - \deg(A_{ii})/2$, for some large enough n. If they are all odd, then we set $w_i := n - (\deg(A_{ii}) - 1)/2$. If one of them, say $\deg(A_{00})$, is even, and the others are odd, then we set $w_0 := n - \deg(A_{00})/2$ and $w_i := n - (\deg(A_{ii}) - 1)/2$ for $i = 1, 2$. If $\deg(A_{00})$ is odd and the others are even, then we set $w_0 := n - (\deg(A_{00}) + 1)/2$ and $w_i := n - \deg(A_{ii})/2$ for $i = 1, 2$. □

Remark 1. With some more effort, one can even achieve degree defect zero always. The idea is to move the infinite point of the t-line to the finite by some birational transformation effecting the t-coordinate. However, lemma 1 is sufficient for our purpose.

Any $\mathbb{Q}(t)$-linear transformation of standard forms can be viewed as a birational map between the corresponding surfaces—the the above proof provides several examples of this principle. If Q' is the transform of Q by T, then we say that the standard forms Q and Q' are equivalent.

In the following, we study the behavior of the degree defect under $\mathbb{Q}(t)$-linear transformations. Note that the degree defects of two standard forms that differ by a factor in $\mathbb{Q}(t)$ is the same. Hence we can always change to the primitive part of the numerator.

Let $W = (w_0, w_1, w_2)$ and $U = (u_0, u_1, u_2)$ be two weight vectors. Let $T \in \mathrm{M}(3\times 3, \mathbb{Q}[t])$, such that $\det(T)$ does not vanish identically. Then we define the weighted degree of T as

$$\deg_{W,U}(T) := \max_{i,j}(\deg(T_{ij}) - w_i + u_j).$$

The weighted homogenization is the matrix T_h defined by

$$(T_h)_{ij}(t_0, t_1) := t_0^{\deg_{W,U}(T) + w_i - u_j} T_{ij}(t_1/t_0).$$

The inequalities $\deg(T_{ij}) \le \deg_{W,U}(T) + w_i - u_j$ imply

$$\deg(\det(T)) \le 3\deg_{W,U}(T) + \sum_i w_i - \sum_i u_i. \qquad (4)$$

The difference between the right hand side and the left hand side of (4) is called the degree defect $\mathrm{def}_{W,U}(T)$. It is easy to see that $\det(T_h)(1, t) = \det(T)$ and that the degree defect is the exponent of t_0 in $\det(T_h)$.

Lemma 2. *Let Q be a standard form. Let W and U be weight vectors. Let $T \in \mathrm{M}(3\times 3, \mathbb{Q}[t])$ such that $\mathrm{def}_{W,U}(T) = 0$. Let Q' be the transformed standard form. Then $\mathrm{def}_U(Q') \le \mathrm{def}_W(Q)$.*

Proof. By definition of Q', we have $H(Q') = T^*\ H(Q)\ T$. The U-homogeneous standard form $Q'' := T_h^*\ H(Q_h)\ T_h$ satisfies $Q''(1, t) = Q'$, hence we have $Q'' = t_0^n Q'_h$ for some nonnegative n. Taking determinants, we get

$$\det(H(Q_h))(\det(T_h))^2 = t_0^{3n} \det(H(Q'_h)).$$

By the interpretation of the degree defect as the exponent of t_0 in the homogeneous determinant, we get

$$\mathrm{def}_U(Q') = \mathrm{def}_W(Q) + 2\mathrm{def}_{W,U}(T) - 3n \le \mathrm{def}_W(Q).$$

□

The next question is: suppose that we have applied a transformation that has positive degree defect, which means that we may have deteriorated the original situation. Can we reverse this effect by applying another transformation? The answer is yes, by the next theorem.

