Computing the System of Adjoint Plane Curves by Puiseux Expansion

Part II: Algorithm [1]

Peter Stadelmeyer

Research Institute for Symbolic Computation
Johannes Kepler Universität
Linz, Austria

Peter.Stadelmeyer@risc.uni-linz.ac.at

Abstract

Let $\mathcal{C} \subset \mathbf{A}_k^2$ be an irreducible algebraic curve in the affine plane. We present an algorithm to compute the linear system of adjoint curves of $\mathcal{C}$ of some given degree based on Puiseux-series expansions.

An important part of the algorithm is, that a priori to the adjoint computation a degree bound for the Puiseux-series expansions can be given, so that lazy evaluation of the coefficients of the Puiseux-series can be avoided.

Contents

[1] Supported by the Austrian Fonds zur Förderung der wissenschaftlichen Forschung project number P11160-TEC "HySaX".

1. Introduction

Let $\mathcal{C} \subset \mathbf{A}^2_k$ be an irreducible algebraic curve in the affine plane. An important and often used concept in the theory of plane algebraic curves is the one of *adjoint curves*. Adjoint curves are not only of theoretical interest (e.g. "Brill–Noethersche Restsatz"), see e.g. [Brieskorn and Knörrer, 1986], [van der Waerden, 1973], [Severi, 1921], but they are also of practical interest, e.g. in parameterization of curves [Sendra and Winkler, 1991].

In this paper we present a fast algorithm for computing the linear system of adjoint curves of $\mathcal{C}$ of some given degree based on truncated Puiseux-series expansions. The basic idea of the algorithm is already described in Part I, see [Stadelmeyer and Winkler, 1997], but no degree bounds for the Puiseux-series expansions are given there. Hence, an implementation would have to use a kind of lazy evaluation for the Puiseux-series. This can now be avoided using the degree bounds (which are also the smallest possible ones) given in this paper. The computation of the truncated Puiseux-expansions of $\mathcal{C}$ at some curve point is done by Duval's algorithm ([Duval, 1989]).

The paper has the following structure. In Section 2 we fix notation and define basic objects such as curve-branches and Puiseux-series and we quote some well known facts about them. Section 3 deals with resolution of curve-singularities. Here it is shown in detail, how curve-branches behave w.r.t. quadratic transformations. This is necessary in order to find optimal degree bounds for the Puiseux-series expansions needed to compute the system of adjoint curves. In Section 4 adjoint curves are defined and a first (informal) version of an algorithm to compute adjoints is presented. A refined algorithm using degree bounds is then described in Section 5 and its correctness is shown in section 6. The paper ends with some examples and timing tables given in Section 7.

2. Curves and Curve Branches

Throughout this paper, let k be a commutative, algebraically closed field with characteristic 0, e.g. the complex numbers. The ring of formal power-series over k in the indeterminate t is written as $k[[t]]$. By $\mathbf{A}^2_k$ we denote the affine plane over k.

2.1. Plane Algebraic Curves.

Definition 1. Let $f \in k[x, y]$ be a polynomial of degree d. The *affine algebraic plane curve* $\mathcal{C} \subset \mathbf{A}^2_k$ *defined by* f is the set

$$\mathcal{C} = \{(x_0, y_0) \in \mathbf{A}^2_k \mid f(x_0, y_0) = 0\}.$$

The polynomial f is called the *defining polynomial of* $\mathcal{C}$. The *degree of* $\mathcal{C}$ is defined as the degree of its defining polynomial.

Definition 2. A curve C is called *irreducible* if its defining polynomial is irreducible over k.

2.2. Plane Curve-branches.

Definition 3. Let $A(t)$, $B(t) \in k[[t]]$ be two power-series not both constant. The *affine curve-branch* $\mathfrak{z} \subset \mathbf{A}_k^2$ *defined by* $(A(t), B(t))$ is the set

$$\mathfrak{z} = \{(A(t), B(t)) \mid t \in k\}.$$

The point $q = (A(0), B(0)) \in \mathbf{A}_k^2$ is called the *origin of* $\mathfrak{z}$.

Definition 4. Let C be a curve defined by $f \in k[x, y]$ and $\mathfrak{z}$ a curve-branch with origin q defined by $(A(t), B(t))$, where $A(t)$, $B(t) \in k[[t]]$. We say that $\mathfrak{z}$ is a *curve-branch of the curve C at* q, written as $\mathfrak{z} \subset C$, if $f(A(t), B(t)) = 0$.

Due to the following theorem [Newton, 1680], we have an algorithmic method to compute all curve-branches of a curve at some curve point.

Theorem 1 (Newton's Theorem on Puiseux expansion). *Let $f \in k[[x]][y]$ be a monic polynomial of degree $n > 0$ in y. Then there exists a positive integer m , such that f can be written as*

$$f(t^m, y) = \prod_{i=1}^{n} (y - B_i(t)) \quad , \textit{ with } B_i(t) \in k[[t]].$$

If in addition f is irreducible in $k[[x]][y]$ then $m = n$ and the roots $B_1(t), \ldots, B_n(t)$ can be labeled such that $B_i(t) = B_1(\zeta^i t)$, where ζ is a primitive n-th root of unity.

