Improvements of the Algorithm for Resolution of Singularities

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Abstract

This paper contains several improvements of Villmayor's algorithm for the problem of resolution of singularities of hypersurfaces. First, a condition is presented for avoiding superfluous intermediate resolution problems to be generated (computation of coefficient objects with respect to many hypersurfaces). Then a new representation of algebraic sets is developed to avoid rising the generators of their vanishing ideals to extremely high powers. Based on this improvement, a new representation of basic objects (i.e. resolution problems) is introduced, enabling the algorithm to exploit special structures appearing in them. Finally, a technique is presented for the efficient management of charts of the describing atlas of blown up varieties.

1 Introduction

The problem of resolution of singularities, briefly, is: Given a variety X, find a *desingularization* of it, which means a nonsingular variety Y together with a proper, birational morphism $\pi : Y \to X$. Its solution has many applications in computational algebraic geometry, such as the analysis and classification of singularities, or the parametrization of algebraic curves and surfaces (see [Sch98]).

The resolution problem has remarkable history: For curves, the existence of a resolution was already known in the 19th century. In the surface case, Walker gave an analytic proof [Wal35], and Zariski gave an algebraic proof [Zar39]. Five years later, Zariski proved resolution of 3-folds [Zar44]. Then it took quite some time until Hironaka came up with his famous result, proving the existence of resolution in arbitrary dimension [Hir64]. Hironaka's result holds only for algebraic sets over fields of characteristic zero. In positive characteristic, there are some partial results [Abh65, Lip78], but the general case is still open. For an account of the difficulties in positive characteristic, the reader should consult [Hau98].

Hironaka's existence proof is not constructive. Later, [Vil89, Vil96, EV00, EV98] and independently [BM91, BM97] have come up with strategies to find canonical resolution. More precisely, these authors define *stratifying functions* (or *discrete local invariants*) with the property that, if one repeatedly blows up the closed subset where the function attains its maximum, one eventually achieves a resolution. From an algorithmic point of view, these results were major breakthroughs, nevertheless the implementation of such algorithms is far from easy. The definitions of these functions are rather complicated in both approaches: It is not at all trivial how to compute them and how to determine the closed subsets where the maximum is attained.

Recently [BS00b, BS00a] came up with an algorithm based on the results of Villamayor (we felt justified to name it Villamayor's algorithm), which copes with the above mentioned problems. The algorithm has also been implemented in Maple V; however, the implementation pointed out many disadvantages of the original formulation. Even the simplest examples behaved extremely bad in terms of computational complexity, due to the inappropriate representation of algebraic sets, and the great amount of superfluous computations indicated by some "unoptimized" operations.

The implementation, of course, contains all the solutions described in this paper, and uses even more new ideas, which we don't deal with here. In the algorithm, various geometric subtasks like computing the singularities, computing hypersurfaces of maximal contact etc., are reduced to Gröbner bases computations (see [Buc65, Buc85, Win88, BW93]).

Apart from above mentioned algorithms, we are not aware of any other complete resolution algorithms or implementations for singularities in dimension higher than one, except [ALR88, CCLM98] which treat particular classes of surface singularities.

Villamayor's algorithm proceeds inductively: in order to improve a resolution problem, a new resolution problem is generated, represented by an ideal and a prescribed threshold (and some other data which are irrelevant here). Every point where the order of the ideal is greater than or equal to the threshold is a singularity that must be resolved.

In section 4 we present when and how we can avoid generating superfluous resolution problems, by computing coefficient objects with respect to many hypersurfaces of maximal contact. This way we can skip those intermediate stages that would have been introduced if we got to the final stage in several steps taking the hypersurfaces one by one.

A computational problem arises from the fact that the ideals describing the new resolution problem are constructed by taking polynomials to powers. This powering seems necessary to give different components of the ideal different weights, by which the order of the singularity is divided. As this powering accumulates, the expression swell gets enormous and makes it impossible to even compute the smallest nontrivial examples.

The only way to avoid the powering is to give up the idea that the singular situation has to be described by ideals. In section 5, we introduce "generalized ideals" in order to avoid this powering process, comparable with the "idealistic exponents" introduced in [Hir77] for a different purpose.

The second improvement arose from the observation that Villamayor's algorithm often performs expensive computations on ideals (such as computing zero sets, partial derivatives, and so on) in situations where simple combinatorial reasoning would suffice. In section 6 we present a representation of basic objects which builds on the notion of generalized ideals. In section 7 we define all the necessary operations for the new representation.

Another improvement, suggested in [EV98], has its origins in [Abh88]. One observes that there are resolution problems which could be resolved by blowing ups on centers that are found on the ground of combinatorial reasoning. Such a situation is called “good”. However, Villamayor’s algorithm follows the recursive structure, trying to improve the situation by creating new resolution problems. The “good points strategy” is simply to trap these situations and to resolve them directly. Using the above representation, it was very easy to implement this improvement.

In every section, well definedness and correctness proofs are presented for the newly introduced notions and operations. At last, in section 8, we prove that the original algorithm and the new, using all the improvements we presented, compute equivalent resolutions.

Villamayor’s algorithm computes a resolution by repeatedly blowing up non-singular subvarieties. There are implementations for performing blowing ups in various computer algebra systems like [GPS98] and [BSS93], but these computations are time consuming. In [BS00a], the blown up varieties are represented more efficiently by an atlas of charts.

The chart representation also has its inherent redundancy, which comes from the presence of isomorphic subsets (we simply call them chart overlaps). Let the charts of the covering of the blownup be uniquely indexed by natural numbers. A relevant point of a chart with a given index is one that lies in no chart of smaller index. Now it is easy to see that a chart that has no relevant point is redundant, while to detect such a chart efficiently (without examining chart change maps explicitly) is not at all trivial.

Another source of redundancy comes from multiple resolution of singularities that lie in overlaps of charts. Obviously, this multiple resolution should be avoided for the sake of efficiency. However, we do not want to simply leave unresolved singularities in the output of the algorithm, even if at one of the overlapping charts the singularity is resolved.

In section 9 we present a surprisingly simple method that copes with both problems, requiring a minimal bookkeeping during the resolution. The method seems to be applicable whenever we deal with representations of varieties by covering charts, not just for the resolution problem.

It is still an open problem to give a complexity bound for the algorithm. The difficulty is that we do not have an upper bound for the number of required blowing up steps in Villamayor’s algorithm. We could compute full resolutions for various surface singularities (see `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup/`), but we have not yet succeeded to resolve any nontrivial singularities in higher dimension.

As a concluding demonstration, we present an example in section 10, the first steps of the resolution of a surface, in which we follow the operations in both the old and the new representation.

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The notation and structure of *charts* and *objects* of the original algorithm come from the paper [BS00a].

2 Preliminary definitions

Henceforth k denotes a computable field of characteristic zero. We consider varieties over k. A *morphism* means a regular map. A morphism is called *proper* iff it is not a restriction of a map from some larger domain to an open dense subset. A morphism is *birational* if it has a rational inverse, i.e. an inverse map on some open dense subset of the image of the morphism, that can be given by rational functions.

Problem 1: Given a variety X, find a *desingularization* of it, which means a nonsingular variety Y together with a proper, birational morphism $\pi : Y \to X$.

At a nonsingular point p of a variety W of dimension n, $f_1, \ldots, f_n \in k(W)$ form a *regular system of parameters* iff they all vanish at p, $f_1, \ldots, f_n$ are regular on some open $U \subseteq W$ containing p, and $df_1, \ldots, df_n$ generate the module of differential forms $\Omega(U)$.

An algebraic set, contained in the nonsingular variety W, has at most *normal crossings*, iff for all points $x \in W$, there exists a system of regular parameters such that each irreducible component of the algebraic set is given by a subset of this parameter set locally at x.

Let W be a nonsingular variety, and $X \subset W$ be a variety of codimension 1. By abusing the language we call the pair (W, X) a *hypersurface*. If $W = \mathbb{A}^n$, (W, X) is an affine hypersurface, if $W = \mathbb{P}^n$, (W, X) is a projective hypersurface.

Problem 2: Given a hypersurface (W, X), find an *embedded desingularization* of it, which is a nonsingular variety W' together with a proper, birational morphism $\pi : W' \to W$, such that the *total transform*, $\pi^{-1}(X)$, has at most normal crossings.

An embedded desingularization of (W, X) provides indeed a desingularization of X. The rational inverse is defined on an open dense subset of X, and the closure of its image, say, X'—also called as the *proper transform* of X along π—is nonsingular in W' (it is one of the components of the total transform). Now X' together with the map $\pi|_{X'} : X' \to X$ is a desingularization of X.

Such a desingularization is usually constructed using the technique of blowing up, which we now describe.

Let X be an affine variety over k with $a_1, \ldots, a_r$ generators for its coordinate ring. Let $C \subset X$ be a variety with vanishing ideal $\mathbf{I}(C) = \langle g_0, \ldots, g_s \rangle$. The *blowing up* of X at C is defined as a proper, birational morphism $\pi : Y \to X$, where $Y \subset \mathbb{A}^r \times \mathbb{P}^s$ with ideal

$$\mathbf{I}(Y) = \{P \in k[x_1, \ldots, x_r, y_0, \ldots, y_s] \mid P(a_1, \ldots, a_r, tg_0, \ldots, tg_s) = 0 \text{ in } k[X][t]\}.$$

The morphism π is just the projection on the first r coordinates. The ideal $\mathbf{I}(Y)$ is homogeneous in the y's. The preimage of C in Y, denoted by E, is called the *exceptional divisor* of the blown up variety.

Example 2.1. Let $X = \mathbb{A}^n$, $C = \mathbf{Z}(x_1, \ldots, x_r)$, $r \leq n$. The blowing up of X along C is $\pi : Y \to X$, where $Y \subset \mathbb{A}^n \times \mathbb{P}^{r-1}$,

$$\mathbf{I}(Y) = \langle x_i y_j - x_j y_i \mid i, j = 1, \ldots, r \rangle_{k[x_1, \ldots, x_n, y_1, \ldots, y_r]},$$

π is the projection on the x_i's. The blown up variety can be covered by r affine charts of dimension n, covering $\mathbb{P}^{r-1}$ by its principal open subsets. So the first chart ($y_1 \neq 0$) is generated by $x_1, y_2, \ldots, y_r, x_{r+1}, \ldots, x_n$, where the x_j's were eliminated by the equations $x_1 y_j = x_j,\ j = 2, \ldots, r$.

For a hypersurface S containing C in its singular locus with multiplicity d, the total transform in the first chart is of the form $x_1^d \cdot g(x_1, y_2, \ldots, y_r, x_{r+1}, \ldots, x_n)$, where $x_1 = 0$ is the equation of E, the exceptional divisor, and $g = 0$ is of the proper transform.

In this paper, if not defined explicitly, U denotes an n-dimensional affine chart of the manifold W, with global regular system of local parameters IND $= (P_1, \ldots, P_n)$ where 'global' means that they give rise to a regular system of parameters at each point of U.

3 The summation of the original algorithm

In this section, we describe the overall structure of Villamayor's resolution algorithm. It has been obtained by rewriting the inductive definition of Villamayor's stratifying function [Vil89, Vil96, EV00] in an algorithmic way. The comprehensive treatment of the details can be found in [BS00a].

Villamayor's definition of the stratifying function is based on an intermixed induction on the order and the dimension, also involving the exceptional divisors that have arisen from previous blowing ups. He introduces *basic objects* of the kind (W, J, b, E), where

- W is a nonsingular algebraic set,
- J is a nonzero ideal in $k[W]$ (initially the ideal of the hypersurface),
- b is a positive integer,
- E is a finite sequence of regular hypersurfaces of W with normal crossing (the exceptional divisors up to the current stage).

We sometimes omit components of basic objects when they are not needed.

In the following, we use "partial derivatives". If $W = \mathbb{A}^n$, these have the usual meaning. Otherwise, we can define the partial derivatives only locally with respect to a system of parameters.

The *singular locus* of a basic object (W, J, b, E) is defined by

$$\mathrm{Sing}(J, b) := \{x \mid \mathrm{ord}_x(J) \geq b\},$$

where

$$\mathrm{ord}_x(J) := \min\{\mathrm{ord}_x(f) \mid f \in J\},$$

and $\mathrm{ord}_x(f)$ is the smallest integer m such that there exists a partial derivative of f of order m not vanishing at x. The order of J at x is positive iff x is in the zero set of J.

A basic object is resolved iff its singular locus is empty. When trying to resolve a basic object (W, J, b, E), we do not want to change anything outside its singular locus. This motivates the third restriction: the blowing up center must be contained in the singular locus. It ensures that the total transform of

J (i.e. the ideal that is generated by the pullbacks of elements in J) can be written as $\mathbf{I}(F)^b \cdot J'$, where F is the exceptional divisor.

Let W' be the blown up. Let E' be the sequence of all proper transforms of divisors in E, together with the new exceptional divisor. Let J' be as above. Then the basic object (W', J', b, E') is called the *transform* of (W, J, b, E). In the algorithm, this is going to be the next basic object to be resolved.

We say that a blowing up operation is *permissible* with respect to the basic object (W, J, b, E) iff the blowing up center fulfills the three restrictions above (nonsingular, normal crossing with E, contained in the singular locus).

Basic objects come in three variants: weighted objects, tagged objects, and balanced objects. We have resolution algorithms for these three types, which call each other recursively.

3.1 Weighted objects

A *weighted object* is a basic object (W, J, b, E), together with a function $a : E \to \mathbb{N}$, such that the following holds. Let $E = (H_1, \ldots, H_r)$, and let $(a_1, \ldots, a_r)$ be the sequence of values of a. Define

$$M = \mathbf{I}(H_1)^{a_1} \cdot \ldots \mathbf{I}(H_r)^{a_r}$$

(locally, this ideal is generated by a monomial in a system of parameters). Then the ideal J can be written as $J = M \cdot \overline{J}$. We call M the monomial part and $\overline{J}$ the proper part of J. The largest order of $\overline{J}$ at points within $\mathrm{Sing}(J, b)$ is called the *maxorder*.

We will blowup weighted objects only at centers which are contained in the subset of $\mathrm{Sing}(J, b)$ where the maximal order is achieved. The transform of the new component a is done as follows. For any hypersurface $H \in E$, the value of the proper transform of a is defined to be $a(H)$, i.e. old hypersurfaces remain with old values. The value of a at the new exceptional divisor F is defined to be

$$a(H_1) + \ldots + a(H_r) + \overline{b} - b,$$

where $\overline{b}$ is the maxorder, and $\{H_1, \ldots, H_r\}$ is the set of all hypersurfaces in E containing the blowing up center. To ensure that this assignment is valid, we have to show that the transformed ideal J' contains $\mathbf{I}(F)^{a(H_1)+\ldots+a(H_r)+\overline{b}-b}$. But this follows from the fact that the order of any point in the blowing up center is at least $a(H_1) + \ldots + a(H_r) + \overline{b}$.

A *simple object* is a basic object (W, J, b, E) satisfying $\mathrm{ord}_x(J) = b$ for all $x \in \mathrm{Sing}(J, b)$. One can easily show that the transform of a simple object is again simple.

A *tagged object* is a simple object (W, J, b, E) together with a subset E_- of E, satisfying an additional "condition T", which will be explained in subsection 3.3. For the moment, it suffices to know two things about condition T:

1. If (W, J, b, E) is simple, then $(W, J, b, E, E_- = E)$ is tagged.

2. The transform of a tagged object is tagged.

Here, the transform of the component E_- is done in the obvious way: E'_- is defined as the set of proper transforms of the hypersurfaces in E_-.

Two tagged objects are called *similar* iff their underlying simple objects coincide (in other words, they may only differ in the E_- component). Obviously, two similar tagged objects have the same singular locus. Moreover, the transforms of similar tagged objects are again similar. Therefore, any sequence of blowing ups which is permissible for one tagged object is also permissible for a similar tagged object.

For any weighted object (W, J, b, E, a) with nontrivial proper part, the *equiorder* is defined as the tagged object constructed by the algorithm *EquiOrder*.

Algorithm *EquiOrder*(W, J, b, E, a);
  $M, \overline{J}$:= the monomial part and the proper part;
  $\overline{b}$:= the maxorder;
  if $\overline{b} \geq b$ then
    $I := \overline{J}$;
    $w := \overline{b}$;
  else
    $I := \overline{J}^{b-\overline{b}} + M^{\overline{b}}$;
    $w := (b - \overline{b})\overline{b}$;
  $E_- := E$;
  return (W, I, w, E, E_-);

Weighted objects with zero maxorder are also called *monomial objects.* One can easily show that the transform of a monomial object is again monomial. Monomial objects can be resolved by the algorithm *ResolveMonomial.* We say that the hypersurfaces $H_1, \ldots, H_r$ have *common singularities* iff the intersection of $H_1, \ldots, H_r$ and the singular locus of the basic object is non-empty.

Algorithm *ResolveMonomial*$(\mathcal{T})$;
  while not resolved do
    r := the smallest number such that r hypersurfaces in E have common singularities;
    $\overline{b}$:= the largest possible value of $a(H_1) + \ldots + a(H_r)$, where $H_1, \ldots, H_r \in E$ have common singularities;
    $H_1, \ldots, H_r$:= the first r hypersurfaces for which the maximum $\overline{b}$ is achieved (with respect to lexicographical ordering);
    Y := the intersection of $H_1, \ldots, H_r$;
    BlowUp(Y);

Remark. The statement *BlowUp*(Y) implies that we pass from the given basic object to its transform. Moreover, we will have in general a whole stack of "current objects" of all three types, due to the recursive calls. In this case, Y will be permissible with respect to all of them, and the statement *BlowUp*(Y) means that we pass to the proper transform for all basic objects in the stack.

The algorithm *ResolveWeighted* is for the resolution of weighted objects (it calls the resolution algorithm for tagged objects).

