Difference Forms
and Hypergeometric Summation

Diplomarbeit

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Abstract

The method of *difference forms* (*WZ forms*) was invented by Zeilberger in order to discover and prove hypergeometric summation identities. To prove multisum identities by this method, one needs nontrivial closed difference forms of higher degree. Almost no such forms were known so far. To find some, we develop a new method for transforming difference forms in a way that preserves their closedness, which can be seen as a discrete variant of change of variables in differential forms. Our final goal is to discover new multisum identities; examples are given.

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Contents

1 Introduction

1.1 Some Applications of WZ Forms

To prove hypergeometric summation identities, H. Wilf and D. Zeilberger introduced *WZ pairs* [WZ90] and, more generally, *WZ forms* [Zei93]. Their work was honored with the 1998 Steele Prize for Seminal Contribution to Research from the AMS. Before explaining (defining) *WZ forms*, we list some of their applications.

Proving known combinatorial identities. WZ pairs prove most known hypergeometric single sum identities like, for example, Saalschütz's Theorem [GKP89, p.171]: Let m and n be natural numbers. Then

$$\sum_{k=0}^{n} \binom{r+k}{n+m} \binom{-s+r+n}{n-k} \binom{s-r+m}{k} = \binom{r}{m} \binom{s}{n},$$

or, in hypergeometric notation,

$$\binom{r}{n+m} \binom{-s+r+n}{n} {}_3F_2\left[\begin{matrix} -n,\, r+1,\, -s+r-m \\ r-n-m+1,\, -s+r+1 \end{matrix}; 1\right] = \binom{r}{m} \binom{s}{n}.$$

Wilf and Zeilberger [WZ90] prove all "classical" hypergeometric summation identities using the WZ method. Section 3 contains examples of binomial coefficient identities proved by the WZ method.

Finding new combinatorial identities. Given a hypergeometric single sum identity, WZ forms allow us to produce new single sum identities from it. Starting from Vandermonde's identity, Zeilberger [Zei95] derives the identity

$$\sum_{n=0}^{k} (3\,n-2\,k) \binom{k}{n}^2 \binom{2\,n}{n} = 0 \quad \text{for } k \geq 0,$$

or, equivalently,

$${}_4F_3\left[\begin{matrix} -k,\, -k,\, 1-\frac{2\,k}{3},\, \frac{1}{2} \\ 1,\, \frac{-2\,k}{3},\, 1 \end{matrix}; 4\right] = 0 \quad \text{for } k \geq 0.$$

by his *dualize and specialize* method and comments on it:

> "This is *a brand new identity*, unknown to Askey. It has a q-analog derived from the q-version of WZ, that was unknown to Andrews, and even whose limiting case was brand new, and it took George Andrews three densely packed pages, using five different identities, to prove."

In the dualize and specialize method, a certain miracle remains unexplained; new light is shed on it by closedness preserving substitutions – see section 4.2. A long list of new single sum identities found by (a variant of) the dualize and specialize method is contained in [Ges95].

A fast series for Apéry's constant. Using a certain WZ pair, Zeilberger [Zei93] proves Apéry's celebrated identity

$$\zeta(3) = \frac{5}{2} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3 \binom{2n}{n}}, \tag{1}$$

where $\zeta(3) := \sum_{n=1}^{\infty} \frac{1}{n^3}$.

Proving known multisum identities. Consider the identity [Den96]

$$\sum_b \sum_s (-1)^b \binom{-s+k}{2\,v-b} \binom{s}{b} \binom{-2\,v+k}{s-b} = 2^{-2\,v+k} \binom{-v+k}{-2\,v+k} \qquad (2)$$

which holds for $0 \le 2v \le k$. By H. Wilf's amazingly successful *method of dividing by the right hand side* [WZ90] and a subsequent call to K. Wegschaider's variant [Weg97] of Sister Celine Fasenmyer's algorithm [Fas47, Fas49], we find a particularly simple recursion for the summand which can be immediately translated to a WZ form of degree two; see section 3.5.

Finding new multisum identities. The same WZ form that proves identity 2 immediately leads to the identity

$$\sum_b \sum_k (-1)^b\, 2^{-k} \binom{s+b-2}{s-1} \binom{s+b-1}{2\,v-k} \binom{s+k}{2\,v-b+1} \binom{v-1}{-v+k} = 0$$

which is valid for $1 \le s$, $1 \le v$ and to the identity

$$\sum_k \sum_s (-2)^k \binom{b}{s} \binom{2\,v-b-2}{-s+k-1} \binom{2\,v-k-1}{-s+b-1} \binom{2\,v-k-1}{v-k} = 0$$

which is valid for $b \ge 0$, $1 \le v$, $b+2 \le 2\,v$.

Testing simplifiers for hypergeometric multisums. Multisum identities found by WZ forms, as, for example,

$$\underset{m\ge 0}{\forall} \sum_{i=0}^{m} \sum_{j=0}^{m} \binom{m+i+j}{m,\ i,\ j} 3^{-i-j} \;=\; 3^m,$$

(which is derived in section 5.2), are useful for checking and comparing summation algorithms.

Challenging simplifiers for expressions involving hypergeometric sums. Using WZ forms, we can easily find and prove identities like

$$\begin{aligned}
\underset{p\ge 0}{\forall}\ \underset{q\ge 0}{\forall}\ \underset{r\ge 0}{\forall} \qquad & 3^{-p} \sum_{j=0}^{q} \sum_{k=0}^{r} \binom{p+j+k}{p,\ j,\ k} 3^{-j} 3^{-k} \\
+ \;& 3^{-q} \sum_{i=0}^{p} \sum_{k=0}^{r} \binom{i+q+k}{i,\ q,\ k} 3^{-i} 3^{-k} \\
+ \;& 3^{-r} \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{i+j+r}{i,\ j,\ r} 3^{-i} 3^{-j} \;=\; 3.
\end{aligned}$$

Given the left hand side expression of this identity, a simplifier should reduce it to the right hand side. As far as I know, no simplifier that can handle the sum of more than one sum has been invented yet.

Challenging automatic provers. Even if given both sides of an identity like the one above as input, there is no reasonably fast algorithm that computes a proof of the given identity. There is no dedicated proof algorithm for such problems, and general (predicate logic) provers are way too inefficient to tackle them.

1.2 WZ Form Transformations

The need for WZ form transformations. Much is known about hypergeometric single sum identities. There is an amazingly successful database of a few general summation identities that cover most sums encountered in combinatorial practice as special cases. It is listed in Appendix III of [Sla66] and implemented in Ch. Krattenthaler's Mathematica package `HYP.m`[Kra95].

Even better, the problem of expressing a single sum over a hypergeometric summand as a hypergeometric expression is algorithmically solved [1] by finding a recurrence of the sum with Zeilberger's fast algorithm [Zei91] and subsequently solving this recurrence with M. Petkovšek's algorithm [PWZ96].

Much less is known about hypergeometric *multi*sums. Sister Mary Celine Fasenmyer [Fas47, Fas49] invented an algorithm for finding recurrences for multisums [PWZ96]. Her algorithm was improved by P. Verbaeten and Kurt Wegschaider, who implemented it in Mathematica. Note that her algorithm does not help us to *find* multisum identities: Given a sum, we do not know a priori if it finds a closed form evaluation. A randomly chosen sum is unlikely to find a closed form evaluation.

To find more multisum identities by the WZ form mehtod, we need nontrivial WZ forms of higher degree. K. Wegschaider's Mathematica package `multisum` can be used to construct such forms. We were successful to do so only 3 times; in all other cases we interrupted the program `multisum` after running for some long time.

To get more forms from a few known forms, we would like to *transform a known WZ form into an essentially different WZ form.* By "essentially different" we mean that identities produced by the new form should not follow directly (i.e. by substitution) from identities of the original form.

Early WZ pair transformations. The first WZ pair transformation is introduced in the very first article on WZ pairs [WZ90] as "Theorem B". I. Gessel generalizes it slightly ([Ges95, Theorem 3.1]) and uses the basic WZ method together with his generalization to discover an abundance of new hypergeometric single sum identities. These transformations do not produce essentially different WZ forms.

Fast series for Apéry's constant. T. Amdeberhan and D. Zeilberger [AZ97] present a certain new WZ form transformation which does indeed produce essentially different WZ forms. It is skillfully used in [Amd96] to obtain an infinite sequence of faster and faster converging series for $\zeta(3)$. The first series in this sequence gives Apéry's celebrated identity (1); the second series gives

$$\zeta(3) = \frac{1}{4}\sum_{n=1}^{\infty} \frac{(56n^2-32n+5)}{(2n-1)^2 n^3} \frac{(-1)^{n+1}}{\binom{2n}{n}\binom{3n}{n}}$$

which enjoys much faster convergence than the first series.

More general WZ form transformations. We aim to transform WZ forms by applying arbitrary integer linear substitutions. For WZ pairs and WZ 1-forms, you can easily find these transformations yourself using a simple new trick that is explained in section 4.1 and that was discovered independently by I. Gessel [Ges99]

[1] at least up to a small gap

Unfortunately, this trick cannot be carried over to r-forms where $r > 1$, i.e. to the more interesting multisum case. Section 4.3, the main part of the thesis, solves the transformation problem in the multisum case. The transformation algorithm is implemented in Mathematica.

Finding new multisum identities. Given a hypergeometric multisum identity, we may hope to find a WZ form of higher degree from it. This form usually leads to new multisum identities. However, we can do better. The method of *closedness preserving substitutions* allows us to construct new WZ forms from it which in turn allow us to discover ever more – essentially different – multisum identities. For example, we obtain the identity

$$\sum_k \sum_s 2^{-k}(2v+s-b-k)\binom{b}{s}\binom{b}{2v-k}\binom{k+s}{2v+s-b}\binom{v}{k-v} = 0$$

(provided that $b \geq 0$ and $v \geq 0$) in this way, starting from S. Dent's identity (section 5.3).

Symmetry as a bonus All WZ pairs [2] $F(n,k)\,dk + G(n,k)\,dn$ satisfy the WZ equation

$$F(n+1,k) - F(n,k) = G(n,k+1) - G(n,k)$$

whose symmetry is praised in [PWZ96] on page 123:

> "When the WZ equation holds, there is complete symmetry between the indices n and k, especially for terminating identities, which previously had seemed to be playing seemingly different roles. The revelation of symmetry in nature has always been one of the main objectives in science."

Of course, WZ pairs should match this appealing symmetry. So far, no nontrivial nicely symmetric WZ pairs were known. We obtain the WZ form

$$2^{-\mathsf{a}-\mathsf{b}}\binom{\mathsf{a}+\mathsf{b}}{\mathsf{a},\,\mathsf{b}}\frac{1}{\mathsf{a}+\mathsf{b}}\,(\mathsf{b}\,d\mathsf{a} - \mathsf{a}\,d\mathsf{b}),$$

from the well known binomial form by our method of closedness preserving substitutions in the sections 5.1. By the same method, the WZ forms

$$\binom{r}{\mathsf{a}}\binom{s}{\mathsf{b}}\binom{r+s}{\mathsf{a}+\mathsf{b}}^{-1}\frac{1}{\mathsf{a}+\mathsf{b}}\,(\mathsf{b}\,d\mathsf{a} - \mathsf{a}\,d\mathsf{b})$$

and

$$4^{-\mathsf{b}-\mathsf{a}}\binom{2\,\mathsf{a}}{\mathsf{a}}\binom{2\,\mathsf{b}}{\mathsf{b}}\frac{1}{\mathsf{a}+\mathsf{b}}\,(\mathsf{b}\,d\mathsf{a} - \mathsf{a}\,d\mathsf{b})$$

are obtained. In the language of recurrences, the closedness of these forms means that

$$(\Delta_{\mathsf{a}}\,\mathsf{a} + \Delta_{\mathsf{b}}\,\mathsf{b})\frac{1}{\mathsf{a}+\mathsf{b}}\,2^{-\mathsf{a}-\mathsf{b}}\binom{\mathsf{a}+\mathsf{b}}{\mathsf{a},\,\mathsf{b}} = 0,$$

$$(\Delta_{\mathsf{a}}\,\mathsf{a} + \Delta_{\mathsf{b}}\,\mathsf{b})\frac{1}{\mathsf{a}+\mathsf{b}}\binom{r}{\mathsf{a}}\binom{s}{\mathsf{b}}\binom{r+s}{\mathsf{a}+\mathsf{b}}^{-1} = 0$$

and

$$(\Delta_{\mathsf{a}}\,\mathsf{a} + \Delta_{\mathsf{b}}\,\mathsf{b})\frac{1}{\mathsf{a}+\mathsf{b}}\,4^{-\mathsf{b}-\mathsf{a}}\binom{2\,\mathsf{a}}{\mathsf{a}}\binom{2\,\mathsf{b}}{\mathsf{b}} = 0.$$

[2] For uniformity, we write the WZ pair $(F(n,k), G(n,k))$ as $F(n,k)\,dk + G(n,k)\,dn$

2 Difference Forms

The purpose of this section is to give a complete proof of the Theorem of Stokes–Zeilberger. It is deliberately dissected into a collection of small definitions and propositions in order to allow for automatic proving by a system like Theorema [BJK+97] in the future. All proofs are obvious; some are left out while others are included. Our interest lies in the dissection itself, and in the particular choice of definitions. A struggle for combining formal correctness with traditional (and useful) notation forced us to redefine a few most familiar notions like *variable* and *(hypergeometric) term*. Of course one might see re-introducing these concepts as overdoing; to our defense, we cite from the preface of the textbook "Advanced Calculus" on differential forms by M. Spivak:

> "There are good reasons why the theorems should all be easy and the definitions hard. As the evolution of Stokes' Theorem revealed, a single simple principle can masquerade as several difficult results; the proofs of many theorems involve merely stripping away the disguise. The definitions, on the other hand, serve a twofold purpose: they are rigorous replacements for vague notions, and machinery for elegant proofs."

We do not give a tutorial on WZ pairs and WZ forms here, since the original papers [WZ90] and [Zei93] are very readable. For a quick start, you may also look at my slides for the 44e session du Séminaire Lotharingien de Combinatoire.

2.1 Labels, Lattice Vectors and Terms

Established notation for difference equations is quite readable. Consider the equation

$$(\Delta_{\mathsf{a}}\,\mathsf{a} + \Delta_{\mathsf{b}}\,\mathsf{b})\,\frac{1}{\mathsf{a}+\mathsf{b}}\binom{r}{\mathsf{a}}\binom{s}{\mathsf{b}}\binom{r+s}{\mathsf{a}+\mathsf{b}}^{-1} = 0. \tag{3}$$

The *difference operators* Δ_{a} and Δ_{b} are defined by $\Delta_{\mathsf{a}} = S_{\mathsf{a}} - 1$ and $\Delta_{\mathsf{b}} = S_{\mathsf{b}} - 1$ where S_{a} and S_{b} are the *shift operators* w.r.t. a and b: for example,

$$S_{\mathsf{a}}\,\frac{1}{\mathsf{a}+\mathsf{b}}\binom{r}{\mathsf{a}}\binom{s}{\mathsf{b}} = \frac{1}{\mathsf{a}+\mathsf{b}+1}\binom{r}{\mathsf{a}+1}\binom{s}{\mathsf{b}}$$

and

$$S_{\mathsf{b}}\,\frac{1}{\mathsf{a}+\mathsf{b}}\binom{r}{\mathsf{a}}\binom{s}{\mathsf{b}} = \frac{1}{\mathsf{a}+\mathsf{b}+1}\binom{r}{\mathsf{a}}\binom{s}{\mathsf{b}+1}.$$

How to define the shift operators S_{a} and S_{b}? A moment of thought shows that attempts like

$$(S_{\mathsf{a}}f)(a,b) := f(a+1,b)$$

lead to confusing consequences; for example, we would get

$$(S_{\mathsf{a}}f)(b,a) = f(b+1,a) \tag{4}$$

and

$$(S_{\mathsf{a}}f)(-3a,b) = f(-3a+1,b) \tag{5}$$

Looking at (4) and (5) may suggest that S_{a} should be renamed to S_1 since it shifts a function in its first argument. We define shift operators $(S_1 f)(a,b) :=$

$f(a+1,b)$ and $(S_2 f)(a,b) := f(a,b+1)$ as well as multiplication operators $(m_1 f)(a,b) := af(a,b)$ and $(m_2 f)(a,b) := bf(a,b)$. In this notation, we would write equation (3) as

$$(\Delta_1 m_1 + \Delta_2 m_2) f = 0 \tag{6}$$

where

$$f(a,b) = \frac{1}{a+b}\binom{r}{a}\binom{s}{b}\binom{r+s}{a+b}^{-1}.$$

Note that equation (3) is more readable than equation (6). We prefer the operators $\Delta\mathsf{a}$, $\Delta\mathsf{b}$, a, b that act on terms like

$$\frac{1}{\mathsf{a}+\mathsf{b}}\binom{r}{\mathsf{a}}\binom{s}{\mathsf{b}}\binom{r+s}{\mathsf{a}+\mathsf{b}}^{-1}$$

over the operators S_1, S_2, m_1, m_2 that act on functions like f. However, we do not want shift operators S_a and S_b to behave as (4) and (5); we want to get

$$S_\mathsf{a} f(\mathsf{b},\mathsf{a}) = f(\mathsf{b},\mathsf{a}+1) \tag{7}$$

and

$$S_\mathsf{a} f(-3\mathsf{a},\mathsf{b}) = f(-3\mathsf{a}-3,\mathsf{b}) \tag{8}$$

instead.

Our use of the word "term" comes from "hypergeometric term" by dropping hypergeometricity. None of our proofs assume hypergeometricity, but all our examples involve hypergeometric terms only.

An obvious candidate for the notion of "term" might be "purely syntactical term" as used, for example, in the literature on term rewriting systems. To avoid fixing a certain "signature", introducing an "evaluation function" and defining equivalence of terms modulo this evaluation function, we do not adopt that particular notion of term.

To us, the term $\binom{\mathsf{n}}{\mathsf{k}}^2\binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}}^2$ is simply the function that maps, for example, the "vector" $\{(\mathsf{n},4), (\mathsf{k},1)\}$ to the number $\binom{4}{1}^2\binom{4+1}{1}^2 = 400$.

Labels. In the sequel, we will frequently assume that some finite set L has been fixed. Its elements, called *labels*, model the integer variables – like n and k – occurring in hypergeometric terms. Labels will be set in sans serif font, like n and k. The variables x, y and z range over L. For example, $\sum_x$ means $\sum_{x\in\mathsf{L}}$. By convention, different letters in sans serif denote different labels. Thus, for example, $\mathsf{k} \neq \mathsf{n}$ holds.

Remark: Where do we need the convention that different letters in sans serif denote different labels? Let's compute a difference:

$$\Delta_\mathsf{k} \mathsf{k}\mathsf{n}^3 = (\mathsf{k}+1)\mathsf{n}^3 - \mathsf{k}\mathsf{n}^3 = \mathsf{n}^3.$$

One might be tempted to "generalize" this to

$$\Delta_x x y^3 = y^3$$

which, of course, is wrong as can be seen by considering the particular case $x = y = \mathsf{k}$:

$$\Delta_\mathsf{k} \mathsf{k}\mathsf{k}^3 = 4\mathsf{k}^3 + 6\mathsf{k}^2 + 4\mathsf{k} + 1.$$

instead of

$$\Delta_{\mathsf{k}}\mathsf{k}^4 = \mathsf{k}^3.$$

Thus one really needs to assume $\mathsf{k} \neq \mathsf{n}$ for calculating $\Delta_{\mathsf{k}}\mathsf{k}\mathsf{n}^3 = \mathsf{n}^3$. □

The labels are assumed to be totally ordered by $\prec$. We assume that $\mathsf{L} := \{\mathsf{k}, \mathsf{n}\}$ where $\mathsf{k} \prec \mathsf{n}$ in all examples of this section.

Vectors. $X \to Y$ denotes the set of all functions from X to Y. We will often denote function application by using subscripts: f_x means $f(x)$.

Def. 1 (lattice vector). *The set of all* (lattice) vectors *is* $\mathsf{V} := \mathsf{L} \to \mathbb{Z}$. *Vectors will also be called "points" or "lattice points". By convention, p and p' range over* V. *As usual, vector addition and scalar multiplication are defined pointwise:*

1. $(p + p')_x := p_x + p'_x$
2. $(cp)_x := cp_x$.

An example of a vector is p where $p_{\mathsf{k}} = 3$ and $p_{\mathsf{n}} = 5$. Alternatively, we write $p = \{(\mathsf{k}, 3), (\mathsf{n}, 5)\}$.

Remark: Remark: Definition 1 may seem strange, but in fact it is natural. Usually, $\mathbb{Z}^n$ is used as the set of all lattice vectors. Since $n = \{0, 1, \ldots, n-1\}$ in the usual (set theoretic) construction of $\mathbb{N}$, we have $\mathbb{Z}^n = \mathbb{Z}^{\{0,1,\ldots,n-1\}} = \{0, 1, \ldots, n-1\} \to \mathbb{Z}$.

All we do is to change the index set from $\{0, 1, \ldots, n-1\}$ to something like $\{\mathsf{k}, \mathsf{n}\}$. □

Intuitively, a *term* (or *expression*) is an object like $\binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}}2^{-\mathsf{k}}$ that allows plugging in integers for the labels appearing in it: For example, plugging $\mathsf{k} = 1$ and $\mathsf{n} = 7$ into $\binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}}2^{-\mathsf{k}}$ yields 4. Thus we can view a *term* as a function from V to $\mathbb{C}$. For example, if

$$T = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}}2^{-\mathsf{k}}$$

and

$$p = \{(\mathsf{k}, 1), (\mathsf{n}, 7)\}$$

then we can evaluate the term T at the point p to get

$$T(p) = \binom{7+1}{1}2^{-1} = 4.$$

When speaking of terms, we always have (multivariate) hypergeometric terms [Weg97, pp. 12–14] in mind. However, we don't restrict ourselves to hypergeometric terms (which we even do not define), since our theorems do not depend on this restriction. In practice this means that all our theorems hold for multivariate q-hypergeometric terms [Rie95] as well. However, our software is presently restricted to (multivariate) hypergeometric terms.

Def. 2 (term). *The set of* terms *is* $\mathsf{T} := \mathsf{V} \to \mathbb{C}$. *We recursively define:*

1. *Let x be a label in* L. *Then $\underline{x}$ in* T *is defined by* $\underline{x}(p) := p_x$.

2. *Let c be number in $\mathbb{C}$. Then $\underline{c}$ in T is defined by $\underline{c}(p) := c$.*

3. *Let f be function in $\mathbb{C}^n \to \mathbb{C}$. Then $\underline{f}$ in $\mathsf{T}^n \to \mathsf{T}$ is defined by*

$$\left(\underline{f}(T_1, T_2, \ldots, T_n)\right)(p) := f(T_1(p), T_2(p), \ldots, T_n(p)).$$

(In our applications, the function f will typically be addition, subtraction, multiplication, division, the binomial coefficient function or the factorial function.)

To avoid heavy notation, we allow ourselves to omit underlining. We hope that context always resolves ambiguities.

Example: Let $p = \{(\mathsf{k}, 1), (\mathsf{n}, 7)\}$. Then

1. $\mathsf{k}(p) = 1$ and $\mathsf{n}(p) = 7$,
2. $(\mathsf{k} + \mathsf{n})(p) = 8$, and
3. $\binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}}(p) = 8$.

□

Example: $\binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}} 2^{-\mathsf{k}}$ is a term. It is the function that assigns to a point p the number $\binom{p_\mathsf{n}+p_\mathsf{k}}{p_\mathsf{k}} 2^{-p_\mathsf{k}}$, as can be calculated as follows:

$$\binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}} 2^{-\mathsf{k}}(p) = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}}(p) \cdot (2^{-\mathsf{k}})(p) =$$
$$= \binom{\mathsf{n}(p)+\mathsf{k}(p)}{\mathsf{k}(p)} 2^{-\mathsf{k}(p)} = \binom{p_\mathsf{n}+p_\mathsf{k}}{p_\mathsf{k}} 2^{-p_\mathsf{k}}.$$

□

2.2 Sums of Terms

To produce summation identities, we need to sum terms.

Def. 3 (support and sum). *The* support of T, $\mathsf{supp}\ T$, *consists of all lattice points where T does not vanish. T has* finite support *iff* $\mathsf{supp}\ T$ *is finite. In this case the* sum of T *is defined by*

$$\mathsf{sum}\ T := \sum_{p \,\in\, \mathsf{supp}\ T} T(p)\,;$$

otherwise, $\mathsf{sum}\ T$ *is left undefined.*

Remark: We do *not* lose any infinite sum identities by insisting on the finiteness of $\mathsf{supp}\ T$. We just prove finite versions of our identities first and take appropriate limits afterwards. □

Example: Let $\mathsf{L} = \{\mathsf{a}\}$ and let n be a natural number. Then $\mathsf{sum}\ \binom{n}{\mathsf{a}} = 2^n$. □

Example: Let $\mathsf{L} = \{\mathsf{a}, \mathsf{n}\}$. Then $\binom{\mathsf{n}}{\mathsf{a}}$ has infinite support and $\mathsf{sum}\ \binom{\mathsf{n}}{\mathsf{a}}$ is left undefined. □

Example: Let $\mathsf{L} = \{\mathsf{a}, \mathsf{n}\}$ and let n be a natural number. Then $\mathsf{sum}\ [\mathsf{n} = n]\binom{\mathsf{n}}{\mathsf{a}}$ is left undefined. □

The function sum is $\mathbb{C}$-linear:

Proposition 1. *If T, T_1 and T_2 have finite support and $c \in \mathbb{C}$ then*

1. $\mathsf{sum}\ (c \cdot T) = c \cdot \mathsf{sum}\ T$ *and*
2. $\mathsf{sum}\ (T_1 + T_2) = \mathsf{sum}\ (T_1) + \mathsf{sum}\ (T_2)$.

Remark: Recalling the definition of terms, we should have written $\mathsf{sum}\ (c \cdot T)$ as $\mathsf{sum}\ (\underline{\mathsf{c}} \cdot T)$ in order to distinguish between the complex number $c \in \mathbb{C}$ and the term $\underline{\mathsf{c}} \in \mathsf{T}$. □

2.3 Forms

Def. 4 (difference form). *Let $\mathcal{P}(\mathsf{L})$ denote the set of all subsets of L. The set F of all* (difference) forms *is defined by $\mathsf{F} := \mathcal{P}(\mathsf{L}) \to \mathsf{T}$.*

Throughout this section, we stick to the following type conventions:

- i, j, k, l, m and n are integers.
- x, y and z are labels (in L).
- X, Y and Z are sets of labels, i.e. $X, Y, Z \in \mathcal{P}(\mathsf{L})$.
- T is a term (in T).
- ρ, ω and η are forms (in F).

For example, we abbreviate $\sum_{X \subseteq \mathsf{L}}$ by $\sum_X$, we abbreviate $\underset{x \in \mathsf{L}}{\forall}$ by $\underset{x}{\forall}$ and so on. For all examples in this section we fix $\mathsf{L} = \{\mathsf{a}, \mathsf{b}, \mathsf{c}\}$.

Def. 5. *For $X \subseteq \mathsf{L}$, the form dX is defined by $dX\,(Y) := [X = Y]$. For $x \in \mathsf{L}$, we abbreviate $d\{x\}$ by dx.*

Example: $d\mathsf{a}\,(\{\mathsf{a}\}) = 1$ and $d\mathsf{a}\,(\{\mathsf{a}, \mathsf{b}\})=0$. □

Def. 6. *Forms can be multiplied by terms and added pointwise:*

1. $(T \cdot \omega)\,(X) := T \cdot \omega(X)$.
2. $(\omega + \eta)\,(X) := \omega(X) + \eta(X)$.

The multiplication dot may be dropped. The form $T\,d\emptyset$ is abbreviated by T; context resolves ambiguities.

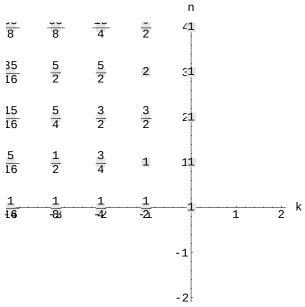
Example: $\left(\binom{\mathsf{b}}{\mathsf{a}} 2^{-\mathsf{b}} d\mathsf{a}\right)(X)$ equals $\binom{\mathsf{b}}{\mathsf{a}} 2^{-\mathsf{b}}$ if $X = \{\mathsf{a}\}$ and 0 otherwise. □

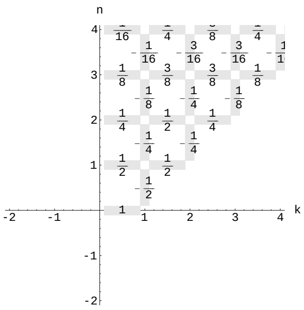
Example: $(f(\mathsf{a}, \mathsf{b})\, d\mathsf{a} + g(\mathsf{a}, \mathsf{b})\, d\mathsf{b})\ (\{\mathsf{a}\})$ equals $f(\mathsf{a}, \mathsf{b})$. □

2.4 Plotting Forms

To train our intuition about difference forms, we plot them.