To simplify things, we assume for the rest of this section that C is an irreducible curve defined by some polynomial $f \in k[x, y]$ monic in y and $\mathfrak{z}$ is a curve-branch of C with origin q.

What we know from Newton's theorem is, that $\mathfrak{z}$ can be defined by $(a_0 + t^m, B(t))$, where $a_0 \in k$, m is a positive integer and $B(t) \in k[[t]]$. But until now, m is not uniquely determined, e.g. $\mathfrak{z}$ could be also defined by $(a_0 + t^{2m}, B(t^2))$. If we take the smallest possible m we get the so-called Puiseux-expansion of $\mathfrak{z}$.

Definition 5. A pair $(a_0 + t^m, B(t))$, with $m \in \mathbb{N}$, $m \geq 0$ and $B(t) = \sum_{k \geq 0} b_k t^k \in k[[t]]$, is called a *Puiseux-expansion of* $\mathfrak{z}$ if

$$\mathfrak{z} = \{(a_0 + t^m, B(t)) \mid t \in k\}$$

and $\gcd(\{k \mid b_k \neq 0\}) = 1$.

We are now able to define some simple properties of curve-branches, where we assume that $(a_0 + t^m, B(t))$ is a Puiseux-expansion of $\mathfrak{z}$.

Definition 6. We say that the curve-branch $\mathfrak{z}$ is *linear (or regular)* if $\min(m, \mathbf{o}_t(B(t) - b_0)) = 1$.

Definition 7. The curve-branch $\mathfrak{z}$ is said to be in *general position* if $m \leq \mathbf{o}_t(B(t) - b_0)$.

If the curve-branch $\mathfrak{z}$ is not in general position then by a linear change of coordinates $\mathfrak{z}$ can be transformed into a curve-branch which is in general position w.r.t. the new coordinate system.

In the following we will define what we mean by the characteristic exponents and characteristic terms of curve-branches. For more details we refer to [Abhyankar, 1990] or [Brieskorn and Knörrer, 1986].

Definition 8. We assume that the coordinate system is chosen in such a way that the curve-branch $\mathfrak{z}$ is in general position. If $\mathfrak{z}$ is linear then no characteristic exponents and terms are defined.

Otherwise, let $B(t) = \sum_{k \geq 0} b_k t^k$, $e_0 = m$. The first *characteristic exponent* of $\mathfrak{z}$ is defined as the smallest natural number $k_1 > 0$ such that $e_1 = \gcd(\{e_0\} \cup \{k \mid b_k \neq 0, k \leq k_1\}) < e_0$. The term $b_{k_1} t^{k_1}$ is called the first *characteristic term of* $\mathfrak{z}$.

The characteristic exponents k_i and characteristic terms $a_{k_i} t^{k_i}$ are defined recursively by: $k_i > k_{i-1}$ is the smallest natural number such that $e_i = \gcd(\{e_{i-1}\} \cup \{k \mid b_k \neq 0, k \leq k_i\}) < e_{i-1}$. By g we denote the total number of characteristic exponents of $\mathfrak{z}$.

The *singular part of the Puiseux-expansion of* $\mathfrak{z}$ is then defined as

$$(a_0 + t^m, \sum_{k=0}^{k_g} b_k t^k)$$

and the *characteristic part of the Puiseux-expansion of* $\mathfrak{z}$, or short the *characteristic Puiseux-expansion of* $\mathfrak{z}$, is defined as

$$(t^m, b_{k_1} t^{k_1} + b_{k_2} t^{k_2} + \cdots + b_{k_g} t^{k_g}).$$

Example 1. The singular and characteristic part of the Puiseux-expansion of the curve-branch defined by

$$(1 + t^{12}, t^{24} + t^{30} + t^{36} + t^{42} + t^{45} + t^{50} + t^{52} + t^{53})$$

are

$$(1 + t^{12}, t^{24} + t^{30} + t^{36} + t^{42} + t^{45} + t^{50})$$

and

$$(t^{12}, t^{30} + t^{45} + t^{50}).$$

2.3. Intersection Multiplicity.

Definition 9. Let $\mathfrak{z}$, $\mathfrak{z}' \subset \mathbf{A}_k^2$ be two curve-branches with origin q and Puiseux-expansions $(a_0 + t^m, B(t))$ and $(c_0 + t^n, D(t))$. We define the *intersection multiplicity* $\nu_q(\mathfrak{z}, \mathfrak{z}')$ *of* $\mathfrak{z}$ *and* $\mathfrak{z}'$ *at* q as

$$\nu_q(\mathfrak{z}, \mathfrak{z}') = \mathbf{o}_t\left(\prod_{i=1}^{n}\left(B(t) - D(\zeta^i t^{\frac{m}{n}})\right)\right),$$

where ζ is a primitive n-th root of unity.