Algorithm *ResolveWeighted*($\mathcal{W}$);
    while not resolved and maxorder>0 do
        $\mathcal{T} :=$ *EquiOrder*(current object);
        ResolveTagged($\mathcal{T}$);
    if not resolved then
        ResolveMonomial(current object);

3.2 Balanced objects

Let (W, J, b, E) be a simple object. The *delta-ideal* $\Delta(J, b)$ is defined as the ideal generated by all partial derivatives of elements in J of order less than b. Its zero set contains the singular locus. For every point $x \in \mathrm{Sing}(J, b)$, there exists a $z \in \Delta(J, b)$ which has order 1 at x (because for each point x in the singular locus, there is a partial derivative of order b not vanishing at x).

The simple object (W, J, b, E) is called *balanced* iff the following condition holds. For every point $x \in \mathrm{Sing}(J, b)$, there exists a $z \in \Delta(J, b)$ which has order 1 at x and $\mathbf{V}(z)$ has normal crossing with E in a neighborhood of x.

Let (W, J, b, E) be a balanced simple object, and let $z \in \Delta(J, b)$ be an element which has order 1 and normal crossing with E at any point of its zero set. Then we say that this zero set (called V below) is a *hypersurface of maximal contact.*

Suppose that $W = \mathbb{A}^n$, and z is one of the coordinate functions. Then we define the *coefficient ideal* $C(J, b, V)$ as follows. Let

$$f_{0,1} + \ldots + f_{d_1,1} z^{d_1}, \ldots, f_{0,r} + \ldots + f_{d_r,r} z^{d_r}$$

be generators of J, where $f_{i,j}$ are polynomials in the remaining $n-1$ variables. The ideal C is generated by

$$f_{0,1}^{\frac{b!}{b}}, \ldots, f_{b-1,1}^{\frac{b!}{1}}, \ldots, f_{0,r}^{\frac{b!}{b}}, \ldots, f_{b-1,r}^{\frac{b!}{1}}.$$

In the general case, one can construct the coefficient ideal by partial differentiation.

The coefficient ideal is zero iff V is equal to the singular locus. In this case, V is permissible for blowing up. Since V is a hypersurface, the blowing up map is an isomorphism on W. It is easy to check that the transform of (W, J, b, E) is then resolved.

Remark. In general, V may be reducible, and there might be components of V where the coefficient ideal is zero, and other components where it is not. In this section, we assume for simplicity that V is irreducible. For the general case, refer to section 5 of [BS00a].

If the coefficient ideal is not zero, then the *coefficient object* is defined as

$$(V, C(J, b, V), b!, E \cap V, 0_E),$$

where 0_E is the constant zero function. It enjoys the following properties.

The algorithm *ResolveBalanced* is for the resolution of balanced objects. It calls a resolution algorithm for weighted objects of smaller dimension.

An obstacle is that the ideal $\Delta(J, b)$ may have no global element of order 1, even though it has elements of order 1 at any point by definition of being "simple". But then we may replace W by open subsets which cover, and in each of the subsets we have a global element of order 1.

Algorithm *ResolveBalanced*($\mathcal{B}$);
  $V :=$ a hypersurface of maximal contact;
  $C :=$ the coefficient ideal of $\mathcal{B}$ with respect to V;
  if C is the zero ideal then
    BlowUp(V);
  else
    $\mathcal{W} :=$ the coefficient of object of $\mathcal{B}$ with respect to V;
    ResolveWeighted($\mathcal{W}$);

3.3 Tagged objects

A tagged object is a simple object (W, J, b, E) together with a subset E_- of E, such that the simple object $(W, J, b, E - E_-)$—which we call the *shadow*—is a balanced object. (This condition was referred to as "property T" in subsection 3.1.)

Let $\mathcal{T}$ be a tagged basic object. Then the transform of $\mathcal{T}$ along some permissible blowing up is again tagged, because its shadow is the transform of the shadow of $\mathcal{T}$, which is balanced. This shows a fact that we have used in section 3.1: the transform of a tagged object is again tagged. The second statement that we have used in section 3.1 is that $E = E_-$ and being simple implies being tagged; but this is obvious.

Let r be the largest number such that r hypersurfaces in E_- have common singularities. If $r > 0$, let m be the number of such r-subsets of E_-. The pair (r, m) (or (0, "undefined")) is called the *balance* of (W, J, b, E, E_-). It is a measure of the severeness of the tagged object. Balances are compared by the lexicographic order.

The *support* is defined as the basic object constructed by the algorithm *Support*.

Algorithm *Support*(W, J, b, E, E_-);
  $r :=$ the largest number such that r hypersurfaces in E_- have common
    singularities;
  $H_1, \ldots, H_r :=$ the first r hypersurfaces in E_- having common singularities
    (with respect to lexicographical ordering);
  $I := J + \mathbf{I}(H_1)^b + \ldots + \mathbf{I}(H_r)^b$;
  $E_+ := E - E_-$;
  return (W, I, b, E_+);

By enlarging the ideal, one does not destroy the property of being simple. Hence the support is a simple object. If $r > 0$, then the hypersurface H_1 has maximal contact and normal crossing with E_+, therefore the support is balanced. If $r = 0$, then the support is balanced by property T.

The algorithm *ResolveTagged* is for the resolution of tagged objects. It calls the algorithm for resolution of balanced objects.

Algorithm *ResolveTagged*($\mathcal{T}$);
  while not resolved do
    $\mathcal{B}$:= *Support*(current object);
    ResolveBalanced($\mathcal{B}$);

3.4 Desingularization

The problem of embedded desingularization of hypersurfaces can be reduced to the resolution of basic objects in the following way. Given a nonsingular variety U and a hypersurface X in it, we just have to resolve the weighted basic object $(U, \mathbf{I}(X), 1, \emptyset, \text{empty function})$. Then the final transform has ideal $\langle 1 \rangle$ and therefore the total transform is a product of ideals of exceptional divisors (i.e. elements in E), which always have normal crossing.

A desingularization (not necessarily embedded) is already obtained as soon as the maxorder has become less than 2. This follows from the fact that the proper part is nothing but the equation of the proper transform.

We try to outline the work of the resolution algorithm in the diagram of figure 1. The objects of U_i form a finite sequence which starts with the first weighted object, WO_1^i, being the transform of the initial weighted object, WO_1^1, along the sequence of blowing ups made so far, on the branch of the resolution tree with the given list of exceptional divisors. The "descendant" object of WO_1^i is TO_1^i via equiorder computation, followed by BO_1^i via support computation, followed by WO_2^i via coefficient object computation, and so on.

The maximum length of such an object list (it is depth in the diagram) can be $3n$, where n is the dimension of U.

The progress of the resolution along the branch of the resolution tree corresponding to the exceptional divisor list $P^{(1)}, \dots, P^{(N-1)}$ of consecutive blowing ups, appears in the horizontal direction in the figure. We also assume that over the charts $U_1, \dots, U_{N-1}$ there exist global order 1 elements generating hypersurfaces of maximal contacts in each balanced objects. That is, we need not to pass to local subcases.

The double crosses denote commutative subdiagrams, where similar tagged objects are considered to be equivalent. It has to be noted, that basic objects don't get transformed: they are recomputed from the tagged objects of the same dimension. Moreover, when a tagged object gets resolved, and the weighted object of the same dimension is not resolved yet, a new tagged object is created which is not the transform of the previous tagged object along the last blowing up; therefore the subdiagram fails to commute (e.g. the situation of TO_1^j and TO_1^i).

4 Support and coefficient ideal revisited

In this section we present new definitions for the algorithm Support and we show how to gain speedup in the algorithm *ResolveBalanced*, by computing the coefficient ideal with respect to an algebraic set of codimension greater than one.

Lemma 4.1. *Given a tagged object (W, J, b, E, E_-) in a chart U, then there*

$$\begin{tikzcd}
U_1 = U & & & & & U_i & & \\
\mathrm{WO}_1^1 \arrow[r, "P^{(1)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] \arrow[dr, phantom, "\#"] & \mathrm{WO}_1^2 \arrow[r, "P^{(2)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] \arrow[drr, phantom, "\#"] & \cdots \arrow[r, "P^{(j-1)}", "\text{tr.}"'] & \mathrm{WO}_1^j \arrow[r, "P^{(j)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] & \cdots \arrow[r, "P^{(i-1)}", "\text{tr.}"'] & \mathrm{WO}_1^i \arrow[r, "P^{(i)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] & \cdots \arrow[r, "P^{(N-1)}", "\text{tr.}"'] & \mathrm{WO}_1^N \\
\mathrm{TO}_1^1 \arrow[r, "P^{(1)}", "\text{tr.}"'] \arrow[d, "\text{supp.}"] & \mathrm{TO}_1^2 \arrow[r, "P^{(2)}", "\text{tr.}"'] \arrow[d, "\text{supp.}"] & \cdots \arrow[r, "P^{(j-1)}", "\text{tr.}"'] & \mathrm{TO}_1^j \arrow[d, "\text{supp.}"] & & \mathrm{TO}_1^i \arrow[d, "\text{supp.}"] & & \text{resolved} \\
\mathrm{BO}_1^1 \arrow[r, phantom, "\#"] \arrow[d, "\text{coeff.}"] & \mathrm{BO}_1^2 \arrow[d, "\text{coeff.}"] & & \mathrm{BO}_1^j & & \mathrm{BO}_1^i \arrow[d, "\text{coeff.}"] & & \\
\mathrm{WO}_2^1 \arrow[r, "P^{(1)}", "\text{tr.}"'] \arrow[d, "\text{eqo.}"] & \mathrm{WO}_2^2 \arrow[d, "\text{eqo.}"] & & \text{blowup}_j & & \mathrm{WO}_2^i \arrow[d, "\text{eqo.}"] & & \\
\vdots & \vdots & & & & \vdots \arrow[d, "\text{supp.}"] & & \\
 & & & & & \mathrm{BO}_k^i & \text{blowup}_i &
\end{tikzcd}$$

Figure 1: A local branch of the resolution of the weighted object WO_1^1.

always exists an open covering of U, where in each open subset either the object is resolved or all the exceptional divisors from E_- have common singularities.

Proof. We construct the open covering: If the singular locus and E_- has no common singularities, we perform a *cover* operation (see section 4.1 in [BS00a]) with E_- and a finite set of generators of $\mathbf{I}(\mathrm{Sing}(J, b))$.

In the new charts either one of the exceptional divisors from E_- is absent, or the singular locus gets empty. This way we reduced the situation to a simpler one in each of the charts, and since there are always finitely many members of E_-, repeating the previous step finitely many times, we reach a state that fulfills the claim of the lemma. $\square$

The result of the lemma is used in the *Support2* algorithm. The instruction "fork" means that the computation continues for each new chart independently and that the current *Support2* routine and its caller (a *ResolveTagged* procedure) is linked out from the call tree of each new thread. Actually, they get replaced by the branch started at the fork command.

In the following we define the coefficient ideal with respect to finitely many hypersurfaces of maximal contact, and we present a technique which uses this new definition to "skip" redundant weighted, tagged and balanced objects during the resolution.

Definition 4.1. *Let $V_1, \ldots, V_r$ be hypersurfaces of maximal contact with The coefficient ideal of (J, b) with respect to all of them is*

$$\mathrm{Coeff}(J, b, (V_1, \ldots, V_r)) = C(C(\ldots C(J, b, V_1) \ldots), m_r, V_r).$$

The corresponding coefficient object is

$$(W \cap V_1 \cap \cdots \cap V_r, \mathrm{Coeff}(J, b, (V_1, \ldots, V_r)), b(!)^r, E \setminus (V_1 \cup \cdots \cup V_r), 0_E),$$

Algorithm *Support2*(W, J, b, E, E_-);
  drop all elements from E and E_- which are outside the current chart;
  let $H_1, \ldots, H_s$ be the remaining exceptional divisors in E_-;
  if E_- has no common singularities then
    let $f_1, \ldots, f_r$ be generators of $\mathbf{I}(\mathrm{Sing}(J, b))$;
    let $g_1, \ldots, g_s$ generate $\mathbf{I}(H_1), \ldots, \mathbf{I}(H_s)$ respectively;
    New_charts := *Cover*(current_chart, $(f_1, \ldots, f_r, g_1, \ldots g_s)$);
    fork *ResolveTagged*$((W, J, b, E, E_-))$ for all the charts in *New_charts*;
  else
    $I := J + \mathbf{I}(H_1)^b + \ldots + \mathbf{I}(H_s)^b$;
    $E_+ := E - E_-$;
    return (W, I, b, E_+);

where $b(!)^r$ denotes the computation of r-accumulated factorials $((b!)!...)!$, and 0_E is the zero function on E.

Consider the following situation in the resolution process: Given a tagged object $\mathcal{T}_0 = (W_0, J_{0(t)}, b_{0(t)}, E_{0(t)}, E_{-0})$ with $E_- = E = \{H_1, \ldots H_r\}$. Without the loss of generality, we can assume that all the members of E_- have common singularities. The support of $\mathcal{T}_0$ is the balanced object $\mathcal{B}_0 = (W_0, J_{0(b)} = J_{0(t)} + \mathbf{I}(H_1)^{b_{0(t)}} + \cdots + \mathbf{I}(H_r)^{b_{0(t)}}, b_{0(b)} = b_{0(t)}, E_{0(b)} = \emptyset)$. By some bookkeeping, i.e. storing the H_i hypersurfaces, we can achieve that in this situation the algorithm (by a systematic choice of hypersurfaces of maximal contact) computes the following sequence of objects:

$$\begin{aligned}
\mathcal{W}_1 &= (W_1 = W_0 \cap H_1, J_{1(w)} = \mathrm{Coeff}(J_{0(b)}, b_{0(b)}, H_1), \\
&\qquad\qquad\qquad b_1 = b_0!, E_{1(w)} = \emptyset, a_1 = \emptyset), \\
\mathcal{T}_1 &= (W_1, J_{1(t)} = J_{1(w)}, b_{1(t)} = b_{1(w)}, E_{1(t)} = \emptyset, E_{-1} = \emptyset), \\
\mathcal{B}_1 &= (W_1, J_{1(b)} = J_{1(t)}, b_{1(b)} = b_{1(t)}, E_{1(b)} = \emptyset), \\
\mathcal{W}_2 &= (W_2 = W_1 \cap H_2, J_{2(w)} = \mathrm{Coeff}(J_{1(b)}, b_{1(b)}, H_2), \\
&\qquad\qquad\qquad b_{2(w)} = b_{1(b)}!, E_{2(w)} = \emptyset, a_2 = \emptyset), \\
&\vdots \\
\mathcal{B}_{r-1} &= (W_{r-1}, J_{r-1(b)} = J_{r-1(t)}, b_{r-1(b)} = b_{r-1(t)}, E_{r-1(b)} = \emptyset), \\
\mathcal{W}_r &= (W_r = W_{r-1} \cap H_r, J_{r(w)} = \mathrm{Coeff}(J_{r-1(b)}, b_{r-1(b)}, H_r), \\
&\qquad\qquad\qquad b_{r(w)} = b_{r-1(b)}!, E_{r(w)} = \emptyset, a_r = \emptyset).
\end{aligned}$$

This is possible because all the objects with index less than r are simple, and the list of exceptional divisors are empty; moreover for $i = 0, \ldots, r-1$, H_{i+1} is a hypersurface of maximal contact in $\mathcal{B}_i$.

Lemma 4.2. *In the above situation we have*

$$\begin{aligned}
\mathcal{W}_r = (W_r = W_0 \cap H_1 \cap \cdots \cap H_r, J_{r(w)} &= \mathrm{Coeff}(J_{0(b)}, b, (H_1, \ldots, H_r)), \\
b_{r(w)} &= b(!)^r, E_{r(w)} = \emptyset, a_r = \emptyset).
\end{aligned}$$

Proof. By the definition 4.1, the statement is obvious. ∎

In the next lemma we mark the transforms of objects along a finite sequence of blowing ups in a chart, U, of the affine covering with an asterisk.

Lemma 4.3. *Consider the situation as in lemma 4.2. The transform of the first coefficient object* $\mathcal{W}_{1*}$ *is resolved iff* $\mathcal{W}_{r*}$ *is resolved.*

Proof. For $r = 1$ the statement is trivial. Assume now that the lemma is true for $r-1$ ($< \dim(U)$).

We have that $\mathbf{I}(H_{r*})^{b_{r-1}}$ is a subideal of $J_{r-1(w)*}$, and since $\mathcal{W}_{r-1}$ is simple $\mathcal{W}_{r-1*}$ is also simple. It follows that $J_{r-1(t)*} = J_{r-1(w)*}$ and $b_{r-1(t)} = b_{r-1(w)}$ in $\mathcal{T}_{r-1*}$, whence $\mathcal{W}_{r-1*}$ is resolved iff $\mathcal{T}_{r-1*}$ is resolved.

This also means that whenever $\mathcal{W}_{r-1*}$ is not resolved, its weighted order hasn't drop, so E_- hasn't changed in $\mathcal{T}_{r-1*}$; therefore $J_{r-1(b)*} = J_{r-1(t)*}$ in $\mathcal{B}_{r-1*}$. Thus $\mathcal{B}_{r-1*}$ is resolved iff $\mathcal{T}_{r-1*}$ is resolved.

The singular loci of $\mathcal{B}_{r-1*}$ is the same as of $\mathcal{W}_{r*}$, because coefficient ideal computation and transform along a chain of blowing ups are commutative operations. Finally, because for all $x \in \mathrm{Sing}(J_{r-1(w)*}, b_{r-1(w)})$ we must have $x \in H_{r*}$, we have $x \in \mathrm{Sing}(J_{r-1(w)*}, b_{r-1(w)})$ iff $x \in \mathrm{Sing}(J_{r(w)*}, b_{r(w)})$. □

Remark. According to lemma 4.3, the claim of lemma 4.2 also holds after finitely many blowing up operations, provided that the sequence of objects is not resolved. In this case E_- does not change in the tagged objects, but E gets extended with new exceptional divisors.