Theorem 2. *Let $T \in \mathrm{M}(3\times 3, \mathbb{Q}[t])$ such that $\det T$ does not vanish identically. Let W be a weight vector. Then there exists a matrix $S \in \mathrm{GL}(3, \mathbb{Q}[t])$ and a weight vector U, such that $\mathrm{def}_{W,U}(TS) = 0$.*

Proof. Let $A \subset \mathbb{Q}[t]^3$ be the $\mathbb{Q}[t]$-module generated by the columns of T. We write $e_0 := (1, 0, 0)^*$, $e_1 = (0, 1, 0)^*$, $e_2 := (0, 0, 1)^*$. We compute a Gröbner basis [3, 4] of A with respect to the following term order: $t^m e_i \prec t^n e_j$ iff $m - w_i < n - w_j$ or $m - w_i = n - w_j$ and $i < j$. When applying Buchberger's algorithm, we never need to compute an S-polynomial, because e_i and e_j are not multiplied, and for any two leading monomials containing e_i, we have one that is divisible by the other. Thus, the Gröbner basis can be computed by inter-reduction alone. On the matrix level, these inter-reductions correspond to elementary column operations or to the multiplication by Frobenius matrices from the right.

After termination, the leading monomials are $t^{n_0} e_0$, $t^{n_1} e_1$, $t^{n_2} e_2$ for suitable n_0, n_1, n_2. We permute the generators such that the leading monomials appear in this order. This corresponds to the multiplication with a permutation matrix from the right. We set S to be the product of the Frobenius matrices and the permutation matrix.

Since the columns of TS form a Gröbner basis with the above leading monomials, we have $\deg((TS)_{ij}) \le n_j + w_i - w_j$, $\deg((TS)_{ii}) = n_i$, and $\deg((TS)_{ij}) < n_j + w_i - w_j$ for $j < i$. We set U by $u_j := w_j - n_j + n$, where n is large enough to make all weights positive. Then $\deg_{W,U}(TS) = n$. The degree of $\det(TS)$ is equal to the degree of the product of the diagonals, because all other products in the determinant formula have lower degree. Thus we have equality in (4), i.e. the degree defect is zero. □

Example 3. Let $W := (1, 1, 1)$. The matrix

$$T := \begin{bmatrix} t^2+1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

has W-degree 2. Since $\det(T)$ has degree 2, the degree defect is 4.

The columns do not form a Gröbner basis, because the first column $t^2 e_1 + e_1$ can be reduced by the second column $te_1 + e_2$. We add the second column times $-t$ to the first column. The result is $e_1 - te_2, te_1 + e_2, e_3$, which is a Gröbner basis. After exchange of the first two elements, the leading monomials are te_1, te_2, e_3. Thus, we have $n_0 = n_1 = 1, n_2 = 0$.

In terms of matrices, we get

$$S = \begin{bmatrix} 1 & 0 & 0 \\ -t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & -t & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$TS = \begin{bmatrix} t & 1 & 0 \\ 1 & -t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Choosing $n := 1$, we get the new weights $U = (1,1,2)$. Then $\deg_{W,U}(TS) = 1$ and $\deg(\det(TS)) = 2$, hence the degree defect is indeed zero.

The proof of theorem 2 leads to the following algorithm.

Input: a matrix $T \in \mathrm{M}(3\times 3, \mathbb{Q}[t])$ such that $\det(T)$ does not vanish identically;
a weight vector $W \in \mathbb{Z}_+^3$.

Output: a matrix $T' \in \mathrm{M}(3\times 3, \mathbb{Q}[t])$, such that $T' = TS$ for some $S \in \mathrm{GL}(3, \mathbb{Q}[t])$;
a weight vector $U \in \mathbb{Z}_+^3$, such that $\mathrm{def}_{W,U}(T') = 0$.

We say that the pair (i,j) is a *leader* iff $\deg(T_ij) - w_j \le \deg(T_ik) - w_k$ for $k = 0, \dots, 2$, and $\deg(T_ij) - w_j < \deg(T_ik) - w_k$ for $k < j$. It is obvious that every column has exactly one leader.

Algorithm *ReduceColumns*:
while there exist leaders (i,j) and (i,k) do
  choose i,j,k such that (i,j) and (i,k) are leaders;
  if $\deg(T_{ij}) < \deg(T_{ik})$ then
    interchange i and j;
  $C := \frac{\mathrm{leadmon}(T_{ij})}{\mathrm{leadmon}(T_{ik})}$;
  for l from 0 to 2 do
    $T_{lj} := T_{lj} - CT_{lk}$;
for all leaders (i,j) do
  $u_j := w_i - \deg(T_{ij})$;
$n := -\min(u_0, u_1, u_2) + 1$;
for j from 0 to 2 do
  $u_j := u_j + n$;
$U := (u_0, u_1, u_2)$;
return (T, U)

The correctness of the algorithm *ReduceColumns* follows by the proof of theorem 2.