The difference form $2^{\mathsf{k}}\binom{\mathsf{n}-\mathsf{k}}{\mathsf{n}}d\emptyset$ is plotted as follows: The color of a point p is determined by the number $v = 2^{\mathsf{k}}\binom{\mathsf{n}-\mathsf{k}}{\mathsf{n}}(p)$. If v is 0, the point is left white. Otherwise, a grey dot is plotted at p and the value of v is written down near p. (Unfortunately some of these numbers “collide” with the numbers placed on the axes.)





The difference form

$$\frac{-1}{2}\,2^{-\mathsf{n}}\binom{\mathsf{n}}{\mathsf{k}-1}\,d\mathsf{n} + 2^{-\mathsf{n}}\binom{\mathsf{n}}{\mathsf{k}}\,d\mathsf{k}$$

is plotted as follows: The color of a *unit cube in direction* k, i.e. a horizontal bar, is determined by the value of the term $2^{-\mathsf{n}}\binom{\mathsf{n}}{\mathsf{k}}$ at the left corner of the unit cube. Similarly the color of a *unit cube in direction* n, i.e. a vertical bar, is determined by the value of the term $\frac{-1}{2}\,2^{-\mathsf{n}}\binom{\mathsf{n}}{\mathsf{k}-1}$ at the lower corner of the unit cube.

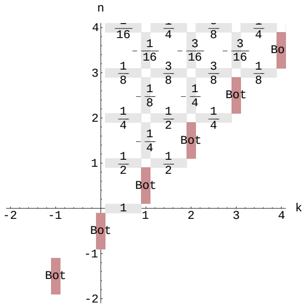
Strictly speaking,

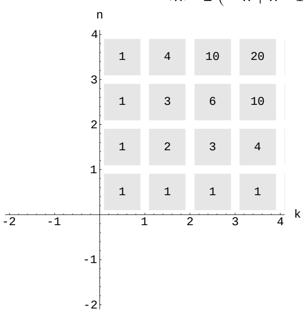
$$2^{-\mathsf{n}}\binom{\mathsf{n}}{\mathsf{k}}\left(\frac{\mathsf{k}}{2\,(-\mathsf{n}+\mathsf{k}-1)}\,d\mathsf{n} + 1\,d\mathsf{k}\right)$$

is no difference form at all since

$$2^{-\mathsf{n}}\binom{\mathsf{n}}{\mathsf{k}}\frac{\mathsf{k}}{2\,(-\mathsf{n}+\mathsf{k}-1)}$$

is no total function in $\mathsf{V} \to \mathbb{C}$ and therefore no term. To warn about a term that is not defined at a point p, we plot this term at p in red color. The vertical red bars in the plot stem from the denominator $-\mathsf{n}+\mathsf{k}-1$ that occurs in $2^{-\mathsf{n}}\binom{\mathsf{n}}{\mathsf{k}}\frac{\mathsf{k}}{2\,(-\mathsf{n}+\mathsf{k}-1)}$.





The difference form $\binom{\mathsf{n}+\mathsf{k}}{\mathsf{n}}\,d\mathsf{k}\,d\mathsf{n}$ is plotted as follows: The color of a *unit cube extending in directions* k *and* n, i.e. a square, is determined by the number $\binom{\mathsf{n}+\mathsf{k}}{\mathsf{n}}(p)$ where p is the lower left corner of that square.

2.5 Forms, Part 2

With respect to the operations $\cdot$ and $+$ of Definition 6 we have:

Proposition 2. *Multiplication is associative and distributes over addition:*

1. $(T_1 \cdot T_2) \cdot \omega = T_1 \cdot (T_2 \cdot \omega)$.
2. $(T_1 + T_2) \cdot \omega = T_1 \cdot \omega + T_2 \cdot \omega$.
3. $T \cdot (\omega + \eta) = T \cdot \omega + T \cdot \eta$.

Thus F is a T-module.

Def. 7 (sign). *The* sign *of a pair of labels is defined by*

$$s(x, y) = \begin{cases} 1 & \text{if} \quad x \prec y \\ 0 & \text{if} \quad x = y \\ -1 & \text{if} \quad x \succ y \end{cases}$$

and the sign *of a pair of label sets is defined by*

$$s(X, Y) = \prod_{x \in X} \prod_{y \in Y} s(x, y).$$

Proposition 3. *The sign function satisfies the following skew commutation laws:*

1. $s(x, y) = -s(y, x)$.
2. $s(X, Y) = (-1)^{\#X \#Y} s(Y, X)$.

Proposition 4. *The sign function distributes over $\cup$ in the following restricted sense:*

1. $s(X, Y)\, s(X \cup Y, Z) = s(X, Y)\, s(X, Z)\, s(Y, Z)$.
2. $s(X, Y \cup Z)\, s(Y, Z) = s(X, Y)\, s(X, Z)\, s(Y, Z)$.

Proof. We prove 1. If $X \cap Y \neq \emptyset$, then $s(X, Y) = 0$ and (1) follows. If $X \cap Y = \emptyset$, then $s(X \cup Y, Z) = s(X, Z)\, s(Y, Z)$ by splitting the range of the product quantifier: $\prod_{x \in X \cup Y} \prod_{y \in Z} s(x, y) = \left(\prod_{x \in X} \prod_{y \in Z} s(x, y)\right) \left(\prod_{x \in Y} \prod_{y \in Z} s(x, y)\right)$. $\square$

Def. 8 (exterior product). *The* exterior product *of two forms is defined by*

$$(\omega \wedge \eta)\,(Z) := \sum_{\substack{X, Y \\ X \cup Y = Z}} s(X, Y)\, \omega(X)\, \eta(Y).$$

For brevity $\omega \wedge \eta$ may be abbreviated by $\omega\eta$.

Remark: Due to the fact that $s(X, Y) = 0$ whenever $X \cap Y \neq \emptyset$, only summands for complementary (with respect to L) X and Y contribute. $\square$

Proposition 5. *As expected we have skew commutation:*

1. $dx\, dy = -dy\, dx$.
2. $dx\, dx = 0$.

Remark: Note that $\omega \wedge \eta = -\eta \wedge \omega$ is *not* true in general. $\square$

Proposition 6. $(\mathsf{F}, \wedge, d\emptyset)$ *is a monoid:*

1. $\omega \wedge d\emptyset = d\emptyset \wedge \omega = \omega$.
2. $(\omega_1 \wedge \omega_2) \wedge \omega_3 = \omega_1 \wedge (\omega_2 \wedge \omega_3)$.

Proof of Proposition 2.2. Let X be arbitrary. Then both $((\omega_1 \wedge \omega_2) \wedge \omega_3)(X)$ and $(\omega_1 \wedge (\omega_2 \wedge \omega_3))(X)$ are equal to

$$\sum_{Y_1 \cup Y_2 \cup Y_3 = X} s(Y_1, Y_2)\, s(Y_1, Y_3)\, s(Y_2, Y_3)\, \omega_1(Y_1)\, \omega_2(Y_2)\, \omega_3(Y_3)$$

as can be shown using Proposition 4. $\square$

Proposition 7. *The exterior product is both left and right* T*-linear:*

1. $(T \cdot \omega) \wedge \eta = \omega \wedge (T \cdot \eta) = T \cdot (\omega \wedge \eta)$.
2. $(\omega_1 + \omega_2) \wedge \eta = \omega_1 \wedge \eta + \omega_2 \wedge \eta$.
3. $\omega \wedge (\eta_1 + \eta_2) = \omega \wedge \eta_1 + \omega \wedge \eta_2$.

Proposition 6 and Proposition 7.1 will be used for dropping parentheses without introducing ambiguities.

Sometimes it is easier to prove theorems for *monomial* forms first and to extend them to arbitrary forms afterwards.

Def. 9 (monomial). *A difference form* ω *is called* monomial *iff there is a set* Z *of labels such that*

$$\omega(X) \neq 0 \implies X = Z.$$

For example, $T\, d\mathsf{a}\, d\mathsf{b}$ is monomial while $T_1\, d\mathsf{a}\, d\mathsf{b} + T_2\, d\mathsf{a}\, d\mathsf{c}$ is not, whenever $T_1 \neq 0$ and $T_2 \neq 0$.

Some proofs (as for example the proof of Proposition 43) proceed by induction on the *degree*.

Def. 10 (degree). *A difference form* is homogeneous of degree r *(or* has degree r*) iff*

$$\omega(X) \neq 0 \implies \#X = r.$$

An r-form *is a difference forms of degree* r*. We define* F_r *to be the set of all* r*-forms.*

An example of a 2-form is $T_1\, d\mathsf{a}\, d\mathsf{b} + T_2\, d\mathsf{b}\, d\mathsf{c} + T_3\, d\mathsf{a}\, d\mathsf{c}$. The form 0 is homogeneous of degree r, for any natural number r.

2.6 The Inner Product of Forms

Def. 11 (inner product). *The* inner product *of the forms ρ and ω is defined by*

$$\langle \rho, \omega \rangle := \sum_{X \subseteq \mathsf{L}} \rho(X) \cdot \omega(X)$$

Proposition 8. *The inner product is symmetric: $\langle \rho, \omega \rangle = \langle \omega, \rho \rangle$.*

Proposition 9. *The inner product is both left and right T-linear:*

1. $\langle T\rho, \omega \rangle = T\langle \rho, \omega \rangle$
2. $\langle \rho_1 + \rho_2, \omega \rangle = \langle \rho_1, \omega \rangle + \langle \rho_2, \omega \rangle$
3. $\langle \rho, T\omega \rangle = T\langle \rho, \omega \rangle$
4. $\langle \rho, \omega_1 + \omega_2 \rangle = \langle \rho, \omega_1 \rangle + \langle \rho, \omega_2 \rangle$

2.7 Sums of Forms over Ranges

Def. 12 (Iversons bracket). *We define* Iverson's bracket *function from booleans to integers by*

1. $[true] := 1,$
2. $[false] := 0.$

For example, $[2 = 2] = 1$ and $[2 < 2] = 0$.

Def. 13 (sum). *Let ρ have finite support. The* sum of ω over ρ *is defined by*

$$\sum_{\rho} \omega := \mathsf{sum}\ \langle \rho, \omega \rangle.$$

The form ρ appearing under the $\sum$ sign in $\sum_\rho \omega$ is said to be used as *summation range* and corresponds to a manifold in the continuous case. The form ω appearing on the right of the $\sum$ sign in $\sum_\rho \omega$ is said to be used as *summand*. We typically use summation ranges involving Iverson brackets as, for example,

$$\rho = [\mathsf{n} = n][0 \leq \mathsf{k} < 0]$$

satisfying

$$\rho(X)(p) \in \{-1, 0, 1\},$$

but we never use this assumption. Using summation ranges involving hypergeometric terms may lead to new summation identities.

Proposition 10. *Let ρ, ρ_1 and ρ_2 have finite support and let $c \in \mathbb{C}$. Then*

1. $\sum_\rho c \cdot \omega = c \cdot \sum_\rho \omega,$
2. $\sum_\rho (\omega_1 + \omega_2) = \sum_\rho \omega_1 + \sum_\rho \omega_2,$
3. $\sum_{c \cdot \rho} \omega = c \cdot \sum_\rho \omega,$

4. $\sum_{\rho_1+\rho_2} \omega = \sum_{\rho_1} \omega + \sum_{\rho_2} \omega$,

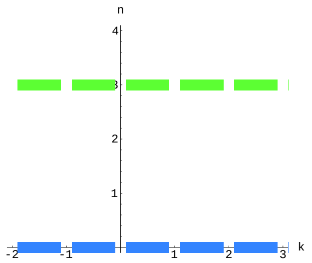
The following proposition illustrates that we treat forms and ranges uniformly, in contrast to [Zei93] and to the differential forms case.

Proposition 11.

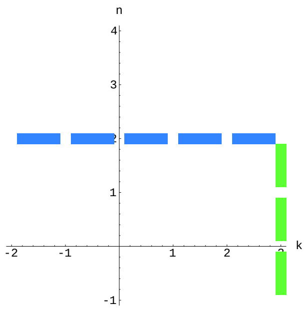
$$\sum_{\rho} \omega = \sum_{\omega} \rho$$

2.8 Plotting Summation Ranges

As we encode summation ranges as forms, we plot them like forms; see Section 2.4. Terms in summation ranges typically take on the values 1, 0 and −1 only. We encode these values by colors: Green denotes 1, white denotes 0 and blue denotes −1.

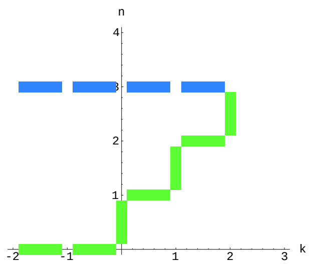


$([\mathsf{n} = 3] - [\mathsf{n} = 0])\, d\mathsf{k}$



$[\mathsf{k} = 3]\,[\mathsf{n} < 2]\, d\mathsf{n} - [\mathsf{n} = 2]\,[\mathsf{k} < 3]\, d\mathsf{k}$

$[\mathsf{k} = \mathsf{n}]\,[\mathsf{n} < 3]\,[0 \le \mathsf{n}]\, d\mathsf{n} + ([\mathsf{k} = \mathsf{n}-1]\,[\mathsf{n} < 3+1]\,[0 \le \mathsf{n}-1] - [\mathsf{n} = 3]\,[\mathsf{k} < \mathsf{n}]\,[0 \le \mathsf{n}-1] + [0 = \mathsf{n}]\,[\mathsf{k} < \mathsf{n}]\,[\mathsf{n} < 3])\, d\mathsf{k}$

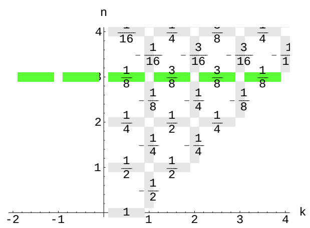


2.9 Plotting Sums of Forms over ranges

The plot shows the sum of the form $\omega = \frac{-1}{2} 2^{-\mathsf{n}} \binom{\mathsf{n}}{\mathsf{k}-1} d\mathsf{n} + 2^{-\mathsf{n}} \binom{\mathsf{n}}{\mathsf{k}} d\mathsf{k}$ over the range $\rho = [\mathsf{n} = n]\, d\mathsf{k}$ where $n = 3$. (The range ρ is depicted on top of the form ω.) By definition 13,

$$\sum_{\rho} \omega = \sum_{k} 2^{-n} \binom{n}{k}$$

(which of course equals 1).



2.10 Operators on Terms

Def. 14. *Let $x \in \mathsf{L}$. The unit vector in x-direction e_x is defined by*

$$(e_x)_y := [x = y].$$

For example, $e_{\mathsf{k}} = \{(\mathsf{k}, 1), (\mathsf{n}, 0)\}$ and $e_{\mathsf{n}} = \{(\mathsf{k}, 0), (\mathsf{n}, 1)\}$.

Def. 15. *Let $x \in \mathsf{L}$. The shift operator S_x in $\mathsf{T} \to \mathsf{T}$ is defined by*

$$(S_x T)(p) := T(p + e_x)$$

To compute $S_x T$ in our notation of terms, it suffices to replace each occurrence of $\underline{x}$ textually by $\underline{x+}1$, as justified by the following obvious proposition. (Note that x and y are labels, not integers; thus $[x = y]$ below tests equality of labels, not equality of integers.)

Proposition 12. *1. $S_x \underline{y} = \underline{x} + [x = y]$,*

2. $S_x \underline{c} = \underline{c}$,

3. $S_x \underline{f}(t_1, \ldots, t_n) = \underline{f}(S_x t_1, \ldots, S_x t_n)$.

For example, $S_{\mathsf{k}} \binom{\mathsf{n+k}}{\mathsf{k}} 2^{-\mathsf{k}} = \binom{\mathsf{n+k+1}}{\mathsf{k+1}} 2^{-\mathsf{k}-1}$.

Def. 16. *We define the difference operator Δ_x and the dual difference operator $\bar{\Delta}_x$ by*

1. $\Delta_x := -I + S_x$,

2. $\bar{\Delta}_x := -I + S_x^{-1}$,

where operator addition and subtraction is, of course, defined pointwise.

For example, $\Delta_{\mathsf{k}} \mathsf{k}! = (\mathsf{k} + 1)! - \mathsf{k}! = \mathsf{k}\mathsf{k}!$. and $\bar{\Delta}_k [\mathsf{k} < \mathsf{n}] = [\mathsf{k} - 1 < \mathsf{n}] - [\mathsf{k} < \mathsf{n}] = [\mathsf{k} \leq \mathsf{n}] - [\mathsf{k} < \mathsf{n}] = [\mathsf{k} = \mathsf{n}]$. (Computing $\bar{\Delta}_x$ on Iversons allows us to compute boundaries – even of infinite ranges – symbolically.)

Proposition 13. *Let T , T_1 and T_2 have finite support. Then*

1. $\mathsf{sum}\ S_x^k T = \mathsf{sum}\ T$,

2. $\mathsf{sum}\ (T_1 \cdot S_x T_2) = \mathsf{sum}\ (S_x^{-1} T_1 \cdot T_2)$,

3. $\mathsf{sum}\ (T_1 \cdot \Delta_x T_2) = \mathsf{sum}\ (\bar{\Delta}_x T_1 \cdot T_2)$.

Def. 17. *The set of* shift polynomials SP *is the smallest subset in $\mathsf{T} \to \mathsf{T}$ satisfying*

1. All shift operators S_x^k are in SP,

2. SP is closed under composition: If A and B are in SP, then AB is in SP,

3. SP is closed under addition: If A and B are in SP, then $A + B$ and $-A$ are in SP,

4. 0 is in SP.

Thus $\mathsf{SP} = \mathbb{Z}[S_{x_1}, \ldots, S_{x_n}]$ where $\mathsf{L} = \{x_1, \ldots, x_n\}$.

Examples of shift polynomials include Δ_x and G_x^k.

Proposition 14. *Let A and B be shift polynomials. Then*

1. $A\,(T_1 + T_2) = A\,T_1 + A\,T_2$,
2. $A\,B = B\,A$,

Proof. Shift operators commute with each other. □

2.11 Operators on Forms

Operators acting on terms induce corresponding operators acting on forms in a natural way:

Def. 18. *Let A be a function in $\mathsf{T} \to \mathsf{T}$. Then $\hat{A}$ in $\mathsf{F} \to \mathsf{F}$ is defined by*

$$(\hat{A}\omega)\,(X) := A(\omega(X)).$$

Overloading notation, we frequently abbreviate $\hat{A}$ by A.

We will use Definition 18 to lift shift operators S_x^k, difference operators Δ_x and multiplication operators $(T\cdot)$ from $\mathsf{T} \to \mathsf{T}$ to $\mathsf{F} \to \mathsf{F}$. Of course, lifting distributes over composition:

Proposition 15. *Let A and B be functions in $\mathsf{T} \to \mathsf{T}$. Then*

$$(AB)\hat{} = \hat{A}\hat{B}$$

Proposition 16. *Difference operators in $\mathsf{F} \to \mathsf{F}$ commute:*

1. $\hat{\Delta}_x \hat{\Delta}_y = \hat{\Delta}_y \hat{\Delta}_x$.
2. $\hat{\bar{\Delta}}_x \hat{\bar{\Delta}}_y = \hat{\bar{\Delta}}_y \hat{\bar{\Delta}}_x$.

Proof. We prove 1 by reducing it to the commutation of difference operators in $\mathsf{T} \to \mathsf{T}$ via Proposition 15:

$$\hat{\Delta}_x \hat{\Delta}_y = (\Delta_x \Delta_y)\hat{} = ((-1\cdot)\,\Delta_y \Delta_x)\hat{} = -\hat{\Delta} y \hat{\Delta} x.$$

□

Def. 19. *Overloading the meaning of dx, we define the operator dx in $\mathsf{F} \to \mathsf{F}$ by $dx\,(\omega) := dx \wedge \omega$.*

Thus dx may denote a form in F or an operator in $\mathsf{F} \to \mathsf{F}$ depending on context.

Proposition 17. $(dx\,\omega)(Z) = \sum_{\substack{Y \\ \{x\} \cup Y = Z}} s(\{x\}, Y)\,\omega(Y)$.

Remark: The summand in Proposition 17 is nonvanishing for $Y = Z \setminus \{x\}$ only; thus we could dispense of using a sum. Refraining from doing so helps in proving Proposition 23. □

Proposition 18. *The operators dx and dy satisfy the skew commutation law $dx\,dy = -dy\,dx$.*

Proof. By applying both operators to ω and using the associativity of $\wedge$, Proposition 18 can be reduced to the skew commuation law $dx\,dy = -dy\,dx$ for the *forms* dx and dy. $\square$

Def. 20. $(\bar{d}x\,\rho)(X) := s(x, X)\,\rho(\{x\} \cup X)$.

The operators $\bar{d}x$ and $\bar{d}y$ skew commute with each other. To see this, we need a technical lemma:

Lemma 1. *The sign function satisfies*

$$s(\{x\}, Z)\,s(\{y\}, \{x\} \cup Z) = -s(\{y\}, Z)\,s(\{x\}, \{y\} \cup Z)$$

Proof. By Proposition 4 2, the left hand side is equal to

$$s(\{x\}, Z)\,s(\{y\}, \{x\})\,s(\{y\}, Z)$$

and the right hand side is equal to

$$-s(\{y\}, Z)\,s(\{x\}, \{y\})\,s(\{x\}, Z).$$

Since $s(\{y\}, \{x\}) = -s(\{x\}, \{y\})$ by Proposition 3 2, both sides are equal. $\square$

As an immediate corollary of this lemma, we get:

Proposition 19. *The operators $\bar{d}x$ and $\bar{d}y$ skew commute with each other:*

$$\bar{d}x\,\bar{d}y = -\bar{d}y\,\bar{d}x.$$

Proposition 20. *Let A be a shift polynomial. Then*

1. $A\,dx = dx\,A$.
2. $A\,\bar{d}x = \bar{d}x\,A$.

Proof. Shift operators commute with the operator dx. $\square$

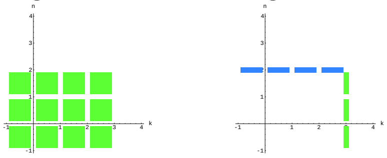
Def. 21. *The* exterior derivative operator d *in* $\mathsf{F} \to \mathsf{F}$ *and the* boundary operator ∂ *in* $\mathsf{F} \to \mathsf{F}$ *are defined by*

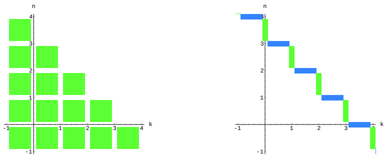
1. $d := \sum_x dx\,\Delta_x$,
2. $\partial := \sum_x \bar{d}x\,\bar{\Delta}_x$.

Remark: The letter d is overloaded. We use it both for constructing forms and for denoting the exterior derivative. Since the exterior derivative of the form a is indeed the form $d\mathsf{a}$, both interpretations of d agree. $\square$

Remark: The operator ∂ is called “boundary operator” since it computes “signed boundaries” of summation ranges; evidence is given by the following pictures:

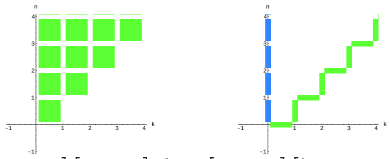
$\partial\,[\mathsf{k} < k]\,[\mathsf{n} < n]\,d\mathsf{k}\,d\mathsf{n} \;=\; [\mathsf{k} = k]\,[\mathsf{n} < n]\,d\mathsf{n} - [\mathsf{n} = n]\,[\mathsf{k} < k]\,d\mathsf{k}$

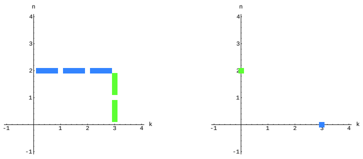




$\partial\,[\mathsf{n} + \mathsf{k} < k]\,d\mathsf{k}\,d\mathsf{n} \;=\; [\mathsf{n} + \mathsf{k} = k]\,d\mathsf{n} - [\mathsf{n} + \mathsf{k} = k]\,d\mathsf{k}$.

$\partial\,[0 \le \mathsf{k}]\,[\mathsf{k} \le \mathsf{n}]\,d\mathsf{k}\,d\mathsf{n} \;=\; (-[0 = \mathsf{k}]\,[\mathsf{k} \le \mathsf{n} + 1] + [\mathsf{k} = \mathsf{n} + 1]\,[0 \le \mathsf{k}])\,d\mathsf{n} + [\mathsf{k} = \mathsf{n}]\,[0 \le \mathsf{k}]\,d\mathsf{k}$.





$\partial\,[\mathsf{k} = k]\,[\mathsf{n} < n]\,[0 \le \mathsf{n}]\,d\mathsf{n} - [\mathsf{n} = n]\,[\mathsf{k} < k]\,[0 \le \mathsf{k}]\,d\mathsf{k} \;=\; [\mathsf{k} = k]\,[\mathsf{n} = n]\,[0 \le \mathsf{n} - 1] - [\mathsf{k} = k]\,[\mathsf{n} = n]\,[0 \le \mathsf{k} - 1] - [0 = \mathsf{n}]\,[\mathsf{k} = k]\,[\mathsf{n} < n] + [0 = \mathsf{k}]\,[\mathsf{n} = n]\,[\mathsf{k} < k]$. $\square$

Proposition 21. *Both d and ∂ are additive:*

1. $d(\omega + \eta) = d(\omega) + d(\eta)$.
2. $\partial(\omega + \eta) = \partial(\omega) + \partial(\eta)$.

Proposition 22. *We have $dd = 0$ as well as $\partial\partial = 0$.*

Proof. We prove $dd = 0$. Using the additivity of Δ_y and dy we expand dd to $\sum_y \sum_x A(y,x)$ where $A(y,x) = dy\,dx\,\Delta_y\Delta_x$. Now $A(y,x) = -A(x,y)$ as can be shown using the commutation $\Delta_y\Delta_x = \Delta_x\Delta_y$ and the skew commutation $dy\,dx = -dx\,dy$. Hence $\sum_y \sum_x A(y,x) = 0$. $\square$

Proposition 23. *In inner products, the operator dx can be moved from the right side to left side, getting $\bar{d}x$:*

$$\langle \rho, dx\,\omega \rangle = \langle \bar{d}x\,\rho, \omega \rangle.$$

Proof. We transform the left hand side stepwise to the right hand side. By definition of the inner product it is

$$\sum_Z \rho(Z)\,(dx\,\omega)(Z).$$

By Proposition 17, $(dx\,\omega)(Z)$ equals $\sum_{\{x\}\cup Y = Z}^{Y} s(\{x\},Y)\,\omega(Y)$. Thus the left hand side equals

$$\sum_{\substack{Y,Z\\ \{x\}\cup Y = Z}} s(\{x\},Y)\,\rho(Z)\,\omega(Y).$$

The condition $\{x\} \cup Y = Z$ allows us to eliminate the sum on Z, yielding

$$\sum_Y s(\{x\}, Y)\, \rho(\{x\} \cup Y)\, \omega(Y).$$

As $s(\{x\}, Y)\, \rho(\{x\} \cup Y) = (\bar{d}x\, \rho)(Y)$, this simplifies to

$$\sum_Y (\bar{d}x\, \rho)(Y)\, \omega(Y)$$

which, by the definition of the inner product, is the right hand side of Proposition 23. □

In sums, operators can be moved from the summand to the summation range by the following Proposition:

Proposition 24. *Let ρ have finite support. Then*

1. $\sum_\rho dx\, \omega = \sum_{\bar{d}x\, \rho} \omega$,
2. $\sum_\rho \Delta_x \omega = \sum_{\bar{\Delta}_x \rho} \omega$,
3. $\sum_\rho dx\, \Delta_x \omega = \sum_{\bar{d}x\, \bar{\Delta}_x \rho} \omega$.

Proof. We prove 1. It is true since $\langle \rho, dx\, \omega \rangle = \langle \bar{d}x\, \rho, \omega \rangle$ by Proposition 23. Next we prove 2. By Definition 13, Definition 3 and the additivity of sum it is equivalent to

$$\sum_X \rho(X)\, \Delta_x(\omega(X)) \;=\; \sum_X \bar{\Delta}_x(\rho(X))\, \omega(X)$$

which is true by Proposition 13. Finally 3 can be proved using 1, 2 and the commutation $\bar{\Delta}_x \bar{d}x = \bar{d}x\, \bar{\Delta}_x$ (Proposition 20(2)). □

2.12 The Theorem of Stokes–Zeilberger

We are now ready for the central theorem about difference forms.