Further properties of the intersection multiplicity of two curve-branches, e.g. symmetry, or other ways to define it, can be found in [Abhyankar, 1990] or [Brieskorn and Knörrer, 1986].

The next Lemma provides an easy way to compute the intersection multiplicity of two curve-branches.

Lemma 1. *Let* $\mathfrak{z}$, $\mathfrak{z}' \subset \mathbf{A}_k^2$ *be two different curve-branches in general position with origin* q *and Puiseux-expansions* $(a_0 + t^m, B(t))$ *and* $(c_0 + t^n, D(t))$.

The power-series $B(t)$ *and* $D(t)$ *can be written as*

$$B(t) = b_0 + b_1 t^{m_1} + b_2 t^{m_1+m_2} + \cdots + b_s t^{m_1+m_2+\cdots+m_s} + \cdots,$$
$$D(t) = d_0 + d_1 t^{n_1} + d_2 t^{n_1+n_2} + \cdots + d_s t^{n_1+n_2+\cdots+n_s} + \cdots,$$

where $m_1 = m$, $n_1 = n$ *and* b_i, $d_i \neq 0$ *and* m_i, $n_i > 0$, *for* $i \geq 2$.

Further let s *be defined as the number for which*

$$a_i = b_i \quad \textit{and} \quad \frac{m_i}{m} = \frac{n_i}{n},$$

for $0 \leq i \leq s$, *and*

$$a_{s+1} \neq b_{s+1} \quad \textit{or} \quad \frac{m_{s+1}}{m} \neq \frac{n_{s+1}}{n}$$

holds.

Then the intersection multiplicity of $\mathfrak{z}$ *and* $\mathfrak{z}'$ *at* q *is equal to*

$$\nu_q(\mathfrak{z}, \mathfrak{z}') = \min(\lambda_1, \lambda_2),$$

where

$$\lambda_1 = n\left(m + m_1 + \frac{m_2}{l_1} + \frac{m_3}{l_1 l_2} + \cdots + \frac{m_{s+1}}{l_1 \cdots l_s}\right),$$
$$\lambda_2 = m\left(n + n_1 + \frac{n_2}{l_1} + \frac{n_3}{l_1 l_2} + \cdots + \frac{n_{s+1}}{l_1 \cdots l_s}\right),$$

and the l_i *are recursively defined by*

$$
\begin{array}{ll}
\dfrac{m_1}{m} = \dfrac{n_1}{n} = \dfrac{k_1}{l_1} & , \ \gcd(k_1, l_1) = 1, \\
\dfrac{m_2}{m} = \dfrac{n_2}{n} = \dfrac{k_2}{l_1 l_2} & , \ \gcd(k_1, l_2) = 1, \\
\vdots & \quad \vdots \\
\dfrac{m_s}{m} = \dfrac{n_s}{n} = \dfrac{k_s}{l_1 \cdots l_s} & , \ \gcd(k_s, l_s) = 1.
\end{array}
$$

Proof. For a proof we refer to [van der Waerden, 1973]. □

Based on the definition of intersection multiplicity of two curve-branches we define the intersection multiplicity of a curve-branch and a curve, and the intersection multiplicity of two curves.

Definition 10. Let $\mathcal{C}$ and $\mathcal{D} \subset \mathbf{A}_k^2$ be curves and $\mathfrak{z} \subset \mathbf{A}_k^2$ a curve-branch with origin q. The *intersection multiplicity* $\nu_q(\mathfrak{z}, \mathcal{C})$ *of* $\mathfrak{z}$ *and* $\mathcal{C}$ *at* q is defined as

$$\nu_q(\mathfrak{z}, \mathcal{C}) = \sum_{\mathfrak{z}' \subset \mathcal{C} \text{ with origin } q} \nu_q(\mathfrak{z}, \mathfrak{z}')$$

and the *intersection multiplicity* $\nu_q(\mathcal{C}, \mathcal{D})$ *of* $\mathcal{C}$ *and* $\mathcal{D}$ *at* q is defined as

$$\nu_q(\mathcal{C}, \mathcal{D}) = \sum_{\mathfrak{z}' \subset \mathcal{C} \text{ with origin } q} \nu_q(\mathfrak{z}', \mathcal{D}).$$

Lemma 2. *Given a curve* $\mathcal{C} \subset \mathbf{A}_k^2$ *defined by* $f \in k[x, y]$ *and a curve-branch* $\mathfrak{z} \subset \mathbf{A}_k^2$ *defined by* $(A(t), B(t))$.