Now we have, by lemma 4.3 and lemma 4.2, that whenever we create the support of a tagged object with all members of E_- having common singularities, we can compute the next coefficient object with respect to all the exceptional divisors of E_-. This way we save all the computations needed in the intermediate objects. The results are used in the *ResolveTagged2* and *ResolveBalanced2* algorithms.

Algorithm *ResolveTagged2*($\mathcal{T}$);
    while not resolved do
        $\mathcal{B}$:= *Support*(current object);
            ResolveBalanced2($\mathcal{B}, E_-$);

Algorithm *ResolveBalanced2*($\mathcal{B} = (W, J, b, E), H$);
    if $H = \emptyset$ or $E \neq \emptyset$ then
        V := a hypersurface of maximal contact;
        C := Coeff$(J, b, (V))$;
    else
        C := Coeff(J, b, H);
        V := H;
    if C is the zero ideal then
        BlowUp(V);
    else
        $\mathcal{W}$:= the coefficient of object of $\mathcal{B}$ with respect to V;
        ResolveWeighted($\mathcal{W}$);

Finally, we consider a revised version of the algorithm presented in [BS00a] to be our reference algorithm; let's call it *Villamayor_resolution2*. This modified version uses the ideas and algorithms (*Support2*, *ResolveTagged2*, *ResolveBalanced2*) presented in this section, instead of the old counterparts.

Theorem 4.1. *The* Villamayor_resolution2 *algorithm terminates within finitely many steps.*

Proof. We give a relative argument relying on the termination of the original algorithm. By lemma 4.2 and lemma 4.3 the replacement of *ResolveTagged2* and *ResolveBalanced2* does not alter the termination of the original algorithm.

By lemma 4.1, in the resolution of a tagged object with $k = |E_-|$, the *Support2* algorithm may be called at most k times before the execution goes to the 'else' branch, where termination is ensured by the original proofs given in [BS00a]. □

5 Generalized ideals

The purpose of the introduction of generalized ideals is to simplify the representation of vanishing ideals of algebraic sets in the resolution process. This section presents the first stages of the theory of generalized ideals.

We define the following functions on finite sets of ordered pairs from $k[U] \times \mathbb{Q}_{\geq 0}$.

Definition 5.1. *The zeroset of* $\{(F_1, b_1), \ldots, (F_k, b_k)\}$ *in* U *is* $\bigcap_{b_i > 0} \mathbf{Z}(F_i)$.

Definition 5.2. *The* singular locus *of* $a = \{(F_1, b_1), \ldots, (F_k, b_k)\}$, *denoted as* $\mathrm{Sing}(a)$, *is*

$$\mathbf{Z}\Big(\Big\langle \frac{\partial^j F_i}{\partial P_1^{j_1} \cdots \partial P_n^{j_n}} \,\Big|\, i = 1, \ldots, k,\ j = 0, \ldots, \lceil b_i \rceil - 1,\ j_l \in \mathbb{N},\ \sum\nolimits_{l=1}^{n} j_l = j \Big\rangle\Big),$$

when there is at least one $b_i > 0$*; otherwise, if all* $b_i = 0$*, we set* $\mathrm{Sing}(a) = \emptyset$.

Definition 5.3. *Let* $a = \{(F_1, b_1), \ldots, (F_k, b_k)\}$ *and* $c \in \mathbb{Q}_{\geq 0}$*, the* update *of* a *with* c *is defined as* $a_c = \{(F_1, b_1 c), \ldots, (F_k, b_k c)\}$.

Definition 5.4. *The* order, $\mathrm{ord}(a)$, *is the greatest* $c \in \mathbb{Q}_{\geq 0}$*, such that* $\mathrm{Sing}(a_c) \neq \emptyset$*; or 0 if* c *does not exist. Analogously, we can define the order of* a *at a point* p, $\mathrm{ord}_p(a)$, *requiring* $p \in \mathrm{Sing}(a_c)$.

Definition 5.5. *A* generalized ideal *is an equivalence class of the equivalence relation* $\sim$ *on finite sets of pairs from* $k[U] \times \mathbb{Q}_{\geq 0}$ *defined as:*

$$\begin{aligned} a \sim b :\Longleftrightarrow\ & \text{for all } p \in U : \mathrm{ord}_p(a) = \mathrm{ord}_p(b) \text{ and} \\ & \text{if } \pi : W \to U \text{ is a blowing up, where } W \text{ is nonsingular with} \\ & \text{open cover } W = \bigcup U_i, \text{ then for all } U_i \text{ we have} \\ & \mathrm{ord}_p(a_*) = \mathrm{ord}_p(b_*) \text{ for all } p \in U_i. \end{aligned}$$

Lemma 5.1. *The relation* $\sim$ *of definition 5.5 is indeed an equivalence.*

Proof. Since the definition is based on equality, the lemma is obvious. □

We denote the class of a representant a as $\langle a \rangle$. We show now that the update function, the pullback operation, $\mathbf{Z}$, Sing and ord_p are well defined with respect to $\sim$; therefore their definitions can naturally be generalized for generalized ideals.

Lemma 5.2. *If* $\{(F_1, b_1), \ldots, (F_f, b_k)\} \sim \{(G_1, c_1), \ldots, (G_m, c_m)\}$ *then*

$$\mathbf{Z}(\{(F_1, b_1), \ldots, (F_f, b_k)\}) = \mathbf{Z}(\{(G_1, c_1), \ldots, (G_m, c_m)\}).$$

Proof. By the definition of $\sim$ we have $\mathrm{ord}_p(\{(F_1, b_1), \ldots, (F_f, b_k)\}) > 0$ iff $\mathrm{ord}_p(\{(G_1, c_1), \ldots, (G_m, c_m)\}) > 0$. ◻

Lemma 5.3. *If* $\{(F_1, b_1), \ldots, (F_f, b_k)\} \sim \{(G_1, c_1), \ldots, (G_m, c_m)\}$ *then*

$$\mathrm{Sing}(\{(F_1, b_1), \ldots, (F_f, b_k)\}) = \mathrm{Sing}(\{(G_1, c_1), \ldots, (G_m, c_m)\}).$$

Proof. We have for all $p \in U$

$$\mathrm{ord}_p(\{(F_1, b_1), \ldots, (F_k, b_k)\}) = \mathrm{ord}_p(\{(G_1, c_1), \ldots, (G_m, c_m)\});$$

therefore

$$\begin{aligned}\mathrm{Sing}(\{(F_1, b_1), \ldots, (F_f, b_k)\}) =& \\ &= \{p \in U \mid \mathrm{ord}_p(\{(F_1, b_1), \ldots, (F_f, b_k)\}) \geq 1\} = \\ &= \{p \in U \mid \mathrm{ord}_p(\{(G_1, c_1), \ldots, (G_m, c_m)\}) \geq 1\} = \\ &= \mathrm{Sing}(\{(G_1, c_1), \ldots, (G_m, c_m)\}).\end{aligned}$$

◻

Lemma 5.4. *If* $\pi : W \to U$ *is a blowing up as required in definition 5.5 and* $\{(F_1, b_1), \ldots, (F_f, b_k)\} \sim \{(G_1, c_1), \ldots, (G_m, c_m)\}$ *then for any* U_i *we have* $\{(F_{1*}, b_1), \ldots, (F_{f*}, b_k)\} \sim \{(G_{1*}, c_1), \ldots, (G_{m*}, c_m)\}$.

Proof. Let's fix a chart $U_i \subseteq W$. By the definition of $\sim$ we have for all $p \in U_i$

$$\mathrm{ord}_p(\{(F_{1*}, b_1), \ldots, (F_{f*}, b_k)\}) = \mathrm{ord}_p(\{(G_{1*}, c_1), \ldots, (G_{m*}, c_m)\}).$$

We should also have for any blowing up $\sigma : Z \to W$ that fulfills the requirements of definition 5.5, with open cover $\bigcup V_j = Z$, at any point $q \in V_j$ in any V_j

$$\mathrm{ord}_q(\{((F_{1*})_*, b_1), \ldots, ((F_{f*})_*, b_k)\}) = \mathrm{ord}_q(\{((G_{1*})_*, c_1), \ldots, ((G_{m*})_*, c_m)\}).$$

But this is immediate, since $\sigma \circ \pi : Z \to U$ is also a blowing that fulfills the requirements of definition 5.5. ◻

Lemma 5.5. *If* $\{(F_1, b_1), \ldots, (F_f, b_k)\} \sim \{(G_1, c_1), \ldots, (G_m, c_m)\}$ *then for any* $c \in \mathbb{Q}_{\geq 0}$*,*

$$\{(F_1, b_1), \ldots, (F_f, b_k)\}_c \sim \{(G_1, c_1), \ldots, (G_m, c_m)\}_c.$$

Proof. We have for all $p \in U$

$$c_p = \mathrm{ord}_p(\{(F_1, b_1), \ldots, (F_k, b_k)\}) = \mathrm{ord}_p(\{(G_1, c_1), \ldots, (G_m, c_m)\});$$

therefore if $c > 0$ we have for any $p \in U$

$$c_p/c = \mathrm{ord}_p(\{(F_1, b_1), \ldots, (F_k, b_k)\}_c) = \mathrm{ord}_p(\{(G_1, c_1), \ldots, (G_m, c_m)\}_c).$$

If $c = 0$, both orders are 0 by definition. Since we have the same argument for the pullbacks the lemma is true. ◻

Lemma 5.6. *If* $\{(F_1,b_1),\dots,(F_f,b_k)\}\sim\{(G_1,c_1),\dots,(G_m,c_m)\}$ *then for any* $p\in U$

$$\mathrm{ord}_p(\{(F_1,b_1),\dots,(F_f,b_k)\})=\mathrm{ord}_p(\{(G_1,c_1),\dots,(G_m,c_m)\}).$$

Proof. By the definition of $\sim$, the statement is obvious. $\square$

Remark. The statement of lemma 5.6 holds also for the ord function.

Here we need to do some technical preparation. Let an ordinary ideal, J be given by a finite generating set. The ideal $\Delta(J,i)$ is generated by all the partial derivatives of the generators of J of order less than $i\in\mathbb{N}$ (by definition $\Delta(J,0)$ is the zero ideal).

Lemma 5.7. *For* $J_1,J_2\subseteq k[U]$ *and* $i_1,i_2\in\mathbb{N}$*, we have*

1. $\mathbf{Z}(\Delta(J_1\cdot J_2,i_1+i_2))\subseteq\mathbf{Z}(\Delta(J_1,i_1))\cup\mathbf{Z}(\Delta(J_2,i_2))$
2. $\mathbf{Z}(\Delta(J_1\cdot J_2,i_1+i_2))\supseteq\mathbf{Z}(\Delta(J_1,i_1))\cap\mathbf{Z}(\Delta(J_2,i_2))$.

Proof.

1. Consider a point x in the left hand side, then in $k_x[U]$ with maximal ideal $\mathfrak{m}_x$ we have $J_1\cdot J_2\subseteq\mathfrak{m}_x^{i_1+i_2}$. This means, we must have either $J_1\subseteq\mathfrak{m}_x^{i_1}$ or $J_2\subseteq\mathfrak{m}_x^{i_2}$, which corresponds to the right hand side.
2. Consider a point x in the right hand side, then in $k_x[U]$ with maximal ideal $\mathfrak{m}_x$ we have $J_1\subseteq\mathfrak{m}_x^{i_1}$ and $J_2\subseteq\mathfrak{m}_x^{i_2}$. It follows by the monotonicity of the product with respect to the inclusion that we also have $J_1\cdot J_2\subseteq\mathfrak{m}_x^{i_1}\cdot\mathfrak{m}_x^{i_2}=\mathfrak{m}_x^{i_1+i_2}$.

$\square$

Lemma 5.8. *We have for any point* x*:* $\mathrm{ord}_x(F^{q\cdot l})\geq p\cdot l$ *iff* $\mathrm{ord}_x(F^q)\geq p$*, for* $l,q\in\mathbb{N}_+$*,* $p\in\mathbb{N}$*.*

Proof. The formula $\mathrm{ord}_x(F^q)\geq p$ means that $x\in\mathbf{Z}(\Delta(F^q,p))$. For $l=1$ the lemma is trivial. Assume it now for $l-1$, $(l>1)$.

By Lemma 5.7 we have that $\mathbf{Z}(\Delta(F^q\cdot F^{(l-1)q},p+(l-1)p))\subseteq\mathbf{Z}(\Delta(F^q,p))\cup\mathbf{Z}(\Delta(F^{(l-1)q},(l-1)p))$. That is $x\in\mathbf{Z}(\Delta(F^{l\cdot q},l\cdot p))$ implies that either $x\in\mathbf{Z}(\Delta(F^{(l-1)q},(l-1)p))$ or $x\in\mathbf{Z}(\Delta(F^q,p))$, but by the induction hypothesis both cases coincide.

Along the same lines, we have that $\mathbf{Z}(\Delta(F^q\cdot F^{(l-1)q},p+(l-1)p))\supseteq\mathbf{Z}(\Delta(F^q,p))\cap\mathbf{Z}(\Delta(F^{(l-1)q},(l-1)p))$. And since $x\in\mathbf{Z}(\Delta(F^{(l-1)q},(l-1)p))$ and $x\in\mathbf{Z}(\Delta(F^q,p))$ coincides by the induction hypothesis, we have $x\in\mathbf{Z}(\Delta(F^{l\cdot q},l\cdot p))$. $\square$

Remark. Lemma 5.8 holds in particular for equality, i.e. $\mathrm{ord}_x(F^{q\cdot l})=p\cdot l$ iff $\mathrm{ord}_x(F^q)=p$. The statement of the lemma holds also for the ord function.

Lemma 5.9. *We have* $\{(F,b)\}\sim\{(F^l,bl)\}$ *for* $l\in\mathbb{N}_+$*.*

Proof. By lemma 5.8 we have $\mathrm{ord}_p(F)=b$ iff $\mathrm{ord}_p(F^l)=bl$. This implies $\mathrm{ord}_p(\{(F,b)\})=\mathrm{ord}_p(\{(F^l,bl)\})$, because $\mathrm{ord}_p(\{(F,b)\})\cdot b=\mathrm{ord}_p(F)$.

On the other hand, since the pullback operation is compatible with polynomial multiplication, we also have $\mathrm{ord}_p(\{(F_*,b)\})=\mathrm{ord}_p(\{(F_*^l,bl)\})$ for all $p\in U_i$ in all charts U_i of the blownup for any admissible blowing up. $\square$

Lemma 5.10. *We have* $\{(F_1,b_1),(F_2,b_2)\} \sim \{(F_1,b_1),(F_2,b_2),(F_1F_2,b_1{+}b_2)\}$,

Proof. We have $\mathrm{ord}_p(F_1F_2) = \mathrm{ord}_p(F_1) + \mathrm{ord}_p(F_2)$, and from definition 5.4 we have $\mathrm{ord}_p(\{(F_1,b_1)\}) = \mathrm{ord}_p(F_1)/b_1$. Therefore

$$\begin{aligned}\mathrm{ord}_p(\{(F_1,b_1),(F_2,b_2)\}) = \min\Big(\frac{\mathrm{ord}_p(F_1)}{b_1}, \frac{\mathrm{ord}_p(F_2)}{b_2}\Big) =\\ = \min\Big(\frac{\mathrm{ord}_p(F_1)}{b_1}, \frac{\mathrm{ord}_p(F_2)}{b_2}, \frac{\mathrm{ord}_p(F_1)+\mathrm{ord}_p(F_2)}{b_1+b_2}\Big) =\\ = \mathrm{ord}_p(\{(F_1,b_1),(F_2,b_2),(F_1F_2,b_1+b_2)\})\end{aligned}$$

because the third quantity of the minimum function of the second line is always between the first two.

We have the same for the pullbacks along blowing ups as in definition 5.5, so the lemma is true. □

Lemma 5.11. *If* $\langle F_1,\dots,F_k\rangle = \langle G_1,\dots,G_m\rangle$ *then for any* $b \in \mathbb{Q}_{\geq 0}$

$$\{(F_1,b),\dots,(F_k,b)\} \sim \{(G_1,b),\dots,(G_m,b)\}.$$

Proof. We have for any $p \in U$

$$\begin{aligned}\mathrm{ord}_p(\{(F_1,b),\dots,(F_k,b)\}) \cdot b = \mathrm{ord}_p(\langle F_1,\dots,F_k\rangle) =\\ = \mathrm{ord}_p(\langle G_1,\dots,G_m\rangle) = \mathrm{ord}_p(\{(G_1,b),\dots,(G_m,b)\}) \cdot b;\end{aligned}$$

therefore $\mathrm{ord}_p(\{(F_1,b),\dots,(F_k,b)\}) = \mathrm{ord}_p(\{(G_1,b),\dots,(G_m,b)\})$.