Theorem 1 (Stokes–Zeilberger). *Let ρ have finite support. Then*

$$\sum_\rho d\omega = \sum_{\partial \rho} \omega.$$

Proof. Summing Proposition 24.3 over all $x \in \mathsf{L}$ gives

$$\sum_x \sum_\rho dx\, \Delta_x \omega = \sum_x \sum_{\bar{d}x\, \bar{\Delta}_x \rho} \omega.$$

The bilinearity of $\sum_\rho \omega$ allows us to move the sums on x inside:

$$\sum_\rho \sum_x dx\, \Delta_x \omega = \sum_{\sum_x \bar{d}x\, \bar{\Delta}_x \rho} \omega.$$

Since $\sum_x dx\, \Delta_x = d$ and $\sum_x \bar{d}x\, \bar{\Delta}_x = \partial$ Theorem 1 follows. □

The omnipresent telescoping trick [GKP89, p.50] is the simplest special case of Stokes' Theorem:

Proposition 25 (telescoping). *Suppose that the functions f and F in $\mathbb{Z} \to \mathbb{C}$ satisfy $F(n+1) - F(n) = f(n)$ for all n. Assume $a < b$. Then $\sum_{k=a}^{b-1} f(k) = F(b) - F(a)$.*

Proof. Let $\mathsf{L} := \{\mathsf{k}\}$, $\omega := F(\mathsf{k})$ and $\rho := [a \leq \mathsf{k} < b] d\mathsf{k}$. Use Stokes' Theorem. □

Needless to say, Proposition 25 is a discrete analog of $\int_a^b F'(x)dx = F(b) - F(a)$.

Def. 22. *We define* closedness *and* exactness *as follows:*

1. *The form ω is d-*closed*, or, shorter, ω is a* closed form*, iff $d\omega = 0$.*
2. *The form ρ is ∂-*closed*, or, shorter, ρ is a* closed range*, iff $\partial\rho = 0$.*
3. *The form ω is d-*exact*, or, shorter, ω is an* exact form*, iff $\underset{\tilde{\omega}}{\exists}\, d\tilde{\omega} = \omega$.*
4. *The form ρ is ∂-*exact*, or, shorter, ρ is an* exact range*, iff $\underset{\tilde{\rho}}{\exists}\, \partial\tilde{\rho} = \rho$.*

For the purpose of producing interesting identities, we usually do not use the Theorem of Stokes–Zeilberger in full generality. Instead, we only use the following immediate corollary of it.

Theorem 2 (identity mill). *Let ρ be an exact range having finite support and let ω be closed form. Then*

$$\sum_{\rho} \omega = 0.$$

Proof. Since ρ is an exact form, there is a form $\tilde{\rho}$ such that $\rho = \partial\tilde{\rho}$. Thus

$$\sum_{\rho} \omega = \sum_{\partial\tilde{\rho}} \omega = \sum_{\tilde{\rho}} d\omega = \sum_{\tilde{\rho}} 0 = 0.$$

The essential step is the use of the Theorem of Stokes-Zeilberger. □

2.13 WZ Pairs

Def. 23 (WZ pair). *Let f and g be functions in $\mathbb{Z}^2 \to \mathbb{Z}$. The pair (f,g) is called a* WZ pair *[WZ90] iff*

$$f(n+1,k) - f(n,k) = g(n,k+1) - g(n,k).$$

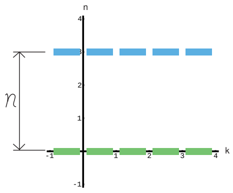
WZ pairs can be encoded as closed difference forms:

Proposition 26. *Let $\mathsf{L} = \{\mathsf{k}, \mathsf{n}\}$. Then (f,g) is a WZ pair iff*

$$f(\mathsf{n},\mathsf{k})\, d\mathsf{k} + g(\mathsf{n},\mathsf{k})\, d\mathsf{n}$$

is closed.

Well known propositions about WZ pairs naturally follow from the Identity Mill Theorem. The following proposition is part of Theorem A of [WZ90]:



Proposition 27. *Let* (f, g) *be a WZ pair. Suppose that*

$$\underset{n\geq 0}{\forall} \lim_{k\to\pm\infty} g(n,k) = 0.$$

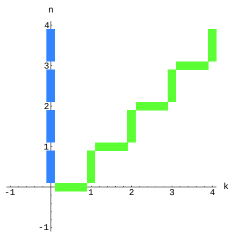
Then

$$\underset{n\geq 0}{\forall} \sum_k f(n,k) = \sum_k f(0,k)$$

whenever both sums converge.

Proof. Let n be a fixed natural number, $\rho_K := \partial[-K \leq \mathsf{k} < K][0 \leq \mathsf{n} < n]d\mathsf{k}d\mathsf{n}$ and $\omega := f(\mathsf{n},\mathsf{k})\,d\mathsf{k} + g(\mathsf{n},\mathsf{k})\,d\mathsf{n}$. The Identity Mill Theorem shows that $\underset{K}{\forall} \sum_{\rho_K} \omega = 0$ which implies $\lim_{K\to\infty} \sum_{\rho_K} \omega = 0$ i.e. $\sum_k f(n,k) = \sum_k f(0,k)$. $\square$

The following proposition is used in [Zei93] to prove Apérys series for $\zeta(3)$:



Proposition 28. *Let* (f, g) *be a WZ pair. Suppose that*

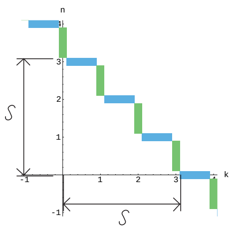
$$\underset{\epsilon>0}{\forall} \underset{N}{\exists} \underset{n\geq N}{\forall} \underset{k}{\forall} |f(n,k)| < \epsilon.$$

Then

$$\sum_{n=0}^{\infty} g(n,0) = \sum_{n+0}^{\infty} (f(n,n) + g(n,n+1))$$

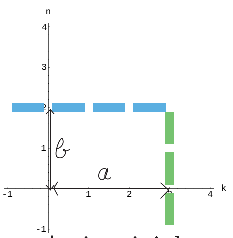
whenever both sums converge.

Remark: The original version of this proposition, Theorem 7 of [Zei93], misses a condition like $\underset{\epsilon>0}{\forall} \underset{N}{\exists} \underset{n\geq N}{\forall} \underset{k}{\forall} |f(n,k)| < \epsilon$. Plugging $\omega = d\frac{2\,k}{k+n+1}$ into the original version would yield a WZ proof of $0 = 1$. $\square$



Proposition 29. *Let* (f, g) *be a WZ pair and let* s *be a fixed integer. Suppose that both* f *and* g *vanish for negative arguments. Then*

$$\sum_{a+b=s} (g(a,b) - f(a,b)) = 0.$$



Proposition 30. *Let* (f, g) *be a WZ pair and let* a *and* b *be fixed integers. Suppose that both* f *and* g *vanish for negative arguments. Then*

$$\sum_{k<a} f(b,k) = \sum_{n<b} g(n,a).$$

As is trivial to get such propositions by plugging some exact range ρ into the Identity Mill Theorem (even automatically!), we stop doing this.

2.14 WZ Forms and the Residue Calculus

In his foreword to [PWZ96], D. E. Knuth mentions the identity

$$\sum_k \binom{2k}{k}\binom{2n-2k}{2-k} = 4^n.$$

Of course, it is immediately proved by comparing coefficients at z^n in

$$\frac{1}{\sqrt{1-4z}}\frac{1}{\sqrt{1-4z}} = \frac{1}{1-4z}.$$

We prove it in a different way in order to point at an analogy between the WZ method and the residue calculus.

To prove

$$\sum_k 4^{-n}\binom{2k}{k}\binom{2n-2k}{n-k} = 1,$$

we start by searching for a range ρ and a closed form ω such that

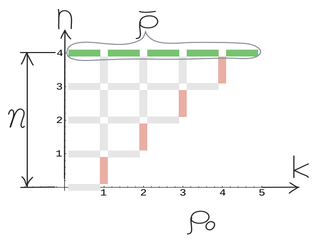
$$\sum_\rho \omega = \sum_k 4^{-n}\binom{2k}{k}\binom{2n-2k}{n-k}.$$

Gosper's algorithm helps us to find

$$\omega = 4^{-\mathsf{n}}\binom{2\mathsf{k}}{\mathsf{k}}\binom{2\mathsf{n}-2\mathsf{k}}{\mathsf{n}-\mathsf{k}} \times \left(1d\mathsf{k} - \frac{\mathsf{n}-\mathsf{k}+1/2}{\mathsf{n}-\mathsf{k}+1}\frac{\mathsf{k}}{\mathsf{n}+1}d\mathsf{n}\right).$$

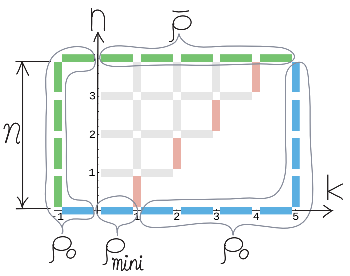
We fix an integer n and the range $\bar\rho$ where

$$\bar\rho = [\mathsf{n}=n]d\mathsf{k}.$$



Next we extend $\bar\rho$ to an exact range ρ,

$$\rho = \bar\rho + \rho_0 + \rho_{\text{mini}}.$$



To prove

$$\int_{-\infty}^{\infty}\frac{1}{1+x^2}dx = \pi,$$

we start by searching for a meromorphic function f and a path γ such that

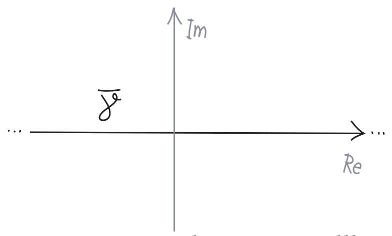
$$\int_\gamma f(z)dz = \int_{-\infty}^{\infty}\frac{1}{1+x^2}dx.$$

Finding f is immediate:

$$f(z) = \frac{1}{1+z^2}.$$

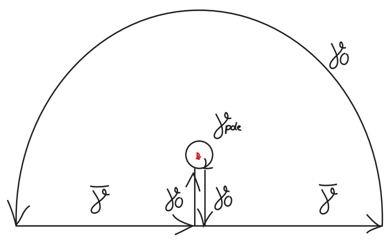
We fix the path $\bar\gamma$ where

$$\bar\gamma(t) = t.$$



Next we extend $\bar\gamma$ to a nullhomotopic closed path γ,

$$\gamma = \bar\gamma + \gamma_0 + \gamma_{\text{pole}}.$$



We carefully choose a range ρ_{mini} that is nonvanishing on a handful of points only, and a range ρ_0 that lies outside the support of ω. Thus

$$\sum_{\rho_{\text{mini}}} \omega$$
$$= \text{trivial to compute},$$

and

$$\sum_{\rho_0} \omega = 0.$$

Since

$$\sum_{\rho} \omega = -\sum_{\rho_{\text{mini}}} \omega$$

we prove

$$\sum_k 4^{-n} \binom{2k}{k} \binom{2n-2k}{n-k} = 1.$$

We carefully choose a path ρ_{mini} that consists of cycles around poles only, and a path γ_0 that does not contribute to the integral. Thus

$$\int_{\gamma_{\text{pole}}} f(z)dz$$
$$= \text{revealed by the residue}$$

and

$$\int_{\gamma_0} f(z)dz = 0.$$

Since

$$\int_{\gamma} f(z)dz = -\int_{\gamma_{\text{pole}}} f(z)dz$$

we prove

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2} dx = \pi.$$

3 WZ Forms

Sometimes "WZ form" is used as a synonym for "closed difference form". To stress the close correspondence between differential forms and difference forms, we reuse common terminology from differential forms and reserve the notion *WZ form* for the hypergeometric case.

Def. 24 (WZ form). *A* WZ form *is a closed difference form whose coefficient terms are hypergeometric or q-hypergeometric.*

Presently, our package `wz.m` is restricted to WZ forms; however, we plan to support other coefficient domains as well in the future.

Def. 25 (trivial WZ form). *A WZ form is called* trivial*, iff it is the exterior derivative of another WZ form.*

To find nontrivial identities, we need nontrivial forms.

3.1 Gosper's Algorithm Constructs WZ Pairs

Given a hypergeometric term and a label, Gosper's algorithm finds out if there is a hypergeometric antidifference to the term. In the affirmative case, the algorithm returns this antidifference.

As shown in [WZ90], Gosper's algorithm solves the problem of constructing WZ pairs: Consider the Binomial Theorem. Its natural WZ-style proof is to sum a WZ form

$$\omega := -\binom{\mathsf{n}}{\mathsf{k}}\frac{x^{\mathsf{k}}y^{\mathsf{n}-\mathsf{k}}}{(x+y)^{\mathsf{n}}}\,d\mathsf{k} + G(\mathsf{k},\mathsf{n})\,d\mathsf{n}$$

(where $\mathsf{L} = \{\mathsf{k}, \mathsf{n}\}$) over the exact range

$$\rho := \partial\,[0 \le \mathsf{n} < n]\,d\mathsf{k}\,d\mathsf{n}$$

getting

$$-1 + \sum_k \binom{n}{k}\frac{x^k y^{n-k}}{(x+y)^n} = 0.$$

The form ω is closed iff

$$\Delta_{\mathsf{k}} G(\mathsf{k},\mathsf{n}) = -\Delta_{\mathsf{n}}\binom{\mathsf{n}}{\mathsf{k}}\frac{x^{\mathsf{k}}y^{\mathsf{n}-\mathsf{k}}}{(x+y)^{\mathsf{n}}};$$

Gosper's algorithm computes $G(\mathsf{k},\mathsf{n})$ and we finally obtain the WZ form

$$\omega = \binom{\mathsf{n}}{\mathsf{k}}\frac{x^{\mathsf{k}}y^{\mathsf{n}-\mathsf{k}}}{(x+y)^{\mathsf{n}}}\left(\frac{\mathsf{k}y}{(\mathsf{n}-\mathsf{k}+1)(x+y)}\,d\mathsf{n} - \,d\mathsf{k}\right).$$

Remark: The form ω receives some treatment in section 5.1. □

3.2 Incompleteness of the WZ Forms Method

Unfortunately, the method of WZ forms is not complete for proving identities of the form

$$\sum_{k=-\infty}^{\infty} f(n,k) = 1$$

where $f(n,k)$ is hypergeometric in n and k. Note however that Zeilberger's fast algorithm [Zei91] *is* complete for proving all these identities.

A counterexample [PS95] for the completeness of the WZ method is

$$f(n,k) = (-1)^k \binom{n}{k} \binom{3k}{n} (-3)^{-n}.$$

More generally, Paule and Schorn [PS95] proved that

$$f_d(n,k) = (-1)^k \binom{n}{k} \binom{dk}{n} (-3)^{-d}$$

is a counterexample for each integer $d \geq 3$.

3.3 Singlesum Identities

All identities in this section were generated by a computer program; citations were added by hand. All parameters appearing in the identities are assumed to be integers. The program annotates identities with inequality constraints on the parameters. While these constraints are sufficient, some inequalities occurring in them may be redundant: For example, the program does not simplify "for $a \geq 0$, $2\,a \geq 0$, $a \leq n$, $n+a \geq 0$" to "for $0 \leq a \leq n$" in identity 1 below. Redundant inequalities could be detected by the *simplex algorithm* and subsequently removed; we have not implemented this so far.

Identity 1 (a "Moriarity" identity of Davis [Ego84, p. 52]).

$$\sum_{k=a}^{n} (-1)^k \; 2^{2\,k} \binom{k}{a} \binom{n+k}{2\,k} = \frac{2\,n+1}{2\,a+1} \; (-1)^n \; 2^{2\,a} \binom{n+a}{2\,a}$$

$$\textit{for} \quad a \geq 0,\; 2\,a \geq 0,\; a \leq n,\; n+a \geq 0.$$

Remark: The sum equals

$${}_2F_1\left[\begin{matrix} n+a+1,\; -n+a \\ a+\frac{1}{2} \end{matrix}; 1\right] \; (-1)^a \; 2^{2\,a} \binom{n+a}{2\,a}.$$

It can be evaluated by Vandermonde's Theorem and Gauss's Theorem. □

Identity 2 (→1).

$$\sum_{n=a+1}^{k} \frac{n}{(2\,n-1)\;(2\,n+1)} \binom{2\,k}{n+k} \binom{n+a}{2\,a+1}$$

$$= \frac{a+1}{(2\,a+1)\;(2\,k+1)} \, 2^{2\,k-2\,a-2} \binom{k}{a+1}$$

$$\textit{for} \quad a+1 \geq 0,\; 2\,a+1 \geq 0,\; k \geq 0,\; 2\,k \geq 0,\; a+1 \leq k.$$

Remark: The sum equals

$$
{}_4F_3\left[\begin{matrix}2\,a+2,\,a+2,\,a+\frac{1}{2},\,-k+a+1\\ a+1,\,a+\frac{5}{2},\,k+a+2\end{matrix};-1\right]
\times\frac{a+1}{(2\,(a+1)-1)\,(2\,(a+1)+1)}\binom{2\,k}{k+a+1}.
$$

It can be evaluated by Slater III.10. □

Identity 3 ([Ego84, p. 52], corrected).

$$
\sum_{i=r+1}^{r+n}\binom{n-1}{-r+i-1}\binom{r}{2\,r-i}=\binom{r+n-1}{r-1}
$$

$$
\textit{for}\quad 1\le n,\,n\ge 0,\,1\le r,\,r\ge 0,\,1\le r+n.
$$

Remark: The sum is a certain ${}_2F_1\left[\begin{matrix}\cdots\\ \cdots\end{matrix};1\right]$ and can be evaluated by Vandermonde's Theorem and Gauss's Theorem. □

Identity 4 (→3).

$$
\sum_{r=-i+1}^{n-1}\frac{3\,r^2-2\,n\,r+3\,i\,r+r-i\,n+i^2}{2\,r+i}\binom{n-1}{r+i-1}\binom{n}{-r+n-1}\binom{2\,r+i}{r}
=0
$$

$$
\textit{for}\quad 1\le n,\,n\ge 0.
$$

Identity 5 (of Dixon, [GKP89]).

$$
\sum_{k=-a}^{a}(-1)^k\binom{b+a}{k+a}\binom{c+a}{k+c}\binom{c+b}{k+b}=a!^{-1}\,b!^{-1}\,c!^{-1}\,(c+b+a)!
$$

$$
\textit{for}\quad a\ge 0,\,b\ge 0,\,b+a\ge 0,\,c\ge 0,\,c+a\ge 0,\,c+b\ge 0,\,c+b+a\ge 0.
$$

Remark: The sum equals

$$
{}_3F_2\left[\begin{matrix}-2\,a,\,-b-a,\,-c-a\\ b-a+1,\,c-a+1\end{matrix};1\right]\,(-1)^a\binom{c+a}{c-a}\binom{c+b}{b-a}.
$$

It can be evaluated by Dixon's Theorem (Slater III.8, terminated in the first variable), Dixon's Theorem (Slater III.8) and Dixon's Theorem (Slater III.9). □

Identity 6 (→5).

$$
\sum_{c=k+1}^{b}(-1)^c\binom{c+a-1}{-k+c-1}\binom{k+b}{-c+b}(c-1)!\,(c-b+a)!^{-1}
=(-1)\,(-1)^{b+1}\binom{b-a-1}{k-a}a!^{-1}\,(b-1)!
$$

$$
\textit{for}\quad a\ge 0,\,1\le b,\,a+1\le b,\,k+1\le b,\,a\le k,\,k+a\ge 0,\,k+b\ge 0.
$$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix} k-b+1,\, k+1,\, k+a+1 \\ k-b+a+2,\, 2\,k+2 \end{matrix};1\right] \\ \times\ (-1)^{k+1}\binom{k+b}{-k+b-1}\,k!\,(k-b+a+1)!^{-1}. $$

It can be evaluated by Saalschütz's Theorem (Slater III.2) and Saalschütz's Theorem (Slater III.31). □

Identity 7 (→5).

$$ \sum_{a=k+1}^{c}(-1)^a\binom{a-1}{-c+b+a}\binom{b+a-1}{-k+a-1}\binom{k+c}{c-a} \\ = \frac{k-c}{c}\,(-1)^{c+1}\binom{c}{c-b}\binom{c-b}{k-b} $$
$$ \textit{for}\quad b\geq 0,\, c\geq 0,\, b+1\leq c,\, b\leq c,\, k\leq c,\, b\leq k,\, k+b\geq 0,\, k+c\geq 0. $$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix} k-c+1,\, k+1,\, k+b+1 \\ k-c+b+2,\, 2\,k+2 \end{matrix};1\right](-1)^{k+1}\binom{k}{k-c+b+1}\binom{k+c}{-k+c-1}. \tag{9} $$

It can be evaluated by Saalschütz's Theorem (Slater III.2). □

Identity 8 (→5).

$$ \sum_{b=0}^{k}(-1)^b\binom{a+1}{-c-b+a+1}\binom{-b+a}{-k+a}\binom{c+b}{b}\binom{k+b}{c+b}=0 $$
$$ \textit{for}\quad a+1\geq 0,\, c\leq a+1,\, c\geq 0,\, k\leq a,\, k+a+1\geq 0,\, c\leq k,\, k+c\geq 0. $$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix} c-a-1,\, -k,\, k+1 \\ c+1,\, -a \end{matrix};1\right]\binom{a}{-k+a}\binom{a+1}{-c+a+1}\binom{k}{c}. \tag{10} $$

It can be evaluated by Saalschütz's Theorem (Slater III.2) □

Identity 9 (→5).

$$ \sum_{c=0}^{k}(-1)^c\binom{c+b}{b}\binom{c+b+a-1}{a-1}\binom{k+b}{c+b}\binom{k+c}{c+a}=0 $$
$$ \textit{for}\quad 1\leq a,\, b\geq 0,\, 1\leq b+a,\, k+1\leq b,\, a\leq k,\, k+a\geq 0,\, k+b\geq 0. $$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix} b+a,\, k+1,\, -k \\ a+1,\, b+1 \end{matrix};1\right]\binom{b+a-1}{a-1}\binom{k}{a}\binom{k+b}{b}. \tag{11} $$

It can be evaluated by Saalschütz's Theorem (Slater III.2). □

Identity 10 (→5).

$$\sum_{k=-a}^{a} (-1)^k \binom{b+a}{k+a}\binom{c+a}{k+c}\binom{c+b}{k+b} = \binom{c+b}{b}\binom{c+b+a}{a}$$

$$\textit{for}\quad a \geq 0,\, b \geq 0,\, b+a \geq 0,\, c \geq 0,\, c+a \geq 0,\, c+b \geq 0,\, c+b+a \geq 0.$$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix}-2\,a,\, -b-a,\, -c-a \\ b-a+1,\, c-a+1\end{matrix};1\right] (-1)^a \binom{c+a}{c-a}\binom{c+b}{b-a}. \tag{12}$$

It can be evaluated by Dixon's Theorem (Slater III.8, terminated in the first variable) . □

Identity 11 (of Carlitz [Ego84, p. 170]).

$$\sum_{k=0}^{m} \binom{m}{k}\binom{n}{k}\binom{p+n+m-k}{n+m} = \binom{p+m}{m}\binom{p+n}{n}$$

$$\textit{for}\quad m \geq 0,\, n \geq 0,\, n+m \geq 0,\, p \geq 0,\, p+m \geq 0,\, p+n \geq 0.$$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix}-m,\, -n,\, -p \\ 1,\, -p-n-m\end{matrix};1\right] \binom{p+n+m}{n+m}. \tag{13}$$

It can be evaluated by Saalschütz's Theorem (Slater III.2) and Saalschütz's Theorem (Slater III.31). □

Identity 12 (→11).

$$\sum_{n=p}^{p-m+k} (-1)^n \binom{n}{k-1}\binom{n}{-p+n}\binom{p+k}{p-n-m+k}$$

$$= (-1)^{p+m+k} \binom{k-1}{m-1}\binom{p}{p-m+1}$$

$$\textit{for}\quad 1 \leq k,\, m \leq k,\, 1 \leq m,\, p \geq 0,\, p+k \geq 0,\, m \leq p+1.$$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix}m-k,\, p+1,\, p+1 \\ p-k+2,\, p+m+1\end{matrix};1\right] (-1)^p \binom{p}{k-1}\binom{p+k}{-m+k}. \tag{14}$$

It can be evaluated by Saalschütz's Theorem (Slater III.2) and Saalschütz's Theorem (Slater III.31). □

Identity 13 (→11).

$$\sum_{p=0}^{k+1} (-1)^p \binom{n+m+1}{p+n+m-k}\binom{p+m}{m}\binom{p+n}{n} = (-1)^{k+1}\binom{m}{k+1}\binom{n}{k+1}$$

$$\textit{for}\quad k+1 \geq 0,\, m \geq 0,\, k+1 \leq m,\, n \geq 0,\, k+1 \leq n,\, n+m+1 \geq 0.$$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix}-k-1,\, m+1,\, n+1\\ n+m-k+1,\, 1\end{matrix};1\right]\binom{n+m+1}{n+m-k}. \tag{15}$$

It can be evaluated by Saalschütz's Theorem (Slater III.2) and Saalschütz's Theorem (Slater III.31). $\square$

Identity 14 (of Saalschütz [GKP89, (5.28)]).

$$\sum_{k=0}^{n}\binom{r+k}{n+m}\binom{-s+r+n}{n-k}\binom{s-r+m}{k}=\binom{r}{m}\binom{s}{n}$$
$$\textit{for}\quad m\geq 0,\, n\geq 0,\, m\leq r,\, s\leq r+n,\, n\leq s,\, r\leq s+m.$$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix}-n,\, r+1,\, -s+r-m\\ r-n-m+1,\, -s+r+1\end{matrix};1\right]\binom{r}{n+m}\binom{-s+r+n}{n}. \tag{16}$$

The identity is equivalent to Saalschütz's Theorem (Slater III.2). $\square$

Identity 15 (→14).

$$\sum_{m=0}^{n}\binom{-r+k}{n-m}\binom{r}{m}\binom{s+r-m}{k}=\binom{s}{n}\binom{s+r-n}{-n+k}$$
$$\textit{for}\quad k\geq 0,\, n\leq k,\, n\geq 0,\, r\leq k,\, r\geq 0,\, s\geq 0,\, n\leq s,\, k\leq s+r,\, n\leq s+r.$$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix}-n,\, -r,\, -s-r+k\\ -r-n+k+1,\, -s-r\end{matrix};1\right]\binom{-r+k}{n}\binom{s+r}{k}. \tag{17}$$

The identity is equivalent to Saalschütz's Theorem (Slater III.2). $\square$

Identity 16 (→14).

$$\sum_{n=0}^{k}\binom{r-k}{-n+m}\binom{s}{n}\binom{s+r-n}{-n+k}=\binom{r}{m}\binom{s+r-m}{k}$$
$$\textit{for}\quad k\geq 0,\, m\geq 0,\, k\leq r,\, m\leq r,\, s\geq 0,\, k\leq s+r,\, m\leq s+r,\, m+k\leq s+r.$$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix}-k,\, -m,\, -s\\ r-m-k+1,\, -s-r\end{matrix};1\right]\binom{r-k}{m}\binom{s+r}{k}. \tag{18}$$

The identity is equivalent to Saalschütz's Theorem (Slater III.2). $\square$

Identity 17 (→14).

$$\sum_{r=k}^{s+k} (-s+2\,r-m-k) \binom{n+k}{-s+r+n} \binom{n+m}{r-k} \binom{r}{m} \binom{s-r+m+k}{s-r+m} = 0$$

$$\textit{for} \quad k \ge 0,\, m \ge 0,\, 1 \le n,\, n+k \ge 0,\, n+m \ge 0,\, s+1 \le n,\, s \ge 0.$$

Remark: The sum equals

$$ {}_5F_4\left[\begin{matrix} -s-m+k,\, -\frac{1}{2}\,s-\frac{1}{2}\,m+\frac{1}{2}\,k+1,\, -n-m,\, k+1,\, -s \\ -\frac{1}{2}\,s-\frac{1}{2}\,m+\frac{1}{2}\,k,\, -s+n+k+1,\, -s-m,\, -m+k+1 \end{matrix}; 1\right] \\ \times\ (-s-m+k) \binom{k}{m} \binom{n+k}{-s+n+k} \binom{s+m}{s+m-k}.$$

It can be evaluated by Dixon's Theorem as stated in Slater III.12. □

Identity 18 (→14).