For the intersection multiplicity $\nu_q(\mathfrak{z}, \mathcal{C})$ *of* $\mathfrak{z}$ *and* $\mathcal{C}$ *at* q *we have the equality*

$$\nu_q(\mathfrak{z}, \mathcal{C}) = \mathbf{o}_t\left(f(A(t), B(t))\right).$$

Proof. A proof can be found in [Brieskorn and Knörrer, 1986]. □

Definition 11. For a curve-branch $\mathfrak{z} \subset \mathbf{A}_k^2$ with origin q the *multiplicity of* q *w.r.t.* $\mathfrak{z}$ is defined to be the minimal intersection multiplicity of $\mathfrak{z}$ and $\mathcal{L}$ at q, where $\mathcal{L} \subset \mathbf{A}_k^2$ is a line through q, i.e.

$$\nu_q(\mathfrak{z}) = \min_{\text{line } \mathcal{L} \text{ through } q} (\nu_q(\mathfrak{z}, \mathcal{L})).$$

For a curve $\mathcal{C} \subset \mathbf{A}_k^2$ the *multiplicity of* q *w.r.t.* $\mathcal{C}$ is defined to be

$$\nu_q(\mathcal{C}) = \sum_{\mathfrak{z}' \subset \mathcal{C} \text{ with origin } q} \nu_q(\mathfrak{z}').$$

It follows immediately from the above definitions that for a curve-branch $\mathfrak{z} \subset \mathbf{A}_k^2$ with origin q and Puiseux-expansion $(a_0 + t^m, B(t))$ the multiplicity $\nu_q(\mathfrak{z})$ is equal to $\min(m, \mathbf{o}_t(B(t)))$.

Definition 12. A line $\mathcal{L} \subset \mathbf{A}_k^2$ is called a *tangent* of a curve $\mathcal{C} \subset \mathbf{A}_k^2$ at q if $\nu_q(\mathcal{C}) < \nu_q(\mathcal{C}, \mathcal{L})$.

Definition 13. Let $\mathcal{C} \subset \mathbf{A}_k^2$ be a curve. All points of $\mathcal{C}$ with multiplicity greater than one are called *singular points.*

A singular point q with multiplicity m is called an *ordinary singular point* if there are m different tangents to $\mathcal{C}$ at q, otherwise q is called a *non-ordinary singular point.*

Non-singular points of $\mathcal{C}$ are also called *regular* or *simple points.*

3. Resolution of Singularities

One possible way to describe resolution of singularities of plane algebraic curves is using multiplicity sequences and multiplicity graphs. In this section we will define them and we will show how they can easily be computed once the curve-branches are given by their Puiseux-expansions.

Let $C \subset \mathbf{A}_k^2$ be a curve and q a point on it. We will distinguish two cases:

- First we assume that C is *locally irreducible at* q, i.e. there is only one curve-branch of C with origin q. In this case we will describe the resolution of q by a *multiplicity sequence.*
- On the other hand, if the curve C is *locally reducible at* q, i.e. C has more than one curve-branch at q, in a first step we will resolve each curve-branch separately getting multiplicity sequences for each curve-branch. Since these multiplicity sequences do not suffice to describe the structure of the singularity completely, we also need a sequence of equivalence relations describing how the branches interact with respect to each other during the resolution process. What we finally get is the so called *multiplicity graph.*

Resolution of singularities of plane curves is based in the following fundamental theorem.

Theorem 2. *Each singularity* q *of a plane algebraic curve* $C \subset \mathbf{A}_k^2$ *can be transformed by a finite number of quadratic transformations such that the proper transform of* C *cuts the exceptional divisor transversally in regular points only.*

Proof. For a proof we again refer to [Brieskorn and Knörrer, 1986]. □

3.1. Irreducible Singularities.

Let $C \subset \mathbf{A}_k^2$ be a curve and $q \in \mathbf{A}_k^2$ a point on it. Assume that there is only one curve-branch $\mathfrak{z} \subset \mathbf{A}_k^2$ of C with origin q. We set $q_0 = q$, $\mathfrak{z}_0 = \mathfrak{z}$ and denote by m_0 the multiplicity of q_0 w.r.t. to $\mathfrak{z}_0$.

Applying a quadratic transformation (QT) to $\mathfrak{z}_0$ at q_0 we get a new branch $\mathfrak{z}_1$ with origin q_1, the proper transform of $\mathfrak{z}_0$, and an exceptional divisor E_1. The set $\mathfrak{z}_1 \cap E_1$ consists of exactly one point, namely q_1. The point q_1 is called a *neighbouring point of* q_0 and its multiplicity w.r.t. $\mathfrak{z}_1$ is denoted by m_1 (for details see [Brieskorn and Knörrer, 1986]). If we apply another QT, now at q_1, we get a curve-branch $\mathfrak{z}_2$, an exceptional divisor E_2 and another neighbouring point q_2, whose multiplicity we will denote with m_2.

From Theorem 2 we know that after a finite number of such QTs we get a linear curve-branch whose origin does not lie on the intersection of two exceptional divisors and which intersects the exceptional divisor transversally. In this case we say that the curve-branch has *normal crossings* with the exceptional divisors. Let K be the smallest possible number of QTs which are needed such that the curve-branch only has normal crossings.