Also, since for any blowing up fulfilling the requirement of definition 5.5 we have $\langle F_{1*},\dots,F_{k*}\rangle = \langle G_{1*},\dots,G_{m*}\rangle$, by the same argument as above, we have for any $p \in U_i$ $\mathrm{ord}_p(\{(F_{1*},b),\dots,(F_{k*},b)\}) = \mathrm{ord}_p(\{(G_{1*},b),\dots,(G_{m*},b)\})$ in any U_i of the blownup. □

Definition 5.6. *The sum of generalized ideals is defined as*

$$\begin{aligned}\langle(F_1,b_1),\dots,(F_k,b_k)\rangle + \langle(G_1,c_1),\dots,(G_m,c_m)\rangle =\\ \langle(F_1,b_1),\dots,(F_k,b_k),(G_1,c_1),\dots,(G_m,c_m)\rangle.\end{aligned}$$

Lemma 5.12. *The sum of generalized ideals is well defined.*

Proof. Assume $\{(F_1,b_1),\dots,(F_k,b_k)\} \sim \{(G_1,c_1),\dots,(G_m,c_m)\}$, then we need to show

$$\begin{aligned}\{(F_1,b_1),\dots,(F_k,b_k),(H_1,d_1),\dots,(H_l,d_l)\} \sim\\ \{(G_1,c_1),\dots,(G_m,c_m),(H_1,d_1),\dots,(H_l,d_l)\}.\end{aligned}$$

We have

$$\begin{aligned}\mathrm{ord}_p(\{(F_1,b_1),\dots,(F_k,b_k),(H_1,d_1),\dots,(H_l,d_l)\}) =\\ \min(\mathrm{ord}_p(\{(F_1,b_1),\dots,(F_k,b_k)\}), \mathrm{ord}_p(\{(H_1,d_1),\dots,(H_l,d_l)\}))\end{aligned}$$

and the same for $\{(G_1,c_1),\dots,(G_m,c_m)\}$. Therefore

$$\begin{aligned}\mathrm{ord}_p(\{(F_1,b_1),\dots,(F_k,b_k),(H_1,d_1),\dots,(H_l,d_l)\}) =\\ \mathrm{ord}_p(\{(G_1,c_1),\dots,(G_m,c_m),(H_1,d_1),\dots,(H_l,d_l)\})\end{aligned}$$

We also have $\{(F_{1*}, b_1), \ldots, (F_{k*}, b_k)\} \sim \{(G_{1*}, c_1), \ldots, (G_{m*}, c_m)\}$; thus we can apply the same argument for the pullbacks, getting

$$\begin{aligned}\operatorname{ord}_p(\{(F_{1*}, b_1), \ldots, (F_{k*}, b_k), (H_{1*}, d_1), \ldots, (H_{l*}, d_l)\}) = \\ \operatorname{ord}_p(\{(G_{1*}, c_1), \ldots, (G_{m*}, c_m), (H_{1*}, d_1), \ldots, (H_{l*}, d_l)\})\end{aligned}$$

for all $p \in U_i$ in all U_i. □

Lemma 5.13. *We have* $\{(F_1, b), (F_2, b)\} \sim \{(F_1, b), (F_2, b), (F_1 + F_2, b)\}$, *and for any* $G \in k[U]$ *we have* $\{(F, b)\} \sim \{(F, b), (GF, b)\}$.

Proof. Since $\langle F_1, F_2\rangle = \langle F_1, F_2, F_1 + F_2\rangle$ and $\langle F\rangle = \langle F, GF\rangle$, by lemma 5.11 the statement holds. □

Remark. From lemma 5.13 it is immediate that $\{(F, b)\} \sim \{(F, b), (F, b/l)\}$ for $l \in \mathbb{N}_+$.

Definition 5.7. *Let* (I, b) *be a Villamayor-pair, where* $I = \langle F_1, \ldots, F_k\rangle$ *is an ideal in* $k[U]$, *and* $b \in \mathbb{N}$. *The translation of this pair is the generalized ideal* $\mathrm{Tr}(I, b) = \langle (F_1, b), \ldots, (F_k, b)\rangle$.

Lemma 5.14. *The* Tr *function is well defined.*

Proof. For two different representations $\langle F_1, \ldots, F_k\rangle = \langle G_1, \ldots, G_m\rangle$, we have $\mathrm{Tr}(\langle F_1, \ldots, F_k\rangle, b) = \mathrm{Tr}(\langle G_1, \ldots, G_m\rangle, b)$, by lemma 5.11. □

Lemma 5.15. *Let* (I, b) *be a Villamayor-pair, then* $\mathrm{Sing}(I, b) = \mathrm{Sing}(\mathrm{Tr}(I, b))$.

Proof. Since the ideal generated by the same polynomials in both cases, the statement is true. □

Lemma 5.16. *Consider a Villamayor-pair* (I, b), *then* $\operatorname{ord}(I) = \operatorname{ord}(\mathrm{Tr}(I, b)) \cdot b$.

Proof. By lemma 5.15, the definition of ord and the definition of the Tr function the lemma holds. □

Let the chart U be blown up along $\{P_j \in \mathrm{IND}\}_{j \in I}$, and let the chart with exceptional divisor P_i be U_i. We also assume that the blowing up is permissible, which implies that the divisions can be performed.

Definition 5.8. *In the above described situation the* transform *of the generalized ideal* $J = \langle (F_1, p_1/q_1), \ldots, (F_k, p_k/q_k)\rangle$ *is*

$$\mathrm{Transform}(J, P_i) = \langle (F_{1*}^{q_1} : P_{i*}^{p_1}, p_1), \ldots, (F_{k*}^{q_k} : P_{i*}^{p_k}, p_k)\rangle \subseteq k[U_i].$$

Lemma 5.17. *The* Transform *of generalized ideals is well defined.*

Proof. Without the loss of generality, we can assume that all the rational components of the generators are actually integers, because by lemma 5.9 and lemma 5.12 we can always move non-unit denominators to the exponents of the polynomials.

Assume $\{(F_1, b_1), \ldots, (F_k, b_k)\} \sim \{(G_1, c_1), \ldots, (G_m, c_m)\}$ and fix a chart $U_i \subseteq W$ of the blown up. The pullbacks are also equivalent by lemma 5.4. We have for all $p \in U_i$

$$\operatorname{ord}_p(\{(F_{j*} : P_{i*}^{b_j}, b_j)\}) = \operatorname{ord}_p(\{(F_{j*}, b_j)\}) - \operatorname{ord}_p(\{(P_{i*}^{b_j}, b_j)\}) = \\ \operatorname{ord}_p(\{(F_{j*}, b_j)\}) - \operatorname{ord}_p(\{(P_{i*}, 1)\}),$$

for all $j = 1, \ldots, k$, and, of course, we have the same equations for the other representation. Since the order at p drops by the same constant for all generators, and the order of the generalized ideal at p is the minimum of the orders of the principal ideals of the generators, we must have

$$\operatorname{ord}_p(\operatorname{Transform}(\{(F_1, b_1), \ldots, (F_k, b_k)\}, P_i)) = \\ \operatorname{ord}_p(\operatorname{Transform}(\{(G_1, c_1), \ldots, (G_m, c_m)\}, P_i)).$$

Since any chain of blowing ups can be substituted by a single one, we can apply the same argument for the pullbacks of the two representations along a second blowing up. $\square$

Lemma 5.18. *Consider a Villamayor-pair, (I, b), and perform blowing up operation as described above, then* $\operatorname{Transform}(\operatorname{Tr}(I, b), P_i) = \operatorname{Tr}(\operatorname{Transform}(I, b, P_i), b)$.

Proof. Consider a permissible blowing up so that the divisions can be carried out in $k[U_i]$. Because $\langle P_i^b \rangle$, the principal ideal of the exceptional divisor, is a factor of I_* we have $I_* \subseteq \langle P_i^b \rangle$, so by Theorem 11. p195 in [CLO92], whose method coincides with the one of definition 5.8, the lemma is true. $\square$

Definition 5.9. *Let $\pi : \mathfrak{W} \to U$ be a blowing up as in definition 5.8, $F = \{(F_1, b_1), \ldots, (F_k, b_k)\}$. Let $c = \min_{a \in Z} \operatorname{ord}_a(F)$ and let $b_i c = r_i / q_i$ (if $c = 0$, $r_i = 0, q_i = 1$). The* proper transform *of F in the chart U_i of $\mathfrak{W}$, with exceptional divisor defined by P_i^* is*

$$\overline{F} = \{(P_1^{*q_1} : P_i^{*r_1}, b_1 q_1), \ldots, (P_k^{*q_k} : P_i^{*r_k}, b_k q_k)\} \subset k[U_i] \times \mathbb{Q}_{\geq 0}.$$

Lemma 5.19. *If $\pi : \mathfrak{W} \to U$ is a blowing up as in definition 5.9 and $F = \{(F_1, b_1), \ldots, (F_k, b_k)\} \sim \{(G_1, c_1), \ldots, (G_m, c_m)\} = G$ then for any chart U_i of $\mathfrak{W}$ we have $\overline{F} \sim \overline{G}$.*

Proof. Because $F \sim G$, we must have $\min_{a \in Z} \operatorname{ord}_a(F) = \min_{a \in Z} \operatorname{ord}_a(G)$, thus c is the same for both representants. If $c = 0$, by lemma 5.4, we have $\overline{F} = F^* \sim G^* = \overline{G}$. Otherwise we have $\overline{F} = (\operatorname{Transform}(F_c))_{1/c} \sim (\operatorname{Transform}(G_c))_{1/c} = \overline{G}$, by lemmas 5.5, 5.17. $\square$

Definition 5.10 and lemma 5.20 is taken from [Bod00].

Definition 5.10. *Let $F = \{(F_1, b_1), \ldots, (F_k, b_k)\}$ have the property* $\operatorname{ord}(F) = 1$, *let P be a regular parameter over U, $V = \mathbf{Z}(P)$ with* $\operatorname{Sing}(F) \subseteq V$. *The* coefficient ideal *of F with respect to V, is*

$$\operatorname{Coeff}(F, V) = \{(\partial_P^j F_i, b_i - j) \mid j = 0, \ldots, \lceil b_i \rceil - 1,\ i = 1, \ldots, k\}$$

where the partial differentiation is performed in $k[U]$, but the generators are considered as elements from $k[V] \times \mathbb{Q}_{\geq 0}$.

Lemma 5.20. *Using the notation and the caveats of definition 5.10, the* Coeff *function is well defined over* $\mathrm{Sing}(F)$ *with respect to* $\sim$.

Proof. For $F = \{(F_1, b_1), \ldots, (F_k, b_k)\}$, $G = \{(G_1, c_1), \ldots, (G_m, c_m)\}$ we have to show that if $F \sim G$, we have $\mathrm{Coeff}(F, V) \sim \mathrm{Coeff}(G, V)$ over $\mathrm{Sing}(F)(= \mathrm{Sing}(G))$, which is nonsingular in U by the assumption $\mathrm{ord}(F) = \mathrm{ord}(G) = 1$.

We need to check for a blowing up as in definition 5.8 having $Z \subseteq \mathrm{Sing}(F)$, considering a chart U_i with exceptional divisor coming from $P_i \neq P$, that the diagram

$$\begin{tikzcd} F \arrow[r] \arrow[d, "\mathrm{Coeff}"] \arrow[dr, phantom, "\#"] & \mathrm{Transform}(F, P_i) \arrow[d, "\mathrm{Coeff}"] \\ \mathrm{Coeff}(F, V) \arrow[r] & \mathrm{Transform}(\mathrm{Coeff}(F, V), P_i), \end{tikzcd}$$

commutes in the sense that

$$\mathrm{Transform}(\mathrm{Coeff}(F, V), P_i) \sim \mathrm{Coeff}(\mathrm{Transform}(F, P_i), V'),$$

where $V' = \mathbf{Z}(\overline{P})$. Notice that because $\mathrm{ord}(F) = 1$, $\mathrm{Transfrom}(F, P_i) = \overline{F}$, the proper transform in U_i.

Consider that $\partial_p^j F_t$ in $k[V]$ is nothing else than the coefficient F_{t,p^j} of P^j in the power series expansion of F_t in P. Let us set up the parameter system IND so that both P and P_i are members of it, that is P_i becomes a constant in P. For the (F_t, b_t) generator of F in the jth power, denoted as Coeff_j, of P, we have at any point $a \in V'$,

$$\begin{aligned} &\mathrm{ord}_a(\mathrm{Coeff}_j(\mathrm{Transform}(\{(F_t, b_t)\}, P_i), V')) = \\ &\quad = \mathrm{ord}_a(\{((F_t^*)_{\overline{P}^j}, b_t - j)\}) - \mathrm{ord}_a(\{(P_i^*, (b_t - j)/b_t)\}) = \end{aligned}$$

and because $(F_t^*)_{\overline{P}^j} = P_i^{*j} F^*_{t,P^j}$, we have

$$\begin{aligned} &= \mathrm{ord}_a(\{(F^*_{t,P^j}, b_t - j)\}) + \mathrm{ord}_a(\{(P_i^*, (b_t - j)/j)\}) - \\ &\qquad\qquad \mathrm{ord}_a(\{(P_i^*, (b_t - j)/b_t)\}) = \\ &= \mathrm{ord}_a(\{(F^*_{t,P^j}, b_t - j)\}) - \mathrm{ord}_a(\{((P_i^*)^{\frac{b_t}{b_t - j} - \frac{j}{b_t - j}}, 1)\}) = \\ &= \mathrm{ord}_a(\{(F^*_{t,P^j}, b_t - j)\}) - \mathrm{ord}_a(\{(P_i^*, 1)\}) = \\ &= \mathrm{ord}_a(\mathrm{Transform}(\mathrm{Coeff}_j(\{(F_t, b_t)\}, V), P_i)). \end{aligned}$$

Since the order of a set of ordered pairs at a point is the minimum of the orders of the elements, we have

$$\mathrm{ord}_a(\mathrm{Transform}(\mathrm{Coeff}(F, V), P_i)) = \mathrm{ord}_a(\mathrm{Coeff}(\mathrm{Transform}(F, P_i), V')).$$

Let us perform a second blowing up, now a general one, as permitted by definition 5.5, getting the pullbacks of the coefficient ideals computed along the two paths of the diagram. In the following the inner asterisks denote pullback along the first blowing up, while the outer ones along the second. Because pullback is compatible with polynomial multiplication, we have $(F_t^*)^*_{\overline{P}^j} = (P_i^*)^{*j}(F^*_{t,P^j})^*$.

For a point $a \in V'' = \mathbf{Z}(\overline{P}^*) \subset W_l$, W_l being a chart of the second blownup, we have

$$\begin{aligned}\operatorname{ord}_a(&\operatorname{Coeff}_j(\operatorname{Transform}(\{(F_t, b_t)\}, P_i), V')^*) = \\ &= \operatorname{ord}_a(\{((F^*_{t,P^j})^*, b_t - j)\}) - \operatorname{ord}_a(\{((P_i^*)^*, 1\}) = \\ &= \operatorname{ord}_a(\operatorname{Transform}(\operatorname{Coeff}_j(\{(F_t, b_t)\}, V), P_i)^*).\end{aligned}$$

Again, equality of the orders of the generators at a implies the equality of the orders of the sets of ordered pairs at the a; therefore we have

$$\operatorname{Transform}(\operatorname{Coeff}(F, V), P_i) \sim \operatorname{Coeff}(\operatorname{Transform}(F, P_i), V').$$

We also need to show that $\operatorname{Sing}(F) = \operatorname{Sing}(\operatorname{Coeff}(F, V))$. Let $a \in \operatorname{Sing}(F)$; this means for all $t = 1, \dots, k$, $j = 0, \dots, \lceil b_t \rceil - 1$, by the definition of Sing, $\operatorname{ord}_a(\{(F_{t,P^j}, b_t - j)\}) \geq 1$, that is $a \in \operatorname{Sing}(\operatorname{Coeff}(F, V))$. On the other hand, let $a \in \operatorname{Sing}(\operatorname{Coeff}(F, V))$, that is $\operatorname{ord}_a(\{(F_{t,P^j}, b_t - j)\}) \geq 1$, and because $\operatorname{ord}_a(\{(P, 1)\}) = 1$, we have $\operatorname{ord}_a(\{(F_{t,P^j} P^j, b_t)\}) \geq 1$ also for $j \geq \lceil b_t \rceil$, for all $t = 1, \dots, k$, so $\operatorname{ord}_a(\{(\sum_{j=0}^{\infty} F_{t,P^j} P^j, b_t)\}) \geq 1$, thus $a \in \operatorname{Sing}(F)$.

Now we apply Hironaka's approach (see [Hir77], or [EV00, EV98]) to express $c = \operatorname{ord}_a(\operatorname{Coeff}(F, V)) \geq 1$ for $a \in \operatorname{Sing}(F)$ in terms of quantities which are invariant for the generalized ideal $\langle F \rangle$.

Let us set up $U_0 = U \times \mathbb{A}^1$, $V_0 = V \times \mathbb{A}^1$, and $F_{(0)} = Fk[U_0]$ $(\subset k[U_0] \times \mathbb{Q}_{\geq 0})$, $P_0 = P$ considered in $k[U_0]$, $V_0 = \mathbf{Z}(P_0)$, with $\operatorname{Sing}(F_{(0)}) \subseteq V_0$. We have $C_0 = \operatorname{Coeff}(F_{(0)}, V_0) = \operatorname{Coeff}(F, V)k[U_0]$, thus we have $\operatorname{Sing}(F_{(0)}) = \operatorname{Sing}(C_0)$. Let $L_0 = a \times \mathbb{A}^1$, $a_0 = (a, 0) \in \operatorname{Sing}(F_{(0)})$. We have $c = \operatorname{ord}_{a_0}(\operatorname{Coeff}(F_{(0)}, V_0))$.

Blow up U_0 at a_0, taking any chart of the covering of the blownup, say U_1 with exceptional divisor defined by $P_1 = P_0^*$. Compute $F_{(1)} = \operatorname{Transform}(F_{(0)}, P_0)$ $(= \overline{F_{(0)}}$ over $U_1)$ and $C_1 = \operatorname{Transform}(\operatorname{Coeff}(F_{(0)}, V_0), P_0)$, which is equivalent to $\operatorname{Coeff}(\operatorname{Transfrom}(F_{(0)}, P_0), V_1)$, where $V_1 = \mathbf{Z}(P_1)$, so we have $\operatorname{Sing}(F_{(1)}) = \operatorname{Sing}(C_1)$. Let L_1 be the proper transform of L_0 and $a_1 = L_1 \cap V_1$. We have $\operatorname{ord}_{a_1}(C_1) = \operatorname{ord}_{a_1}(\overline{C_0}) + c - 1$, where the increment comes from the fact that we divided out the exceptional divisor with too little power. Because a_1 is a closed point of L_1, where the order remained constant, by upper-semicontinuity $\operatorname{ord}_{a_1}(F_{(1)})$ could not drop, and because $F_{(1)} = \overline{F_{(0)}}$ it also could not rise; therefore $\operatorname{ord}_{a_1}(F_{(1)}) = \operatorname{ord}_{a_0}(F_{(0)})$.