$$\sum_{s=n}^{-r+m+k+1} (-1)^s \binom{k+1}{s+r-m} \binom{s}{n} \binom{s+r-n}{-n+k} \\ = (-1)^{r+m+k+1} \binom{-r+k}{n-m-1} \binom{r}{m+1}$$

for $m+1 \ge 0,\, n \le k,\, m+1 \le n,\, r \le k,\, r+n \le m+k+1,\, r \ge 0,\, m+1 \le r.$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix} r+n-m-k-1,\, n+1,\, r+1 \\ r+n-m+1,\, r+n-k+1 \end{matrix}; 1\right] (-1)^n \binom{k+1}{r+n-m} \binom{r}{-n+k}. \quad (19)$$

The identity is equivalent to Saalschütz's Theorem (Slater III.2). □

Identity 19 (of Kummer [GKP89, (5.30)]).

$$\sum_{k=-a}^{a} (-1)^k \binom{b+a}{k+a} \binom{b+a}{k+b} = \binom{b+a}{a}$$

$$\textit{for} \quad a \ge 0,\, b \ge 0,\, b+a \ge 0.$$

Remark: The sum equals

$$ {}_2F_1\left[\begin{matrix} -2\,a,\, -b-a \\ b-a+1 \end{matrix}; -1\right] (-1)^a \binom{b+a}{b-a}. \quad (20)$$

It can be evaluated by Kummer's Theorem. □

Identity 20 (→19).

$$\sum_{b=0}^{k} (-1)^b \binom{b+a}{a} \binom{k+a}{b+a} \binom{k+b}{b+a} = 0 \quad \textit{for} \quad a \ge 0,\, a \le k,\, k+a \ge 0. \quad (21)$$

Remark: The sum equals

$$ {}_2F_1\left[\begin{matrix} k+1,\, -k \\ a+1 \end{matrix}; 1\right] \binom{k}{a} \binom{k+a}{a}. \tag{22}$$

It can be evaluated by Vandermonde's Theorem and Gauss's Theorem. □

Identity 21 (→19).

$$\sum_{a=0}^{k} (-1)^a \binom{b+a}{a} \binom{k+a}{b+a} \binom{k+b}{b+a} = 0 \quad \textit{for} \quad b \geq 0,\, b \leq k,\, k+b \geq 0. \tag{23}$$

Remark: The sum equals

$$ {}_2F_1\left[\begin{matrix} -k,\, k+1 \\ b+1 \end{matrix}; 1\right] \binom{k}{b} \binom{k+b}{b}. \tag{24}$$

It can be evaluated by Vandermonde's Theorem and Gauss's Theorem. □

Identity 22 (of Moriarty [Ego84, p. 11]).

$$\sum_{k=m}^{n} \frac{1}{n+k} (-4)^k \binom{k}{m} \binom{n+k}{2k} = \frac{1}{n+m} (-1)^n\, 4^m \binom{n+m}{2m}$$
$$\textit{for} \quad m \geq 0,\, 2m \geq 0,\, m \leq n,\, n+m \geq 0.$$

Remark: The sum equals

$$ {}_2F_1\left[\begin{matrix} n+m,\, -n+m \\ m+\frac{1}{2} \end{matrix}; 1\right] \frac{1}{n+m} (-4)^m \binom{n+m}{2m}. \tag{25}$$

It can be evaluated by Vandermonde's Theorem and Gauss's Theorem. □

Identity 23 (→22).

$$\sum_{n=m}^{k-1} \binom{2k-1}{n+k} \binom{n+m}{2m} = 4^{-m+k-1} \binom{k-1}{m}$$
$$\textit{for} \quad 1 \leq k,\, 1 \leq 2k,\, m+1 \leq k,\, m \geq 0,\, 2m \geq 0.$$

Remark: The sum equals

$$ {}_2F_1\left[\begin{matrix} 2m+1,\, m-k+1 \\ m+k+1 \end{matrix}; -1\right] \binom{2k-1}{m+k}. \tag{26}$$

It can be evaluated by Kummer's Theorem. □

Identity 24.

$$\sum_{n} \left(18n^2 - 9kn + 3n - 8k - 12\right) \binom{k+4}{3n-k} = 2\,(k+3)\,(k+4)\,(-1)^k$$
$$\textit{for} \quad k+4 \geq 0.$$

Remark: The sum is a certain ${}_5F_4$ with argument -1. There are integer distances between upper and lower entries; contiguous relations may apply. □

Exercise: Try to evaluate the sum with Mathematica 3.0.1. □

Identity 25 (of Grosswald [Ego84, p. 27]).

$$\sum_{v=0}^{2m} \left(\frac{-1}{2}\right)^{v} \binom{r+2m}{v+r} \binom{v+2r+2m}{v} = (-1)^m\, 2^{-2m} \binom{r+2m}{m}$$

$$\textit{for} \quad m \geq 0,\, r+m \geq 0,\, r+2m \geq 0,\, 2r+2m \geq 0.$$

Remark: The sum equals

$${}_2F_1\left[\begin{matrix} 2r+2m+1,\ -2m \\ r+1 \end{matrix}; \frac{1}{2}\right] \binom{r+2m}{r}. \tag{27}$$

It can be evaluated by Gauss's Second Theorem . □

Identity 26 (→25).

$$\sum_{r} \frac{(-1)^r}{-2r+2m-1} \binom{-2r+2m}{-v-2r+2m-1} \binom{-r+2m}{m} \binom{v+2m}{-r+2m} = 0$$

$$\textit{for} \quad m \geq 0,\, v+1 \geq 0,\, v+2m \geq 0.$$

Remark: The sum equals

$${}_2F_1\left[\begin{matrix} -\frac{1}{2}v-m+\frac{1}{2},\ -\frac{1}{2}v-m+1 \\ -v-m+\frac{3}{2} \end{matrix}; 1\right] \times \frac{1}{2v+2m-1}\,(-1)^v \binom{v+2m}{m} \binom{2v+2m}{v+2m-1}.$$

It can be evaluated by Vandermonde's Theorem, Gauss's Theorem and S2105. □

Identity 27 ([Ego84, p. 27]).

$$\sum_{k=0}^{n} \frac{1}{(k+1)\,(-n+k-1)} \binom{2k}{k} \binom{2n-2k}{n-k} = \frac{-1}{n+2} \binom{2n+2}{n+1}$$

$$\textit{for} \quad n+1 \geq 0,\, 2n+2 \geq 0.$$

Remark: The sum equals

$${}_3F_2\left[\begin{matrix} -n-1,\ \frac{1}{2},\ 1 \\ 2,\ -n+\frac{1}{2} \end{matrix}; 1\right] \frac{-1}{n+1} \binom{2n}{n}. \tag{28}$$

It can be transformed by T3204, T3205, T3206, T3207, T3217, T3237, u, T3240, T3261, T3262, T3263, T3264, T3267 and T3268. Since there are integer distances, contiguous relations may apply. □

Identity 28 (→27).

$$\sum_{n=0}^{k} \frac{3\,n-2\,k-1}{(-n+k+1)\,(n+1)} \binom{-2\,n+2\,k}{-n+k} \binom{2\,n}{n} = \frac{-1}{2} \binom{2\,k+2}{k+1}$$

for $k+1 \geq 0,\, 2\,k+2 \geq 0.$

Remark: The sum equals

$$ {}_4F_3\left[\begin{matrix} -\frac{2}{3}\,k+\frac{2}{3},\, -k-1,\, \frac{1}{2},\, 1 \\ -\frac{2}{3}\,k-\frac{1}{3},\, 2,\, -k+\frac{1}{2} \end{matrix}; 1\right] \frac{-2\,k-1}{k+1} \binom{2\,k}{k}. \tag{29}$$

It can be transformed by T4301, T4302, T4303, T4304 and T4362. Since there are integer distances, contiguous relations may apply. □

Identity 29 (a companion of [Ego84, p. 49]).

$$\sum_{n=0}^{k} \frac{1}{q+2\,n} \binom{-2\,n+2\,k}{-n+k} \binom{q+2\,n}{n} = \frac{1}{q} \binom{q+2\,k}{k}$$

for $k \geq 0,\, q+k \geq 0,\, q+2\,k \geq 0.$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix} \frac{1}{2}\,q,\, -k,\, \frac{1}{2}\,q+\frac{1}{2} \\ -k+\frac{1}{2},\, q+1 \end{matrix}; 1\right] \frac{1}{q} \binom{2\,k}{k}. \tag{30}$$

It can be evaluated by Saalschütz's Theorem (Slater III.2) and Saalschütz's Theorem (Slater III.31). □

Identity 30 (a companion of [Ego84, p. 49]).

$$\sum_{q=-k}^{n} \binom{-q+2\,n-1}{n-1} \binom{q+2\,k}{k} = \frac{n+k+1}{2\,(2\,n+2\,k+1)} \binom{2\,n+2\,k+2}{n+k+1}$$

for $k \geq 0,\, 1 \leq n,\, n+k+1 \geq 0,\, 2\,n+2\,k+2 \geq 0.$

Remark: The sum equals

$$ {}_2F_1\left[\begin{matrix} -n-k,\, k+1 \\ -2\,n-k+1 \end{matrix}; 1\right] \binom{2\,n+k-1}{n-1}. \tag{31}$$

It can be evaluated by Vandermonde's Theorem and Gauss's Theorem. □

Identity 31 (of Le-Jen Shoo [Ego84, p. 52]).

$$\sum_{k=0}^{m} \binom{m}{k}^2 \binom{n+2\,m-k}{2\,m} = \binom{n+m}{n}^2$$

for $m \geq 0,\, 2\,m \geq 0,\, n \geq 0,\, n+m \geq 0.$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix} -m,\, -m,\, -n \\ 1,\, -n-2\,m \end{matrix};1\right] \binom{n+2\,m}{2\,m}. \tag{32} $$

It can be evaluated by Saalschütz's Theorem (Slater III.2) and Saalschütz's Theorem (Slater III.31). □

Identity 32 (→31). *Assume* $0 \le n \le k$ *and*

$$ p(m) = 3\,m\,n - 2\,k\,n - 2\,n + 4\,m^2 - 3\,k\,m - 2\,k - 2. $$

Then

$$ \sum_m p(m) \binom{k+1}{m}^2 \binom{2\,m}{n+2\,m-k} \binom{n+m}{n}^2 = 0. $$

Remark: The sum is a certain ${}_7F_6$ with integer distances; contiguous relations may apply. □

Identity 33 (→31).

$$ \sum_{n=0}^{k+1} (-1)^n \binom{2\,m+1}{n+2\,m-k} \binom{n+m}{n}^2 = (-1)^{k+1} \binom{m}{k+1}^2 $$
$$ \textit{for} \quad k+1 \ge 0,\, m \ge 0,\, k+1 \le m,\, 2\,m+1 \ge 0. $$

Remark: The sum equals

$$ {}_3F_2\left[\begin{matrix} m+1,\, m+1,\, -k-1 \\ 1,\, 2\,m-k+1 \end{matrix};1\right] \binom{2\,m+1}{2\,m-k}. \tag{33} $$

It can be evaluated by Saalschütz's Theorem (Slater III.2) and Saalschütz's Theorem (Slater III.31). □

3.4 Partial Sums of Hypergeometric Series

Some identities involving partial sums of hypergeometric series can be proven by the WZ-forms method. As an example, consider the following identity (it appears as (2.6.4) in [Sla66]):

Identity 34 (Bailey 1931).

$$ \frac{\Gamma(x+m)\,\Gamma(y+m)}{\Gamma(m)\,\Gamma(x+y+m)} {}_3F_2\left[\begin{matrix} x,\, y,\, v+m-1 \\ v,\, x+y+m \end{matrix};1\right] \textit{ to } n \textit{ terms} $$
$$ = \frac{\Gamma(x+n)\,\Gamma(y+n)}{\Gamma(n)\,\Gamma(x+y+n)} {}_3F_2\left[\begin{matrix} x,\, y,\, v+n-1 \\ v,\, x+y+n \end{matrix};1\right] \textit{ to } m \textit{ terms} $$

Proof. We aim to find a WZ style proof of Identity 34. We rewrite Identity 34 using $a^{\overline{k}} = \Gamma(a+k) \,/\, \Gamma(a)$:

$$\begin{aligned} &\frac{\Gamma(x+m)\,\Gamma(y+m)}{\Gamma(m)\,\Gamma(x+y+m)} \\ \times \sum_{0\le i<n} &\frac{\Gamma(x+i)}{\Gamma(x)} \frac{\Gamma(y+i)}{\Gamma(y)} \frac{\Gamma(v+m-1+i)}{\Gamma(v+m-1)} \frac{\Gamma(v)}{\Gamma(v+i)} \frac{\Gamma(x+y+m)}{\Gamma(x+y+m+i)} \frac{1}{i!} \\ &= \frac{\Gamma(x+n)\,\Gamma(y+n)}{\Gamma(n)\,\Gamma(x+y+n)} \\ \times \sum_{0\le j<m} &\frac{\Gamma(x+j)}{\Gamma(x)} \frac{\Gamma(y+j)}{\Gamma(y)} \frac{\Gamma(v+n-1+j)}{\Gamma(v+n-1)} \frac{\Gamma(v)}{\Gamma(v+j)} \frac{\Gamma(x+y+n)}{\Gamma(x+y+n+j)} \frac{1}{j!} \end{aligned} \tag{34}$$

Inspection shows that both summands differ by a rational factor only. Exploiting this observation we rewrite Equation 34 as

$$\sum_{0\le i<n} m \,\frac{v+m-1}{v+m-1+i}\, t(i,m) = \sum_{0\le j<m} n \,\frac{v+n-1}{v+n-1+j}\, t(n,i) \tag{35}$$

where

$$t(i,j) = \frac{\Gamma(x+i)\,\Gamma(y+i)}{i!\,\Gamma(v+i)} \frac{\Gamma(x+j)\,\Gamma(y+j)}{j!\,\Gamma(v+j)} \frac{\Gamma(v+i+j)\,\Gamma(v)}{\Gamma(x+y+i+j)\,\Gamma(x)\,\Gamma(y)}. \tag{36}$$

Equation 35 suggests a WZ style proof which indeed works. Let $L = \{i, j\}$ (we allow ourselves to use italic letters for label constants from now on) and

$$\omega = j\,\frac{v+j-1}{v+i+j-1}\, t(i,j)\, di + i\,\frac{v+i-1}{v+i+j-1}\, t(i,j)\, dj.$$

with the motive of writing Equation 35 as

$$\sum_{\rho_1} \omega = \sum_{\rho_2} \omega \tag{37}$$

where

$$\rho_1 = [0 \le i < n][j = m]\, di \text{ and } \rho_2 = [i = n][0 \le j < m]\, dj.$$

Fortunately, ω is closed (as can be checked by our package `wz.m`). Therefore,

$$\sum_{\partial\ [i<n][j<m]\ di\, dj} \omega = 0. \tag{38}$$

By the support of ω only the edges ρ_1 and ρ_2 of the rectangle $\partial\ [i < n][j < m]\, di\, dj$ contribute to the sum in Equation 38. Equation 37is equivalent to Equation 38 and therefore proved. $\square$

Note that finding WZ style proofs requires some luck: A proposed form ω might well turn out to be non-closed. We do not know an algorithm for finding WZ style proofs.

Open Problem: Find a closed multivariate analog to the difference form ω of the proof above. $\square$

Following [Sla66, p. 81], we consider the special case $m \to \infty$ of Identity 34. To do so, we use:

Lemma 2. *Let x and y be complex numbers and let k be a natural number. Then*

1. $\lim_{n\to\infty} \left(\frac{\Gamma(n+x)}{\Gamma(n+y)}\right) / \left(\frac{n^x}{n^y}\right) = 1$ *and*

2. $\lim_{n\to\infty} \frac{(n+x)^{\bar{k}}}{(n+y)^{\bar{k}}} = 1.$

As $m \to \infty$, Identity 34 reduces to:

Identity 35.

$$ {}_2F_1\left[\begin{matrix} x,\, y \\ v \end{matrix};1\right] \textit{ to } n \textit{ terms } = \frac{\Gamma(x+n)\,\Gamma(y+n)}{\Gamma(n)\,\Gamma(x+y+n)} {}_3F_2\left[\begin{matrix} x,\, y,\, v+n-1 \\ v,\, x+y+n \end{matrix};1\right] $$

Plugging in $x = y = \frac{1}{2}$ and $v = 1$ yields the following identity of Ramanujan (which appears as (2.6.1) in [Sla66]):

Identity 36.

$$ {}_2F_1\left[\begin{matrix} \frac{1}{2},\, \frac{1}{2} \\ 1 \end{matrix};1\right] \textit{ to } n \textit{ terms } = \frac{1}{n}\left(\frac{\Gamma(n+\frac{1}{2})}{\Gamma(n)}\right)^2 {}_3F_2\left[\begin{matrix} \frac{1}{2},\, \frac{1}{2},\, n \\ 1,\, n+1 \end{matrix};1\right] $$

Note that the special case $n = 1$ gives us a series for $1/\pi$:

$$ 1 = \frac{\pi}{4} \cdot {}_2F_1\left[\begin{matrix} \frac{1}{2},\, \frac{1}{2} \\ 2 \end{matrix};1\right]. $$

3.5 Wegschaider's Algorithm Constructs WZ r-forms

Given an r-fold hypergeometric summation identity, Kurt Wegschaider's algorithm allows us to construct a WZ form of degree r in $r{+}1$ variables from it.

Consider the identity [Den96]

$$ \sum_b \sum_s (-1)^b \binom{-s+k}{2\,v-b}\binom{s}{b}\binom{-2\,v+k}{s-b} = 2^{-2\,v+k}\binom{-v+k}{-2\,v+k} $$

which is valid for $0 \le 2v \le k$. Let

$$ f(b,k,s) := (-1)^b \binom{-s+k}{2\,v-b}\binom{s}{b}\binom{-2\,v+k}{s-b} \Big/ \left(2^{-2\,v+k}\binom{-v+k}{-2\,v+k}\right) $$

be the result of dividing the summand by the right hand side. To show that

$$ \sum_b \sum_s f(b,k,s) = 1 $$

we run Wegschaider's algorithm by typing

```
 <<MultiSum.m; summand=(-1)^b
Binomial[s,b]Binomial[k-s,2v-b]Binomial[k-2v,s-b];
rhs=Binomial[k-v,k-2v]2^(k-2v); fbks=summand/rhs;
rek=FindCertificate[fbks,k,0,{b,s},{1,0},1]; rek1=rek[[1]];
```

at the Mathematica command line, getting the answer

```
    2 (1+k-2 v) (1+k-v) F[k,-1+b,s]
    - 2 (1+k-2 v) (1+k-v) F[1+k,-1+b,s]
== Delta[b,- (1+k-2 v) (b+k-2 v) F[k,-1+b,s]
            - 2 (-2+b-s) (1+k-v) F[1+k,-1+b,1+s]]
  +Delta[s,- 2 (-1+b-s) (1+k-v) F[1+k,-1+b,s]].
```

Up to a shift in b this means that

$$(p_1 S_\mathsf{k} - p_2 I + \Delta_\mathsf{b} A_1 + \Delta_\mathsf{s} A_2) f(\mathsf{b},\mathsf{k},\mathsf{s}) = 0$$

where $\mathsf{L} = \{\mathsf{b},\mathsf{k},\mathsf{s}\}$ (note that $v \notin \mathsf{L}$),

$$p_1 = p_2 = -2(1+\mathsf{k}-2v)(1+\mathsf{k}-v),$$

$$A_1 = -(1+\mathsf{k}-2v)(1+\mathsf{b}+\mathsf{k}-2v)I - 2(-1+\mathsf{b}-\mathsf{s})(1+\mathsf{k}-v)S_\mathsf{k}S_\mathsf{s}$$

and

$$A_2 = -2(\mathsf{b}-\mathsf{s})(1+\mathsf{k}-v)S_\mathsf{k}.$$

Both p_1 and p_2 are free of b and s (by design of Wegschaider's algorithm). Furthermore, $p_1 = p_2$ (since $\sum_b \sum_s f(b,k,s) = 1$). These two properties of p_1 and p_2 allow us to transform the recurrence to

$$(\Delta_\mathsf{k} + \Delta_\mathsf{b} \frac{1}{p_1} A_1 + \Delta_\mathsf{s} \frac{1}{p_1} A_2) f(\mathsf{b},\mathsf{k},\mathsf{s}) = 0.$$

The latter recurrence asserts the closedness of

$$\omega := -f(\mathsf{b},\mathsf{k},\mathsf{s}) d\mathsf{b} d\mathsf{s} + \frac{1}{p_1} A_1 f(\mathsf{b},\mathsf{k},\mathsf{s}) d\mathsf{k} d\mathsf{s} + \frac{1}{p_1} A_2 f(\mathsf{b},\mathsf{k},\mathsf{s}) d\mathsf{b} d\mathsf{k}.$$

Straightforward computation gives

$$\begin{aligned}\omega = (-1)^{\mathsf{b}}\, 2^{-\mathsf{k}+2\,v} \binom{\mathsf{k}-2\,v}{\mathsf{s}-\mathsf{b}} \binom{\mathsf{k}-v}{\mathsf{k}-2\,v}^{-1} \binom{-\mathsf{s}+\mathsf{k}}{-\mathsf{b}+2\,v} \binom{\mathsf{s}}{\mathsf{b}} \\ \times \Bigg(\frac{(\mathsf{b}-1)\ \mathsf{b}\ (-\mathsf{s}+\mathsf{k}+\mathsf{b}-2\,v-1)\ (-\mathsf{s}+\mathsf{k}+\mathsf{b}-2\,v)^2}{2\ (\mathsf{b}-2\,v-1)\ (-\mathsf{k}+v-1)\ (-\mathsf{s}+\mathsf{b}-2)\ (-\mathsf{s}+\mathsf{b}-1)^2}\, d\mathsf{k}\, d\mathsf{s} \\ + \frac{-\mathsf{b}\ (-\mathsf{s}+\mathsf{k}+\mathsf{b}-2\,v)^2}{(\mathsf{b}-2\,v-1)\ (-\mathsf{s}+\mathsf{b}-1)^2}\, d\mathsf{b}\, d\mathsf{s} \\ + \frac{\mathsf{b}\ (\mathsf{k}-2\,v+1)\ (-\mathsf{s}+\mathsf{k}+1)}{2\ (\mathsf{b}-2\,v-1)\ (-\mathsf{k}+v-1)\ (-\mathsf{s}+\mathsf{b}-1)}\, d\mathsf{b}\, d\mathsf{k} \Bigg).\end{aligned}$$

Remark: The idea of dividing by the right hand side before running Wegschaider's algorithm is due to Wilf [Wil98]. □

3.6 Some Multisum Identities

Identity 37 ([Den96]).

$$\sum_b \sum_s (-1)^b \binom{-s+k}{2\,v-b} \binom{s}{b} \binom{-2\,v+k}{s-b} = 2^{-2\,v+k} \binom{-v+k}{-2\,v+k}$$

for $0 \le 2v \le k$.

Identity 38 (→37).

$$\sum_b \sum_k (-1)^b\, 2^{-k} \binom{s+b-2}{s-1} \binom{s+b-1}{2\,v-k} \binom{s+k}{2\,v-b+1} \binom{v-1}{-v+k} = 0$$

$$\textit{for} \quad 1 \le s,\, 1 \le v.$$

Remark: The annotation "(→37)" above indicates that identity 38 is obtained as a companion of identity 37. □

Identity 39 (→37).

$$\sum_k \sum_s (-2)^k \binom{b}{s} \binom{2\,v-b-2}{-s+k-1} \binom{2\,v-k-1}{-s+b-1} \binom{2\,v-k-1}{v-k} = 0$$

$$\textit{for} \quad b \ge 0,\, 1 \le v,\, b+2 \le 2\,v.$$

Identity 40 ([AP93]).

$$\sum_i \sum_j \binom{j+i}{j}^2 \binom{n+m-j-i}{n-j}^2 = \frac{1}{2} \binom{2\,n+2\,m+2}{2\,n+1}$$

$$\textit{for} \quad m \ge 0,\, n \ge 0.$$

Identity 41 (→40). *Let j and m be natural numbers. Define the polynomial $p(i,n)$ by*

$$\begin{aligned} p(i,n) \quad = \quad & -2\,j\,n^3 + i\,n^3 - n^3 + 5\,j\,m\,n^2 - 2\,i\,m\,n^2 + 2\,m\,n^2 - 4\,j^2\,n^2 \\ & -2\,i\,j\,n^2 - 2\,j\,n^2 + 2\,i^2\,n^2 - 2\,i\,n^2 - 4\,j\,m^2\,n - m^2\,n + 6\,j^2\,m\,n \\ & +4\,i\,j\,m\,n + 2\,j\,m\,n + 2\,i\,m\,n - 2\,j^3\,n - 3\,i\,j^2\,n - j^2\,n - 2\,i\,j\,n \\ & -i^2\,n - 2\,j^2\,m^2 + j^3\,m + 2\,i\,j^2\,m. \end{aligned}$$

Then

$$\sum_i \sum_n p(i,n) \frac{1}{(n+j)^2} \binom{j+i-1}{j}^2 \binom{2\,m}{-2\,n+2\,m} \binom{n+j}{n-m+j+i}^2 = 0.$$

4 Transformations

4.1 WZ Pairs Yield New WZ Pairs

Tewodros Amdeberhan [AZ97], Ira Gessel[Ges95], Herbert Wilf and Doron Zeilberger [WZ90] found transformations of known WZ pairs to new WZ pairs. They are stated as Propositions 33–36 on pp. 42–42. Applications of these transformations range from discovering new summation identities to obtaining faster and faster convergent series for $\zeta(3)$ [Amd96]. A common feature of all known transformations is that they do not mix the labels n and k. We aim to find more general transformations that do mix n and k. Note that the transformations we find were independently discovered by Ira Gessel [Ges99].

A naive attempt that fails. Consider the form

$$\omega = \binom{\mathsf{n}}{\mathsf{k}} 2^{-\mathsf{n}} \left(\frac{\mathsf{n}-2\mathsf{k}-1}{\mathsf{k}+1} d\mathsf{k} + \frac{\mathsf{n}-2\mathsf{k}+1}{2(\mathsf{k}-\mathsf{n}-1)} d\mathsf{n} \right)$$

which is closed as could be checked by computation. To obtain a new closed form, we try to apply the substitution $\mathsf{n} \to \mathsf{n}+\mathsf{k}$ to ω. A reasonable guess is to proceed just as in the case of differential forms, replacing $d\mathsf{n}$ by $d\mathsf{n}+d\mathsf{k}$.

$$\omega' = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}} 2^{-\mathsf{n}-\mathsf{k}} \left(\frac{\mathsf{n}-\mathsf{k}-1}{\mathsf{k}+1} d\mathsf{k} + \frac{\mathsf{n}-\mathsf{k}+1}{2(-\mathsf{n}-1)} (d\mathsf{n}+d\mathsf{k}) \right) \tag{39}$$

$$= \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}} 2^{-\mathsf{n}-\mathsf{k}} \left(\frac{\mathsf{n}-\mathsf{k}-1}{\mathsf{k}+1} d\mathsf{k} + \frac{\mathsf{n}-\mathsf{k}+1}{2(-\mathsf{n}-1)} d\mathsf{n} + \frac{\mathsf{n}-\mathsf{k}+1}{2(-\mathsf{n}-1)} d\mathsf{k} \right) \tag{40}$$

$$= \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}} 2^{-\mathsf{n}-\mathsf{k}} \left(\frac{2\mathsf{n}^2+\mathsf{k}^2-2\mathsf{n}\mathsf{k}-\mathsf{n}-2\mathsf{k}-3}{2(\mathsf{n}+1)(\mathsf{k}+1)} d\mathsf{k} + \frac{\mathsf{n}-\mathsf{k}+1}{2(-\mathsf{n}-1)} d\mathsf{n} \right) \tag{41}$$

Since computation reveals that ω' is not closed, our substitution has not preserved closedness. Thus we learn that difference forms require a different method for substituting closedness-preservingly.