Definition 14. The sequence

$$(m_0, m_1, m_2, \dots) = (m_k)_{0 \le k}$$

of natural numbers as defined above is called the *multiplicity sequence of the curve-branch* $\mathfrak{z}$. The number K is called the *proper resolution length of* $\mathfrak{z}$. If we truncate the multiplicity sequence after K steps we get the so called *proper multiplicity sequence* $(m_k)_{0 \le k \le K}$.

The corresponding sequences $(q_k)_{0 \le k}$ and $(q_k)_{0 \le k \le K}$ of neighboring points are called the *neighboring point sequence* and *proper neighboring point sequence of* $\mathfrak{z}$, respectively.

If we look at the QTs and the corresponding proper transforms more closely we see that we are always in one of the following two cases.

Let $\mathfrak{z} \subset \mathbf{A}_k^2$ be a curve-branch with origin q defined by $(A(t), B(t))$, where

$$A(t) = a_0 + a_{k_1} t^{k_1} + a_{k_2} t^{k_2} + \cdots,$$
$$B(t) = b_0 + b_{l_1} t^{l_1} + b_{l_2} t^{l_2} + \cdots,$$

a_{k_i}, $b_{l_i} \neq 0$, for $i > 0$. The two cases are (we denote the proper transform of $\mathfrak{z}$ by $\mathfrak{z}'$):

(i) **Case I:** $k_1 \le l_1$.
In this case the QT can be described by (QT of type I)

$$x = x' + a_0$$
$$y = x'y' + b_0$$

and $\mathfrak{z}'$ is given by

$$A'(t) = a_{k_1} t^{k_1} + a_{k_2} t^{k_2} + \cdots,$$
$$B'(t) = b'_{l_1 - k_1} t^{l_1 - k_1} + \cdots.$$

(ii) **Case II:** $k_1 > l_1$.
Here the QT can be described by (QT of type II)

$$x = x'y' + a_0$$
$$y = y' + b_0$$

and $\mathfrak{z}'$ is given by

$$A'(t) = a'_{k_1 - l_1} t^{k_1 - l_1} + \cdots,$$
$$B'(t) = b_{l_1} t^{l_1} + b_{l_2} t^{l_2} + \cdots.$$

In both cases the multiplicity m of $q = (a_0, b_0)$ w.r.t. $\mathfrak{z}$ is equal to $\min(k_1, l_1)$. And as we can see, for the curve-branch $\mathfrak{z}'$ we are in the same situation as we were for $\mathfrak{z}$.

Hence, by iterating the above procedure we obtain an algorithm for computing the multiplicity sequence together with the neighbouring point sequence of an irreducible singularity.

Example 2. Let $\mathfrak{z} \subset \mathbf{A}_k^2$ be the curve-branch (from Example 1) defined by the Puiseux-expansion

$$(1 + t^{12}, t^{24} + t^{30} + t^{36} + t^{42} + t^{45} + t^{50} + t^{52} + t^{53}).$$

Its multiplicity sequence is of the form

$$(12, 12, 6, 6, 6, 6, 3, 3, 3, 2, 1, 1, 1, 1, \dots).$$

The following theorem of Enriques-Chisini shows that the multiplicity sequence, is already determined by the characteristic part of a Puiseux-expansion.

For reasons of convenience we introduce the following notation for sequences:

$$\begin{aligned} [a] &= (a), \\ [a]^n &= (\underbrace{a, \dots, a}_{n \text{ times}}) \qquad , \text{for } n \in \mathbb{N}, \\ [a]^* &= (a, a, \dots) \end{aligned}$$

If A and B are two sequences then their concatenation is simply written as AB.

Theorem 3 (Enriques – Chisini). *For a curve-branch $\mathfrak{z} \subset \mathbf{A}_k^2$ with origin q and characteristic Puiseux-expansion*

$$(t^m, b_{k_1} t^{k_1} + b_{k_2} t^{k_2} + \dots + b_{k_g} t^{k_g})$$

*we define the following **Euclidean division chains** ($i = 1, \dots, g$):*

$$\begin{aligned} d_i &= \mu_{i,1} m_{i,1} + m_{i,2} \\ m_{i,1} &= \mu_{i,2} m_{i,2} + m_{i,3} \\ &\vdots \\ m_{i,l_i - 1} &= \mu_{i,l_i} m_{i,l_i} \end{aligned}$$

with

$$\begin{aligned} d_0 &= 0, \\ d_i &= k_i - k_{i-1} \quad , \textit{for } i = 1, \dots, g, \\ m_{1,1} &= m, \\ m_{i,1} &= m_{i-1,l_i} \quad , \textit{for } i > 1. \end{aligned}$$