Next we blow up $a_1 \in U_1$ and continue in the same way. In the lth step we blow up $a_{l-1} \in U_{l-1}$, getting U_l with exceptional divisor $P_l = P^*_{l-1}$, $F_{(l)} = \operatorname{Transform}(F_{(l-1)}, P_{l-1})$ $(= \overline{F_{(l-1)}}$ over $U_l)$, and $C_l = \operatorname{Transform}(\operatorname{Coeff}(F_{(l-1)}, V_{l-1}), P_{l-1})$, which is equivalent to $\operatorname{Coeff}(\operatorname{Transfrom}(F_{(l-1)}, P_{l-1}), V_l)$, where $V_l = \mathbf{Z}(P_l)$, thus $\operatorname{Sing}(F_{(l)}) = \operatorname{Sing}(C_l)$. Let L_l be the proper transform of L_{l-1} and $a_l = L_l \cap V_l$. We have $\operatorname{ord}_{a_l}(C_l) = \operatorname{ord}_{a_l}(\overline{C_{l-1}}) + l(c-1)$, while $\operatorname{ord}_{a_l}(F_{(l)}) = \operatorname{ord}_{a_0}(F_{(0)})$.

If $c > 1$, for large enough l we must have $l(c-1) \geq 1$, which means that the equation of the exceptional divisor, P_l, is accumulated with high enough power in C_l to have $\mathbf{Z}(P_l) \subseteq \operatorname{Sing}(C_l) = \operatorname{Sing}(F_l)$.

We can set the center of the next blowing up to $\mathbf{Z}(P_l)$, which will be an isomorphism over U_l. We get $U_{l+1} = U_l$, $P_{l+1} = P_l$, $V_{l+1} = V_l$, $a_{l+1} = a_l$, $F_{(l+1)} = \operatorname{Transform}(F_{(l)}, P_l)$ $(= \overline{F_{(l)}}$ over $U_{l+1})$, and $C_{l+1} = \operatorname{Transform}(\operatorname{Coeff}(F_{(l)}, V_l), P_l)$, which is equivalent to $\operatorname{Coeff}(\operatorname{Transfrom}(F_{(l)}, P_l), V_{l+1})$, so as before, $\operatorname{Sing}(F_{(l+1)}) =$

$\mathrm{Sing}(C_{l+1})$. Moreover, we have $\mathrm{ord}_{a_{l+1}}(C_{l+1}) = \mathrm{ord}_{a_{l+1}}(\overline{C_l}) + l(c-1) - 1$ and $\mathrm{ord}_{a_{l+1}}(F_{(l+1)}) = \mathrm{ord}_{a_0}(F_{(0)})$.

We can continue blowing up P_{l+o} as long as $l(c-1)-o \geq 1$, that is, $\lfloor l(c-1) \rfloor$ times when $\mathbf{Z}(P_{l+o})$ disappears from $\mathrm{Sing}(C_{l+o}) = \mathrm{Sing}(F_{(l+o)})$. Because of the equality of the singular loci, o depends only on the singular locus of F and its transforms along the presented blowing ups, and on l. On the other hand we have $\lim_{l\to\infty} o/l = \lim_{l\to\infty} \lfloor l(c-1) \rfloor / l = c-1$. If $c = 1$, there is no l for which we can perform blowing up along $\mathbf{Z}(P_l)$. We also remark that the blowing ups we performed in the previous process fulfill the requirements of definition 5.5.

Since the singular loci of any representant of a generalized ideal along any sequence of blowing ups are the same, we have

$$\mathrm{ord}_a(\mathrm{Coeff}(F,V)) = \mathrm{ord}_a(\mathrm{Coeff}(G,V)).$$

For a blowing up $\pi : \mathfrak{W} \to U$ fulfilling the requirements of definition 5.5 over $\mathrm{Sing}(F)$, we have $\pi = \pi_q \circ \cdots \circ \pi_1$, such that $\pi_j : \mathfrak{W}_j \to \mathfrak{W}_{j-1}$ with $\mathfrak{W}_0 = U_0 = U$, $\mathfrak{W}_q = \mathfrak{W}$, so that over some open dense subset U_{j-1} of $\mathfrak{W}_{j-1}$, π_j is a blowing up with center generated by a subset of a system of regular parameters $P_{j-1,1}, \ldots, P_{j-1,r_{j-1}}$ as in definition 5.8. Let

$$T_1 = \mathrm{Transform}(F, P_{0,i_0}), \quad TC_1 = \mathrm{Transform}(\mathrm{Coeff}(F,V), P_{0,i_0}),$$

We have for $a \in \pi_1^{-1}(\mathrm{Sing}(F))$ ($\subseteq \mathrm{Sing}(F^{*\pi_1})$) over U_1 of $\mathfrak{W}_1$ with exceptional divisor P^*_{0,i_0},

$$\begin{aligned}\mathrm{ord}_a(\mathrm{Coeff}(F,V)^{*\pi_1}) &= \mathrm{ord}_a(TC_1) + \mathrm{ord}_a(\{(P_{0,i_0}^{*\pi_1}, 1)\}) = \\ &= \mathrm{ord}_a(\mathrm{Coeff}(T_1, V_1)) + \mathrm{ord}_a(\{(P_{0,i_0}^{*\pi_1}, 1)\}),\end{aligned}$$

where $V_1 = \mathbf{Z}(\overline{P})$ in U_1. We know that $\mathrm{ord}_a(\mathrm{Coeff}(T_1, V_1))$ can be expressed in terms of inherent quantities of $\langle T_1 \rangle$ ($= \langle \mathrm{Transform}(G, P_{0,i_0}) \rangle$); hence we have $\mathrm{ord}_a(\mathrm{Coeff}(F,V)^{*\pi_1}) = \mathrm{ord}_a(\mathrm{Coeff}(G,V)^{*\pi_1})$.

For $k \leq j$, let $\pi_{(j,k)} = \pi_j \circ \cdots \circ \pi_k$ and for $k > j$ let $\pi_{(j,k)} = \mathrm{id}_{\mathfrak{W}_j}$. Assume that for $j \geq 1$ we have at any $a \in \pi_{(j,1)}^{-1}(\mathrm{Sing}(F))$ over U_j of $\mathfrak{W}_j$ that belongs to the sequence of exceptional divisors coming from $P_{j-1,i_{j-1}}, \ldots, P_{0,i_0}$, $\mathrm{ord}_a(\mathrm{Coeff}(F,V)^{*\pi_{(j,1)}}) = \mathrm{ord}_a(\mathrm{Coeff}(G,V)^{*\pi_{(j,1)}})$. Let us define T_j and TC_j in a recursive way,

$$T_j = \mathrm{Transform}(T_{j-1}, P_{j-1,i_{j-1}}), \quad TC_j = \mathrm{Transform}(TC_{j-1}, P_{j-1,i_{j-1}}).$$

Then we have for any $a \in \pi_{(j+1,1)}^{-1}(\mathrm{Sing}(F))$ over U_{j+1} of $\mathfrak{W}_{j+1}$ that belongs to the sequence of exceptional divisors coming from $P_{j,i_j}, P_{j-1,i_{j-1}}, \ldots, P_{0,i_0}$,

$$\begin{aligned}&\mathrm{ord}_a(\mathrm{Coeff}(F,V)^{*\pi_{j+1,1}}) = \\ &\quad = \mathrm{ord}_a(TC_{j+1}) + \mathrm{ord}_a(\{(P_{j,i_j}^{*\pi_{j+1}}, 1)\}) + \sum_{l=0}^{j-1} \mathrm{ord}_a(\{(P_{l,i_l}^{*\pi_{(j+1,l+1)}}, 1)\}) = \\ &\quad = \mathrm{ord}_a(\mathrm{Coeff}(T_{j+1}), V_{j+1}) + \sum_{l=0}^{j} \mathrm{ord}_a(\{(P_{l,i_l}^{*\pi_{(j+1,l+1)}}, 1)\}),\end{aligned}$$

where V_{j+1} is the proper transfrom of V along $\pi_{(j+1,1)}$. Eventually we got that $\mathrm{ord}_a(\mathrm{Coeff}(F,V)^{*\pi})$ can be expressed in terms of $\langle F \rangle$ and π at any point $a \in \pi^{-1}(\mathrm{Sing}(F))$, so $\mathrm{ord}_a(\mathrm{Coeff}(F,V)^{*\pi}) = \mathrm{ord}_a(\mathrm{Coeff}(G,V)^{*\pi})$. $\square$

In the resolution algorithm we need well definedness for coefficient ideals only over the singular locus of the base ideal, because all blowing ups affecting the coefficient ideal are along centers within its singular locus (or within the singular locus of its transform along a sequence of blowing ups).

Remark. One subtle point of the well definedness of the Coeff function is to show $\mathrm{Coeff}(\langle(F,b)\rangle,V) = \mathrm{Coeff}(\langle(F^l,bl)\rangle,V)$. We can try to use the previously proven lemmata to check this in a simple example.

Let $F = a_3x^3 + a_2x^2 + a_1x + a_0$, and consider $J_1 = \mathrm{Coeff}(\langle(F,2)\rangle,\mathbf{Z}(x)) = \langle(a_1,1),(a_0,2)\rangle$, and $J_2 = \mathrm{Coeff}(\langle(F^2 = a_3^2x^6 + 2a_2a_3x^5 + (2a_1a_3 + a_2^2)x^4 + (2a_0a_3+2a_1a_2)x^3+(2a_0a_2+a_1^2)x^2+2a_0a_1x+a_0^2,4)\rangle,\mathbf{Z}(x)) = \langle(2a_0a_3+2a_1a_2,1), (2a_0a_2 + a_1^2,2),(2a_0a_1,3),(a_0^2,4)\rangle$.

Let's try to generate J_2 with the generators of J_1: $(a_0^2,4) \in J_1$ by lemma 5.9, $(2a_0a_1,3) \in \langle(a_0,2),(a_1,1),(2a_0,2),(2a_0a_1,3)\rangle = J_1$ by lemmas 5.13, 5.10, $(2a_0a_2 + a_1^2,2) \in \langle((a_0,2),(a_1^2,2),(2a_2a_0,2),((2a_2a_0 + a_1^2,2)\rangle = J_1$ by lemmas 5.13, 5.9, $(2a_0a_3+2a_1a_2,1) \in \langle(a_0,2),(a_1,1),(a_0,1),(2a_3a_0,1),(2a_2,a_1,1)\rangle = J_1$ by lemma 5.13.

Now let's try to generate J_1 with the generators of J_2: $(a_0,2) \in J_2$ by lemma 5.9, and finally $(a_1,1) \in \langle(2a_0a_3 + 2a_1a_2,1),(2a_0a_2 + a_1^2,2),(2a_0a_1,3), (a_0^2,4),(-2a_2a_0,2),(a_1^2,2)\rangle = J_2$ by lemmas 5.13, 5.9.

However, the general case is much deeper, requiring more powerful arguments, like in lemma 5.20. The reader might try to show $\mathrm{Coeff}(\langle(F,3)\rangle,\mathbf{Z}(x)) = \mathrm{Coeff}(\langle(F^2,6)\rangle,\mathbf{Z}(x))$ with the above approach, where $F = \sum_{i=0}^5 a_ix^i$ (we didn't succeed).

Lemma 5.21. *Let (I,b) be a Villamayor-pair, and $V = \mathbf{Z}(P)$ be a hypersurface of maximal contact as above, then* $\mathrm{Coeff}(\mathrm{Tr}(I,b),V) \sim \mathrm{Tr}(\mathrm{Coeff}(I,b,V),b!)$.

Proof. Consider

$$\begin{aligned}\mathrm{Coeff}(\mathrm{Tr}(I,b),V) = \langle(F_1,b),(\partial_PF_1,b-1),\dots,(\partial_P^{b-1}F_1,1),\dots,\\ (F_k,b),(\partial_PF_k,b-1),\dots,(\partial_P^{b-1}F_k,1)\rangle.\end{aligned}$$

By writing the coefficient ideal as the sum of the principal ideals of the generators (by lemma 5.12), and using lemma 5.9 we have that the above ideal is the same as

$$\begin{aligned}\langle(F_1^{b!/b},b!),((\partial_PF_1)^{b!/(b-1)},b!),\dots,((\partial_P^{b-1}F_1)^{b!},b!),\dots,\\ (F_k^{b!/b},b!),((\partial_PF_k)^{b!/(b-1)},b!),\dots,((\partial_P^{b-1}F_k)^{b!},b!)\rangle,\end{aligned}$$

which is $\mathrm{Tr}(\mathrm{Coeff}(I,b,V),b!)$. □

If we are given a list of IND elements $P_1,\dots,P_r$ generating hypersurfaces of maximal contact $V_1,\dots,V_r$ respectively, the coefficient ideal of J with respect to all of them is just

$$\mathrm{Coeff}(J,(V_1,\dots,V_r)) = \mathrm{Coeff}(\mathrm{Coeff}((\dots\mathrm{Coeff}(J,V_1)\dots),V_{r-1}),V_r).$$

Lemma 5.22. *For an ordinary ideal I and $b \in \mathbb{N}_+$ we have for any $n \in \mathbb{N}_+$,* $\mathrm{Tr}(I,b) = \mathrm{Tr}(I^n,bn)$.

Proof. We have for any $p \in U$, $\mathrm{ord}_p(\mathrm{Tr}(I,b)) = \mathrm{ord}_p(\mathrm{Tr}(I^n,bn))$ by lemma 5.16 and the fact $\mathrm{ord}_p(I^n) = \mathrm{ord}_p(I)\cdot n$. Since the pullback is compatible with ring operations, by the definition of product of ideals and by lemma 5.10 we have the above equation for the pullbacks along any permissible blowing up. □

6 New representation of objects

We define the new representations of weighted, tagged and balanced objects, which rely on the notion of generalized ideals, presented in the section 5. Exceptional divisors can always be represented locally by regular parameters in the algorithm; therefore we assume that they are generated by elements from IND.

Definition 6.1. *A* weighted object, $\mathcal{W} = (W, J, C, c, E, a)$ *contains the following data:*

- *W is a nonsingular algebraic set,*
- *J is a nonzero generalized ideal over $k[W] \times \mathbb{Q}_{\geq 0}$,*
- *C is list of functions, $C = (a_i : \mathrm{IND} \to \mathbb{Q}_{\geq 0} \mid i = 1, \ldots, m)$,*
- *$c \in \mathbb{Q}_{\geq 0}$, the weighted order (see section 7.2),*
- *E is a sublist of* IND*, the list of generators of exceptional divisors,*
- *a is a function, $a : E \to \mathbb{Q}_{\geq 0}$.*

For the sake of simpler presentation we assume $c = 1$. Let's construct the vanishing generalized ideal of such a weighted object from the data of J, C, E, a in several steps.

Definition 6.2. *Let $E = (P_{i_1}, \ldots, P_{i_r})$, $(i_1 < \cdots < i_r)$, $a = (d_{i_1}/d, \ldots, d_{i_r}/d)$, and let $M = \prod_{j=1}^{r} P_{i_j}^{d_{i_j}}$. The* monomial part *of $\mathcal{W}$, denoted as E^a, is $\langle (M, d) \rangle$.*

Definition 6.3. *Let $J = \langle (F_1, b_1), \ldots, (F_k, b_k) \rangle$, and apply the translation of definition 6.2 for all $a_i \in C$ getting $\langle (G_1, c_1), \ldots, (G_m, c_m) \rangle$. The* proper part *of $\mathcal{W}$ is $\langle (F_1, b_1), \ldots, (F_k, b_k), (G_1, c_1), \ldots, (G_m, c_m) \rangle$.*

To compute the product of the monomial and the proper part, take an $l \in \mathbb{N}$ such that l/d, l/b_i, l/c_j are all natural numbers (whenever the fraction is defined).

Definition 6.4. *The generalized ideal of the weighted object $\mathcal{W}$, using the notation introduced in definitions 6.2, 6.3, is*

$$\langle (F_1^{l/b_1} M^{l/d}, l), \ldots, (F_k^{l/b_k} M^{l/d}, l), (G_1^{l/c_1} M^{l/d}, l), \ldots, (G_m^{l/c_m} M^{l/d}, l) \rangle.$$

Actually, we will never have to compute this ideal in the algorithm explicitly.

Definition 6.5. *A* tagged object, $\mathcal{T} = (W, J, C, E, E_-)$, *contains the data: W, J, C, E with types as in a weighted object, and*

- *E_- a sublist of E.*

Definition 6.6. *The ideal of a tagged object $\mathcal{T}$ is computed from J and C as the proper part of a weighted object in definition 6.3.*

Definition 6.7. *A* balanced object, $\mathcal{B} = (W, J, C, E, H)$, *contains the data: W, J, C, E with types as in a weighted object, and*

- *H a sublist of* IND.

Definition 6.8. *Let the generalized ideal computed from J and C as in definition 6.3 be $\langle (F_1, b_1), \ldots, (F_k, b_k) \rangle$ and let $H = (P_1, \ldots, P_r)$. Then the generalized ideal of the balanced object is $\langle (F_1, b_1), \ldots, (F_k, b_k), (P_1, 1), \ldots, (P_r, 1) \rangle$.*

7 Computations in the new representation

We define the operations of the resolution algorithm on the new data structures given in the previous section. We overload function names like Sing, Coeff or Transform.

The point in using the complicated formulae presented in this section instead of computing the ideal of definition 6.4 and using the corresponding functions for generalized ideals, is to avoid the powering of the generators prescribed in definition 6.4. Although in general we cannot avoid Gröbner basis computation, the new data structure allows us, for instance, to detect resolvedness by means of combinatorial methods in certain cases. It is also beneficial to keep the generators of the input ideals of Gröbner basis computations simple, which is not the case in the construction of definition 6.4.

We would like to bring to the reader's attention a simple but useful way to check the correctness of the operations defined in this section. The sometimes seemingly complicated formulae of certain operations were gotten by constructing the generalized ideal of the given object using definition 6.4, 6.6 or 6.8 then applying the corresponding generalized ideal operation and reverse engineering what happens with the function values of C or a.

7.1 Singular locus

The singular locus of generalized ideals is defined in terms of finite sets of generators in definition 5.2.