Theorem 3. *Let n be a bound for the degrees of T_{ij} and for the w_i. Then the algorithm ReduceColumns requires at most $O(n^2)$ field operations.*

Proof. A single elementary column operation requires $O(n)$ field operations. In a column operation, the numbers $l_j := \max_i(\deg(T_{ij}) - w_i)$ cannot grow. After 3 column operations changing the same column, the corresponding number l_j actually drops. Since each of the l_j is bounded between $\max_j(\deg(T_{ij})$ and w_j, there are at most $O(n)$ column operations. □

Remark 2. Theorem 2 holds for $m\times m$-matrices, with the same proof. The complexity in the general case is $O(mn^2)$.

It is clear that a transformation by a matrix in $\mathrm{GL}(3, \mathbb{Q}[t])$ does not change the Hessian (except for a constant factor). In the following, we consider transformations which do change the Hessian—in fact, they make it smaller.

Let P be an irreducible factor of $H(Q)$. Then the standard form Q is degenerate in $\mathbb{Q}[t]/\langle P\rangle$. Thus, Q factors into two linear forms in the algebraic closure of $\mathbb{Q}[t]/\langle P\rangle$. If this factorization is possible in $\mathbb{Q}[t]/\langle P\rangle$, then we say that P is a *removeable factor*. Here is the motivation for this name.

Theorem 4. *Let Q be a standard form such that $H(Q)$ is squarefree. Let P be a removeable factor of $H(Q)$. Then there exists a matrix $T \in \mathrm{M}(3\times 3, \mathbb{Q}[t])$, such that $H(Q') = H(Q)/P$ (up to a constant factor), where Q' is the transform of Q by T.*

Proof. Since P is removeable, there exist $E_i, F_i \in \mathbb{Q}[t]$, such that

$$(\sum_i E_i x_i)(\sum_i F_i x_i) \equiv Q(x_0, x_1, x_2) \mod P.$$

Without loss of generality, we assume that $E_0 \not\equiv 0 \mod P$. Then, we divide the left factor by E_i and multiply the right by E_i in the field $\mathbb{Q}[t]/\langle P\rangle$. Thus, we may assume that $E_0 = 1$. We define

$$T := \begin{bmatrix} P & -E_1 & -E_2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Then $\det(T^* \; M(Q) \; T) = P^2 H(Q)$. Modulo P, we compute

$$Q(Py_0 - E_1 y_1 - E_2 y_2, y_1, y_2) \equiv$$

$$(Py_0)(F_0(Py_0 - E_1 y_1 - E_2 y_2) + F_1 y_1 + F_2 y_2) \equiv 0,$$

hence $T^* \; M(Q) \; T$ is divisible by P. The content of $T^* \; M(Q) \; T$ is then P, because $P^2 H(Q)$ has no other cubic factors. Therefore $M(Q') = T^* \; M(Q) \; T/P$ and consfquently $H(Q') = H(Q)/P$. □

We are now able to show the main theorem of this section. We say that a standard surface is *of minimal degree* iff it has squarefree Hessian without removeable factors and if its degree defect is less than or equal to 1, with respect to a suitable weight vector.

Theorem 5. *Any standard form is equivalent to a standard form of minimal degree.*

Proof. By lemma 1, it suffices to show the theorem for standard forms with squarefree Hessian and with degree defect less than or equal to 1.

Let P be a removeable factor. By theorem4, there is a transformation T such that the transformed standard form Q' has Hessian $H(Q') = H(Q)/P$. By theorem 2, there is an invertible transformation S such that TS has degree defect zero. By lemma 2, the transformed standard form Q'' has degree defect less than or equal to $\mathrm{def}(Q)$. Moreover, $H(Q'') = H(Q') = H(Q)/P$.