A method based on cheating. Note that $\omega = d(\phi(\mathsf{n},\mathsf{k}))$ where

$$\phi(\mathsf{n},\mathsf{k}) = \binom{\mathsf{n}}{\mathsf{k}} 2^{-\mathsf{n}}.$$

All we need to do is to apply $\mathsf{n} \to \mathsf{n}+\mathsf{k}$ to the *potential term* $\phi(\mathsf{n},\mathsf{k})$ of ω instead of applying it directly to ω getting

$$\phi'(\mathsf{n},\mathsf{k}) = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}} 2^{-\mathsf{n}-\mathsf{k}}$$

and choose $\omega' := d(\phi'(\mathsf{n},\mathsf{k}))$. Calculation yields

$$\omega' = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}} 2^{-\mathsf{n}-\mathsf{k}} \left(\frac{\mathsf{n}-\mathsf{k}-1}{2(\mathsf{k}+1)} d\mathsf{k} + \frac{\mathsf{k}-\mathsf{n}-1}{2(\mathsf{n}+1)} d\mathsf{n} \right)$$

As ω' is exact by its definition, it is closed. Summarizing, we reached ω' by a detour via $\phi(\mathsf{n},\mathsf{k})$ and $\phi'(\mathsf{n},\mathsf{k})$:

$$\begin{tikzcd}
\phi(\mathsf{n},\mathsf{k}) \arrow[r, "\text{substitute}"] \arrow[d, "d"'] & \phi'(\mathsf{n},\mathsf{k}) \arrow[d, "d"] \\
\omega \arrow[r] & \omega'
\end{tikzcd}$$

Unfortunately, our method depends on having a hypergeometric potential term of ω. Thus it works for trivial forms only. [3] But we should not give up too early, we just need an additional trick.

A general method. Consider Example 1 of [WZ90]:

$$\omega = f(\mathsf{n},\mathsf{k})d\mathsf{k} + g(\mathsf{n},\mathsf{k})d\mathsf{n}$$

where

$$\begin{aligned} f(\mathsf{n},\mathsf{k}) &= \binom{\mathsf{n}}{\mathsf{k}} 2^{-\mathsf{n}} \\ g(\mathsf{n},\mathsf{k}) &= \binom{\mathsf{n}}{\mathsf{k}} 2^{-\mathsf{n}-1} \frac{\mathsf{k}}{\mathsf{k}-\mathsf{n}-1} \end{aligned}$$

Since Gosper's algorithm [Gos78] shows that there is no hypergeometric term $\phi(\mathsf{n},\mathsf{k})$ satisfying $d(\phi(\mathsf{n},\mathsf{k})) = \omega$ we cannot cheat any more. A discrete counterpart of Poincaré's Lemma (which we do not prove) assures us that there is some term $\phi(\mathsf{n},\mathsf{k})$ satisfying $d(\phi(\mathsf{n},\mathsf{k})) = \omega$. Of course, $\phi(\mathsf{n},\mathsf{k})$ might well fail to be hypergeometric; it seems that we loose in that case. A simple trick rescues us. Assume that $\omega = d(\phi(\mathsf{n},\mathsf{k}))$, this is, $\omega = f(\mathsf{n},\mathsf{k})d\mathsf{k} + g(\mathsf{n},\mathsf{k})d\mathsf{n}$ where

$$\begin{aligned} f(\mathsf{n},\mathsf{k}) &= \phi(\mathsf{n},\mathsf{k}+1) - \phi(\mathsf{n},\mathsf{k}), \\ g(\mathsf{n},\mathsf{k}) &= \phi(\mathsf{n}+1,\mathsf{k}) - \phi(\mathsf{n},\mathsf{k}). \end{aligned}$$

We define $\phi'(\mathsf{n},\mathsf{k}) := \phi(\mathsf{n}+\mathsf{k},\mathsf{k})$ and $\omega' := d(\phi'(\mathsf{n},\mathsf{k}))$ in order to imitate the substitution $\mathsf{n} \to \mathsf{n}+\mathsf{k}$ somehow. Straightforward computation gives

$$\omega' = f'(\mathsf{n},\mathsf{k})d\mathsf{k} + g'(\mathsf{n},\mathsf{k})d\mathsf{n}$$

where

$$\begin{aligned} f'(\mathsf{n},\mathsf{k}) &= \Delta_{\mathsf{k}}\phi(\mathsf{n}+\mathsf{k},\mathsf{k}) \\ &= \phi(\mathsf{n}+\mathsf{k}+1,\mathsf{k}+1) - \phi(\mathsf{n}+\mathsf{k},\mathsf{k}), \\ g'(\mathsf{n},\mathsf{k}) &= \Delta_{\mathsf{n}}\phi(\mathsf{n}+\mathsf{k},\mathsf{k}) \\ &= \phi(\mathsf{n}+\mathsf{k}+1,\mathsf{k}) - \phi(\mathsf{n}+\mathsf{k},\mathsf{k}). \end{aligned}$$

Next we simply eliminate all occurrences of the unknown potential function ϕ by expressing differences of ϕ by f and g only:

$$f'(\mathsf{n},\mathsf{k}) = \underbrace{\phi(\mathsf{n}+\mathsf{k}+1,\mathsf{k}+1) - \phi(\mathsf{n}+\mathsf{k}+1,\mathsf{k})}_{f(\mathsf{n}+\mathsf{k}+1,\mathsf{k})} + \underbrace{\phi(\mathsf{n}+\mathsf{k}+1,\mathsf{k}) - \phi(\mathsf{n}+\mathsf{k},\mathsf{k})}_{g(\mathsf{n}+\mathsf{k},\mathsf{k})}$$

$$g'(\mathsf{n},\mathsf{k}) = g(\mathsf{n}+\mathsf{k},\mathsf{k}).$$

Note that we don't need to know $\phi(\mathsf{n},\mathsf{k})$ *any more!* In a nutshell, our trick is to *pretend* to know the potential function ϕ.

[3] [Zei93] calls a form *trivial* iff there is a hypergeometric term $\phi(\mathsf{n},\mathsf{k})$ satisfying $d(\phi(\mathsf{n},\mathsf{k})) = \omega$.

Proposition 31. *If*

$$f(\mathsf{n},\mathsf{k})d\mathsf{k} + g(\mathsf{n},\mathsf{k})d\mathsf{n}$$

is closed, then

$$(f(\mathsf{n}+\mathsf{k}+1,\mathsf{k}) + g(\mathsf{n}+\mathsf{k},\mathsf{k}))d\mathsf{k} + g(\mathsf{n}+\mathsf{k},\mathsf{k})d\mathsf{n}$$

is closed, too.

Proof. The proof is an easy calculation (that does not use the Lemma of Poincaré). Lemma. ☐

Let's go back to Example 1 of [WZ90],

$$\omega = \binom{\mathsf{n}}{\mathsf{k}} 2^{-\mathsf{n}} d\mathsf{k} + \binom{\mathsf{n}}{\mathsf{k}} 2^{-\mathsf{n}-1} \frac{\mathsf{k}}{\mathsf{k}-\mathsf{n}-1} d\mathsf{n}.$$

Proposition 31 gives

$$\omega' = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}} 2^{-\mathsf{n}-\mathsf{k}} \left(\frac{1}{2} d\mathsf{k} - \frac{\mathsf{k}}{2(\mathsf{n}+1)} d\mathsf{n} \right)$$

which is closed indeed. To get rid of ugly rational factors ω' we try some shifts on it, and S_{n}^{-1} succeeds in the sense that $\omega'' := S_{\mathsf{n}}^{-1}\omega'$ looks nice:

$$\omega'' = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{n}\ ,\ \mathsf{k}} 2^{-\mathsf{n}-\mathsf{k}} \left(\frac{\mathsf{n}}{\mathsf{n}+\mathsf{k}} d\mathsf{k} - \frac{\mathsf{k}}{\mathsf{n}+\mathsf{k}} d\mathsf{n} \right).$$

Grasping the pattern in ω'' allows us to find an infinite sequence of closed forms; see page 53.

We close with two remarks on Proposition 31.

Remark: Naive substitution of $\mathsf{n} \to \mathsf{n}+\mathsf{k}$ into $f(\mathsf{n},\mathsf{k})d\mathsf{k} + g(\mathsf{n},\mathsf{k})d\mathsf{n}$ yields

$$\begin{aligned} \omega' &= f(\mathsf{n}+\mathsf{k},\mathsf{k})d\mathsf{k} + g(\mathsf{n}+\mathsf{k},\mathsf{k})d(\mathsf{n}+\mathsf{k}) \\ &= (f(\mathsf{n}+\mathsf{k},\mathsf{k}) + g(\mathsf{n}+\mathsf{k},\mathsf{k}))\, d\mathsf{k} + g(\mathsf{n}+\mathsf{k},\mathsf{k}) \end{aligned}$$

which differs from the form in Proposition 31 just by a shift. ☐

Herb Wilf [Wil99] obtains the following Proposition by iterating the transformation of Proposition 31. To obtain it directly via potential functions, we use $\phi'(\mathsf{n},\mathsf{k}) = \phi(\mathsf{n}+rk,\mathsf{k})$.

Proposition 32. *Let* $f(\mathsf{n},\mathsf{k})d\mathsf{k} + g(\mathsf{n},\mathsf{k})d\mathsf{n}$ *be closed and let* r *be a natural number. Define*

$$\begin{aligned} f'(\mathsf{n},\mathsf{k}) &:= f(\mathsf{n}+rk+r,\mathsf{k}) + \sum_{0\le j<r} g(\mathsf{n}+rk+j,\mathsf{k}), \\ g'(\mathsf{n},\mathsf{k}) &:= g(\mathsf{n}+rk,\mathsf{k}). \end{aligned}$$

Then $f'(\mathsf{n},\mathsf{k})d\mathsf{k} + g'(\mathsf{n},\mathsf{k})d\mathsf{n}$ *is closed.*

A transformation of Ira Gessel can be obtained via $\phi'(\mathsf{n},\mathsf{k}) := \phi(-\mathsf{n},\mathsf{k})$.

Proposition 33 (Theorem 3.1 (iv) of [Ges95]). *If*

$$f(\mathsf{n},\mathsf{k})d\mathsf{k} + g(\mathsf{n},\mathsf{k})d\mathsf{n}$$

is closed, then

$$f(-\mathsf{n},\mathsf{k})d\mathsf{k} - g(-\mathsf{n}-1,\mathsf{k})d\mathsf{n}$$

is closed.

Similarly, $\phi'(\mathsf{n},\mathsf{k}) := -\phi(\mathsf{n},-\mathsf{k}+1)$ yields:

Proposition 34 ([Ges95], Theorem 3.1 (v)). *If*

$$f(\mathsf{n},\mathsf{k})d\mathsf{k} + g(\mathsf{n},\mathsf{k})d\mathsf{n}$$

is closed, then

$$f(\mathsf{n},-\mathsf{k})d\mathsf{k} - g(\mathsf{n},-\mathsf{k}+1)d\mathsf{n}$$

is closed.

A transformation from *Rational Functions Certify Combinatorial Identities* [WZ90] can be found using $\phi'(\mathsf{n},\mathsf{k}) := -\phi(-\mathsf{k},-\mathsf{n})$.

Proposition 35 ([WZ90], part of Theorem B). *If*

$$f(\mathsf{n},\mathsf{k})d\mathsf{k} + g(\mathsf{n},\mathsf{k})d\mathsf{n}$$

is closed, then

$$g(-\mathsf{k}-1,\mathsf{n})d\mathsf{k} + f(-\mathsf{k},-\mathsf{n}-1)d\mathsf{n}$$

is closed.

A transformation of Tewodros Amdeberhan can be obtained via $\phi'(\mathsf{n},\mathsf{k}) := \phi(sn,\mathsf{k})$.

Proposition 36 ([AZ97]). *Let s be a positive integer and let*

$$f(\mathsf{n},\mathsf{k})d\mathsf{k} + g(\mathsf{n},\mathsf{k})d\mathsf{n}$$

be a closed form. Then

$$f(sn,\mathsf{k})d\mathsf{k} + \sum_{0\le i<s} g(sn+i,\mathsf{k})d\mathsf{n}$$

is closed.

4.2 WZ 1-Forms Yield New WZ 1-Forms

The method of substituting in potential functions extends to 1-forms in an arbitrary number of variables and any integer linear substitutions in a straightforward way.

As a first application, we show that closedness preserving substitution in 1-forms partially explain the *dualize and specialize miracle* [Zei95]. Consider the Vandermonde identity

$$\sum_k \binom{a}{k}\binom{n}{k} = \binom{a+n}{a} \quad \text{for } a \ge 0 \text{ and } n \ge 0. \tag{42}$$

Its associate identities are just other instances of the Vandermonde identity. However, its special case

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2\,n}{n} \quad \text{for } n \geq 0 \tag{43}$$

yields, as an associate identity,

$$\sum_{n=0}^{k} (3\,n - 2\,k) \binom{k}{n}^2 \binom{2\,n}{n} = 0 \quad \text{for } k \geq 0, \tag{44}$$

or, equivalently,

$$ {}_4F_3\left[\begin{matrix} -k,\ -k,\ 1 - \frac{2\,k}{3},\ \frac{1}{2} \\ 1,\ \frac{-2\,k}{3},\ 1 \end{matrix};4\right] = 0 \quad \text{for } k \geq 0, \tag{45}$$

which first appears in [Zei95]:

> This is *a brand new identity*, unknown to Askey. It has a q-analog derived from the q-version of WZ, that was unknown to Andrews, and even whose limiting case was brand new, and it took George Andrews three densely packed pages, using five different identities, to prove.

Let's look at this from the point of view of closedness preserving substitutions. To prove identity 42, we could use the form

$$\binom{\mathsf{a}}{\mathsf{k}} \binom{\mathsf{n}}{\mathsf{k}} \binom{\mathsf{a}+\mathsf{n}}{\mathsf{a}}^{-1} \left(1\, d\mathsf{k} + \frac{\mathsf{k}^2}{(\mathsf{n}-\mathsf{k}+1)\ (\mathsf{a}+\mathsf{n}+1)}\, d\mathsf{n} \right) \tag{46}$$

which is closed with respect to $\mathsf{L} = \{\mathsf{k}, \mathsf{n}\}$. However, let us use the form

$$\binom{\mathsf{a}}{\mathsf{k}} \binom{\mathsf{n}}{\mathsf{k}} \binom{\mathsf{a}+\mathsf{n}}{\mathsf{a}}^{-1} \times \left(1\, d\mathsf{k} + \frac{\mathsf{k}^2}{(\mathsf{n}-\mathsf{k}+1)\ (\mathsf{a}+\mathsf{n}+1)}\, d\mathsf{n} + \frac{\mathsf{k}^2}{(\mathsf{a}-\mathsf{k}+1)\ (\mathsf{a}+\mathsf{n}+1)}\, d\mathsf{a} \right)$$

– which is closed with respect to $\mathsf{L} = \{\mathsf{k}, \mathsf{n}, \mathsf{a}\}$ – instead. (It is a remarkable fact that it is usually possible to extend hypergeometric WZ 1-forms in two variables to hypergeometric WZ 1-forms in more than two variables. In our example this is obvious since

$$\binom{\mathsf{a}}{\mathsf{k}} \binom{\mathsf{n}}{\mathsf{k}} \binom{\mathsf{a}+\mathsf{n}}{\mathsf{a}}^{-1}$$

is symmetric under exchanging a and n.) Closedness preserving substitution $\{\mathsf{a} \to \mathsf{n}\}^*$ yields

$$\{\mathsf{a} \to \mathsf{n}\}^*\omega = \binom{\mathsf{n}}{\mathsf{k}}^2 \binom{2\,\mathsf{n}}{\mathsf{n}}^{-1} \left(1\, d\mathsf{k} + \frac{\mathsf{k}^2\ (2\,\mathsf{k} - 3\,\mathsf{n} - 3)}{2\,(\mathsf{n}-\mathsf{k}+1)^2\ (2\,\mathsf{n}+1)}\, d\mathsf{n} \right)$$

and a shadow of this form proves identity 44. It remains to disclose what is going on in the closedness preserving substitution $\{\mathsf{a} \to \mathsf{n}\}^*$.

Proposition 37. *Assume that*

$$f(\mathsf{k},\mathsf{n},\mathsf{a})\,d\mathsf{k} + g(\mathsf{k},\mathsf{n},\mathsf{a})\,d\mathsf{n} + h(\mathsf{k},\mathsf{n},\mathsf{a})\,d\mathsf{a}$$

is closed with respect to $\mathsf{L} = \{\mathsf{k},\mathsf{n},\mathsf{a}\}$*. Then*

$$f(\mathsf{k},\mathsf{n},\mathsf{n})\,d\mathsf{k} + (g(\mathsf{k},\mathsf{n},\mathsf{n}+1) + h(\mathsf{k},\mathsf{n},\mathsf{n}))\,d\mathsf{a}$$

is closed with respect to $\mathsf{L} = \{\mathsf{k},\mathsf{n}\}$.

4.3 Transforming Forms of Arbitrary Degree

Unfortunately, the simple substitution trick (as described in [Ges99] or section 4.1) does not generalize to forms of higher degree. Therefore we needed to develop a completely different method; it is presented in the following subsections.

Note that substitution is more important in the domain of higher degree forms than in the domain of forms of degree 1. This is due to a lack of known nontrivial higher degree forms. In fact, as far as I know, only a single higher degree forms has been known so far

> To find higher degree forms, we start with a well known multisum closed form identity, and we would need a multivariate analog of Gosper's algorithm. I am presently developing such an algorithm, but until I succeed, all I can present is the r-form arising out of the multinomial identity ... which produces ....

To find other higher degree forms, we use two methods:

1. We use Kurt Wegschaider's algorithm [Weg97] as shown in section 3.5.
2. We transform known closed difference forms to new ones. The transformation algorithm developed in this section is implemented in our Mathematica package `wz.m`. Using this package, transformation theorems can be produced by pressing a few keys.

4.4 Substitutions

A *substitution* σ assigns a term to each label in L. If the term t is assigned to the label x (this is, if $\sigma(x) = t$), then we say that t *is substituted for* x. The set of all substitutions is $\mathsf{L} \to \mathsf{T}$. We adopt special notation for substitutions: $\{(x_1, t_1), \ldots, (x_n, t_n)\}$ is written $\{x_1 \to t_1, \ldots, x_n \to t_n\}$. Furthermore, $x \to \underline{x}$ may be dropped. Thus the substitution $\{(\mathsf{n}, \mathsf{n}+\mathsf{k}), (\mathsf{k}, \mathsf{k})\}$ can be written $\{\mathsf{n} \to \mathsf{n}+\mathsf{k}\}$.

Applying substitutions to terms. An example of an *application of a substitution to a term* is

$$\underbrace{\{\mathsf{n} \to \widehat{\mathsf{n}+\mathsf{k},}\ \mathsf{k} \to \mathsf{k}\}}_{\text{substitution}} \underbrace{\binom{\mathsf{n}}{\mathsf{k}} 2^{-\mathsf{n}}}_{\text{term}} = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}} 2^{-\mathsf{n}-\mathsf{k}}.$$

If we had introduced terms syntactically, we could easily define the application of a substitution σ to a term t by turning Proposition 38 on page 45 into a definition; it would be the obvious recursive definition of substitution that is used in most functional and logic programming languages, as for example in Prolog, Haskell and Mathematica. For simplicity (to hide unimportant detail) we have not defined terms syntactically; for us, a term is a function in $\mathsf{V} \to \mathbb{C}$. Still, substitutions following Proposition 38 are import to us: We have used them already when computing shifts – the operator S_{n} is nothing but the substitution $\{\mathsf{n} \to \mathsf{n}{+}1\}$ – and we will use them heavily in the method of closedness preserving substitutions. Thus we want to define substitutions. Of course, substitutions should obey Proposition 38. Indeed, knowing Proposition 38, together with Proposition 39, is all one needs to know about substitutions; the technical but trivial rest of this section may be skipped at no risk.

Let us avoid parentheses by two conventions in this section. First, $f\,x := f(x)$ denotes the application of f to x. Second, function application associates to the left: $F\,f\,x$ denotes $(F\,f)\,x$ which would be usually written $(F(f))(x)$. The so-called λ-notation for functions turns out to be handy in this section. Using λ-notation, $\forall x : \mathsf{increment}(x) = x+1$ can be written $\mathsf{increment} = \lambda x.\, x+1$, and $\forall x : \mathsf{square}(x) = x^2$ can be written $\mathsf{square} = \lambda a.\, a^2$ – the name of the variable "bound by λ" does not matter. λ-notation allows us to use functions in intermediate steps of proofs without giving names to these functions. For example, we might calculate $(\lambda x.\, x-1)\,4 = 3$ and $(\lambda y.\, y^3)\,4 = 64$.

Def. 26. *Let σ be a substitution and t be a term. The* application of the substitution σ to the term t *is denoted by $\hat{\sigma}\,t$ and defined by*

$$\hat{\sigma}\,t\,p := t\,(\lambda x.\,\sigma\,x\,p).$$

Example: Let $\sigma = \{\mathsf{n} \to \mathsf{n}+\mathsf{k},\ \mathsf{k} \to \mathsf{k}\}$ and $t = \binom{\mathsf{n}}{\mathsf{k}}2^{-\mathsf{n}}$. Let us check if Definition 26 gives the expected result $\hat{\sigma}\,t = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}}2^{-\mathsf{n}-\mathsf{k}}$. Let n and k be arbitrary but fixed integers and let $p = \{(\mathsf{n}, n), (\mathsf{k}, k)\}$. We want to see if indeed $\hat{\sigma}\,t\,p = \binom{n+k}{k}2^{-n-k}$.

Note that $\sigma\,\mathsf{n}\,p = n+k$ and $\sigma\,\mathsf{k}\,p = k$. Taken together, these equations show that $\lambda x.\,\sigma\,x\,p$ is the function $\{(\mathsf{n}, n+k),\ (\mathsf{k}, k)\}$. By definition of $\hat{\sigma}\,t$ we thus have $\hat{\sigma}\,t\,p = t\,(\lambda x.\,\sigma\,x\,p) = \binom{n+k}{k}2^{-n}$. □

The result $\hat{\sigma}\,t = \binom{\mathsf{n}+\mathsf{k}}{\mathsf{k}}2^{-\mathsf{n}-\mathsf{k}}$ could have been obtained immediately by the following proposition; it shows the close analogy of our notion of term to syntactical terms and justifies the our use of the word "substitution".

Proposition 38. *Let $c \in \mathbb{C}$, $x \in \mathsf{L}$, $f \in \mathbb{C}^n \to \mathbb{C}$ and $\sigma \in \mathsf{L} \to \mathsf{T}$. Then*

1. $\hat{\sigma}\,\underline{c} = \underline{c}$.
2. $\hat{\sigma}\,\underline{x} = \underline{\sigma(x)}$.
3. $\hat{\sigma}\,(\underline{f}(t_1, \ldots, t_n)) = \underline{f}(\hat{\sigma}\,t_1, \ldots, \hat{\sigma}\,t_n)$.

Proof. We prove (1). Let p be an arbitrary point. Then $\hat{\sigma}\,\underline{c}\,p = \underline{c}\,(\lambda x.\,\sigma\,x\,p) = \underline{c}\,p$.

We prove (2). Let p be an arbitrary point. Then $\hat{\sigma}\,\underline{x}\,p = \underline{x}\,(\lambda z.\,\sigma\,z\,p) = (\lambda z.\,\sigma\,z\,p)\,(x) = \sigma\,x\,p = \underline{\sigma(x)}\,p$.

We prove (3). Let p be an arbitrary point. Then $\hat{\sigma}\,(\underline{f}(t_1, \ldots, t_n))\,p$
$= \underline{f}(t_1, \ldots, t_n)\,(\lambda x.\,\sigma\,x\,p)$
$= f(t_1\,(\lambda x.\,\sigma\,x\,p), \ldots, t_n\,(\lambda x.\,\sigma\,x\,p))$
$= f(\hat{\sigma}\,t_1\,p, \ldots, \hat{\sigma}\,t_n\,p)$
$= \underline{f}(\hat{\sigma}\,t_1, \ldots, \hat{\sigma}\,t_n)\,p$. □

Def. 27. *A term t is* integer linear *iff it is integer-valued and additive:*

1. $t \in \mathsf{V} \to \mathbb{Z}$.
2. $\underset{p,q \in \mathsf{V}}{\forall}\ t(p+q) = t(p) + t(q)$.

For example, $2\mathsf{n} - 3\mathsf{k}$ is integer linear. The term $\frac{1}{2}\mathsf{n} - 3\mathsf{k}$ is not integer linear since it is not integer-valued. The term $2\mathsf{n} - 3\mathsf{k} + 1$ is not integer linear since it is not additive.

Def. 28. *A substitution σ is* integer linear *iff $\sigma(x)$ is an integer-linear term for each label x. In this case we define σ_{xy}, the* coefficient of y in $\sigma(x)$ *by* $\sigma_{xy} := \sigma\, x\, e_y$.

For example, $\sigma = \{\mathsf{n} \to 2\mathsf{n}+3\mathsf{k},\ \mathsf{k} \to 4\mathsf{n}+5\mathsf{k}\}$ is an integer linear substitution and $\sigma_{\mathsf{nk}} = 3$.

One more convention: $\prod_x f_x$ denotes the composition of the functions f_{x_1}, $\dots$, f_{x_n} where $\{x_1, \dots, x_n\} = \mathsf{L}$ and $x_1 \prec \cdots \prec x_n$. It does *not* denote a product. Shifts can be "moved to the right of substitutions" as follows:

Proposition 39. *Let $y \in \mathsf{L}$ and let σ be an arbitrary integer linear substitution in $\mathsf{L} \to \mathsf{T}$. Then*

$$S_y \circ \hat{\sigma} = \hat{\sigma} \circ \prod_x S_x^{\sigma_{xy}}. \tag{47}$$

For example,

$$S_{\mathsf{k}} \circ (\{\mathsf{n} \to 2\mathsf{n}+3\mathsf{k},\ \mathsf{k} \to 4\mathsf{n}+5\mathsf{k}\})\hat{} = S_{\mathsf{k}} \circ (\{\mathsf{n} \to 2\mathsf{n}+3\mathsf{k},\ \mathsf{k} \to 4\mathsf{n}+5\mathsf{k}\})\hat{} \circ S_{\mathsf{n}}^3 \circ S_{\mathsf{k}}^5$$

as can be checked by applying both operators to the term $f(\mathsf{n}, \mathsf{k})$ – the result is $f(2\mathsf{n}+3\mathsf{k}+3,\ \mathsf{k} \to 4\mathsf{n}+5\mathsf{k}+5)$ either way. Of course, Proposition 39 is a triviality. With an eye towards automatic proof-checking, we prove it anyway. To this end, we need a lemma:

Lemma 3. *Let d be a vector. Then $t(p+d) = (\prod_x S_x^{d(x)} t)(p)$.*

Proof. Expand $d = \sum_x d(x) e_x$ and use $t(p + m e_x) = S_x^m t\, p$. □

Proof of Proposition 39. Let t be an arbitrary but fixed term and p be an arbitrary but fixed vector. We have to show

$$S_y \circ \hat{\sigma}\, t\, p \;=\; \hat{\sigma} \circ \prod_x S_x^{\sigma_{xy}}\, t\, p.$$

We transform the left hand side to the right hand side. By the definition of the shift, the left hand side is equal to $\hat{\sigma}\, t\, (p + e_y)$. By the definition of substitution application, this equals $t(\lambda x.\, \sigma\, x\, (p + e_y))$; from this point on, λ-notation comes in handy. The term $\sigma\, x$ is additive since σ is assumed to be integer linear. We thus obtain $t(\lambda x.\, (\sigma\, x\, p) + (\sigma\, x\, e_y))$. By pointwise vector addition, used "backwards", this equals $t((\lambda x.\, \sigma\, x\, p) + (\lambda x.\, \sigma\, x\, e_y))$. By Lemma 3, this equals $(\prod_x S_x^{\sigma\, x\, e_y})\, t\, (\lambda x.\, \sigma\, x\, p)$. Using the definition of substitution application backwards, this equals $(\hat{\sigma} \circ \prod_x S_x^{\sigma\, x\, e_y})\, t\, p$ which can be written as $(\hat{\sigma} \circ \prod_x S_x^{\sigma_{xy}})\, t\, p$ by definition of σ_{xy}. □

We do not use the convention that $f\,x = f(x)$ any more. In the following pages, fg will denote the composition of f and g most of the times: $(fg)(x) = f(g(x))$. Furthermore we abbreviate $\hat{\sigma}$ by σ; context resolves ambiguities. Thus Proposition 39 will be written

$$S_y\sigma = \sigma\prod_x S_x^{\sigma_{x\,y}}.$$

4.5 Guessing σ^*, Part 1

How to transform a closed difference form ω to a new closed form ω'? Let us look at the continuous counterpart – differential forms – for inspiration. Put loosely, a function f induces a *pullback* f^* that preserves closedness: $d\omega = 0$ implies $df^*\omega = 0$. Closedness preservation is implied by the following properties of the pullback:

1. $df^*\omega = f^*d\omega$,
2. $f^*0 = 0$,

which can be proved as follows: Assuming $d\omega = 0$ and $df^*\omega = f^*d\omega$ and $f^*0 = 0$ we have to show that $df^*\omega = 0$. And indeed, $df^*\omega = f^*d\omega = f^*0 = 0$.