The proper multiplicity sequence of $\mathfrak{z}$ *is of the form*

$$[m_{1,1}]^{\mu_{1,1}}[m_{1,2}]^{\mu_{1,2}}\cdots[m_{1,l_1}]^{\mu_{1,l_1}}[m_{2,1}]^{\mu_{2,1}}\cdots[m_{g,l_g}]^{\mu_{g,l_g}}[1]$$

and the proper resolution length K *is equal to*

$$K = \sum_{i=1}^{g}\sum_{j=1}^{l_i}\mu_{i,j} + 1.$$

Proof. [Brieskorn and Knörrer, 1986]. ☐

Example 3. Using Theorem 3 we can compute proper multiplicity sequence for the curve-branch given in Example 2

$$\begin{aligned} 30 &= 2\cdot 12 + 6\\ 12 &= 2\cdot 6\end{aligned}$$

$$\begin{aligned} 15 &= 2\cdot 6 + 3\\ 6 &= 2\cdot 3\end{aligned}$$

$$\begin{aligned} 5 &= 1\cdot 3 + 2\\ 3 &= 1\cdot 2 + 1\\ 2 &= 2\cdot 1\end{aligned}$$

$$(12, 12, 6, 6, 6, 6, 3, 3, 3, 2, 1, 1, 1).$$

Let $\mathfrak{z} \subset \mathbf{A}^2_k$ be a curve-branch with origin q defined by $(A(t), B(t))$, where $A(t) = \sum_{i\geq 0} a_i t^i$ and $B(t) = \sum_{i\geq 0} b_i t^i$.

We now want to investigate the neighboring point sequence of $\mathfrak{z}$ in more detail. The coordinates of q_0 depend on a_0, b_0 and the multiplicity m_0 is equal to $m_0 = \min(\mathbf{o}_t(A(t) - a_0), \mathbf{o}_t(B(t) - b_0))$.

The coordinates of the first neighboring point q_1 depend continuously on the tangent direction of $\mathfrak{z}$ at q which itself is determined by the coefficients a_{m_0}, b_{m_0}.

Repeating this observation we can divide the points in the neighboring point sequence of $\mathfrak{z}$ into two classes: Points whose coordinates can be changed (i.e. points which can be moved) and points whose coordinates are already fixed by the coordinates (and existence) of preceding neighboring points.

Definition 15. Let $(q_k)_{0\leq k}$ be the sequence of neighboring points of a curve-branch $\mathfrak{z} \subset \mathbf{A}^2_k$ with origin q defined by $(A(t), B(t))$.

A point q_k is called a *free neighboring point of* q if its coordinates depend continuously on the coefficients of $A(t)$ and $B(t)$. All other points of $(q_k)_{0\leq k}$ are called *non-free neighboring points* or *satellite points of* q.

Lemma 3. *Let* $\mathfrak{z} \subset \mathbf{A}_k^2$ *be a linear curve branch with origin* q. *Then all neighboring points of* q *are simple and free.*

Proof. W.l.o.g. we can assume that $\mathfrak{z}$ is in general position. Since $\mathfrak{z}$ is linear, its Puiseux-expansion is of the form $(a_0+t, B(t))$, where $B(t) = \sum_{k\geq 0} b_k t^k \in k[[t]]$. So $q_0 = q$ is simple and free (depending on a_0, b_0). All QTs of $\mathfrak{z}$ are of type I and the coordinates of q_i are determined by a_0, b_0, $\ldots$, b_i, for all $i \geq 1$. □

Lemma 4. *Let* $\mathfrak{z} \subset \mathbf{A}_k^2$ *be a curve-branch with origin* q *and characteristic Puiseux-expansion* $(t^m, b_{k_1}t^{k_1} + b_{k_2}t^{k_2} + \cdots + b_{k_g}t^{k_g})$. *Furthermore, let the numbers* l_i *and* $\mu_{i,j}$, *for* $1 \leq i \leq g$, $1 \leq j \leq l_i$, *be defined as in Theorem 3. The neighboring points of* q *are free points* (F) *respectively satellite points* (S) *of* q *according to the following sequence*

$$[F]^{\mu_{1,1}+1}[S]^{\tau_1-\mu_{1,1}-1}[F]^{\mu_{2,1}+1}[S]^{\tau_2-\mu_{2,1}-1}\ldots[F]^{\mu_{g,1}+1}[S]^{\tau_g-\mu_{g,1}-1}[F][F]^*,$$

where $\tau_i = \sum_{1\leq j\leq l_i} \mu_{i,j}$, *for* $1 \leq i \leq g$.

In addition to that, the coordinates of the last neighboring point in the proper neighboring point sequence depend on a_0, b_0, $\ldots$, b_{k_g}.