If Q'' has removeable factors, we repeat. Clearly, this procedure terminates, because the degree of the determinant drops in every step. □

Example 4. After diagonalization and eliminating multiple factors in the standard form (2) from example 2, we get

$$Q(x_0, x_1, x_2) = (t+1)(t^6 + t^5 - t^4 + t^3 + 2t - 2)x_0^2 \\ -tx_1^2 + (t+2)x_2^2.$$

With weight vector $W = (1,4,4)$, the degree defect is zero. The factor t is removeable, because $Q(t{=}0) = (x_0 - x_2)(-2x_2 - 2x_0)$. The factor can be removed by the transformation

$$T := \begin{bmatrix} t & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We apply the algorithm *ReduceColumns* and obtain

$$T' := \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -t & 0 & 1 \end{bmatrix}$$

and weight vector $U = (3, 4, 1)$. The transformed standard form is

$$Q(y_2, y_1, -ty_0 + y_2)/t = (t^2 + 2t)y_0^2 - y_1^2$$
$$+(t^6 + 2t^5 + t^2 + 2t + 1)y_2^2 - 2(t + 2)y_0y_2,$$

which has indeed degree defect zero and Hessian $H(Q)/t$.

The factor $t + 2$ is also removeable, because $Q(t{=}-2) = (y_1 - y_2)(-y_1 - y_2)$. The factor can be removed by the transformation

$$T := \begin{bmatrix} 1 & 0 & 0 \\ 0 & t & 1 \\ 0 & 0 & 1 \end{bmatrix}.$$

This matrix already has degree defect zero, with new weight vector $V = (3, 3, 1)$. The transformed standard form is

$$tz_0^2 - (t + 2)z_1^2 + (t^5 + t)z_2^2 - 2z_0z_2 - 2z_1z_2.$$

It has degree defect zero with respect to the weight vector $(3, 3, 1)$, and its Hessian is $(t^6+2t^5+t^2+2t+1)(t+1)$. Both factors are not removeable. Thus, the new standard form is of minimal degree.

A result by Iskovskih tells that non-removeable factors cannot be removed, indeed.

Theorem 6. *Let Q and Q' be two equivalent standard forms with squarefree Hessian. Then $H(Q)$ and $H(Q')$ have the same non-removeable factors.*

Proof. See [14]. □

Remark 3. It is easy to show that a irreducible polynomial P is a non-removeable factor of $H(Q)$ iff Q has no non-trivial solution in the completion of $\mathbb{Q}(t)$ with respect to the valuation defined by P. Theorem 6 can be obtained as a consequence of this fact. Moreover, one can use this fact to extend the definition of non-removeability to standard forms with arbitrary Hessian (not necessarily squarefree).

It is interesting to ask whether any standard form is equivalent to a *diagonal* standard form of minimal degree. The answer is no. Here is a counter-example.

Example 5. Consider the standard form

$$Q = x_0^2 + tx_1^2 + (t + 1)x_2^2 + 2x_1x_2.$$

Its Hessian $t^2 + t - 1$ is irreducible and not removeable. By theorem 5, any equivalent diagonal form without removeable factors has the form

$$Q' = (t^2 + t - 1)y_0^2 + ay_1^2 + by_2^2, \tag{5}$$

with $a, b \in \mathrm{Q}^*$.

In order to show that no such form exists, we recall the notion of the *spine*: if Q has squarefree Hessian, then the spine is the set of all $t_0 \in \mathbb{R}$ such that $Q(t{=}t_0)$ has a non-trivial real solution. Two standard forms are equivalent by a matrix in $\mathrm{M}(3{\times}3, \mathbb{R}[t])$ iff their spines coincide (see [28]).

It is easy to check that the spine of Q is $(-\infty, \frac{\sqrt{5}-1}{2}]$, and the spine of Q' is $\mathbb{R}$, if $ab < 0$, $[\frac{-\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}]$, if $a > 0$ and $b > 0$, and $(-\infty, \frac{-\sqrt{5}-1}{2}] \cup [\frac{\sqrt{5}-1}{2}, \infty)$, if $a < 0$ and $b < 0$. Hence none of the forms (5) is equivalent to Q.

In order to make the proof of theorem 4 algorithmic, we need to factor a ternary quadratic form over $K := \mathbb{Q}[t]/\langle P\rangle$. It is easy to show that this problem can be reduced to the factorization of a quadratic polynomial in $K[s]$. For the problem of factoring polynomials over algebraic number fields, re refer to [1, 18, 31].

We are now ready to give an algorithm for transforming a standard form to the smallest possible degree.

Input: a standard form $Q \in \mathbb{Q}[t, x_0, x_1, x_2]$;

Output: an equivalent standard form Q' of minimal degree; a weight vector W such that $\mathrm{def}_W(Q') \leq 1$.