Consider the monomial part E^a of a weighted object with M and d as in definition 6.2. We define $\mathrm{Sing}(E^a) = \mathrm{Sing}(\langle (M,d)\rangle)$.

Lemma 7.1. *With the above notation we have*

$$\mathrm{Sing}(E^a) = \mathbf{Z}\Big(\bigcap_{\substack{\{P_{i_1},\dots,P_{i_l}\}\subseteq E \\ \sum_{j=1}^{l} a(P_{i_j})\geq 1}} \langle P_{i_1},\dots,P_{i_l}\rangle \Big).$$

Proof. Take M and d as in definition 6.2. By definition 5.2 $p \in \mathrm{Sing}(\langle (M,d)\rangle)$ iff $\mathrm{ord}_p(M) \geq d$, and for all $p \in \mathbf{Z}(P_i)$ we have $\mathrm{ord}_p(P_i) = 1$ for all i, moreover the exceptional divisors can have at most normal crossings. Considering this, the construction of the ideal of the lemma just describes the points of $\mathrm{Sing}(E^a)$. $\square$

Let $C = (a_1,\dots,a_m)$ represent $\langle (G_1,c_1),\dots,(G_m,c_m)\rangle$ as in definition 6.3. We define $\mathrm{Sing}(C) = \mathrm{Sing}(\langle (G_1,c_1),\dots,(G_m,c_m)\rangle)$.

Lemma 7.2. *With the above notation we have*

$$\mathrm{Sing}(C) = \bigcap_{i=1}^{m} \mathbf{Z}\Big(\bigcap_{\substack{\{P_{i_1},\dots,P_{i_l}\}\subseteq \mathrm{IND} \\ \sum_{j=1}^{l} a_i(P_{i_j})\geq 1}} \langle P_{i_1},\dots,P_{i_l}\rangle \Big).$$

Proof. The singular locus of C is the intersection of the singular loci of the principal ideals generated by pairs (G_i,c_i), coming from the a_i functions as in definition 6.3. Now by lemma 7.1, the singular locus of $\langle (G_i, c_i) \rangle$ is

$$\mathbf{Z}\Big(\bigcap_{\substack{\{P_{i_1},\dots,P_{i_l}\} \subseteq \mathrm{IND} \\ \sum_{j=1}^{l} a_i(P_{i_j}) \geq 1}} \langle P_{i_1}, \dots, P_{i_l} \rangle \Big).$$

□

7.1.1 Weighted objects

Consider a weighted object as in definition 6.1, and use the notation of definition 6.4. Let the generalized ideal of the weighted object as in definition 6.4 be I with rational component l. We define the singular locus of the weighted object to be $\mathrm{Sing}(I)$.

In the sequel, the function list C_c denotes the list of the functions of C with all the values multiplied by $c \in \mathbb{Q}_{\geq 0}$. The algorithm changes the rational components of J and the function values of C in the weighted order computation (see section 7.2). However, we need the original rational values to compute the singular locus, which we can retrieve by taking $J_{1/c}$, and C_c (the inverse operation). Therefore we can assume, without the loss of generality, that $c = 1$.

Lemma 7.3. *With the notation described above we have*

$$\begin{aligned} \mathrm{Sing}(I) = \ & \mathrm{Sing}(\langle (M,d) \rangle) \cup (\mathrm{Sing}(J) \cap \mathrm{Sing}(C)) \cup \\ & \bigcup_{r=1}^{l-1} \Big(\mathrm{Sing}(J_{\frac{r}{l}}) \cap \mathrm{Sing}(C_{\frac{l}{r}}) \cap \mathbf{Z}\big(\bigcap_{\substack{\{P_{i_1},\dots,P_{i_l}\} \subseteq E \\ \sum_{j=1}^{l} a(P_{i_j}) + \frac{r}{l} \geq 1}} \langle P_{i_1}, \dots, P_{i_l} \rangle \big) \Big). \end{aligned}$$

Proof. Consider that $\mathrm{Sing}(I)$ consists of those points $p \in U$ where $\mathrm{ord}_p(F_i^{l/b_i}) + \mathrm{ord}_p(M^{l/d}) \geq l$ and $\mathrm{ord}_p(G_j^{l/c_j}) + \mathrm{ord}_p(M^{l/d}) \geq l$ for all i and j. The extreme cases are:

1. $\mathrm{ord}_p(M^{l/d}) \geq l$ and,
2. $\mathrm{ord}_p(F_i^{l/b_i}) \geq l$ and $\mathrm{ord}_p(G_j^{l/c_j}) \geq l$ for all i, j.

Points of type 1 are exactly the ones in $\mathrm{Sing}(\langle (M,d) \rangle)$, while of points of type 2 are the ones in $\mathrm{Sing}(J) \cap \mathrm{Sing}(C)$.

In the intermediate cases, for $r = 1, \dots, l-1$, we require $\mathrm{ord}_p(F_i^{l/b_i}) \geq r$, $\mathrm{ord}_p(G_j^{l/c_j}) \geq r$ and $\mathrm{ord}_p(M^{l/d}) \geq l - r$. These points are exactly the ones in

$$\begin{aligned} & \mathrm{Sing}(\langle (F_1^{l/b_1}, r), \dots, (F_k^{l/b_k}, r), (G_1^{l/c_1}, r), \dots, (G_m^{l/c_m}, r) \rangle) \cap \\ & \quad \mathrm{Sing}(\langle (M^{l/d}, l - r) \rangle) = \\ & \qquad \mathrm{Sing}(J_{r/l}) \cap \mathrm{Sing}(C_{l/r}) \cap \mathrm{Sing}(\langle (M, d(1 - r/l)) \rangle). \end{aligned}$$

We take the union of all these points to get the singular locus of I. □

7.1.2 Tagged objects

Consider a tagged object $\mathcal{T}$ as in definition 6.5, with I its ideal as in definition 6.6. The singular locus of $\mathcal{T}$ is $\mathrm{Sing}(I)$.

Lemma 7.4. *With the above notation we have* $\mathrm{Sing}(I) = \mathrm{Sing}(J) \cap \mathrm{Sing}(C)$.

Proof. The lemma is just a special case of lemma 7.3, namely $c = 1$ and the monomial factor is 1. ☐

7.1.3 Balanced objects

Consider a balanced object $\mathcal{B}$ as in definition 6.7, with its ideal I as in definition 6.8. The singular locus of $\mathcal{B}$ is $\mathrm{Sing}(I)$.

Lemma 7.5. *With the above notation we have* $\mathrm{Sing}(I) = \mathrm{Sing}(J) \cap \mathrm{Sing}(C) \cap \bigcap_{P_i \in H} \mathbf{Z}(P_i)$.

Proof. It is clear by lemma 7.4 and the fact that the rational component is 1 for all the generators of I coming from $P_i \in H$. ☐

7.2 Weighted order

Use the notation of definition 6.4, denote the ideal of the weighted object with I. The *weighted order* (w-ord) of the weighted object $\mathcal{W}$ is the (maximal) order of the proper part (see definition 6.3) within $\mathrm{Sing}(I)$.

Lemma 7.6. *With the notation as in the above paragraph, the weighted order of $\mathcal{W}$ is i/l, where i is the maximal natural number so that* $\mathrm{Sing}(J_{i/l}) \cap \mathrm{Sing}(C_{l/i}) \cap \mathrm{Sing}(I)$ *is not empty, or 0 if no such i exists.*

Proof. Consider the representation of J and C as in definition 6.4 (namely $\langle (F_1^{l/b_1}, l), \ldots, (F_k^{l/b_k}, l), (G_1^{l/c_1}, l), \ldots, (G_m^{l/c_m}, l) \rangle$). Since all the generators have the same rational component, by lemma 5.16 we have

$$\operatorname{ord}_{\mathrm{Sing}(I)}(\langle (F_1^{l/b_1}, l), \ldots, (F_k^{l/b_k}, l), (G_1^{l/c_1}, l), \ldots, (G_m^{l/c_m}, l) \rangle) = \\ \operatorname{ord}_{\mathrm{Sing}(I)}(\langle F_1^{l/b_1}, \ldots, F_k^{l/b_k}, G_1^{l/c_1}, \ldots, G_m^{l/c_m} \rangle)/l.$$

Let i be the numerator of the right hand side of the above equation. ☐

Remark. The updated generalized ideal is $J_{i/l}$, the C list gets updated to $C_{l/i}$, and c is updated by setting it to $c \cdot i/l$.

7.3 Equiorder

Let $\mathcal{W} = (W, J, c, C, E, a)$ be a weighted object, the input of the *EquiOrder* algorithm. We assume that the rational numbers in J, C, as well as c are updated by the weighted order computation. Notice that we must have $c > 0$, otherwise $\mathcal{W}$ is either resolved or monomial (see section 3.1). The *EquiOrder* algorithm for the new representation of objects performs the following computations:

- When $c \geq 1$, the tagged object $\mathcal{T} = (W, J, c = 1, C, E, E_- = E)$.

– If $0 < c < 1$, we define a new function a' on IND as:

$$a'(P) = \begin{cases} a(P)/(1-c) & \text{if } P \in E \\ 0 & \text{otherwise.} \end{cases}$$

The new tagged object is $\mathcal{T} = (W, J, c = 1, \text{append}(C, a'), E, E_- = E)$.

Use the notation of definition 6.4, let I denote the proper part, and let I' be the ordinary ideal such that $I = \text{Tr}(I', l)$. Let $\mathcal{T}_0 = \mathit{EquiOrder}(W, I', l, E, a)$ using the old representation, while $\mathcal{T}_1 = \mathit{EquiOrder}(W, J, c, C, E, a)$ using generalized ideals in the new representation. Let (I'_0, l_0) be the Villamayor-pair of $\mathcal{T}_0$, and I_1 be the generalized ideal of $\mathcal{T}_1$ as in definition 6.6.

Lemma 7.7. *With the above notation, we have* $\text{Tr}(I'_0, l_0) = I_1$.

Proof. If $c \geq 1$ the the lemma is trivial.

Assume $c = i/l,\ 0 < i < l$ and consider that for I' we have $\bar{b} = i < l = b$. So applying the compatibility of summation with the Tr function we get

$$\text{Tr}(I'_0, i(l-i)) = \text{Tr}({I'}^{l-i}, i(l-i)) + \text{Tr}(M^{il/d}, i(l-i)) =$$

by lemma 5.22

$$= \text{Tr}(I', i) + \text{Tr}(M^{l/d}, l-i) = I + \langle (M, d(1-c)) \rangle = I_1$$

by the construction of definition 6.6. Since $l_0 = i(l-i)$ the lemma is true. □

7.4 Support

The computation of the support of a tagged object $\mathcal{T} = (W, J, C, E, E_-)$ is done as presented in the algorithm *Support2*, with the difference that the generalized ideal doesn't get modified whenever $E_- = E$. The *Support2* algorithm for the new representation of objects performs the following computations:

– When $E = E_-$, the balanced object is $\mathcal{B} = (W, J, C, E - E_-, H = E_-)$.

– When $E \neq E_-$, assume $E_- = (P_1, \ldots, P_r),\ r \geq 1$, the new balanced object is $\mathcal{B} = (W, I = J + \langle (P_2, 1), \ldots, (P_r, 1) \rangle, C, E - E_-, H = (P_1))$.

Use the notation of definition 6.6, let I be the generalized ideal of $\mathcal{T}$, and let I' be an ordinary ideal such that $I = \text{Tr}(I', l)$. Let $\mathcal{B}_0$ be the resulting balanced object in a chart generated by *Support2*(W, I', l, E, E_-), and let $\mathcal{B}_1$ be the resulting balanced object over the same chart generated by *Support2*(W, J, C, E, E_-).

The previous assumption contains a proposition, namely that the chart structures generated by the two function calls are the same. This is easy to see because we have $\text{Sing}(I', l) = \text{Sing}(I)$ by lemma 5.15 and E_- is the same in both cases.

Let then I'_0 be the ideal of $\mathcal{B}_0$, and let I_1 be the ideal of $\mathcal{B}_1$ as in definition 6.8.

Lemma 7.8. *With the above notation we have* $\text{Tr}(I'_0, l) = I_1$.

Proof. Assume $I' = \langle F_1, \ldots, F_k \rangle$, and we have $I = \mathrm{Tr}(I', l) = \langle (F_1, l), \ldots, (F_k, l) \rangle$, while $E_- = (P_1, \ldots, P_r)$. Without the loss of generality we can assume that E_- has common singularities (i.e. no 'fork' is done in *Support2*). Then either $E = E_-$ or $E \neq E_-$, by definition 6.8 and the construction of the new balanced object above we have

$$\begin{aligned}\mathrm{Tr}({I'}_0, l) = \langle (F_1, l), \ldots, (F_k, l), (P_1^l, l), \ldots, (P_r^l, l) \rangle = \\ = \langle (F_1, l), \ldots, (F_k, l), (P_1, 1), \ldots, (P_r, 1) \rangle = I_1.\end{aligned}$$

□

7.5 Coefficient ideal computation

Let $\mathcal{B} = (W, J, C, E, H)$ be a balanced object, and let V be a list of generators of hypersurfaces of maximal contact (either coming from H or computed explicitly in *ResolveBalanced2*), and let $Z = (\mathbf{Z}(P) \mid P \in V)$. The coefficient object is

$$\mathcal{W} = (W \cap \mathbf{Z}(V), \mathrm{Coeff}(J, Z), c = 1, \mathrm{Coeff}(C, Z), E, 0_E),$$

where 0_E is the zero function on E, and $\mathrm{Coeff}(C, Z)$ is defined below.

For a function $a \in C$ we define $\mathrm{Coeff}(a, Z)$ to be the zero function (on IND) if $\sum_{P \in V} a(P) \geq 1$, else

$$\mathrm{Coeff}(a, Z)(P) = \begin{cases} 0 & \text{if } P \in V, \\ a(P)/(1 - \sum_{P \in V} a(P)) & \text{otherwise.} \end{cases}$$

We define $\mathrm{Coeff}(C, Z) = (\mathrm{Coeff}(a, Z) \mid a \in C,\ \mathrm{Coeff}(a, Z) \neq 0_{\mathrm{IND}})$.

Let I denote the ideal of $\mathcal{B}$ as in definition 6.8, and let I' denote the ideal coming from $\mathrm{Coeff}(J, Z)$ and $\mathrm{Coeff}(C, Z)$ in the same way.

Lemma 7.9. *With the above notation we have* $\mathrm{Coeff}(I, Z) = I'$.

Proof. First we simplify the claim to be proved. Assume that

$$I = \langle (F_1^{l/b_1}, l), \ldots, (F_k^{l/b_k}, l), (G_1^{l/c_1}, l), \ldots, (G_m^{l/c_m}, l) \rangle,$$

and $I' = \mathrm{Coeff}(\langle (F_1, b_1), \ldots, (F_k, b_k) \rangle, Z) + \langle (H_1, d_1), \ldots, (H_m, d_m) \rangle$, where (H_i, d_i) comes from $\mathrm{Coeff}(a_i, Z)$ for $a_i \in C$.

Since coefficient ideal computation is generator-wise, it suffices to show equality on the principal generalized ideals of the generators. By lemma 5.20 we have $\mathrm{Coeff}(\langle (F_i^{l/b_1}, l) \rangle, Z) = \mathrm{Coeff}(\langle (F_i, b_i) \rangle, Z)$ for $i = 1, \ldots, k$.

Along the same lines, we have $\mathrm{Coeff}(\langle (G_i^{l/c_i}, l) \rangle, Z) = \mathrm{Coeff}(\langle (G_i, c_i) \rangle, Z)$, and what we need to show is $\mathrm{Coeff}(\langle (G_i, c_i) \rangle, Z) = \langle (H_i, d_i) \rangle$, for $i = 1, \ldots, m$. It is enough to show the equality for the case when V contains only one generator, and then one can easily proceed by induction to show the general case.

Assume that $G_i = P_1^{e_1} \ldots P_r^{e_r}$, and $V = (P_1)$. If $e_1 \geq c_i$, P_1 is a factor in every polynomial in the generators of $\mathrm{Coeff}(\langle (G_i, c_i) \rangle, Z)$; therefore all of them are zero in $k[U \cap \mathbf{Z}(V)]$.

If $e_1 < c_i$, P_1 is a factor in $\partial^j_{P_1} G_i$ for $j = 0, \ldots, e_1 - 1$, and $\partial^j_{P_1} G_i = 0$ for $j = e_1 + 1, \ldots, \lceil c_i \rceil - 1$. The only nonzero generator in $k[U \cap \mathbf{Z}(V)]$ is $(H_i, d_i) = (\partial^{e_1}_{P_1} G_i, c_i - e_1) = (P_2^{e_2} \ldots P_r^{e_r}, c_i - e_1)$, which is the corresponding generator for the function $\mathrm{Coeff}(a_i, Z) = (0, e_2/(c_i - e_1), \ldots, e_r/(c_i - e_1), 0, \ldots)$. □

7.6 Transform along blowing up

Let the chart U be blown up along $\{P_i \in \text{IND}\}_{i\in I}$, and let the chart with exceptional divisor P_i be U_i. We describe the transform operation only for weighted and tagged objects, since the balanced objects are recomputed from their ancestors every time.

The transform of the generalized ideal is performed by the Transform function (see definition 5.8), its compatibility with the Tr function is shown by lemma 5.18.

In case of a weighted object, in the transformed of the function a is

$$\text{Transform}(a, P_i)(P) = \begin{cases} c - 1 + \sum_{P_j \in E,\, j\in I} a(P_j) & \text{if } P = P_i, \\ a(P) & \text{otherwise.} \end{cases}$$

Consider that by the weighted order computation (see lemma 7.6), $c = i/l \ = \bar{b}/b$, so the operation reflects the transform in the original algorithm: $\bar{b} - b + \sum_{P_j\in E,\, j\in I} a(P_j)$, with the difference that the values of a are divided by b (the reader can easily check how it corresponds to $\langle (M, d)\rangle$ in definition 6.2).