We return to difference forms. The function f corresponds to an integer linear substitution σ in $\mathsf{T} \to \mathsf{T}$; the restriction to integer linearity ensures that σ preserves hypergeometricity. For each substitution σ we aim to construct a nontrivial operator σ^* in $\mathsf{F} \to \mathsf{F}$ such that the following three properties hold:

1. $d\sigma^*\omega = \sigma^*d\omega$,
2. $\sigma^*0 = 0$,
3. $\sigma^*(\omega_1 + \omega_2) = \sigma^*\omega_1 + \sigma^*\omega_2$

By the argument given above, σ^* will be a *closedness preserving substitution*:

$$d\omega = 0 \implies d\sigma^*\omega = 0. \tag{48}$$

Let us try to find a definition for the function $*$ that satisfies $d\sigma^*\omega = \sigma^*d\omega$ by motivated guessing. We first look at Equation $d\sigma^*\omega = \sigma^*d\omega$ in the special case where ω is a 0-form – this is, a term – T.

$$d\sigma^*T = \sigma^*dT \tag{49}$$

Unfortunately we cannot compute either side of Equation 49 since both sides involve the function $*$ whose definition is still unknown to us. However, since 0-forms are just terms, we may reasonably define

$$\sigma^*T := \sigma T \tag{50}$$

for all forms T of degree 0. Thus Equation 49 reduces to

$$d\sigma T = \sigma^*dT \tag{51}$$

whose left hand side does not involve any undefined function.

Our plan is to expand both sides of Equation 51 using the definition of d and to read off a suitable definition of σ^* by "comparing coefficients".

Clearly, the right hand side of Equation 51 equals

$$\sigma^* \sum_x dx \Delta_x T \tag{52}$$

which can be transformed to

$$\sum_x \sigma^* dx \Delta_x T \tag{53}$$

since σ^* is additive.

We turn to the left hand side of Equation 51. By definition of d, it equals

$$\sum_y dy \Delta_y \sigma T. \tag{54}$$

"Comparing coefficients" between 54 and 53 is hindered by the operator Δ which appears on the left of σ in 54 but on the right of σ^* in 53. To make Equation 54 more similar to Equation 53 we try to move the operator Δ to the right of σ in 54; we aim to express $\Delta_y \sigma$ as $\sum_x \sigma A_x \Delta_x$ for suitable operators A_x.

4.6 Differences and Substitutions

Lemma 39 on page 46 shows us how to move a shift to the right of a substitution:

$$S_y \sigma = \sigma \prod_x S_x^{\sigma_{xy}}. \tag{55}$$

We want to move a difference to the right of a substitution. Subtracting $I\sigma$ from both sides of 47 yields

$$\Delta_y \sigma = \sigma(-I + \prod_x S_x^{\sigma_{xy}}). \tag{56}$$

By telescoping according to the pattern

$$
\begin{aligned}
& -I + S_{x_1}^{A_{x_1 y}} S_{x_2}^{A_{x_2 y}} \dots S_{x_n}^{A_{x_n y}} && (57)\\
= & -I + S_{x_1}^{A_{x_1 y}} && (58)\\
& - S_{x_1}^{A_{x_1 y}} + S_{x_1}^{A_{x_1 y}} S_{x_2}^{A_{x_2 y}} && (59)\\
& - S_{x_1}^{A_{x_1 y}} S_{x_2}^{A_{x_2 y}} + S_{x_1}^{A_{x_1 y}} S_{x_2}^{A_{x_2 y}} S_{x_3}^{A_{x_3 y}} && (60)\\
& \dots && (61)\\
& - S_{x_1}^{A_{x_1 y}} S_{x_2}^{A_{x_2 y}} \dots S_{x_{n-1}}^{A_{x_{n-1} y}} + S_{x_1}^{A_{x_1 y}} S_{x_2}^{A_{x_2 y}} \dots S_{x_n}^{A_{x_n y}} && (62)
\end{aligned}
$$

we express $-I + \prod_z S_z^{\sigma_{zy}}$ in terms of "long differences" $-I + S_x^{\sigma_{xy}}$ as follows:

$$-I + \prod_z S_z^{\sigma_{zy}} = \sum_x \left(- \prod_{z \prec x} S_z^{\sigma_{zy}} + \prod_{z \preceq x} S_z^{\sigma_{zy}} \right) = \sum_x \left(\prod_{z \prec x} S_z^{\sigma_{zy}} \right) (-I + S_x^{\sigma_{xy}})$$

By telescoping again, we can reduce "long differences" to differences using

$$-I + S_x^{\sigma_{xy}} = G_x^{\sigma_{xy}} \Delta_x. \tag{63}$$

where the operator G_x^k is defined by:

Def. 29. *The* geometric shift polynomial G_x^k *is defined by*

$$G_x^k = \begin{cases} I + S_x + \cdots + S_x^{k-1} & \textit{if} \quad k > 0 \\ 0 & \textit{if} \quad k = 0 \\ -I - S_x^- 1 - \cdots - S_x^{-k+1} & \textit{if} \quad k < 0 \end{cases}$$

In this subsection we have proved:

Proposition 40. *Let $y \in \mathsf{L}$ and let σ be an arbitrary integer linear substitution in $\mathsf{T} \to \mathsf{T}$. Then*

$$\Delta_y \sigma = \sigma \sum_x \left(\prod_{z \prec x} S_z^{\sigma_{zy}} \right) G_x^{\sigma_{xy}} \Delta_x.$$

4.7 Guessing σ^*, Part 2

Plugging Proposition 40 into Equation 54 on page 48 yields

$$d\sigma T = \sum_y dy \sigma \sum_x \left(\prod_{z \prec x} S_z^{\sigma_{zy}} \right) G_x^{\sigma_{xy}} \Delta_x T \tag{64}$$

which by additivity of σ equals

$$d\sigma T = \sum_x \sum_y dy \sigma \left(\prod_{z \prec x} S_z^{\sigma_{zy}} \right) G_x^{\sigma_{xy}} \Delta_x T. \tag{65}$$

Thus our goal

$$d\sigma T = \sigma^* dT \tag{66}$$

can be restated as

$$\sum_x \sum_y dy \sigma \left(\prod_{z \prec x} S_z^{\sigma_{zy}} \right) G_x^{\sigma_{xy}} \Delta_x T = \sum_x \sigma^* dx \Delta_x T. \tag{67}$$

As a naive first attempt at obtaining equality we try to make both sums equal by equating corresponding summands:

$$\sum_y dy \sigma \left(\prod_{z \prec x} S_z^{\sigma_{zy}} \right) G_x^{\sigma_{xy}} \Delta_x T = \sigma^* dx \Delta_x T. \tag{68}$$

Next we replace $\Delta_x T$ by T'. Note that this requires additional faith since x has occurrences outside Δ_x too.

$$\sum_y dy \sigma \left(\prod_{z \prec x} S_z^{\sigma_{zy}} \right) G_x^{\sigma_{xy}} T' = \sigma^* dx T'. \tag{69}$$

Read from right to left this defines σ^* on all monomial 1-forms

$$\sigma^* dx T := \sum_y dy \sigma \left(\prod_{z \prec x} S_z^{\sigma_{zy}} \right) G_x^{\sigma_{xy}} T, \tag{70}$$

and, by additivity of σ^*, on all 1-forms.

To define σ^* on forms of arbitrary degree we need one more guess (fortunately our last). We extend Equation 70 to the recursion

$$\sigma^* dx\,\omega := \sum_y dy \sigma^* \sum_x \left(\prod_{z \prec x} S_z^{\sigma_{zy}} \right) G_x^{\sigma_{xy}} \omega \tag{71}$$

(Note that σ^* appears on the right hand side of Equation 71).

4.8 Closedness Preserving Substitutions

Equation 71 leads us define:

Def. 30. *Let σ be an integer linear substitution and $x, y \in \mathsf{L}$. Then the shift polynomial $P_{\sigma xy}$ in $\mathsf{F} \to \mathsf{F}$ is defined by*

$$P_{\sigma xy} := G_x^{\sigma_{xy}} \prod_{z \prec x} S_z^{\sigma_{zy}}.$$

Proposition 41. *The following commutation relations hold:*

1. $P_{\sigma x_1 y_1} P_{\sigma x_2 y_2} = P_{\sigma x_2 y_2} P_{\sigma x_1 y_1}$.
2. $P_{\sigma xy} dz = P_{\sigma xy} dz$.

Proof. $P_{\sigma xy}$ is a shift polynomial. $\square$

Proposition 42. *Let σ be an integer linear substitution. Then there is exactly one function σ^* in $\mathsf{F} \to \mathsf{F}$ satisfying*

1. $\sigma^*(\omega_1 + \omega_2) = \sigma^* \omega_1 + \sigma^* \omega_2$.
2. $\sigma^* dx\omega = \sum_y dy \sigma^* P_{\sigma xy} \omega$,
3. $\sigma^* T = \sigma T$,

Proof. Let ω be an arbitrary but fixed form. Since the listed rules allow us to compute $\sigma^*\omega$ in at least one way, there can be at most one such function σ^*.

To show the existence of σ^*, we have to show that the rules listed do not lead to a contradiction. In other words, we have to show that computing $\sigma^*\omega$ in different ways cannot lead to different results. Rules (1) and (3) cannot lead to different results. We show that rule (2) cannot lead to different results either. Since $dx_1 dx_2 = -dx_2 dx_1$, we have to show that $\sigma(dx_1 dx_2) = \sigma(-dx_2 dx_1)$ in order to rule out a contradiction (By checking this transposition, we cover w.l.o.g all permutations). Indeed,

$$\begin{aligned} \sigma^*(dx_1 dx_2 \omega) = \sum_{y_1} dy_1 \sigma^* P_{\sigma x_1 y_1} dx_2 \omega &= \sum_{y_1} dy_1 \sigma^* dx_2 P_{\sigma x_1 y_1} \omega \\ &= \sum_{y_1 y_2} dy_1 dy_2 \sigma^* P_{\sigma x_2 y_2} P_{\sigma x_1 y_1} \omega \end{aligned}$$

agrees with

$$\sigma^*(-dx_2 dx_1)\omega) = -\sum_{y_1 y_2} dy_2 dy_1 \sigma^* P_{\sigma x_1 y_1} P_{\sigma x_2 y_2} \omega$$

where we have used commutation properties of the shift polynomials. $\square$

Proposition 42 allows us to define:

Def. 31. *Let σ be an integer linear substitution. We define the closedness preserving substitution operator σ^* in $\mathsf{F} \to \mathsf{F}$ by*

1. $\sigma^*(\omega_1 + \omega_2) := \sigma^*\omega_1 + \sigma^*\omega_2$,
2. $\sigma^* dx\omega := \sum_y dy\sigma^* P_{\sigma xy}\omega$,
3. $\sigma^* T := \sigma T$.

Remark: Clearly, $\sigma^* dx\omega := \ldots$ is an implicit definition. It would be nice to replace it by an equivalent explicit definition $\sigma^*(\omega)(X) := \ldots$ but we failed to do so. □

Proposition 43 (Lifting Lemma). *Suppose the shift polynomials A and B and the substitution σ satisfy*

$$A\sigma = \sigma B.$$

Then

$$\hat{A}\sigma^* = \sigma^* \hat{B}.$$

Proof. Since both $\hat{A}\sigma^*$ and $\sigma^*\hat{B}$ are additive, it suffices to prove

$$\hat{A}\sigma^*\omega = \sigma^*\hat{B}\omega$$

for any monomial ω. We proceed by induction on the degree of ω.

If the degree of w is zero, then ω is a term and $\hat{A}\sigma^*\omega = \sigma^*\hat{B}\omega$ reduces to $A\sigma = \sigma B$.

If the degree of w is positive, then we can find x and ω' such that $\omega = dx\omega'$. We have to show that

$$\hat{A}\sigma^* dx\omega' = \sigma^*\hat{B} dx\omega'.$$

We compute

$$\hat{A}\sigma^* dx\omega' = \hat{A}\sum_y dy\sigma^* P_{\sigma xy}\omega' = \sum_y dy\hat{A}\sigma^*(P_{\sigma xy}\omega')$$

and

$$\sigma^*\hat{B} dx\omega' = \sigma^* dx\hat{B}\omega' = \sum_y dy\sigma^* P_{\sigma xy}\hat{B}\omega' = \sum_y dy\sigma^*\hat{B}(P_{\sigma xy}\omega')$$

using the definition of σ^* and commutation properties of shift polynomials. Since $P_{\sigma xy}\omega'$ is a monomial form of lesser degree than ω, the induction hypothesis shows that

$$\hat{A}\sigma^*(P_{\sigma xy}\omega') = \sigma^*\hat{B}(P_{\sigma xy}\omega').$$

□

Proposition 43 allows us to lift Proposition 39 to the level of closedness preserving substitutions on forms:

Proposition 44. *Let $y \in \mathsf{L}$ and let σ be an arbitrary integer linear substitution in $\mathsf{T} \to \mathsf{T}$. Then*

$$\Delta_y\sigma^* = \sigma^* \sum_x P_{\sigma xy}\Delta_x.$$

Proof. By Proposition 40,

$$\Delta_y \sigma = \sigma \sum_x P_{\sigma xy} \Delta_x.$$

Since both Δ_y and $\sigma \sum_x P_{\sigma xy} \Delta_x$ are shift polynomials, we can lift this to Proposition 44 by Proposition 43. □

Proposition 45. *The operators d and σ^* commute:*

$$d\sigma^* = \sigma^* d.$$

Proof. By Proposition 44,

$$d\sigma^* = \sum_y dy \Delta_y \sigma^* = \sum_{x,y} dy \sigma^* P_{\sigma xy} \Delta_x.$$

By definition of σ^*,

$$\sigma^* d = \sum_x \sigma^* dx \Delta_x = \sum_{x,y} dy \sigma^* P_{\sigma xy} \Delta_x.$$

Both sides agree. □

Proposition 46 (Closedness Preserving Substitutions). .
The operator σ^* preserves d-closedness: If $d\omega = 0$, then $d\sigma^*\omega = 0$.

Proof. Suppose $d\omega = 0$. Then $d\sigma^*\omega = \sigma^* d\omega = \sigma^* 0 = 0$. □

We have found a tool for constructing new forms and we are ready to apply it.

5 Some New WZ Forms

5.1 The Symmetric Multinomial Form

We use the derivation of the symmetric multinomial form for illustrating how new WZ forms – and therefore summation identities – can be found by the method of closedness preserving substitutions and some guesswork.

Consider the form ([WZ90]; or see section 3.1),

$$\omega = \binom{\mathsf{n}}{\mathsf{k}, \mathsf{n}-\mathsf{k}} \frac{x^{\mathsf{k}} y^{\mathsf{n}-\mathsf{k}}}{(x+y)^{\mathsf{n}}} \left(\frac{\mathsf{k} y}{(\mathsf{n}-\mathsf{k}+1)(x+y)}\, d\mathsf{n} - \, d\mathsf{k} \right).$$

Its asymmetry provokes us to substitute $\{\mathsf{n} \to \mathsf{a}+\mathsf{b},\, \mathsf{k} \to \mathsf{a}\}^*$ (using computer algebra) getting

$$\{\mathsf{n} \to \mathsf{a}+\mathsf{b},\, \mathsf{k} \to \mathsf{a}\}^* \omega = \binom{\mathsf{a}+\mathsf{b}}{\mathsf{a}, \mathsf{b}} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}} \left(\frac{y}{x+y} \frac{\mathsf{a}}{\mathsf{b}+1}\, d\mathsf{b} - \frac{y}{x+y}\, d\mathsf{a} \right),$$

which looks somehow better. At this point it seems that "fine tuning" suffices to reach a nice closed form. We try some substitutions like $\{\mathsf{a} \to \mathsf{a} \pm 1\}^*$ and $\{\mathsf{b} \to \mathsf{b} \pm 1\}^*$; and indeed, one of them is more than successful:

$$\{\mathsf{b} \to \mathsf{b}-1\}^*\{\mathsf{b} \to \mathsf{a}+\mathsf{b}\}^*\omega = \binom{\mathsf{a}+\mathsf{b}}{\mathsf{a},\ \ \mathsf{b}} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}} \left(\frac{\mathsf{a}}{\mathsf{a}+\mathsf{b}}\, d\mathsf{b} - \frac{\mathsf{b}}{\mathsf{a}+\mathsf{b}}\, d\mathsf{a} \right).$$

Impressed by the beauty of our new form, we stop substituting. We absorb rational factors into factorials getting

$$\binom{\mathsf{a}+\mathsf{b}-1}{\mathsf{a},\ \ \mathsf{b}-1} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}}\, d\mathsf{a} - \binom{\mathsf{a}-1+\mathsf{b}}{\mathsf{a}-1,\ \ \mathsf{b}} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}}\, d\mathsf{b}.$$

A pattern pops up now:

$$\begin{aligned}
\omega_2 \quad &= \quad \binom{\mathsf{a}-1+\mathsf{b}}{\mathsf{a}-1,\ \ \mathsf{b}} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}} (-1)^0\ \cancel{d\mathsf{a}}\ d\mathsf{b} \\
&+ \quad \binom{\mathsf{a}+\mathsf{b}-1}{\mathsf{a},\ \ \mathsf{b}-1} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}} (-1)^1\ d\mathsf{a}\ \cancel{d\mathsf{b}}\ ,
\end{aligned}$$

$$\begin{aligned}
\omega_3 \quad &= \quad \binom{\mathsf{a}-1+\mathsf{b}+\mathsf{c}}{\mathsf{a}-1,\ \ \mathsf{b},\ \ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}} (-1)^0\ \cancel{d\mathsf{a}}\ d\mathsf{b}\ d\mathsf{c} \\
&+ \quad \binom{\mathsf{a}+\mathsf{b}-1+\mathsf{c}}{\mathsf{a},\ \ \mathsf{b}-1,\ \ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}} (-1)^1\ d\mathsf{a}\ \cancel{d\mathsf{b}}\ d\mathsf{c} \\
&+ \quad \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}-1}{\mathsf{a},\ \ \mathsf{b},\ \ \mathsf{c}-1} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}} (-1)^2\ d\mathsf{a}\ d\mathsf{b}\ \cancel{d\mathsf{c}}\ ,
\end{aligned}$$

(Alternatively, ω_3 can be found from the trinomial theorem by *dividing through the right hand side* and running Kurt Wegschaider'sWegschaider, Kurt algorithm `FindRecurrence` which is contained in his Mathematica package `Multisum`). We are led to consider an infinite sequence of WZ forms of higher and higher degree:

Def. 32. *Fix a natural number n, let $\mathsf{L} := \{\mathsf{a}_1, \ldots, \mathsf{a}_n\}$ and define the nth symmetric multinomial form ω_n to be*

$$\sum_{\nu=1}^{n} \binom{\mathsf{a}_1 + \cdots + \mathsf{a}_{\nu-1} + \mathsf{a}_\nu - 1 + \mathsf{a}_{\nu+1} + \cdots + \mathsf{a}_n}{\mathsf{a}_1,\ \ldots,\ \mathsf{a}_{\nu-1},\ \mathsf{a}_\nu - 1,\ \mathsf{a}_{\nu+1},\ \ldots,\ \mathsf{a}_n} \frac{{x_1}^{\mathsf{a}_1} \cdots {x_n}^{\mathsf{a}_n}}{(x_1 + \cdots + x_n)^{\mathsf{a}_1 + \cdots + \mathsf{a}_n}}$$
$$\times (-1)^{\nu-1}\ d\mathsf{a}_1 \ldots\ d\mathsf{a}_{\nu-1}\ \cancel{d\mathsf{a}_\nu}\ d\mathsf{a}_{\nu+1} \ldots\ d\mathsf{a}_n,$$

Theorem 3. *For each natural number n, the nth symmetric multinomial form ω_n is closed.*

In order to avoid an abundance of dots, we resist proving Theorem 3 in full generality and confine ourselves to the case $n = 3$.

for $n = 3$. We recall

$$\begin{aligned}
\omega_3 \quad &= \quad \binom{\mathsf{a}-1+\mathsf{b}+\mathsf{c}}{\mathsf{a}-1,\ \ \mathsf{b},\ \ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}} (-1)^0\ \cancel{d\mathsf{a}}\ d\mathsf{b}\ d\mathsf{c} \\
&+ \quad \binom{\mathsf{a}+\mathsf{b}-1+\mathsf{c}}{\mathsf{a},\ \ \mathsf{b}-1,\ \ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}} (-1)^1\ d\mathsf{a}\ \cancel{d\mathsf{b}}\ d\mathsf{c} \\
&+ \quad \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}-1}{\mathsf{a},\ \ \mathsf{b},\ \ \mathsf{c}-1} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}} (-1)^2\ d\mathsf{a}\ d\mathsf{b}\ \cancel{d\mathsf{c}}.
\end{aligned}$$

By definition of the exterior derivative d,

$$\begin{aligned} d(\omega_3) &= \Delta_{\mathsf{a}} \binom{\mathsf{a}-1+\mathsf{b}+\mathsf{c}}{\mathsf{a}-1,\ \mathsf{b},\ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\ d\mathsf{a}\ d\mathsf{b}\ d\mathsf{c} \\ &+ \Delta_{\mathsf{b}} \binom{\mathsf{a}+\mathsf{b}-1+\mathsf{c}}{\mathsf{a},\ \mathsf{b}-1,\ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\ d\mathsf{a}\ d\mathsf{b}\ d\mathsf{c} \\ &+ \Delta_{\mathsf{c}} \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}-1}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}-1} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\ d\mathsf{a}\ d\mathsf{b}\ d\mathsf{c}, \end{aligned}$$

where we have used that "sorting the ds" introduces an alternating sign which cancels the alternating sign appearing in ω_3. For example, in the second line, $(-1)^1\ d\mathsf{b}\ d\mathsf{a}\ d\mathsf{c} = (-1)^{1+1}\ d\mathsf{a}\ d\mathsf{b}\ d\mathsf{c} =\ d\mathsf{a}\ d\mathsf{b}\ d\mathsf{c}$. Computing differences gives

$$\begin{aligned} & d(\omega_3) \\ = & \left(\frac{x}{x+y+z} \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}} - \binom{\mathsf{a}-1+\mathsf{b}+\mathsf{c}}{\mathsf{a}-1,\ \mathsf{b},\ \mathsf{c}} \right) \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\ d\mathsf{a}\ d\mathsf{b}\ d\mathsf{c} \\ + & \left(\frac{y}{x+y+z} \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}} - \binom{\mathsf{a}+\mathsf{b}-1+\mathsf{c}}{\mathsf{a},\ \mathsf{b}-1,\ \mathsf{c}} \right) \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\ d\mathsf{a}\ d\mathsf{b}\ d\mathsf{c} \\ + & \left(\frac{z}{x+y+z} \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}} - \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}-1}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}-1} \right) \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\ d\mathsf{a}\ d\mathsf{b}\ d\mathsf{c}. \end{aligned}$$

Adding up columnwise yields

$$d(\omega_3) = \left(\frac{x+y+z}{x+y+z} \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}} - \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}} \right) \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\ d\mathsf{a}\ d\mathsf{b}\ d\mathsf{c},$$

where we have used

$$\binom{\mathsf{a}-1+\mathsf{b}+\mathsf{c}}{\mathsf{a}-1,\ \mathsf{b},\ \mathsf{c}} + \binom{\mathsf{a}+\mathsf{b}-1+\mathsf{c}}{\mathsf{a},\ \mathsf{b}-1,\ \mathsf{c}} + \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}-1}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}-1} = \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}},$$

which is correct by combinatorial interpretation. Hence $d(\omega_3) = 0$. ☐

Remark: It is trivial to reformulate

$$\underset{n \geq 0}{\forall} \sum_k \binom{n}{k} x^k y^{n-k} = (x+y)^n$$

as

$$\underset{n \geq 0}{\forall} \sum_{\substack{i,j \\ i+j=n}} \binom{i+j}{i,\ j} x^i y^j = (x+y)^n.$$

with the motive of symmetry. This should not mislead us to believe that it is trivial to guess ω_2 directly. ☐

Remark: Note that (an asymmetric version of) the multinomial form appears in the very first paper [Zei93] on WZ forms as equation (7.14):

$$\begin{aligned} \omega_{MULTINOMIAL} &:= \frac{\mathsf{n}!}{\mathsf{k}_1! \cdots \mathsf{k}_r! (\mathsf{k} - \mathsf{k}_1 - \cdots - \mathsf{k}_r + 1)! (r+1)^{\mathsf{n}}} \\ \times & \left((\mathsf{n} - \mathsf{k}_1 - \cdots - \mathsf{k}_r + 1)\ d\mathsf{k}_1 \cdots\ d\mathsf{k}_r + \sum_{i=1}^{r} \mathsf{k}_i\ d\mathsf{n}\ d\mathsf{k}_1 \cdots \cancel{d\mathsf{k}_i} \cdots\ d\mathsf{k}_r \right). \end{aligned}$$

Computation reveals that $\omega_{MULTINOMIAL}$ is not closed for $r = 2$; Hence the definition of $\omega_{MULTINOMIAL}$ must contain an error somewhere. How to fix it quickly? Closedness preserving substitutions help us in this task: We start from ω_n as defined in Definition 32,

$$\sum_{\nu=1}^{n} \binom{\mathsf{a}_1 + \cdots + \mathsf{a}_{\nu-1} + \mathsf{a}_\nu - 1 + \mathsf{a}_{\nu+1} + \cdots + \mathsf{a}_n}{\mathsf{a}_1,\ \ldots,\ \mathsf{a}_{\nu-1},\ \mathsf{a}_\nu - 1,\ \mathsf{a}_{\nu+1},\ \ldots,\ \mathsf{a}_n} \frac{{x_1}^{\mathsf{a}_1} \cdots {x_n}^{\mathsf{a}_n}}{(x_1 + \cdots + x_n)^{\mathsf{a}_1+\cdots+\mathsf{a}_n}}$$
$$\times (-1)^{\nu-1}\, d\mathsf{a}_1 \ldots\, d\mathsf{a}_{\nu-1}\, \cancel{d\mathsf{a}_\nu}\, d\mathsf{a}_{\nu+1} \ldots\, d\mathsf{a}_n,$$

and apply a certain closedness preserving substitution (left as an exercise) to get a debugged definition of $\omega_{MULTINOMIAL}$:

$$\tilde{\omega}_{MULTINOMIAL} := \frac{\mathsf{n}!}{\mathsf{k}_1! \cdots \mathsf{k}_r! (\mathsf{n} - \mathsf{k}_1 - \cdots - \mathsf{k}_r + 1)! (r+1)^{\mathsf{n}}}$$
$$\times \Big((\mathsf{n} - \mathsf{k}_1 - \cdots - \mathsf{k}_r + 1) \underline{(r+1)}\, d\mathsf{k}_1 \cdots d\mathsf{k}_r$$
$$+ \sum_{i=1}^{r} \mathsf{k}_i \underline{(-1)^i}\, d\mathsf{n}\, d\mathsf{k}_1 \cdots \cancel{d\mathsf{k}_i} \cdots d\mathsf{k}_r \Big)$$

(The necessary patches are underlined). □

5.2 Identities from the Symmetric Multinomial Form

To our knowledge, Identities 43, 47, 48, 49 and 50 are new. All calculations in this section can be done by hand almost effortlessly; computer algebra support is superfluous.

We start with an obvious application of ω_2.

Identity 42 (Binomial Theorem).

$$\underset{n \geq 0}{\forall} \sum_{\substack{i,j \\ i+j=n}} \binom{i+j}{i,\ j} x^i y^j = (x+y)^n.$$

Proof of Identity 42. Assume $n \geq 0$. Define

$$\omega := \binom{\mathsf{a} - 1 + \mathsf{b}}{\mathsf{a} - 1,\ \mathsf{b}} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}}\, d\mathsf{b} - \binom{\mathsf{a} + \mathsf{b} - 1}{\mathsf{a},\ \mathsf{b} - 1} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}}\, d\mathsf{a}$$

and

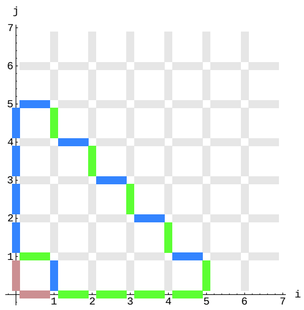
$$\rho := \partial_{\{\mathsf{a},\mathsf{b}\}} \left([0 \leq \mathsf{a}, 0 \leq \mathsf{b}, 1 \leq \mathsf{a} + \mathsf{b} < n]\, d\mathsf{a}\, d\mathsf{b} \right).$$

Since ω is closed by Theorem 3 and ρ is exact we know that

$$\sum_{\mathsf{a},\mathsf{b}} \rho \cdot \omega = 0.$$

We aim to compute $\sum_{\mathsf{a},\mathsf{b}} \rho \cdot \omega$.