Algorithm *DegreeMinimize*:
diagonalize Q by Gauss elimination;
remove squares of the Hessian (see lemma 1);
construct W as in lemma 1;
for each factor P of $H(Q)$ do
  factor Q modulo P;
  if Q modulo P splits into two linear factors then
    define T as in the proof of theorem 4;
    $T' := \mathit{ReduceColumn}(T)$;
    transform Q, W by T';

The correctness is obvious by the proof of theorem 5.

In order to analyze the complexity, we assume that the costs of each field operation is bound by one unit, and the costs for the factorization of a polynomial of degree n is bound by n^k units, where $k \geq 3$.

Theorem 7. *Let n be an upper bound for the degree of the polynomials A_{ij} occuring in Q. Then the computational cost for the algorithm DegreeMinimize is bound by $O(n^k)$ in the above complexity model.*

Proof. The costs and the intermediate expression swell caused by the diagonalization are neglectable. We estimate the costs for the removal of a factor P of degree m: the factorization of Q modulo P can be reduced to factorizing a degree 2 polynomial over the algebraic extension defined by P, and this problem can again be reduced to the problem of factorizing a polynomial of degree $O(m)$ and $O(m^3)$ field operations by Kronecker's reduction (see [18]). Thus, the cost for the factorization of Q modulo P is $O(m^k)$. The cost for subalgorithm *ReduceColumn* is $O(n^2)$, and the cost for the transformation is bound by $O(n^2)$ if one uses ordinary polynomial multiplication.

In order to control the size of the intermediate expressions, we observe that the degree of the Hessian is always $O(n)$. Thus, the degree of the A_{ij} and the weights are also bound by $O(n)$.

Since there are at most $O(n)$ removeable factors, the total costs for *ReduceColumn* and for the transformations are bound by $O(n^3)$. The total costs for the factorizations are maximized when there is one big removeable factor; in this case, the costs are bound by $O(n^k)$. □

Remark 4. All the theorems, algorithms, and proofs in this sections are valid for an arbitrary computational field of characteristic not equal 2, provided that we have an algorithm for univariate factorization. This includes most computable fields occuring in practical computations, for instance all finitely generated fields (see [8]).

3 Construction of Proper Parametrizations

For a minimal standard form and weight vector W, we define the *index*

$$\mathrm{ind}_W(Q) := \deg(\det(H(Q))) + \mathrm{def}_W(Q).$$

The existence and the construction of a proper parametrization of a standard form of minimal degree depends a lot on the index.

If W and W' are weight vectors that differ only by an additive number (i.e. $w_i' = w_i + d$ for some $d \in \mathbb{Z}$), then one observes that the concepts of homogenization, degree defect etc. are common to both weight vectors. From now on, we assume that the weight vector is normed so that at least one of the w_i is equal to 1. Note that the weight vectors produced by *ReduceColumn* are normed in this sense.

Recall that the numbers w_i and $\deg_W(Q)$ fulfill the equation

$$\mathrm{ind}_W(Q) = \sum_i (\deg_W(Q) - 2w_i). \tag{6}$$

If none of the A_{ii} vanishes identically, then the three summands on the right hand side are not negative, because they are upper bounds for $\deg(A_{ii})$. Moreover, they have the same parity. This immediately shows that the index can never be 1. We discuss the first few cases separately.

3.1 Zero Elements in the Diagonal

Without loss of generality, assume $A_{00} = 0$. Then Q is linear in x_0. We can immediately write down a parametrization with parameters $s = x_1/x_2, t$:

$$((x_0 : x_1 : x_2), t) = ((\frac{A_{11}(t)s^2 + A_{12}(t)s + A_{22}(t)}{A_{01}(t)s + A_{02}(t)} : s : 1), t).$$

In the following cases, we assume that there are no zeroes in the diagonal.

3.2 Index 0

All summands in (6) are zero, hence we must have $W = (1,1,1)$ and $\deg_W(Q) = 2$. Hence all A_{ij} are constants, and Q is the equation of a conic C over $\mathbb{Q}$. If C has no point with coordinates in $\mathbb{Q}$, then there is no parametrization. If C has a point p defined over p, we may parametrize C by a well-known method: we intersect C with secants through p with slope s. In order to parametrize the corresponding surface, we just let the t-coordinate be the second parameter.