Proof. Let $(a_0 + t^m, B(t))$, $B(t) = \sum_{k\geq 0} b_k t^k \in k[[t]]$ be the Puiseux-expansion of $\mathfrak{z}$ and let

$$(t^m, b_{k_1}t^{k_1} + b_{k_2}t^{k_2} + \cdots + b_{k_g}t^{k_g})$$

be the characteristic part of $B(t)$. We define the numbers l_i, $\mu_{i,j}$ and $m_{i,j}$, for $1 \leq i \leq g$, $1 \leq j \leq l_i$, as in Theorem 3. We know, that only the following coefficients of the singular part of $B(t)$ may be unequal to zero (where b_{k_1}, $\ldots$, b_{k_g} have to be unequal to zero):

$$\begin{array}{lllll}
b_0, & b_{m_{1,1}}, & b_{2m_{1,1}}, & \cdots, & b_{\mu_{1,1}m_{1,1}}, \\
b_{k_1}, & b_{k_1+m_{2,1}}, & b_{k_1+2m_{2,1}}, & \cdots, & b_{k_1+\mu_{2,1}m_{2,1}}, \\
\vdots & & & & \\
b_{k_{g-1}}, & b_{k_{g-1}+m_{g,1}}, & b_{k_{g-1}+2m_{g,1}}, & \cdots, & b_{k_{g-1}+\mu_{g,1}m_{g,1}}, \\
b_{k_g}. & & & &
\end{array}$$

We denote the proper resolution length of $\mathfrak{z}$ by K and the neighboring point sequence by $(q_k)_{0\leq K}$. Further, let $\mathfrak{z}_0 = \mathfrak{z}$.

It is obvious that q_0 is free (its coordinates depend on a_0, b_0). Applying $\mu_{1,1}$ QTs (of type I) we see that the next $\mu_{1,1}$ neighboring points are also free and the curve-branch $\mathfrak{z}_{\mu_{1,1}}$ is of the form

$$(t^{m_{1,1}}, b_{\mu_{1,1}m_{1,1}} + b_{k_1}t^{m_{1,2}} + \cdots),$$

with $0 \neq m_{1,2} < m_{1,1}$.

The next QT is of type II and $\mathfrak{z}_{\mu_{1,1}+1}$ is given by

$$(t^{m_{1,1}-m_{1,2}} + \cdots, b_{k_1}t^{m_{1,2}} + \cdots).$$

So we see, that $q_{\mu_{1,1}+1}$ is not free. The same is true for the next $\mu_{1,2}-1$ neighboring points. After applying in total $\mu_{1,2}$ QTs of type II, we have

$$\mathfrak{z}_{\mu_{1,1}+\mu_{1,2}} \;:\; (t^{m_{1,3}}+\cdots, b_{k_1}t^{m_{1,2}}+\cdots).$$

If $m_{1,3}\neq 0$ we are basically in the same situation as we were for $\mathfrak{z}_{\mu_{1,1}}$ with the difference that now QT of type I have to be applied.

After repeating the above procedure l_1 times we have applied $\tau_1=\mu_{1,1}+\mu_{1,2}+\cdots+\mu_{1,l_1}$ QTs and the curve-branch $\mathfrak{z}_{\tau_1}$ is of the form

$$\mathfrak{z}_{\tau_1} \;:\; (at^{m_{1,l-1}}+\cdots, b+\cdots),$$

where a and b depend on a_0, b_0, $b_{m_{1,1}}$, $\ldots$, $b_{\mu_{1,1}m_{1,1}}$, b_{k_1}, hence q_{τ_1} is free.

At this stage we have resolved the first characteristic term of $\mathfrak{z}$ and we find ourselves in the same situation as we started with ($\mathfrak{z}_{\tau_1}$ corresponds to $\mathfrak{z}_0$ and q_{τ_1} corresponds to q_0).

Resolving all characteristic terms we have to apply K QTs and we end up with a linear curve-branch $\mathfrak{z}_K$. It is easily seen, that the coordinates of q_K are determined by a_0, b_0, $\ldots$, b_{k_g}.

Putting all together we get the following distribution of free and satellite points for the proper neighboring point sequence

$$[F]^{\mu_{1,1}+1}[S]^{\tau_1-\mu_{1,1}-1}[F]^{\mu_{2,1}+1}[S]^{\tau_2-\mu_{2,1}-1}\ldots[F]^{\mu_{g,1}+1}[S]^{\tau_g-\mu_{g,1}-1}[F].$$

From Lemma 3 we know, that from q_K on all neighboring points are simple and free. $\square$

Corollary 1. *Let $\mathfrak{z}\subset\mathbf{A}^2_k$ be a curve-branch. The coordinates of the points in the proper neighboring point sequence of $\mathfrak{z}$ are determined by the coefficients of the singular part of the Puiseux-expansion of $\mathfrak{z}$.*

Proof. Let $(a_0+t^m, \sum_{k=0}^{S} b_k t^k)$ be the singular part of the Puiseux-expansion of $\mathfrak{z}$. As we saw in the proof of Lemma 4 the terms a_0, b_k, $0\leq k\leq S$, determine the coordinates of the neighboring point q_K, where K is the proper resolution length. Since the coordinates of q_i, for $0\leq i<K$, depend on terms of order less than S the singular part of the Puiseux-expansion of $\mathfrak{z}$ determines the coordinates of all points in the proper neighboring point sequence. $\square$