The diagram shows that ρ decomposes into $\rho = \rho_0 + \rho_1 + \rho_2 + \rho_3$ where



$$
\begin{aligned}
\rho_0 &= [\mathsf{a} = 0, \mathsf{b} = 1]\, d\mathsf{a} - [\mathsf{a} = 1, \mathsf{b} = 0]\, d\mathsf{b}, \\
\rho_1 &= -[0 \leq \mathsf{a}, 0 \leq \mathsf{b}, \mathsf{a} + \mathsf{b} = n]\, d\mathsf{a} \\
&\quad + [0 \leq \mathsf{a}, 0 \leq \mathsf{b}, \mathsf{a} + \mathsf{b} = n]\, d\mathsf{b}, \\
\rho_2 &= [1 \leq \mathsf{a} \leq n, \mathsf{b} = 0]\, d\mathsf{a}, \\
\rho_3 &= -[\mathsf{a} = 0, 1 \leq \mathsf{b} \leq n]\, d\mathsf{b}.
\end{aligned}
$$

The diagram further shows that $\sum_{\mathsf{a},\mathsf{b}} \rho_2 \cdot \omega = 0$, $\sum_{\mathsf{a},\mathsf{b}} \rho_3 \cdot \omega = 0$, and we compute

$$
\sum_{\mathsf{a},\mathsf{b}} \rho_0 \cdot \omega = -\binom{0+0}{0,\ \ 0} \frac{y}{x+y} - \binom{0+0}{0,\ \ 0} \frac{x}{x+y} = -1,
$$

$$
\sum_{\mathsf{a},\mathsf{b}} \rho_1 \cdot \omega = \sum_{\substack{i,j \\ i+j=n}} \binom{i+j}{i,\ \ j} \frac{x^i y^j}{(x+y)^n}.
$$

Adding these four sums gives

$$
\sum_{\mathsf{a},\mathsf{b}} \rho \cdot \omega = -1 + \sum_{\substack{i,j \\ i+j=n}} \binom{i+j}{i,\ \ j} x^i y^j (x+y)^{-n},
$$

which is zero as it is the sum of a closed form over an exact range. Identity 42 follows. $\square$

By summing the same form over different ranges we usually get completely different identities; for example, both Identity 42 and Identity 43 are obtained from ω_2.

Identity 43. *Let p and q be natural numbers. Then*

$$
\left(\frac{x}{x+y}\right)^{p+1} \sum_{k=0}^{q} \binom{p+k}{p} \left(\frac{y}{x+y}\right)^k + \left(\frac{y}{x+y}\right)^{q+1} \sum_{k=0}^{p} \binom{q+k}{q} \left(\frac{x}{x+y}\right)^k = 1.
$$

We postpone the proof of Identity 43 to page 57. It might seem redundant to list special cases of more general identities explicitly. However, this helps us to see that Identity 43 is a generalization of well known identities. Substituting 1 for x and y reduces Identity 43 to Identity 44 which appears in [FC88].

Identity 44.

$$
\underset{q \geq 0}{\forall} \ \underset{p \geq 0}{\forall} \ \sum_{k=0}^{q} \binom{p+k}{k} 2^{-p-k} + \sum_{k=0}^{p} \binom{q+k}{k} 2^{-q-k} = 2.
$$

Finally, substituting m for p and q reduces Identity 44 to Identity 45, which appears as "unexpected identity" (5.20) in [GKP89, p. 167].

Identity 45.

$$
\underset{m \geq 0}{\forall} \ \sum_{k=0}^{m} \binom{m+k}{k} 2^{-k} = 2^m.
$$

Note that Identity 43 and Identity 44 are nontrivial in the sense that their sums cannot be expressed in closed form, as can be proved by Gosper's algorithm.

Proof of Identity 43. Assume $p \geq 0$ and $q \geq 0$. We define

$$\omega := \binom{\mathsf{a}-1+\mathsf{b}}{\mathsf{a}-1,\ \mathsf{b}} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}}\, d\mathsf{b} - \binom{\mathsf{a}+\mathsf{b}-1}{\mathsf{a},\ \mathsf{b}-1} \frac{x^{\mathsf{a}} y^{\mathsf{b}}}{(x+y)^{\mathsf{a}+\mathsf{b}}}\, d\mathsf{a}$$

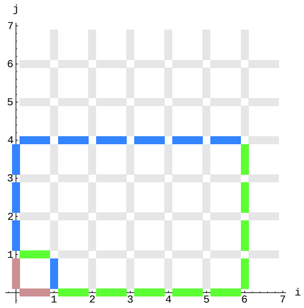
and

$$\rho := \partial\left([0 \leq \mathsf{a} \leq p,\ 0 \leq \mathsf{b} \leq q]\, d\mathsf{a}\, d\mathsf{b} \right).$$

Since ω is closed by Theorem 3 and ρ is exact we know that

$$\sum_{\mathsf{a},\mathsf{b}} \rho \cdot \omega = 0.$$

We aim to compute $\sum_{\mathsf{a},\mathsf{b}} \rho \cdot \omega$.



The diagram shows that ρ decomposes into $\rho = \rho_0 + \rho_1 + \rho_2 + \rho_3 + \rho_4$ where

$$\begin{aligned}
\rho_0 &= [\mathsf{a}=0, \mathsf{b}=1]\, d\mathsf{a} - [\mathsf{a}=1, \mathsf{b}=0]\, d\mathsf{b}, \\
\rho_1 &= [1 \leq \mathsf{a} \leq p, \mathsf{b}=0]\, d\mathsf{a}, \\
\rho_2 &= [\mathsf{a}=p+1, 0 \leq \mathsf{b} \leq q]\, d\mathsf{b}, \\
\rho_3 &= -[0 \leq \mathsf{a} \leq p, \mathsf{b}=q+1]\, d\mathsf{a}, \\
\rho_4 &= -[\mathsf{a}=0, 1 \leq \mathsf{b} \leq q]\, d\mathsf{b}.
\end{aligned}$$

The diagram further shows that $\sum_{\mathsf{a},\mathsf{b}} \rho_1 \cdot \omega = 0$ and $\sum_{\mathsf{a},\mathsf{b}} \rho_4 \cdot \omega = 0$. Computation yields

$$\sum_{\mathsf{a},\mathsf{b}} \rho_0 \cdot \omega = -\binom{0+0}{0,0} \frac{y}{x+y} - \binom{0+0}{0,0} \frac{x}{x+y} = -1,$$

$$\sum_{\mathsf{a},\mathsf{b}} \rho_2 \cdot \omega = \sum_{j=0}^{q} \binom{p+j}{p,\ j} \frac{x^{p+1} y^j}{(x+y)^{p+1+j}},$$

and

$$\sum_{\mathsf{a},\mathsf{b}} \rho_3 \cdot \omega = \sum_{i=0}^{p} \binom{i+q}{i,\ q} \frac{x^i y^{q+1}}{(x+y)^{i+q+1}}.$$

Adding these five sums gives

$$\sum_{\mathsf{a},\mathsf{b}} \rho \cdot \omega = -1 + \sum_{j=0}^{q} \binom{p+j}{p,\ j} \frac{x^{p+1} y^j}{(x+y)^{p+1+j}} + \sum_{i=0}^{p} \binom{i+q}{i,\ q} \frac{x^i y^{q+1}}{(x+y)^{i+q+1}}.$$

which is zero as it is the sum of a closed form over an exact range. Identity 43 follows upon renaming summation indices. □

Of course, ω_3 can be used to prove the Trinomial Theorem.

Identity 46 (Trinomial Theorem).

$$\underset{n\geq 0}{\forall} \sum_{\substack{i,j,k\\ i+j+k=n}} \binom{i+j+k}{i,\ j,\ k} x^i y^j z^k = (x+y+z)^n.$$

Note that the Trinomial Theorem is trivial in the sense that its double sum can be transformed into closed form by iteratively applying the Binomial Theorem, as expected from $(x+y+z)^n = (x+(y+z))^n$.

Proof. Assume $n \geq 0$. We define

$$\begin{aligned}\omega \quad := & \quad \binom{\mathsf{a}-1+\mathsf{b}+\mathsf{c}}{\mathsf{a}-1,\ \mathsf{b},\ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\, \cancel{d\mathsf{a}}\, d\mathsf{b}\, d\mathsf{c} \\ - & \quad \binom{\mathsf{a}+\mathsf{b}-1+\mathsf{c}}{\mathsf{a},\ \mathsf{b}-1,\ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\, d\mathsf{a}\, \cancel{d\mathsf{b}}\, d\mathsf{c} \\ + & \quad \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}-1}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}-1} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\, d\mathsf{a}\, d\mathsf{b}\, \cancel{d\mathsf{c}}\end{aligned}$$

and

$$\rho := \partial\left([0 \leq \mathsf{a}, 0 \leq \mathsf{b}, 0 \leq \mathsf{c}, 1 \leq \mathsf{a}+\mathsf{b}+\mathsf{c} < n]\, d\mathsf{a}\, d\mathsf{b}\, d\mathsf{c}\right);$$

the range ρ is the surface of a "discrete tetrahedron". Since ω is closed by Theorem 3 and ρ is exact we know that $\sum_{\mathsf{a},\mathsf{b},\mathsf{c}} \rho \cdot \omega = 0$. As

$$\sum_{\mathsf{a},\mathsf{b},\mathsf{c}} \rho \cdot \omega = -1 + \sum_{\substack{i,j,k\\ i+j+k=n}} \binom{i+j+k}{i,\ j,\ k} \frac{x^i y^j z^k}{(x+y+z)^n},$$

Identity 46 is proved. ∎

The triviality of the Trinomial Theorem should not mislead us to discard ω_3 which proves Identity 47, a (truly) double sum identity.

Identity 47. *Let p, q and r be natural numbers. Then*

$$\begin{aligned} & \left(\frac{x}{x+y+z}\right)^{p+1} \sum_{j=0}^{q} \sum_{k=0}^{r} \binom{p+j+k}{p,\ j,\ k} \left(\frac{y}{x+y+z}\right)^{j} \left(\frac{z}{x+y+z}\right)^{k} \\ + & \left(\frac{y}{x+y+z}\right)^{q+1} \sum_{i=0}^{p} \sum_{k=0}^{r} \binom{i+q+k}{i,\ q,\ k} \left(\frac{x}{x+y+z}\right)^{i} \left(\frac{z}{x+y+z}\right)^{k} \\ + & \left(\frac{z}{x+y+z}\right)^{r+1} \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{i+j+r}{i,\ j,\ r} \left(\frac{x}{x+y+z}\right)^{i} \left(\frac{y}{x+y+z}\right)^{j} = 1. \end{aligned}$$

Identity 47 is a trivariate analog of Identity 43.

Proof of Identity 47. Assume $p \geq 0$, $q \geq 0$, and $r \geq 0$. We define

$$\begin{aligned}\omega \quad := & \quad \binom{\mathsf{a}-1+\mathsf{b}+\mathsf{c}}{\mathsf{a}-1,\ \mathsf{b},\ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\, \cancel{d\mathsf{a}}\, d\mathsf{b}\, d\mathsf{c} \\ - & \quad \binom{\mathsf{a}+\mathsf{b}-1+\mathsf{c}}{\mathsf{a},\ \mathsf{b}-1,\ \mathsf{c}} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\, d\mathsf{a}\, \cancel{d\mathsf{b}}\, d\mathsf{c} \\ + & \quad \binom{\mathsf{a}+\mathsf{b}+\mathsf{c}-1}{\mathsf{a},\ \mathsf{b},\ \mathsf{c}-1} \frac{x^{\mathsf{a}} y^{\mathsf{b}} z^{\mathsf{c}}}{(x+y+z)^{\mathsf{a}+\mathsf{b}+\mathsf{c}}}\, d\mathsf{a}\, d\mathsf{b}\, \cancel{d\mathsf{c}}.\end{aligned}$$

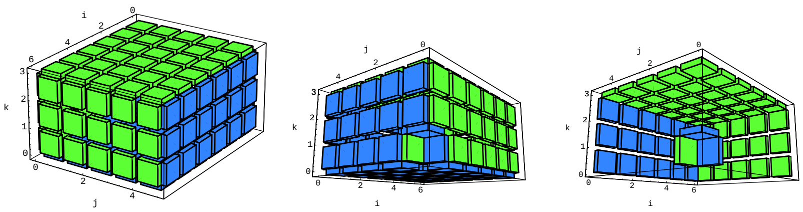
and

$$\rho := \partial\left(\left([0 \le \mathsf{a} \le p,\ 0 \le \mathsf{b} \le q,\ 0 \le \mathsf{c} \le r] - [\mathsf{a}=0,\ \mathsf{b}=0,\ \mathsf{c}=0]\right)\ d\mathsf{a}\ d\mathsf{b}\right).$$

As ω is closed by Theorem 3 and ρ is exact we know that

$$\sum_{\mathsf{a},\mathsf{b},\mathsf{c}} \rho \cdot \omega = 0.$$

We aim to compute $\sum_{\mathsf{a},\mathsf{b},\mathsf{c}} \rho \cdot \omega$.



The range ρ can be found by looking on the diagrams above or by computation. Both methods yield $\rho = \rho_{\mathsf{a}} + \rho_{\mathsf{b}} + \rho_{\mathsf{b}} + \rho_0 + \rho'$ where

$$\begin{aligned}
\rho_{\mathsf{a}} &= [\mathsf{a} = p+1, 0 \le \mathsf{b} \le q, 0 \le \mathsf{c} \le r]\ d\mathsf{b}\ d\mathsf{c},\\
\rho_{\mathsf{b}} &= -[0 \le \mathsf{a} \le p, \mathsf{b} = q+1, 0 \le \mathsf{c} \le r]\ d\mathsf{a}\ d\mathsf{c},\\
\rho_{\mathsf{c}} &= [0 \le \mathsf{a} \le p, 0 \le \mathsf{b} \le q, \mathsf{c} = r+1]\ d\mathsf{a}\ d\mathsf{b},\\
\rho_0 &= -\rho_{\mathsf{a}} - \rho_{\mathsf{b}} - \rho_{\mathsf{c}} \qquad \text{at} \qquad p = q = r = 0,\\
\rho' &= \cdots.
\end{aligned}$$

We compute

$$\sum_{\mathsf{a},\mathsf{b},\mathsf{c}} \rho_{\mathsf{a}} \cdot \omega = \left(\frac{x}{x+y+z}\right)^{p+1} \sum_{j=0}^{q}\sum_{k=0}^{r} \binom{p+j+k}{p,\ j,\ k} \left(\frac{y}{x+y+z}\right)^{j} \left(\frac{z}{x+y+z}\right)^{k},$$

$$\sum_{\mathsf{a},\mathsf{b},\mathsf{c}} \rho_{\mathsf{b}} \cdot \omega = \left(\frac{y}{x+y+z}\right)^{q+1} \sum_{i=0}^{p}\sum_{k=0}^{r} \binom{i+q+k}{i,\ q,\ k} \left(\frac{x}{x+y+z}\right)^{i} \left(\frac{z}{x+y+z}\right)^{k},$$

and

$$\sum_{\mathsf{a},\mathsf{b},\mathsf{c}} \rho_{\mathsf{c}} \cdot \omega = \left(\frac{z}{x+y+z}\right)^{r+1} \sum_{i=0}^{p}\sum_{j=0}^{q} \binom{i+j+r}{i,\ j,\ r} \left(\frac{x}{x+y+z}\right)^{i} \left(\frac{y}{x+y+z}\right)^{j}.$$

Adding yields

$$
\begin{aligned}
&\sum_{\mathsf{a,b,c}} (\rho_{\mathsf{a}} + \rho_{\mathsf{b}} + \rho_{\mathsf{c}}) \cdot \omega \\
= \;& \left(\frac{x}{x+y+z}\right)^{p+1} \sum_{j=0}^{q} \sum_{k=0}^{r} \binom{p+j+k}{p,\; j,\; k} \left(\frac{y}{x+y+z}\right)^{j} \left(\frac{z}{x+y+z}\right)^{k} \\
+ \;& \left(\frac{y}{x+y+z}\right)^{q+1} \sum_{i=0}^{p} \sum_{k=0}^{r} \binom{i+q+k}{i,\; q,\; k} \left(\frac{x}{x+y+z}\right)^{i} \left(\frac{z}{x+y+z}\right)^{k} \\
+ \;& \left(\frac{z}{x+y+z}\right)^{r+1} \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{i+j+r}{i,\; j,\; r} \left(\frac{x}{x+y+z}\right)^{i} \left(\frac{y}{x+y+z}\right)^{j}.
\end{aligned}
$$

As $\rho_0 = -(\rho_{\mathsf{a}} + \rho_{\mathsf{b}} + \rho_{\mathsf{c}})$ at $p = q = r = 0$, we derive $\sum_{\mathsf{a,b,c}} \rho_0 \cdot \omega = -1$ as a particular case of $\sum_{\mathsf{a,b,c}} (\rho_{\mathsf{a}} + \rho_{\mathsf{b}} + \rho_{\mathsf{c}}) \cdot \omega$. Finally, $\sum_{\mathsf{a,b,c}} \rho' \cdot \omega = 0$. Adding these three sums we get

$$
\begin{aligned}
&\sum_{\mathsf{a,b,c}} \rho \cdot \omega \\
= \;& \left(\frac{x}{x+y+z}\right)^{p+1} \sum_{j=0}^{q} \sum_{k=0}^{r} \binom{p+j+k}{p,\; j,\; k} \left(\frac{y}{x+y+z}\right)^{j} \left(\frac{z}{x+y+z}\right)^{k} \\
+ \;& \left(\frac{y}{x+y+z}\right)^{q+1} \sum_{i=0}^{p} \sum_{k=0}^{r} \binom{i+q+k}{i,\; q,\; k} \left(\frac{x}{x+y+z}\right)^{i} \left(\frac{z}{x+y+z}\right)^{k} \\
+ \;& \left(\frac{z}{x+y+z}\right)^{r+1} \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{i+j+r}{i,\; j,\; r} \left(\frac{x}{x+y+z}\right)^{i} \left(\frac{y}{x+y+z}\right)^{j} \\
- \;& 1.
\end{aligned}
$$

which is zero as it is the sum of a closed form over an exact range. Identity 47 follows. ☐

Upon substituting 1 for x, y, and z, Identity 47 reduces to a (truly) double sum analog of Identity 44.

Identity 48.

$$
\begin{aligned}
\underset{p \geq 0}{\forall} \; \underset{q \geq 0}{\forall} \; \underset{r \geq 0}{\forall} \qquad & 3^{-p} \sum_{j=0}^{q} \sum_{k=0}^{r} \binom{p+j+k}{p,\; j,\; k} 3^{-j-k} \\
+ \;& 3^{-q} \sum_{i=0}^{p} \sum_{k=0}^{r} \binom{i+q+k}{i,\; q,\; k} 3^{-i-k} \\
+ \;& 3^{-r} \sum_{i=0}^{p} \sum_{j=0}^{q} \binom{i+j+r}{i,\; j,\; r} 3^{-i-j} \;=\; 3.
\end{aligned}
$$

Upon substituting m for p, q, and r, Identity 48 reduces to a (truly) double sum analog of Identity 45.

Identity 49.

$$
\underset{m \geq 0}{\forall} \; \sum_{i=0}^{m} \sum_{j=0}^{m} \binom{m+i+j}{m,\; i,\; j} 3^{-i-j} \;=\; 3^{m}.
$$

Note that it is easily possible to generalize Identities 43 and 47 to an arbitrary number of summations. In order to save space, we resist this temptation and confine ourselves to looking at the special cases Identity 45 and Identity 49:

$$\begin{aligned} \binom{m}{m} 1^0 &= 1^m, \\ \sum_{i=0}^{m} \binom{m+i}{m,\ i} 2^{-i} &= 2^m, \\ \sum_{i=0}^{m} \sum_{j=0}^{m} \binom{m+i+j}{m,\ i,\ j} 3^{-i-j} &= 3^m. \end{aligned}$$

Grasping a pattern we conjecture a sequence of multisum identities.

Identity 50.

$$\sum_{i_1=0}^{m} \cdots \sum_{i_\nu=0}^{m} \binom{m+i_1+\cdots+i_\nu}{m,\ i_1,\ \ldots,\ i_\nu} (\nu+1)^{-i_1 \cdots - i_\nu} = (\nu+1)^m.$$

5.3 A new WZ form from an identity of S. Dent

An certain identity of S. Dent [Den96] leads to the WZ form

$$\begin{aligned} \omega = (-1)^{\mathsf{b}}\, 2^{-\mathsf{k}+2v} \binom{\mathsf{k}-2v}{\mathsf{s}-\mathsf{b}} \binom{\mathsf{k}-v}{\mathsf{k}-2v}^{-1} \binom{-\mathsf{s}+\mathsf{k}}{-\mathsf{b}+2v} \binom{\mathsf{s}}{\mathsf{b}} \\ \times \left(\frac{(\mathsf{b}-1)\, \mathsf{b}\, (-\mathsf{s}+\mathsf{k}+\mathsf{b}-2v-1)\, (-\mathsf{s}+\mathsf{k}+\mathsf{b}-2v)^2}{2\, (\mathsf{b}-2v-1)\, (-\mathsf{k}+v-1)\, (-\mathsf{s}+\mathsf{b}-2)\, (-\mathsf{s}+\mathsf{b}-1)^2}\, d\mathsf{k}\, d\mathsf{s} \right. \\ + \frac{-\mathsf{b}\, (-\mathsf{s}+\mathsf{k}+\mathsf{b}-2v)^2}{(\mathsf{b}-2v-1)\, (-\mathsf{s}+\mathsf{b}-1)^2}\, d\mathsf{b}\, d\mathsf{s} \\ \left. + \frac{\mathsf{b}\, (\mathsf{k}-2v+1)\, (-\mathsf{s}+\mathsf{k}+1)}{2\, (\mathsf{b}-2v-1)\, (-\mathsf{k}+v-1)\, (-\mathsf{s}+\mathsf{b}-1)}\, d\mathsf{b}\, d\mathsf{k} \right). \end{aligned}$$

Closedness preserving substitution $\{\mathsf{b} \to \mathsf{b}+\mathsf{s}\}^*$ leads to

$$\begin{aligned} (-1)^{\mathsf{s}+\mathsf{b}}\, 2^{-\mathsf{k}+2v} \binom{\mathsf{k}-2v}{-\mathsf{b}} \binom{\mathsf{k}-v}{\mathsf{k}-2v}^{-1} \binom{-\mathsf{s}+\mathsf{k}}{-\mathsf{s}-\mathsf{b}+2v} \binom{\mathsf{s}}{\mathsf{s}+\mathsf{b}} \\ \times \left(\frac{(\mathsf{k}+\mathsf{b}-2v)^2\, (\mathsf{s}+\mathsf{b})\, (\mathsf{s}+\mathsf{k}+\mathsf{b}-2v)}{2\, (\mathsf{b}-1)^2\, (-\mathsf{k}+v-1)\, (\mathsf{s}+\mathsf{b}-2v-1)}\, d\mathsf{k}\, d\mathsf{s} \right. \\ + \frac{-(\mathsf{k}+\mathsf{b}-2v)^2\, (\mathsf{s}+\mathsf{b})}{(\mathsf{b}-1)^2\, (\mathsf{s}+\mathsf{b}-2v-1)}\, d\mathsf{b}\, d\mathsf{s} \\ \left. + \frac{(\mathsf{k}-2v+1)\, (-\mathsf{s}+\mathsf{k}+1)\, (\mathsf{s}+\mathsf{b})}{2\, (\mathsf{b}-1)\, (-\mathsf{k}+v-1)\, (\mathsf{s}+\mathsf{b}-2v-1)}\, d\mathsf{b}\, d\mathsf{k} \right). \end{aligned}$$

Consider the following shadow of the last form:

$$\frac{-\left(\mathsf{k}-2\,v\right)^2\,\mathsf{s}}{\mathsf{b}^2}\,(-1)^{\mathsf{b}}\,2^{-\mathsf{k}+2\,v}\binom{v}{\mathsf{k}-v}\binom{\mathsf{b}}{-\mathsf{k}+2\,v}\binom{\mathsf{b}}{-\mathsf{s}}\binom{-\mathsf{s}+\mathsf{k}}{-\mathsf{s}-\mathsf{b}+2\,v}$$
$$\times\left(\frac{\left(\mathsf{k}+\mathsf{b}-2\,v\right)^2\,\left(\mathsf{s}+\mathsf{b}\right)\,\left(\mathsf{s}+\mathsf{k}+\mathsf{b}-2\,v\right)}{2\,\left(\mathsf{b}-1\right)^2\,\left(-\mathsf{k}+v-1\right)\,\left(\mathsf{s}+\mathsf{b}-2\,v-1\right)}\,d\mathsf{k}\,d\mathsf{s}\right.$$
$$+\frac{-\left(\mathsf{k}+\mathsf{b}-2\,v\right)^2\,\left(\mathsf{s}+\mathsf{b}\right)}{\left(\mathsf{b}-1\right)^2\,\left(\mathsf{s}+\mathsf{b}-2\,v-1\right)}\,d\mathsf{b}\,d\mathsf{s}$$
$$\left.+\frac{\left(\mathsf{k}-2\,v+1\right)\,\left(-\mathsf{s}+\mathsf{k}+1\right)\,\left(\mathsf{s}+\mathsf{b}\right)}{2\,\left(\mathsf{b}-1\right)\,\left(-\mathsf{k}+v-1\right)\,\left(\mathsf{s}+\mathsf{b}-2\,v-1\right)}\,d\mathsf{b}\,d\mathsf{k}\right).$$

It leads, by straightforward manipulation, to the identity

$$\sum_k\sum_s 2^{-k}(2v+s-b-k)\binom{b}{s}\binom{b}{2v-k}\binom{k+s}{2v+s-b}\binom{v}{k-v}=0$$

which holds provided that $b\geq 0$ and $v\geq 0$.

A How to Use the Package wz

This appendix may help the reader to use our package `wz.m`. Explanations apply to the package `wz.m` as of February 00 running under Mathematica 3.x.
Download the files `wz.m` and `wzManual.nb` from
`http://www.risc.uni-linz.ac.at/research/combinat/risc/`.
Under Unix, put these files into some directory (for example, your home directory) and start the Mathematica frontend in that directory by typing
`Mathematica&`
to a shell. From within Mathematica, load the package wz.m by executing (Shift-Return)

In[1] := << wz.m
```
loading wz.m Oct 28, 1999...
```

Under Windows, create a directory `C:\wz` (using the Explorer), put the files `wz.m` and `wzManual.nb` into `C:\wz` and start Mathematica by double-clicking on `C:\wz\wzManual.nb`. From within Mathematica, load the package wz.m by executing (Shift-Return)

In[2] := `$Path = Append[$Path, "c : /wz"];`
`<< wz.m`

(When loading wz.m, you will get some "multiple context" warnings; you can safely ignore them.)
The notebook file `wzManual.nb` contains appendix A. A quick way to get started is to modify and rerun the following examples.

A.1 Constructing Forms

Using precomputed examples. The easiest way to get a closed form is to call a precomputed example; these examples are listed in appendix B.

In[3] := `w1 = example["dixon"]`

$$Out[3] = \frac{(-1)^k \binom{a+b}{a+k}\binom{a+c}{c+k}\binom{b+c}{b+k} a!\, b!\, c!}{(a+b+c)!}\Big(\frac{-(b+k)\,(c+k)}{2\,(1+a+b+c)\,(1+a-k)}da$$
$$-\frac{(a+k)\,(c+k)}{2\,(1+a+b+c)\,(1+b-k)}db + -\frac{(a+k)\,(b+k)}{2\,(1+a+b+c)\,(1+c-k)}dc + 1dk\Big)$$

Using Gosper's algorithm to construct closed 1-forms. If we know a definite single hypergeometric sum identity involving free variables we can try to construct a closed form from it by using Gosper's algorithm in the implementation of Peter Paule and Markus Schorn. Download the file Zb.m from http://www.risc.uni-linz.ac.at/research/combinat/risc/ . Under Unix, copy this file into your *Mathematica* directory; under Windows9x/NT, put it into C:\wz. The function ccf ("complete to a closed form") returns a WZ form. Note that ccf calls the function `Gosper` of Peter Paule and Markus Schorn, which does the difficult part of the computation.