The parametrization problem is therefore reduced to the solution of a ternary from over $\mathbb{Q}$. An algorithmic criterion for the existence is given by the theorem of Legendre. The proof in [13] provides an algorithm for finding a solution in the affirmative case (see [10]).

3.3 Index 1

Since 1 is not the sum of three nonnegative integers of the same parity, the index cannot be 1 under the assumption $A_{00}A_{11}A_{22} \neq 0$.

3.4 Index 2

The only way to get 2 as a sum of three nonnegative integers of the same parity is $2 = 2 + 0 + 0$. Hence $W = (1,2,2)$ and $\deg_W(Q) = 4$. Consequently, A_{11}, A_{12}, A_{22} are constants, A_{01}, A_{02} are at most linear, and A_{00} is at most quadratic. The dehomogenization with respect to x_0

$$A_{00}(t) + A_{01}(t)\frac{x_1}{x_0} + A_{02}(t)\frac{x_2}{x_0} + A_{11}\left(\frac{x_1}{x_0}\right)^2 + A_{12}\frac{x_1}{x_0}\frac{x_2}{x_0} + A_{22}\left(\frac{x_2}{x_0}\right)^2$$

is a quadric surface S in $\mathbb{Q}^3$.

If S is singular, then it is a quadric cone, which can be parametrized iff the base conic can be parametrized. Thus, we have again reduced to the solution of ternary forms over $\mathbb{Q}$, as with index 0.

Let S be nonsingular. If S does have a point p defined over $\mathbb{Q}$, then we can parametrize properly by the well-known method of stereographic projection. Otherwise there is no parametrization. Thus, we have reduced to the solution of quartary quadratic forms over $\mathbb{Q}$. An algorithmic criterion for the existence is given by the Minkowski-Hasse theorem (see [5]).

3.5 Index 3

The only way to get 3 as a sum of three nonnegative integers of the same parity is $3 = 1 + 1 + 1$. Hence $W = (1,1,1)$ and $\deg_W(Q) = 3$. Consequently, Q is linear in t, and we write $Q = Q_0(x_0, x_1, x_2) + tQ_1(x_0, x_1, x_2)$. We can immediately write down a parametrization with parameters $u = x_1/x_0$, $v = x_2/x_0$:

$$((x_0 : x_1 : x_2), t) = ((1 : u : v), -Q_0(1, u, v)/Q_1(1, u, v)).$$

3.6 Index 4 or Higher

Here, the parametrization problem is solved in the negative sense by the following result of Iskovskih.

Theorem 8. *The surface defined by a standard form of minimal degree with index greater than or equal to 4 is not rational over $\mathbb{Q}$.*

Proof. See [14, 15, 16]. ☐

Remark 5. Of course, Iskovskih's terminology is quite different from ours. The precise relation between Iskovskih's ω^2 and our index is $\omega^2 = 8 - \mathrm{ind}(Q)$. Iskovskih's definition does not use the properties of the equation, rather it depends on geometric concepts such as the group of divisor classes.

This ends the discussion of separate cases. As a consequence, we obtain the following algorithm for the parametrization of surfaces with a rational pencil of curves of genus zero.

Input: a surface S over $\mathbb{Q}$, in terms of its equations;
a rational pencil given by a rational function $f \in \mathbb{Q}(S)$;

Output: a parametrization of S over $\mathbb{Q}$, if exists
`NotExist` otherwise.

Algorithm *Parametrize:*
reduce to standard form (see theorem 1);
apply *DegreeMinimize*;
if there are zeroes in the diagonal then
apply the construction for this case;
else
if ind = 0/2/3 then
apply the construction for index 0/2/3;
else
return `NotExist`; exit;
transform the parametrization back

Example 6. Let S be the cubic surface with equation

$$x^2 + 2y^2z + z^3 + z^2 + z + 1 = 0.$$

The equation is quadratic in x, y, hence the sections $z =$ const. have genus zero. In order to establish a standard form, we just have to homogenize and rename variables: $x = x_0/x_2, y = x_1/x_2$, and $z = t$. This yields

$$Q = x_0^2 + 2tx_1^2 + (t^3 + t^2 + t + 1)x_2^2,$$

and we apply *DegreeMinimize* to this equation.