The transformed of a function $a \in C$ is

$$\text{Transform}(a, P_i)(P) = \begin{cases} \sum_{j\in I} a(P_j) \ \ - 1 & \text{if } P = P_i, \\ a(P) & \text{otherwise.} \end{cases}$$

Then we define $\text{Transform}(C, P_i) = (\text{Transform}(a, P_i) \mid a \in C)$.

Let $\langle (G_1, c_1), \ldots, (G_m, c_m)\rangle$ be the generalized ideal represented by C as in definition 6.3, and let $\langle (H_1, d_1), \ldots, (H_m, d_m)\rangle$ be the generalized ideal represented by the $\text{Transform}(C, P_i)$, as in definition 6.3.

Lemma 7.10. *With the above notation we have*

$$\text{Transform}(\langle (G_1, c_1), \ldots, (G_m, c_m)\rangle, P_i) = \langle (H_1, d_1), \ldots, (H_m, d_m)\rangle.$$

Proof. Since the transform computation is generator-wise, it is enough to show that $(G_{k*} : P_{i*}^{c_k}, c_k)$ is represented by $\text{Transform}(a_k, P_i)$ $a_k \in C$ (recall that c_k is a natural number). Without the loss of generality we can assume $G_k = P_1^{e_1} \ldots P_r^{e_r}$ and $i = 1$. Then we have $G_{k*} = P_{1*}^{\sum_{j\in I,\, j\le r} e_j} P_{2*}^{e_2} \ldots P_{r*}^{e_r}$; therefore

$$(G_{k*} : P_{1*}^{c_k}, c_k) = (P_{1*}^{\sum_{j\in I,\, j\le r} e_j - c_k} P_{2*}^{e_2} \ldots P_{r*}^{e_r}, c_k),$$

which is just (H_k, d_k), the generator given by $\text{Transform}(a_k, P_1)$. □

Consider a weighted object $\mathcal{W} = (W, J, c, C, E, a)$, let I be the generalized ideal constructed from $J_{1/c}$, C_c and a as in definition 6.4, and let I' be an ordinary ideal such that $I = \text{Tr}(I', l)$. Keep in mind that the weighted order computation might have updated the rational components of J and the values of the functions in C; therefore with the above order updates we just set back the original values. Recall that we have $\text{Transform}(I', l, P_i) = \text{Transform}(I, P_i)$ by lemma 5.18.

Let I'' be the generalized ideal constructed as in definition 6.4, from the transformed data: $\text{Transform}(J, P_i)_{1/c}$, $\text{Transform}(C, P_i)_c$, $\text{Transform}(a, P_i)$.

Lemma 7.11. *With the above notation we have* $\text{Transform}(I, P_i) = I''$.

Proof. Since Transform is a generator-wise operation, what we have to show is that the generators of Transform(I,P_i), $(F_{j*}^{l/b_j} M_*^{l/d} : P_{i*}^l, l)$ and $(G_{e*}^{l/c_e} M_*^{l/d} : P_{i*}^l, l)$ are in one to one correspondence with the generators of I''.

We have Transform$(J,P_i)_{1/c}$ generated by $(F_{j*} : P_{i*}^{B_j}, b_j)$, $j = 1,\dots,m$, where $B_j = c \cdot b_j$, c being the weighted order. We assume that l is such that l/B_j and l/C_e are also integers, where $C_e = c \cdot c_e$. For the sake of simplicity and without the loss of generality, we can assume (by lemma 5.9) that all the B_j's and C_e's are integers.

Thus we can write $\langle (F_{j*} : P_{i*}^{B_j}, b_j) \rangle = \langle (F_{j*}^{l/b_j} : P_{i*}^{cl}, l) \rangle$. By a similar argument as of lemma 7.10 one gets that Transform(a,P_i) is just the operation $\langle (M_*^{l/d} : P_{i*}^{l(1-c)}, l) \rangle$ on the monomial part, where whenever $c > 1$ the division actually turns into multiplication. Finally, by applying the construction of definition 6.4 for these generators we get $((F_{j*}^{l/b_j} : P_{i*}^{cl})(M_*^{l/d} : P_{i*}^{l(1-c)}), l) = (F_{j*}^{l/b_j} M_*^{l/d} : P_{i*}^l, l)$ the corresponding generator of Transform(I,P_i).

The same argument applies to the generators of Transform(I,P_i) of the form $(G_{e*}^{l/c_e} M_*^{l/d} : P_{i*}^l, l)$, and by lemma 7.10 we get that Transform(C,P_i) performs just the desired operation. $\square$

The point in computing the transform of the weighted object by transforming J, C and a separately is to avoid costly Gröbner basis computations on the generators of I. Though the transform of a generalized ideal is computed by polynomial division on the generators (see definition 5.8), this division is computed in the coordinate ring of the chart U (or in a subset of U), so we cannot avoid using the method of Gröbner bases. Furthermore, this way we transform proper to proper and monomial to monomial part, saving additional computations in the (first line of) algorithm *EquiOrder*.

Lemma 7.12. *The claim of lemma 7.11 in the setup for tagged objects holds.*

Proof. This case is much easier, because of the lack of the monomial part; hence we leave the detailed proof to the reader. We only say that the lemma holds by lemma 5.17 and lemma 7.10. $\square$

In case of a tagged object P_i is dropped from E_- when it contains it.

Remark. Balanced objects do not get transformed.

7.7 Detecting good points

In certain situations one can resolve weighted objects much easier than the way the Villamayor stratifying function prescribes. Such a situation is when the singular locus of the weighted object consists only of so called "exceptional and good" points. Using the original notation of a weighted object (W,J,b,E,a) (see section 3), a point $x \in W$ is *exceptional and good* for $J = \langle P_{i_1}^{a(P_{i_1})} \cdots P_{i_k}^{a(P_{i_k})} \rangle \overline{J}$ with b, if $\mathrm{ord}_x(\langle P_{i_1}^{a(P_{i_1}) \bmod b} \cdots P_{i_k}^{a(P_{i_k}) \bmod b} \rangle \overline{J}) < b$. For more details we refer to section 3 in [EV98].

If we have this condition fulfilled by all the points of the singular locus, we can make the weighted object resolved by blowing up along exceptional divisors. Since blowing ups along hypersurfaces are isomorphisms, we can compute these operations by simply updating the function values of a by subtracting b from the a-value of the exceptional divisor of the blowing up center.

In the new representation, for a weighted object (W, J, C, c, E, a), where we have weighted order computation done (see section 7.2), we can detect easily whether the singular locus of a weighted object consists only of exceptional and good points: We need only to check whether there is $P \in E$ such that $a(P) \geq 1$ and whether $c + \sum_{P \in E}\{a(P)\} < 1$, where $\{\}$ denotes the fraction function.

We also have to remark that if the manifold W of the weighted object is of smaller dimension than the underlying chart, the basic objects, whose manifold contain W are also blown up along centers of codimension higher than one. This means that the blowing up operation for them is not an isomorphism. Nevertheless, it is still much more efficient to resolve the "good" weighted object this way then the one provided by the stratifying function.

8 Equivalence of algorithms

We can formulate the resolution algorithm using the new representation of weighted, tagged and balanced objects, presented in section 6, and applying the new computational techniques presented in section 7 and in section 5. We overload function names like *ResolveWeighted*, so that their new versions accept basic objects in the new representation.

Theorem 8.1. *For $f \in k[x_1, \ldots, x_n]$, the function calls* ResolveWeighted$(W = \mathbb{A}^n, J = \langle f \rangle, b = 1, E = (), a = 0_E)$ *and* ResolveWeighted$(W = \mathbb{A}^n, J = \langle (f, 1) \rangle, c = 1, C = (), E = (), a = 0_E)$ *compute the same sequence of blowing up and cover operations in the resolution of* $\mathbf{Z}(f)$.

Proof. Since $\mathrm{Tr}(\langle f \rangle, 1) = \langle (f, 1) \rangle$, with $C = ()$, and $c = 1$ (the rational components are not modified), the two objects describe exactly the same resolution problem.

Proceed by the steps of the algorithm *Villamayor_resolution2* (see page 14) in both cases and apply the appropriate lemma from the previous sections to show the equivalence of the translation of the ordinary ideal and the generalized ideal computed from object data as in definitions 6.4, 6.6, 6.8.

Since we have equal singular loci at every step of the algorithms, we may assume that its vanishing ideal is generated by the same polynomials in both resolution processes. This way we are sure that we get the same chart structure in cover operations, as well as the same generators of hypersurfaces of maximal contact. □

Finally, let us present some test data run on both the original and the new algorithm. Table 1 shows the benefit of using generalized ideals for representing algebraic sets in the resolution algorithm. Question marks denote computations that failed to complete because the computer algebra system refused to work with the polynomials popped up in intermediate steps of the algorithm. The results show that it is inevitable to use generalized ideals in the algorithm to resolve singularities of higher order.

Equation	t_1	t_2
$x_1x_2x_3$	932.5	52.4
$x_1^2 + x_2^3x_3^3$	1214.1	196.4
$x_1^6 + x_2^6 - x_1x_2x_3$	1551.1	278.7
$x_1^4 - x_1^2x_2(x_3 - 1)$	?	1165.9
$x_1^2x_2^2x_3^2 - x_1^4 + x_2^4 - x_3^4$	?	268.8

Table 1: Notation: $t_1(t_2)$ the used processor time in seconds with ordinary (generalized) ideals.

9 Eliminating chart redundancies

During the resolution process it can often be the case that when the algorithm refines the covering of the nonsingular algebraic set W, some newly generated charts are subsets of already existing ones. This, of course, introduces extra overhead, since singularities lying in multiple charts must be resolved in all of them.

What we can do is trying to avoid such redundant charts to be generated. In this section we present a simple but powerful method that achieves this goal, and even more.

We represent an n-dimensional affine chart in the resolution algorithm by the following data:

- A list of affine variables VAR $= (x_1, \ldots, x_m)$, for $m \geq n$, generating the polynomial ring $k[x_1, \ldots, x_m]$.
- A list DEP $= (D_1, \ldots, D_s) \subset k[\text{VAR}]$. As a subset of the ambient space $\mathbb{A}^m$ the chart is $\mathbf{Z}(\text{DEP})$.
- A list IND $= (P_1, \ldots, P_n) \subset k[U]$ $(= k[\text{VAR}]/\langle\text{DEP}\rangle)$.
- An $m \times n$ matrix PDER $= (p_{ij})$, with elements from $k[U]$.

We postulate that:

1. U is a nonsingular algebraic set in $\mathbb{A}^m$ of dimension n.
2. The elements of IND give rise to a regular system of parameters in every point of U.
3. The matrix PDER is $p_{ij} = \partial_j x_i = \frac{\partial x_i}{\partial P_j}$, $i = 1, \ldots, m$, $j = 1, \ldots, n$.

For more thorough handling of charts we refer to [BS00a]. We do not present here the computation of the chart change maps, which are obviously required for the representation of W. For an example of a chart representing $\mathbb{A}^3$ in the initial stage of the resolution algorithm, please look at U_1 in section 10.

For each pair of charts U_i, U_j, there are principal open subsets $U_{ij} \subset U_i$, $U_{ji} \subset U_j$ and an isomorphism $t_{ij} : U_{ij} \to U_{ji}$, such t_{ii} is the identity on $U_{ii} = U_i$, and the patching conditions $t_{ik} = t_{jk} \circ t_{ij}$ are fulfilled wherever both maps are defined.

The following convention helps to describe the chart transition maps t_{ij}: first, we overload the variables in VAR, such that U_i and U_j have common variables. The remaining variables will always fulfill relations (in DEP) that allows to express them as rational functions in the common coordinates. Thus, there is a unique rational map from U_i to U_j respecting the common coordinates, i.e. the pullback of x under t_{ij} is x if x is a common variable. Then U_{ij} is the domain of definition of this rational map, and t_{ij} is just the restriction to U_{ij} (so that t_{ij} is everywhere defined).

When no confusion can arise, we identify the corresponding points in the different charts. In this sense, we may write the blownup variety W as $\bigcup_{i=1}^{r} U_i$, where the U_i form the atlas describing W.

We extend the chart data structure with a new entry:

- A list FOCUS $= (F_1, \ldots, F_k) \subset k[U]$, which is initially empty.

We also extend the previous list of postulates with the following one:

4 The zerosets of the FOCUS entries of the chart covering form a decomposition of W, where some of the components may be empty. More precisely, assume that $W = \bigcup_{i=1}^{r} U_i$, we require for all $p \in W$ that there is a unique U_i such that $p \in \mathbf{Z}(\mathrm{FOCUS}_{U_i})$.

This postulate is obviously fulfilled by the initial setup.

Since only the cover and the blowing up operations can change the structure of the chart covering, we need to give the transformation of FOCUS only in these operations. The original algorithms with the correctness proofs can be found in section 4 of [BS00a]; therefore we prove compatibility only with the new postulate. In the presented algorithms we use the notation as in the definition of the chart data.

The *cover* operation takes as input a chart and a finite list $F \subset k[U]$, with the property $1 \in \langle F \rangle$. It returns a list of charts with the properties:

1. Each new chart is isomorphic to an affine open subset of the input chart.
2. The new charts cover the input chart.
3. The postulates are fulfilled by the new charts.

The operation is performed by the algorithm *Cover*.

Algorithm *Cover*((VAR, DEP, IND, PDER, FOCUS), F);
    L := (); M := ();
    for each $f \in F$ do
        VAR$'$:= append x_{m+1} to VAR;
        DEP$'$:= append $x_{m+1}f - 1$ to DEP;
        R := $(-x_{m+1}^2 \partial_k f \mid k = 1, \ldots, n)$;
        PDER$'$:= append R as the $(m+1)$-st row to PDER;
        FOCUS$'$:= append M to FOCUS;
        L := append (VAR$'$, DEP$'$, IND, PDER$'$, FOCUS$'$) to L;
        M := append f to M;
    return L;

Lemma 9.1. *Assume that* $W = \bigcup_{i=1}^{r} U_i$ *and the chart covering fulfills postulate 4. Let* $U = U_j$ *be a chart of this covering, let* $f_1, \ldots, f_s \in k[U]$ *with* $\mathbf{Z}(f_1, \ldots, f_s) = \emptyset$*, and let* $(U'_1, \ldots, U'_s) = Cover(U, (f_1, \ldots, f_s))$*. Then* $W = \big(\bigcup_{\substack{i=1,\ldots,r\\ i\neq j}} U_i\big) \cup \big(\bigcup_{k=1}^{s} U'_k\big)$ *also fulfills postulate 4.*

Proof. Let $i_k : U'_k \to U$ be the inclusion maps. We have $\mathbf{Z}(\mathrm{FOCUS}_{U_i}) \subseteq U_i$ for all i, moreover, $\mathbf{Z}(\mathrm{FOCUS}_{U'_k}) \subseteq \mathbf{Z}(\mathrm{FOCUS}_U)$ for all k. Therefore there cannot be any $p \in W$ and $i \neq j$ for which $p \in \mathbf{Z}(\mathrm{FOCUS}_{U_i})$ and $p \in \mathbf{Z}(\mathrm{FOCUS}_{U'_k})$ for some k.

So what we have to show is that $U = \bigcup_{k=1}^{s} U'_k$ fulfills the postulate (formulated now for U). But this is immediate considering that

$$\begin{aligned}
i_1(\mathbf{Z}(\mathrm{FOCUS}_{U'_1})) &= \mathbf{Z}(\mathrm{FOCUS}_U) \setminus \mathbf{Z}(f_1),\\
i_2(\mathbf{Z}(\mathrm{FOCUS}_{U'_2})) &= \mathbf{Z}(\mathrm{FOCUS}_U, f_1) \setminus \mathbf{Z}(f_2),\\
&\vdots\\
i_s(\mathbf{Z}(\mathrm{FOCUS}_{U'_s})) &= \mathbf{Z}(\mathrm{FOCUS}_U, f_1, \ldots, f_{s-1}) \setminus \mathbf{Z}(f_s).
\end{aligned}$$

□

The *blowing up* operation takes as input a chart and a list of different elements of IND. For the sake of better readability we assume that the blowing up center is given by $(P_1, \ldots, P_r)$. The operation returns a list of charts with the following properties:

1. Each new chart is isomorphic to an affine open subset of the blown up algebraic set.
2. The new charts cover the blown up algebraic set.
3. The postulates are fulfilled by the new charts.

The operation is performed by the algorithm *BlowUp*.

Lemma 9.2. *Assume that* $W = \bigcup_{i=1}^{r} U_i$ *and the chart covering fulfills the new postulate. Let* $U = U_j$ *be a chart of this covering, let* $\pi : W' \to W$ *be a blowing up along the center in* U *that is disjoint from* U_i *for* $i \neq j$*, given by* $f_1, \ldots, f_s \in \mathrm{IND}$*, and let* $(U'_1, \ldots, U'_s) = BlowUp(U, (f_1, \ldots, f_s))$*. Then* $W' = \big(\bigcup_{\substack{i=1,\ldots,r\\ i\neq j}} U_i\big) \cup \big(\bigcup_{k=1}^{s} U'_k\big)$ *also fulfills the postulate.*

Proof. We assume that the blowing up center lies only in U in order to simplify the structure of the blownup. In the general case, the intersection of the center and U_i has to be blown up for each i simultaneously to get the global transformation.