In[4] := `term = toPht[Binomial[n,k]2^ - n];`
`w2 = ccf[term, {k}, {n,k}]`

$$Out[4] = (2^{-n} \binom{n}{k})\Big(1dk + \frac{k}{2\,(-1+k-n)}dn\Big)$$

The third argument of ccf[...] determines the set of labels. For example

$In[5] :=$ `term = toPht[Binomial[n,k]Binomial[a,k]/Binomial[a + n,n]];`
`w5 = ccf[term, {k}, {k,n}]`

$$Out[5] = \frac{\binom{a}{k}\binom{n}{k}}{\binom{a+n}{n}}\left(1\mathrm{d}k + \frac{k^2}{(-1+k-n)\ (1+a+n)}\mathrm{d}n\right)$$

is different from

$In[6] :=$ `w6 = ccf[term, {k}, {k, n, a}]`

$$Out[6] = \frac{\binom{a}{k}\binom{n}{k}}{\binom{a+n}{n}}\left(-\frac{k^2}{(1+a-k)\ (1+a+n)}\mathrm{d}a + 1\mathrm{d}k + \frac{k^2}{(-1+k-n)\ (1+a+n)}\mathrm{d}n\right)$$

Using Kurt Wegschaider's package to construct closed forms of higher degree. To execute the examples of this subsection, you need Kurt Wegschaider's package multisum.m. It is available at `http://www.risc.uni-linz.ac.at/research/combinat/risc/`. Under Unix, put the file `multisum.m` into your *Mathematica* directory; under Windows, put it into C:\wz.
Suppose we want to prove the following (trivial) double sum analog of Vandermonde's identity:

$$\sum_{ij}\binom{R}{i}\binom{S}{j}\binom{T}{n-i-j} = \binom{R+S+T}{n}.$$

We divide by the right hand side and enter the resulting summand.
$In[7] :=$ `lhs = bi[R, i] bi[S, j] bi[T, n - i - j];`
`rhs = bi[R + S + T, n];`
`summand = lhs/rhs;`
We compute a recurrence for the summand by Kurt Wegschaider's package.

$In[8] :=$ `rek = FindCertificate[summand, n,`
`{{1, 0, 0}, {0, 0, 0}}, {i, j}, { {{0, 0, 0}}, {{0, 0, 0}} }, 1][[`
`1]]`
$Out[8] =$ `-n MultiSum`F[-1 + n, i, j] + n MultiSum`F[n, i, j] ==`
`Delta[i, -i MultiSum`F[n, i, j]] + Delta[j, -j MultiSum`F[n, i, j]]`

This recurrence yields a closed form:

$In[9] :=$ `w3 = TermRekToForm[summand, rek]`

$$Out[9] = \frac{\binom{R}{i}\binom{S}{j}\binom{T}{-i-j+n}}{\binom{R+S+T}{n}}$$

$$\left(-\frac{(i+j-n)\ (-1+n-R-S-T)}{n\ (-1-i-j+n-T)}\mathrm{d}i\mathrm{d}j + \frac{j}{n}\mathrm{d}n\mathrm{d}i + -\frac{i}{n}\mathrm{d}n\mathrm{d}j\right)$$

Entering forms manually. Suppose we want to enter the form
$$Out[9] = \left(x^a\ y^b\ (x+y)^{-a-b}\ \binom{a+b}{a,\ b}\right)\left(-\frac{b}{a+b}\mathrm{d}a + \frac{a}{a+b}\mathrm{d}b\right)$$
at the keyboard. First we enter the common hypergeometric factor. Note that we have to call the function toPht to convert it to our internal representation for hypergeometric terms.

$In[10] := \texttt{hyp} = \texttt{Multinomial[a,b] x^a y^b (x+y)^(-a-b)// toPht}$

$$Out[10] = \mathtt{x^a\ y^b\ (x+y)^{-a-b}} \left(\begin{array}{c} \mathtt{a+b} \\ \mathtt{a,\ b} \end{array} \right)$$

Next we input the rational function coefficients of the desired form:

$In[11] := \texttt{rat1} = \texttt{toPht[a/(a+b)]}$

$$Out[11] = \frac{\mathtt{a}}{\mathtt{a+b}}$$

and

$In[12] := \texttt{rat2} = \texttt{toPht[-b/(a+b)]}$

$$Out[12] = \frac{\mathtt{b}}{\mathtt{-a-b}}$$

Finally we "assemble" the desired form.

$In[13] := \texttt{w4} = \texttt{hyp} * \texttt{(rat1 d[b] + rat2 d[a])}$

$$Out[13] = \left(\mathtt{x^a\ y^b\ (x+y)^{-a-b}} \left(\begin{array}{c} \mathtt{a+b} \\ \mathtt{a,\ b} \end{array} \right)\right) \left(\frac{\mathtt{b}}{\mathtt{-a-b}}\mathtt{da} + \frac{\mathtt{a}}{\mathtt{a+b}}\mathtt{db}\right)$$

A.2 Closedness Preserving Substitutions

WZ forms yield new WZ forms by closedness preserving substitutions. If <rule> is a substitution, or a list of substitutions, then `cps`[<rule>] is the corresponding closedness preserving substitution. As a first example, we explain a "dualize and specialize-miracle" of D. Zeilberger by a closedness preserving substitution.

$In[14] := \texttt{w6}$

$$Out[14] = \frac{\left(\begin{array}{c} \mathtt{a} \\ \mathtt{k} \end{array} \right) \left(\begin{array}{c} \mathtt{n} \\ \mathtt{k} \end{array} \right)}{\left(\begin{array}{c} \mathtt{a+n} \\ \mathtt{n} \end{array} \right)} \left(-\frac{\mathtt{k^2}}{\mathtt{(1+a-k)\ (1+a+n)}}\mathtt{da} + \mathtt{1dk} + \frac{\mathtt{k^2}}{\mathtt{(-1+k-n)\ (1+a+n)}}\mathtt{dn} \right)$$

$In[15] := \texttt{w-new} = \texttt{w6/.cps[a->n]}$

$$Out[15] = \frac{\left(\left(\begin{array}{c} \mathtt{n} \\ \mathtt{k} \end{array} \right)\right)^2}{\left(\begin{array}{c} \mathtt{2\ n} \\ \mathtt{n} \end{array} \right)} \left(\mathtt{1dk} + \frac{\mathtt{k^2\ (-3+2\ k-3\ n)}}{\mathtt{2\ (-1+k-n)^2\ (1+2\ n)}}\mathtt{dn} \right)$$

For another example, we show how to symmetrize the form

$In[16] := \texttt{w2}$

$$Out[16] = \left(\mathtt{2^{-n}} \left(\begin{array}{c} \mathtt{n} \\ \mathtt{k} \end{array} \right)\right) \left(\mathtt{1dk} + \frac{\mathtt{k}}{\mathtt{2\ (-1+k-n)}}\mathtt{dn} \right)$$

which contains a binomial coefficient $\frac{n!}{k!(n-k)!}$. To get rid of its ugly asymmetric denominator, we substitut n->n+k:

$In[17] := \texttt{w-halfdone} = \texttt{w2/.cps[n->n+k]}$

$$Out[17] = \left(\mathtt{2^{-k-n}} \left(\begin{array}{c} \mathtt{k+n} \\ \mathtt{k} \end{array} \right)\right) \left(\frac{1}{2}\mathtt{dk} + -\frac{\mathtt{k}}{\mathtt{2\ (1+n)}}\mathtt{dn} \right)$$

Now the denominator of the pure hypergeometric factor is symmetric. We wonder if the rational coefficients can be made symmetric too, and we try a couple of shifts. One of them is indeed successful:

$In[18] := \texttt{w-symmetric} = \texttt{shift[n,-1][w-halfdone]}$

$Out[18] = (2^{-k-n} \left(\begin{array}{c} k+n \\ k \end{array} \right)) \left(\frac{n}{k+n} dk + -\frac{k}{k+n} dn \right)$

A.3 Computing Exterior Derivatives

The application d[w] computes the exterior derivative of w (which must be a form). Since all forms introduced so far are closed by construction, their respective exterior derivatives are zero. For example,

In[19] := d[w−symmetric]

Out[19] = 0

A.4 Ranges and Boundaries.

Entering ranges. Suppose we need the range (−[a + b == k])da + ([a + b == k])db. We can either enter it directly at the keyboard

In[20] := r1 = −i[a + b == k]d[a] + i[a + b == k]d[b]

Out[20] = 1 ((−[a + b == k])da + ([a + b == k])db)

or we define it as the boundary of a halfspace:

In[21] := interior−of−r1 = i[a + b < k] d[a, b]

Out[21] = 1 (([a + b < k])dadb)

In[22] := r1 = boundary[interior−of−r1]

Out[22] = 1 ((−[a + b == k])da + ([a + b == k])db)

The second method is recommended for all but the simplest ranges. For example,

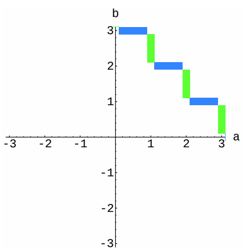
In[23] := interior−of−r2 = i[a + b < k] i[a >= 0]i[b >= 0] d[a, b];
r2 = boundary[interior−of−r2]

Out[23] = 1 ((−[a + b == k] [a ≥ 0] [b ≥ 0] + [b == 0] [a ≥ 0] [a + b < 1 + k])da+
([a + b == k] [a ≥ 0] [b ≥ 0] − [a == 0] [b ≥ 0] [a + b < 1 + k])db)

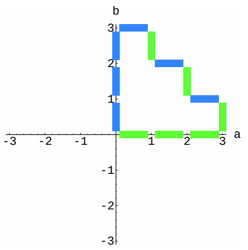
would be hard to enter directly.

Plotting ranges. To check if we get indeed the ranges we have in mind we plot them.

In[24] := Block[{k = 3},
dstPlot[{0, r1, 0}]];



$In[25] := \texttt{Block}[\{\texttt{k} = 3\},$
$\qquad \texttt{dstPlot}[\{0, \texttt{r2}, 0\}]];$



Of course we have to set all parameters, like k in the example above, of the range to be plotted to fixed integers. This can be conveniently done by wrapping a Block[{k=...},...] around.

A.5 Summing Forms over Ranges

We sum the form

$In[26] := \texttt{w4}$

$$Out[26] = \left(\texttt{x}^{\texttt{a}}\ \texttt{y}^{\texttt{b}}\ (\texttt{x}+\texttt{y})^{-\texttt{a}-\texttt{b}}\ \left(\begin{array}{c}\texttt{a}+\texttt{b}\\ \texttt{a},\ \texttt{b}\end{array}\right)\right)\left(\frac{\texttt{b}}{-\texttt{a}-\texttt{b}}\texttt{da}+\frac{\texttt{a}}{\texttt{a}+\texttt{b}}\texttt{db}\right)$$

over the range

$In[27] := \texttt{r1}$

$$Out[27] = 1\ ((-[\texttt{a}+\texttt{b} == \texttt{k}])\texttt{da} + ([\texttt{a}+\texttt{b} == \texttt{k}])\texttt{db})$$

by issuing the command

$In[28] := \texttt{wwSum}[\texttt{r1}, \texttt{w4}]$

$$Out[28] = \sum\nolimits_{\texttt{ab}_{\texttt{a}+\texttt{b}==\texttt{k}}} \texttt{x}^{\texttt{a}}\ \texttt{y}^{\texttt{b}}\ (\texttt{x}+\texttt{y})^{-\texttt{a}-\texttt{b}}\ \left(\begin{array}{c}\texttt{a}+\texttt{b}\\ \texttt{a},\ \texttt{b}\end{array}\right)$$

A.6 Bugs

Please report bugs to `B.Zimmermann@risc.uni-linz.ac.at`.

B Some Closed Forms

All closed forms in the following list are included in the package `wz.m`.

$$\texttt{example["dent"]} = (-1)^{b}\, 2^{-k+2v} \binom{k-2v}{s-b} \binom{k-v}{k-2v}^{-1} \binom{-s+k}{-b+2v} \binom{s}{b}$$
$$\left(\frac{(b-1)\, b\, (-s+k+b-2v-1)\, (-s+k+b-2v)^2}{2\, (b-2v-1)\, (-k+v-1)\, (-s+b-2)\, (-s+b-1)^2}\, dk\, ds + \frac{-b\, (-s+k+b-2v)^2}{(b-2v-1)\, (-s+b-1)^2}\, db\, ds \right.$$
$$\left. + \frac{b\, (k-2v+1)\, (-s+k+1)}{2\, (b-2v-1)\, (-k+v-1)\, (-s+b-1)}\, db\, dk \right)$$

[Den96]

$$\texttt{example["jaegers"]} = 2 \binom{j+i}{j}^2 \binom{n-j-i+m}{n-j}^2 \binom{2n+2m+2}{2n+1}^{-1}$$
$$\left(\frac{-(i-1)^2\, i^2\, (-n+j+i-m-1)^2\, (i\,n-n-m\,j-j)}{(i-m-1)^2\, (j+i-1)^2\, (j+i)^2\, n\, (2n+1)}\, dj\, dn + \frac{-i^2\, j^2\, (-n+j+i-m-1)^2}{(i-m-1)^2\, (j+i-1)^2\, (j+i)^2\, (-n+j-1)^2\, n\, (2n+1)} \right.$$
$$\big(2jn^3 - in^3 - n^3 - 4j^2n^2 - 2ijn^2 + 5mjn^2 + 11jn^2 + 2i^2n^2 - 2min^2 - 2in^2 - 3mn^2 - 5n^2 + 2j^3n + 3ij^2n - 6mj^2n - 11j^2n$$
$$-4mijn - 8ijn + 4m^2jn + 18mjn + 18jn + i^2n + 2min + 3in - 3m^2n - 10mn - 8n + mj^3 + j^3 + 2mij^2 + 2ij^2 - 2m^2j^2$$
$$-7mj^2 - 5j^2 - 4mij - 4ij + 4m^2j + 11mj + 7j + 2mi + 2i - 2m^2 - 5m - 3\big)\, di\, dn$$
$$\left. + \frac{i^2\, j^2\, (-n+j+i-m-1)^2\, (n+m+1)\, (2n+2m+1)}{(i-m-1)^2\, (j+i-1)^2\, (j+i)^2\, n\, (2n+1)}\, di\, dj \right)$$

[,]

$$\texttt{example["hong"]} = (-1)^{i+v+r} \binom{i}{-q+r} \binom{i}{-q+v} \binom{n}{r}^{-1} \binom{n}{v}^{-1} \binom{n}{i} \binom{n-i}{q} \binom{n-i}{q+n-v-r} \left(\frac{-q\, (q+i-r)\, (q+i-v)\, (q+n-v-r)\, (q^2+i\,q-2q-i-r\,v+1)}{(n-r)\, (n-v)\, (q-r-1)\, (q-v-1)\, (q+i-v-r-1)\, (q-n+i-1)}\, dq\, dn \right.$$
$$\left. + \frac{n\, q\, (q+i-r)\, (q+i-v)\, (q+n-v-r-1)\, (q+n-v-r)}{(-n+i)\, (n-r)\, (n-v)\, (-q+r+1)\, (q-v-1)\, (q+i-v-r-1)}\, dq\, di + \frac{-(q-1)\, q\, (q+i-1)\, (q+i-r)\, (q+i-v)\, (q+n-v-r-1)\, (q+n-v-r)}{(-n+i)\, (n-r)\, (n-v)\, (q-r-1)\, (q-v-1)\, (q+i-v-r-1)\, (q-n+i-1)}\, dn\, di \right)$$

[Hon96,]

$$\texttt{example["gkp5.22"]} = \binom{r}{m+k} \binom{s}{n-k} \binom{s+r}{n+m}^{-1} \left(\frac{m+k}{s+r+1}\, ds + \frac{(m+k)\, (s-n+k)}{(-r+m+k-1)\, (s+r+1)}\, dr + \frac{-(m+k)\, (s-n+k)}{(-n+k-1)\, (-s-r+n+m)}\, dn + \frac{s-n+k}{s+r-n-m}\, dm + 1\, dk \right)$$

[GKP89, 5.22]

$$\texttt{example["gkp5.23"]} = \binom{l}{m+k}\binom{s}{n+k}\binom{s+l}{n-m+l}^{-1}\left(\frac{(m+k)\,(n+k)}{(-s+n+k-1)\,(s+l+1)}\,ds + \frac{-m-k}{-s+n-m}\,dn + \frac{-n-k}{-n+m-l}\,dm + \frac{(m+k)\,(n+k)}{(m-l+k-1)\,(s+l+1)}\,dl + 1\,dk\right)$$

[GKP89, 5.23]

$$\texttt{example["gkp5.24"]} = (-1)^{-m-l+k}\binom{l}{m+k}\binom{s+k}{n}\binom{s-m}{n-l}^{-1}\left(\frac{m+k}{-s+m-1}\,ds + \frac{-(m+k)\,(s-n+k)}{(n+1)\,(-s+n+m-l)}\,dn + \frac{-s+n-k}{-s+n+m-l}\,dm + \frac{(m+k)\,(s-n+k)}{(m-l+k-1)\,(-n+l)}\,dl + 1\,dk\right)$$

[GKP89, 5.24]

$$\texttt{example["gkp5.25"]} = (-1)^{-m-l+k}\binom{l-k}{m}\binom{s}{-n+k}\binom{s-m-1}{-n-m+l}^{-1}\left(\frac{(-l+k-1)\,(-n+k)}{(-s+m)\,(s+n-k+1)}\,ds + \frac{-(-l+k-1)\,(-n+k)}{(-n-m+l)\,(s+n-k+1)}\,dn + \frac{(-l+k-1)\,(-n+k)}{(m+1)\,(n+m-l)}\,dm + \frac{-(-l+k-1)\,(-n+k)}{(m-l+k-1)\,(-s-n+l+1)}\,dl + 1\,dk\right)$$

[GKP89, 5.25]

$$\texttt{example["gkp5.26"]} = \binom{l-k}{m}\binom{q+k}{n}\binom{q+l+1}{n+m+1}^{-1}\left(\frac{l-k+1}{q+l+2}\,dq + \frac{-(-l+k-1)\,(q-n+k)}{(n+1)\,(-q+n+m-l)}\,dn + \frac{(-l+k-1)\,(q-n+k)}{(m+1)\,(-q+n+m-l)}\,dm + \frac{-(-l+k-1)\,(q-n+k)}{(m-l+k-1)\,(q+l+2)}\,dl + 1\,dk\right)$$

[GKP89, 5.26]

$$\texttt{example["gkp5.27"]} = \binom{r}{k}\binom{s}{n-k}\binom{s+r}{n}^{-1}\left(\frac{k}{s+r+1}\,ds + \frac{k\,(s-n+k)}{(-r+k-1)\,(s+r+1)}\,dr + \frac{-k\,(s-n+k)}{(-n+k-1)\,(-s-r+n)}\,dn + 1\,dk\right)$$

[GKP89, 5.27]

$$\texttt{example["gkp5.28"]} = \binom{r}{m}^{-1} \binom{r+k}{n+m} \binom{-s+r+n}{n-k} \binom{s}{n}^{-1} \binom{s-r+m}{k}$$

$$\left(\frac{k\ (r-n-m+k)\ (-s+r+k)}{(-s+r-m+k-1)\ (-s+r+n)\ (s+1)}\, ds + \frac{-k\ (-s+2\,r-m+k+1)}{(r+1)\ (-s+r-m)}\, dr + \frac{-k\ (r-n-m+k)\ (-s+r+k)}{(-n+k-1)\ (n+m+1)\ (-s+n)}\, dn + \frac{k\ (r-n-m+k)\ (-s+r+k)}{(n+m+1)\ (-r+m)\ (s-r+m-k+1)}\, dm + 1\, dk \right)$$

[GKP89, 5.28]

$$\texttt{example["gkp5.29"]} = (-1)^{k} \binom{b+a}{k+a} \binom{c+a}{k+c} \binom{c+b}{b}^{-1} \binom{c+b}{k+b} \binom{c+b+a}{a}^{-1}$$

$$\left(1\, dk + \frac{-\,(k+a)\ (k+b)}{2\ (c+b+a+1)\ (-k+c+1)}\, dc + \frac{-\,(k+a)\ (k+c)}{2\ (c+b+a+1)\ (-k+b+1)}\, db + \frac{-\,(k+b)\ (k+c)}{2\ (c+b+a+1)\ (-k+a+1)}\, da \right)$$

[GKP89, 5.29]

$$\texttt{example["gkp5.30"]} = (-1)^{k} \binom{b+a}{a}^{-1} \binom{b+a}{k+a} \binom{b+a}{k+b} \left(1\, dk + \frac{-k-a}{2\,(-k+b+1)}\, db + \frac{-k-b}{2\,(-k+a+1)}\, da \right)$$

[GKP89, 5.30]

$$\texttt{example["ep11"]} = (-1)^{-n+k}\, 4^{-m+k} \binom{k}{m} \binom{n+k}{2\,k} \binom{n+m}{2\,m}^{-1} \left(\frac{-(2\,k-1)\,(-m+k)}{(-n+k-1)\,(n+k)}\, dn + \frac{-(2\,k-1)\,(-m+k)}{2\,(-n+m)\,(n+k)}\, dm + \frac{n+m}{n+k}\, dk \right)$$

[Ego84, p.11]

$$\texttt{example["ep24rest0"]} = (-1)^{-3\,n+k} \binom{3\,n-k}{k} \left(\frac{k\left(-18\,n^2+9\,k\,n-15\,n+k-1\right)}{2\,(-3\,n+2\,k-3)\,(-3\,n+2\,k-2)\,(-3\,n+2\,k-1)}\, dn + \frac{-3\,n}{2\,(-3\,n+k)}\, dk \right)$$

[Ego84, p.24]

$$\texttt{example["ep27"]} = (-1)^{-v-m}\, 2^{-v+2\,m} \binom{r+2\,m}{m}^{-1} \binom{r+2\,m}{v+r} \binom{v+2\,r+2\,m}{v} \left(1\, dv + \frac{v}{-2\,r-2\,m-1}\, dr + \frac{2\,(2\,r+4\,m+3)\,v\,(v+r)}{(2\,r+2\,m+1)\,(-v+2\,m+1)\,(-v+2\,m+2)}\, dm \right)$$

[Ego84, p.27]

example["ep47"] $= (-1)^{k} \binom{r}{k} r!^{-1} \left(\frac{-k\left(-k r^3+a r^3+r^3+k^2 r^2-a k r^2-2 k r^2+r^2+k^2 r-2 k r+r+a k-k-a+1\right)}{(-r+k-1)(r-1) r (r+1)} \, dr + (-k+a) \, dk + \frac{-k}{r} \, da \right)$

[Ego84, p.47]

example["e2.6"] $= \binom{2k}{k} \binom{2n+2}{n+1}^{-1} \binom{2n-2k}{n-k} \left(\frac{-k(-3n+2k-5)(-2n+2k-1)}{(-n+k-2)(-n+k-1)(n+1)(2n+3)} \, dn + \frac{-n-2}{(k+1)(-n+k-1)} \, dk \right)$

[Ego84, 2.6]

example["ep48"] $= 2^{-2n+k} n!^{-1} (n-k)!^{-1} (2n-k)! \left(\frac{-k(-2n+k-1)}{4(-n+k-1)(n+1)} \, dn + 1 \, dk \right)$

[Ego84, p.48]

example["ep49"] $= \binom{2n-2k}{n-k} \binom{q+2k}{k} \binom{q+2n}{n}^{-1} \left(\frac{2k(-2n+2k-1)}{(q+2k)(q+2n+1)} \, dq + \frac{-2k(-2n+2k-1) q (q+k)}{(-n+k-1)(q+2k)(q+2n+1)(q+2n+2)} \, dn + \frac{q}{q+2k} \, dk \right)$

[Ego84, p.49]

example["ep52"] $= \binom{m}{k}^{2} \binom{n+m}{n}^{-2} \binom{n+2m-k}{2m} \left(\frac{-k^2(-n-2m+k-1)}{(-n+k-1)(n+m+1)^2} \, dn + \frac{-k^2(-n-2m+k-1)\left(-3mn+2kn-3n-4m^2+3km-8m+3k-4\right)}{2(-m+k-1)^2(2m+1)(n+m+1)^2} \, dm + 1 \, dk \right)$

[Ego84, p.52]

example["e2.18;1"] $= (-1)^{-n+k} 2^{2k-2a} \binom{k}{a} \binom{n+a}{2a}^{-1} \binom{n+k}{2k} \left(\frac{2(2a+1)(-k+a)(2k-1)(n+1)}{(-n+k-1)(n+a+1)(2n+1)(2n+3)} \, dn + \frac{2a+1}{2n+1} \, dk + \frac{(2a+1)(-k+a)(2k-1)}{2(-n+a)(n+a+1)(2n+1)} \, da \right)$

[Ego84, 2.18;1]

example["ep59"] $= \binom{n-1}{-r+i} \binom{r+1}{2r-i} \binom{r+n}{r}^{-1} \left(\frac{(-r+i)(-r+i+1)\left(3r^2+2nr-3ir+5r-in+2n+i^2-2i+2\right)}{(-2r+i-2)(-2r+i-1)(-r-n+i)(r+n+1)} \, dr + \frac{(-r+i)(-r+i+1)}{(-r-n+i)(r+n+1)} \, dn + 1 \, di \right)$

[Ego84, p.59]

example["gauss"] $= n!^{-1} (\Gamma(c))^{-1} \Gamma(c-a) \Gamma(c-b) (\Gamma(c-b-a))^{-1} a^{\overline{n}} b^{\overline{n}} c^{\overline{n}-1} \left(1 \, dn + \frac{n}{c-b-a} \, dc + \frac{n(n+c-1)}{b(-c+b+1)} \, db + \frac{n(n+c-1)}{a(-c+a+1)} \, da \right)$

example["e3.17"] $= (-1)^{k+j} \binom{p+i}{p+k} \binom{p+k}{p+j} \left(\frac{k-j}{-p-j-1} \, dp + 1 \, dk + \frac{(-k+j)(jp-ip+p+jk-ik+j+1)}{(-j+i-1)(-j+i)(p+j+1)} \, dj + \frac{-(-k+j)(jp-ip+jk-ik-k+i+1)}{(-j+i)(-j+i+1)(-k+i+1)} \, di \right)$

[Ego84, 3.17]

example["e3.4.2;1"] $= (-1)^{k+j} \binom{i}{k} \binom{k}{j} \left(1 \, dk + \frac{(-k+j)(jk-ik+j+1)}{(-j+i-1)(-j+i)(j+1)} \, dj + \frac{-(-k+j)(jk-ik-k+i+1)}{(-j+i)(-j+i+1)(-k+i+1)} \, di \right)$

[Ego84, 3.4.2;1]

$$\texttt{example["e3.4.2;2"]} = (-1)^{k+j} \binom{p-j}{p-k} \binom{p-k}{p-i} \left(\frac{k-j}{-p+i-1}\, dp + 1\, dk + \frac{(-k+j)\ (-j\,p+i\,p-p-j\,k+i\,k+2\,j^2-2\,i\,j+2\,j-i)}{(-j+i-1)\ (-j+i)\ (-p+j)}\, dj + \frac{-(-k+j)\ \left(j\,p-i\,p+j\,k-i\,k-k-2\,i\,j-j+2\,i^2+2\,i+1\right)}{(-j+i)\ (-j+i+1)\ (-k+i+1)}\, di \right)$$

[Ego84, 3.4.2;2]

$$\texttt{example["e3.4.2;3"]} = (-1)^{k+j} \binom{i-1}{k-1} \binom{k-1}{j-1}\, i!\, j!^{-1} \left(1\, dk + \frac{(-k+j)\ \left(j\,k-i\,k+j^3-i\,j^2+j^2+i\right)}{(-j+i-1)\ (-j+i)\ j\ (j+1)}\, dj + \frac{-(-k+j)\ \left(j\,k-i\,k-k+i^2\,j-j-i^3+2\,i+1\right)}{(-j+i)\ (-j+i+1)\ (-k+i+1)}\, di \right)$$

[Ego84, 3.4.2;3]

$$\texttt{example["dixon"]} = (-1)^{k} \binom{b+a}{k+a} \binom{c+a}{k+c} \binom{c+b}{k+b}\, a!\, b!\, c!\ (c+b+a)!^{-1} \left(1\, dk + \frac{-(k+a)\ (k+b)}{2\,(c+b+a+1)\ (-k+c+1)}\, dc + \frac{-(k+a)\ (k+c)}{2\,(c+b+a+1)\ (-k+b+1)}\, db + \frac{-(k+b)\ (k+c)}{2\,(c+b+a+1)\ (-k+a+1)}\, da \right)$$

$$\texttt{example["ep170"]} = \binom{m}{k} \binom{n}{k} \binom{p+m}{m}^{-1} \binom{p+n}{n}^{-1} \binom{p+n+m-k}{n+m} \left(\frac{-k^2\ (-p-n-m+k-1)}{(-p+k-1)\ (p+m+1)\ (p+n+1)}\, dp + \frac{-k^2\ (-p-n-m+k-1)}{(-n+k-1)\ (n+m+1)\ (p+n+1)}\, dn + \frac{-k^2\ (-p-n-m+k-1)}{(-m+k-1)\ (n+m+1)\ (p+m+1)}\, dm + 1\, dk \right)$$

[Ego84, p.170]

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