The form Q is already diagonalized and has squarefree Hessian $H(Q) = t(t+1)(t^2+1)$. The first two factors are not removeable. The third one is removeable, because

$$Q \equiv (x_0 - (t-1)x_1)(x_0 + (t-1)x_1) \mod t^2 + 1.$$

We apply *ReduceColumn* and obtain the matrix

$$T = \begin{bmatrix} t+1 & t-1 & 0 \\ -t & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The transformed form is

$$Q' = (2t+1)y_0^2 - 2y_0y_1 + y_1^2 + (t+1)y_2^2.$$

By subsection 3.5, we get the parametrization

$$((y_0 : y_1 : y_2), t) = ((u : v : 1), \frac{-u^2 + 2uv - v^2 - 1}{2u^2 + 1}).$$

We apply T and get the parametrization for Q:

$$((x_0 : x_1 : x_2), t) = ((T+1)u + (T-1)v : -Tu + v : 1), T),$$

where $T := \frac{-u^2+2uv-v^2-1}{2u^2+1}$. By dehomogenizing and reversing the original renaming of variables, we get

$$(x, y, z) = ((T+1)u + (T-1)v, -Tu + v, T) =$$
$$\left(\frac{u^3 - vu^2 + uv^2 - v^3 - 2v}{2u^2 + 1}, \frac{u^3 + uv^2 + u + v}{2u^2 + 1}, \frac{-u^2 + 2uv - v^2 - 1}{2u^2 + 1}\right),$$

the parametrization for S.

Remark 6. Since we use Legendre's theorem and the Minkowski-Hasse theorem in the constructions for index 0 and 2, the algorithm *Parametrize* cannot be generalized to arbitrary computable fields with factorization. However, these two theorems generalize to number fields, so that we can generalize our algorithm to this case. For finite fields (characteristic not equal 2 because *MinimizeDegree* requires this), the situation is even easier, because the quadratic forms can be solved by checking a finite number of cases.

4 Implications for the Proper Parametrization Problem

The results above are of great relevance for the following problem: "given the equation of a surface, how can we decide rationality over $\mathbb{Q}$? If yes, how can we compute a proper parametrization?" The reason is the following classical theorem by Enriques and Manin [20].

Theorem 9. *A surface with is rational over $\mathbb{C}$ has a rational pencil (defined over $\mathbb{Q}$) or is $\mathbb{Q}$-birationally equivalent to a del Pezzo surface.*

For a modern proof and an algorithm producing the pencil or the birational map, see [27]. For deciding rationality over $\mathbb{C}$, one can use Castelnuovo's criterion $p_a = P_2 = 0$ [6]. By theorem 9 and algorithm *Parametrize*, the proper parametrization problem is reduced to the case of del Pezzo surfaces.

For a definition and an introduction to del Pezzo surfaces, we refer to [21]. In contrast to surfaces with rational pencils, del Pezzo surfaces are bounded in degree, so that the general problem is a problem with only a finite number of parameters involved.

In theory, the proper parametrization problem for del Pezzo surfaces is also solved over $\mathbb{Q}$. Here is an overview on the existing results.

1. Any del Pezzo surface can be birationally reduced to a del Pezzo surface with cyclic Picard group (let's call this a "type I surface") or to a surface with a rational pencil. This reduction requires a constant number of field operations and factorizations of polynomials of constant degree, and it is possible for arbitrary fields with these two operations. For details on the method and examples, see [21].

2. For type I surfaces, one can define (and also compute) an index, similar to the index of standard forms. It ranges from 1 to 9. The type I surfaces with index less than 5 are not rational. The type I surfaces with index greater than or equal to 5 are rational iff they contain a point defined over the working field. These results also hold for arbitrary fields. They can be found in [17, 7].

3. In the case of number fields, the "Hasse principle" holds for the class of type I surfaces of index greater than or equal to 5: the existence of a point defined over K is equivalent to the existence of a point defined over the completion K_ν for all valuations ν of K (see [19]). It is sufficient to check a finite number of valuations. This implies that the existence of K-rational points is decidable.

In practice, the birational reduction to type I surfaces is still a difficult problem, because the polynomials that must be factorized are too large. One can find examples in the literature, but complete algorithms have not yet been worked out.

For the real case and for the nonsingular cubic surfaces (which is the most important subclass of del Pezzo surfaces), one can find algorithms and examples in [2].

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