Again, as in lemma 9.1, since $\pi(\mathbf{Z}(\mathrm{FOCUS}_{U'_k})) \subseteq \mathbf{Z}(\mathrm{FOCUS}_U)$ the new zerosets cannot interfere with the ones in U_i, $i \neq j$. So what we have to show is that $\pi^{-1}(U) = \bigcup_{k=1}^{s} U'_k$ fulfills the postulate (formulated now for $\pi^{-1}(U)$). The answer is analogous to the one in lemma 9.1 since the new charts form the

```
Algorithm BlowUp((VAR, DEP, IND, PDER, FOCUS), (P_1, ..., P_r));
  L := ();
  for i from 1 to r do
    VAR' := append (x_{m+j} | j = 1, ..., r, j ≠ i) to VAR;
    DEP' := append (x_{m+j} P_i − P_j | j = 1, ..., r, j ≠ i) to DEP;
    IND' := replace P_j by x_{m+j} in IND for j = 1, ..., r, j ≠ i;
    for j from 1 to m do
      PDER'[j, i] := p_{ji} + Σ_{k=1,k≠i}^{r} x_{m+k} p_{jk};
      for k from 1 to r except i do
        PDER'[j, k] := P_i p_{jk};
      for k from r + 1 to n do
        PDER'[j, k] := p_{jk};
    for j from 1 to r except i do
      R := a list of zeros of length n with a 1 at the j-th position;
      PDER' := append R as a new row to PDER';
    if i > 1 then
      FOCUS' := append (x_{m+j} | j = 1, ..., i − 1) to FOCUS;
    else
      FOCUS' := FOCUS;
    L := append (VAR', DEP', IND', PDER', FOCUS') to L;
  return L;
```

covering of the preimage of U;

$$\begin{aligned}
i_1(\mathbf{Z}(\mathrm{FOCUS}_{U'_1})) &= \mathbf{Z}(\mathrm{FOCUS}_{\pi^{-1}(U)}) \setminus \mathbf{Z}(f_{1*}),\\
i_2(\mathbf{Z}(\mathrm{FOCUS}_{U'_2})) &= \mathbf{Z}(\mathrm{FOCUS}_{\pi^{-1}(U)}, f_{1*}) \setminus \mathbf{Z}(f_{2*}),\\
&\vdots\\
i_s(\mathbf{Z}(\mathrm{FOCUS}_{U'_s})) &= \mathbf{Z}(\mathrm{FOCUS}_{\pi^{-1}(U)}, f_{1*}, \ldots, f_{s-1*}) \setminus \mathbf{Z}(f_{s*}),
\end{aligned}$$

where $i_k : U'_k \to \pi^{-1}(U)$ are the inclusion maps, and in the chart U_k, $k = 2, \ldots, s$ we have $f_{l*} = x_{m+l}$ for $l = 1, \ldots, k-1$. □

Now we have the tool in our hands to deal with redundant charts. It is easy to see that the construction we presented provides in every chart the subset that is unique in the global structure, in the sense that it is not covered by other charts. This gives rise to the following uses of the FOCUS entry:

1. Whenever a newly generated chart U'_k has $\mathbf{Z}(\mathrm{FOCUS}_{U'_k}) = \emptyset$, we may immediately drop it, because it carries no new information.

2. Whenever the *Villamayor-resolution* algorithm computes the next blowing up center in a chart U, say $f_1, \ldots, f_s \in \mathrm{IND}$, and we have $\mathbf{Z}(f_1, \ldots, f_s) \cap \mathbf{Z}(\mathrm{FOCUS}) = \emptyset$, we may replace the costly blowing up operation with the computationally cheaper cover.

The justification of this latter speedup idea is that, since the blowing up center is disjoint from the unique subset of the chart, there must be other charts also containing it. Furthermore, since every point of W is contained in $\mathbf{Z}(\mathrm{FOCUS}_U)$ for some U, there must be a chart where the singularity gets resolved.

In the case of point 2, since the output charts must contain resolved objects we still have to get rid of the singularity, which we achieve by "tricky" the cover operation. Assume the situation as in point 2 and consider the following function call $(U'_1, \ldots, U'_{s+1}) := \mathit{Cover}(U, (f_1, \ldots, f_s, 1))$. All the output charts U'_k for $k = 1, \ldots, s$ are obviously disjoint from the blowing up center, thus from the singularity we were going to resolve. The only questionable chart is the last one, which is isomorphic to U. Nevertheless we have $\mathrm{FOCUS}_{U'_{s+1}} = \mathrm{FOCUS} + (f_1, \ldots, f_s)$, whose zeroset is empty by the assumption. Therefore the last chart is to be dropped according to point 1.

The results of the tests on the implementation of the algorithm showed that the number of charts in the atlas of the computed manifold dropped on average by 46% when we applied the above speedup criteria. The used processor time dropped proportionally by 36.7% on average, however, the benefit increased with the complexity of the resolution. Table 2 shows that with very simple examples we might find no charts that we can eliminate, thus the computations with FOCUS actually increase the time of the resolution. On the other hand, with nontrivial examples the speedup was over a factor of four.

Equation	n_1	t_1	n_2	t_2
$x_1x_2 - x_3x_4$	8	3.7	8	4.0
$x_1^2x_2 + x_2^2x_3 + x_3^2x_1$	24	13.7	13	12.4
$x_1^2 - x_2x_3^2$	371	204.9	203	117.6
$x_1^6 + x_2^6 - x_1x_2x_3$	344	1103.1	136	278.7
$x_1^2 + x_2^3x_3^3$	748	900.2	334	196.4

Table 2: Notation: $n_1(n_2)$ the number of charts in the atlas without (with) the FOCUS used; $t_1(t_2)$ the used processor time in seconds respectively.

10 Example

In this section we give a comparative examples using both representations. We do not intend to provide full resolutions because of space restrictions, however, the interested reader can find a couple of instances on the URL `http://www.risc.uni-linz.ac.at/projects/basic/adjoints/blowup/examples`.

Consider the surface given by $f = x_1^2(x_2-1) + (x_2-1)^2(x_3-1) + x_1(x_3-1)^2$ in $\mathbb{A}^3$, which has only one singularity, a triple point at $\mathbf{Z}(x_1, x_2-1, x_3-1)$. The charts are the same, of course, for both representations:

$$U_1 = (\mathrm{VAR} = (x_1, x_2, x_3), \mathrm{DEP} = (), \mathrm{IND} = (x_1, x_2, x_3), \mathrm{PDER} = \mathrm{id}_3).$$

In the following steps case 1 describes the old, and case 2 the new representation, while upper indices show the chart index to which the object belongs and lower ones identify their order of creation. The initial configuration is:

1. $\mathrm{WO}_1^1 = (W = \mathbb{A}^3, J = \langle f \rangle, b = 1, E = (), a = ())$,
2. $\mathrm{WO}_1^1 = (W = \mathbb{A}^3, J = \langle (f, 1) \rangle, c = 1, E = (), a = ())$.

The singular locus Sing(J) is generated by f, $\overline{J} = J$ and the order is 3 (in the new representation the generalized ideal and c get updated by $i/n = 3/1$); therefore the new tagged objects are:

1. $\mathrm{TO}_2^1 = (W = \mathbb{A}^3, J = \langle f \rangle, b = 3, E = (), E_- = ())$,
2. $\mathrm{TO}_2^1 = (W = \mathbb{A}^3, J = \langle (f,3) \rangle, c = 3, E = (), E_- = ())$.

Their singular locus is generated by $\langle x_1, x_2 - 1, x_3 - 1 \rangle$. Since E is empty the balanced objects are:

1. $\mathrm{BO}_3^1 = (W = \mathbb{A}^3, J = \langle f \rangle, b = 3, E = ())$,
2. $\mathrm{BO}_3^1 = (W = \mathbb{A}^3, J = \langle (f,3) \rangle, E = (), H = ())$.

The algorithm finds x_1, which is also an IND element, to be a generator of maximal contact, and since it does not generate the singular locus we need to create new weighted objects:

1. $\mathrm{WO}_4^1 = (W = \mathbf{Z}(x_1), J = \langle x_3^2x_2^4 - 4x_3^2x_2^3 - 2x_3x_2^4 + 6x_3^2x_2^2 + 8x_3x_2^3 + x_2^4 - 4x_3^2x_2 - 12x_3x_2^2 - 4x_2^3 + x_3^2 + 8x_3x_2 + 6x_2^2 - 2x_3 - 4x_2 + 1, x_3^6 - 6x_3^5 + 15x_3^4 - 20x_3^3 + 15x_3^2 - 6x_3 + 1, x_2^6 - 6x_2^5 + 15x_2^4 - 20x_2^3 + 15x_2^2 - 6x_2 + 1 \rangle, b = 6, E = (), a = ())$,
2. $\mathrm{WO}_4^1 = (W = \mathbf{Z}(x_1), J = \langle (x_2^2x_3 - x_2^2 - 2x_2x_3 + 2x_2 + x_3 - 1, 3), (x_3^2 - 2x_3 + 1, 2), (x2 - 1, 1) \rangle, c = 1, E = (), a = ())$.

Their singular locus is generated by $\langle x_2 - 1, x_3 - 1 \rangle$ in $k[x_1, x_2, x_3]/\langle x_1 \rangle$. The proper part is the whole ideal in both cases, so the weighted order is 6 in the old representation, and the corresponding order update in the new representation is 1 ($i/n = 6/6$). The new tagged objects (TO_5^1) are the same with $E_- = ()$, and it follows that the balanced objects (BO_6^1) are also determined by these data having $H = ()$.

The next generator of a hypersurface of maximal contact which is found by the algorithm is $x_3 - 1$. Since it is not in IND we have to perform exchange: x_3 gets exchanged by $x_3 - 1$. The modified chart is:

$$U_2 = (\mathrm{VAR} = (x_1, x_2, x_3), \mathrm{DEP} = (), \mathrm{IND} = (x_1, x_2, x_3 - 1), \mathrm{PDER} = \mathrm{id}_3).$$

Which means that $\mathrm{WO}_i^2 = \mathrm{WO}_i^1$, $\mathrm{TO}_{i+1}^2 = \mathrm{TO}_{i+1}^1$, $\mathrm{BO}_{i+2}^2 = \mathrm{BO}_{i+2}^1$ for $i = 1, 4$. Now, since $x_3 - 1$ does not generate the singular locus, we create the new weighted objects:

1. $\mathrm{WO}_7^2 = (W = \mathbf{Z}(x_1, x_3 - 1), J = \langle (x_2^4 - 4x_2^3 + 6x_2^2 - 4x_2 + 1)^{180}, (x_2^6 - 6x_2^5 + 15x_2^4 - 20x_2^3 + 15x_2^2 - 6x_2 + 1)^{120} \rangle, b = 720, E = (), a = ())$,
2. $\mathrm{WO}_7^2 = (W = \mathbf{Z}(x_1, x_3 - 1), J = \langle (x_2^2 - 2x_2 + 1, 2), (x_2 - 1, 1) \rangle, c = 1, E = (), a = ())$.

Remark. Notice that $x_2^4 - 4x_2^3 + 6x_2^2 - 4x_2 + 1 = (x_2^2 - 2x_2 + 1)^2$, and that $x_2^6 - 6x_2^5 + 15x_2^4 - 20x_2^3 + 15x_2^2 - 6x_2 + 1 = (x_2 - 1)^6$.

Their singular locus is generated by $\langle x_2 - 1 \rangle$ in $k[x1, x2, x3]/\langle x_1, x_3 - 1 \rangle$. The proper part is the whole ideal, so the weighted order is 720 in the old representation, and the corresponding order update in the new representation is 1. The new tagged objects (TO_8^2) are again the same with $E_- = ()$, so are the new balanced objects (BO_9^2) with $H = ()$.

The generator of the last hypersurface of maximal contact is $x_2 - 1$, and we need to perform exchange. The modified chart is:

$$U_3 = (\mathrm{VAR} = (x_1, x_2, x_3), \mathrm{DEP} = (), \mathrm{IND} = (x_1, x_2 - 1, x_3 - 1), \mathrm{PDER} = \mathrm{id}_3).$$

Again: $\mathrm{WO}_i^3 = \mathrm{WO}_i^2$, $\mathrm{TO}_{i+1}^3 = \mathrm{TO}_{i+1}^2$, $\mathrm{BO}_{i+2}^3 = \mathrm{BO}_{i+2}^2$ for $i = 1, 4, 7$.

We have $x_2 - 1$ generating the singular locus in the balanced objects BO_9^3; therefore the next operation to be performed is a blowing up with center $\mathbf{Z}(x_1, x_3 - 1, x_2 - 1)$. We follow the resolution in the chart U_4 with exceptional divisor $x_3 - 1$, which is defined as:

- $\mathrm{VAR} = (x_2, x_3, x_4, x_5)$,
- $\mathrm{DEP} = (x_5(x_3 - 1) - (x_2 - 1))$,
- $\mathrm{IND} = (x_1, x_5, x_4)$,
- $\mathrm{PDER} = \begin{pmatrix} 0 & x_3 - 1 & x_5 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$

The variable x_1 has been eliminated by the relation $x_4(x_3 - 1) - x_1$. The proper transforms of the ideals of the first two objects, $\mathrm{WO}_1^3, \mathrm{TO}_2^3$, are the same:

1. $I^1 = \langle x_4^2(x_3 - 1)^2(x_2 - 1) + (x_2 - 1)^2(x_3 - 1) + x_4(x_3 - 1)^3, x_5(x_3 - 1) - (x_2 - 1)\rangle : \langle (x_3 - 1)^3 \rangle = \langle x_5x_3 - x_5 - x_2 + 1, x_4x_3^2 - x_4^2x_3 + x_4^2x_2x_3 - 2x_4x_3 - x_4^2x_2 + x_2^2 + x_4^2 - 2x_2 + x_4 + 1, -x_4x_3 + x_4x_3x_2 - 2x_5x_2 + x_4^2 - x_4x_2 + x_4 + x_4^2x_2^2 + x_5 - 2x_4^2x_2 + x_5x_2^2\rangle$,
2. by the method described in section 7.6 we perform extended Gröbner basis computation: $x_4^2(x_3 - 1)^2(x_2 - 1) + (x_2 - 1)^2(x_3 - 1) + x_4(x_3 - 1)^3 = x_5^2 + x_4^2x_5 + x_4 \cdot (x_3 - 1)^3 + G_1 \cdot x_5(x_3 - 1) - (x_2 - 1)$, so we get $J^1 = \langle (x_5^2 + x_4^2x_5 + x_4, 3)\rangle$.

Balanced objects is recomputed after blowing ups from the transformed tagged objects, so we do not transform them.

Remark. The result in the old representation is just the Gröbner basis of $(x_5^2 + x_4^2x_5 + x_4)(x_3 - 1)^2$ and DEP. Recall that in the new representation we store only the proper part in J, we have to multiply by the monomial part to get the full ideal back.

The fourth and fifth objects, $\mathrm{WO}_4^3, \mathrm{TO}_5^3$, have the same ideals; therefore their transforms coincide:

1. $I^2 = \langle x_3^2x_2^4 - 4x_3^2x_2^3 - 2x_3x_2^4 + 6x_3^2x_2^2 + 8x_3x_2^3 + x_2^4 - 4x_3^2x_2 - 12x_3x_2^2 - 4x_2^3 + x_3^2 + 8x_3x_2 + 6x_2^2 - 2x_3 - 4x_2 + 1, x_3^6 - 6x_3^5 + 15x_3^4 - 20x_3^3 + 15x_3^2 - 6x_3 + 1, x_2^6 - 6x_2^5 + 15x_2^4 - 20x_2^3 + 15x_2^2 - 6x_2 + 1, x_5(x_3 - 1) - (x_2 - 1)\rangle : \langle (x_3 - 1)^6\rangle = \langle 1 \rangle$,
2. using the extended Gröbner basis method for the generators:
   - $x_2^2x_3 - x_2^2 - 2x_2x_3 + 2x_2 + x_3 - 1 = x_5^2 \cdot (x_3 - 1)^3 + G_2 \cdot x_5(x_3 - 1) - (x_2 - 1)$,

- $x_3^2 - 2x_3 + 1 = 1 \cdot (x_3 - 1)^2 + G_3 \cdot x_5(x_3 - 1) - (x_2 - 1)$,
- $x2 - 1 = x_5 \cdot (x_3 - 1) + G_4 \cdot x_5(x_3 - 1) - (x_2 - 1)$;

so the the transformed ideal $J^2 = \langle (x_5^2, 3), (1, 2), (x_5, 1) \rangle$.

Remark. The generalized ideal J^2 also represents $\langle 1 \rangle$.

The rest of the objects are dropped because their ideals were created by coefficient ideal computation with respect to $x_3 - 1$; therefore they are resolved.

The first transformed weighted objects become the following:

1. $\mathrm{WO}_1^4 = (W = \mathbb{A}^3, J = I^1, b = 1, E = (x_3 - 1), a = (2))$,
2. $\mathrm{WO}_1^4 = (W = \mathbb{A}^3, J = J^1, c = 3, E = (x_3 - 1), a = (2))$.

The tagged objects TO_2^4 are the same, without a, having $E_- = ()$.

The fourth weighted objects are:

1. $\mathrm{WO}_4^4 = (W = \mathbf{Z}(x_1), J = I^2, b = 6, E = (x_3 - 1), a = (0))$,
2. $\mathrm{WO}_4^4 = (W = \mathbf{Z}(x_1), J = J^2, c = 1, E = (x_3 - 1), a = (0))$.

The tagged objects TO_5^4 are the same again, without a, and $E_- = ()$.

In the following steps the algorithm will detect the resolved objects, that is, since J^2 contains 1, the objects TO_5^4 get resolved immediately, and since the monomial part $M = E^a = \langle (x_3 - 1)^0 \rangle$ also contains 1, the objects WO_4^4 are also resolved (BO_4^3 is removed by the algorithm). Then the order computation shows the resolvedness of the objects TO_2^4, and the same operation for the first weighted objects WO_1^4 establishes the new situation:

1. $\mathrm{WO}_1^4 = (W = \mathbb{A}^3, J = I^1, b = 1, E = (x_3 - 1), a = (2))$,
2. $\mathrm{WO}_1^4 = (W = \mathbb{A}^3, J = J^1_{1/3}, c = 1, E = (x_3 - 1), a = (2))$.

We stop here, remarking that the computations in the new representation are considerably easier than their counterparts in the old one. We purposefully chose not the chart with exceptional divisor $x_2 - 1$, since in the old representation the ideal quotient computation for WO_7^3 and TO_8^3 are hopeless (in the new representation it is almost trivial).

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