Example 4. The curve-branch defined by $(t^{100}, t^{250}+t^{375}+t^{390}+t^{391})$ has the following free neighboring points

$$[F]^3[S][F]^3[S][F][S]^3[F][S]^4[F][F]^*.$$

For a multiplicity m_k occurring in a multiplicity sequence $(m_k)_{0\leq k}$ of a curve-branch there are according to Theorem 3 three possibilities. Either the multiplicity m_k is equal to m_{k+1} or the multiplicity m_k is greater than m_{k+1} which then implies that either

$$m_k = \mu m_{k+1} + m_{k+\mu+1}$$

or

$$m_k = \mu m_{k+1}.$$

Anyway, for every neighboring point q_k there exist neighboring points, which immediately follow q_k, such that the sum of their multiplicities is equal to m_k.

Definition 16. Let $\mathfrak{z} \subset \mathbf{A}_k^2$ be a curve branch with origin q. For a neighboring point q_k of q with multiplicity m_k we define $l_k \in \mathbb{N}$ in such a way that $m_k = \sum_{k'=k+1}^{k+l_i} m_{k'}$.

The points $q_{k'}$, for $k < k' \leq k + l_k$, are called *proximate points of* q_k.

Lemma 5. *For any proper multiplicity sequence of the form*

$$[m_r]^{\mu_r}[m_{r-1}]^{\mu_{r-1}} \cdots [m_2]^{\mu_2}[m_1]^{\mu_1}[m_0]^{\mu_0}$$

it holds that

$$m_{i-1} + m_i \leq \sum_{j=0}^{i-1} \mu_j m_j \qquad , \textit{for } 1 \leq i \leq r.$$

Proof. We proof this Lemma by induction on i. For the induction base $i = 1$ we know (from Theorem 3) that $m_0 = m_1 = 1$ and $\mu_0 = 1$, $\mu_1 \geq 1$. So it holds that

$$m_0 + m_1 = 2 \leq 1 + \mu_1 = \sum_{j=0}^{1} \mu_j m_j.$$

For the induction step $i \to i+1$ the multiplicity m_{i+1} is either equal to $m_{i+1} = \mu_i m_i$ or $m_{i+1} = \mu_i m_i + m_{i-1}$. In both cases we have

$$m_i + m_{i+1} \leq \mu_i m_i + m_{i-1} + m_i \leq \mu_i m_i + \sum_{j=0}^{i-1} \mu_j m_j = \sum_{j=0}^{i} \mu_j m_j.$$

□

3.2. **Reducible Singularities.** In the case of reducible singularities we also have to take into account that it may happen that neighboring points of different curve-branches coincide up to some QT, i.e. different curve-branches have the same origin for the first QTs.

Definition 17. Let $\mathcal{C} \subset \mathbf{A}^2_k$ be a curve and $q \in \mathbf{A}^2_k$ a point on it. Assume that there are J curve-branches $\mathfrak{z}_j$ of $\mathcal{C}$ with origin q. Let the multiplicity sequences of $\mathfrak{z}_j$, for $1 \leq j \leq J$, be given by

$$(m_{j,k})_{0 \leq k}.$$

For each $k \in \mathbb{N}$ we define an equivalence relation $\sim_k$ in such a way that $j_1 \sim_k j_2$ iff $q_{j_1,k} = q_{j_2,k}$, i.e. the multiplicities $m_{j_1,k}$ and $m_{j_2,k}$ belong to the same neighboring point.

The multiplicity sequences $(m_{j,k})_{0 \leq k}$, $1 \leq j \leq J$, together with the equivalence relations $(\sim_k)_{0 \leq k}$ are called the *multiplicity graph of* $\mathcal{C}$ *at* q.

The corresponding neighboring points together with $(\sim_k)_{0 \leq k}$ are called the *neighboring point graph of* $\mathcal{C}$ *at* q.

As in the case of irreducible singularities only a finite part of the neighborhood graph is non-trivial , which we call again the proper part.

Definition 18. We use the same notation as in Definition 17. Let K'_j, for $1 \leq j \leq J$, be the length of the proper multiplicity sequence belonging to $\mathfrak{z}_j$ if we resolve each curve-branch separately. Furthermore let L_j, for $1 \leq j \leq J$, be defined as the smallest number such that q_{j,L_j} does not coincide with any other neighboring point, i.e. $j_1 \sim_{L_j} j_2$ iff $j_1 = j_2$.

According to Theorem 2 all these numbers are finite and we define the *proper resolution length* K_j *of a curve-branch* $\mathfrak{z}_j$ *of* $\mathcal{C}$ *at* q as $K_j = \max(K'_j, L_j)$. Finally let $K = \max(K_1, \ldots, K_J)$.

The multiplicity sequences $(m_{j,k})_{0 \leq k}$, $1 \leq j \leq J$, truncated at K_j together with the equivalence relations $(\sim_k)_{0 \leq k \leq K}$ are called the *proper multiplicity graph of* $\mathcal{C}$ *at* q.

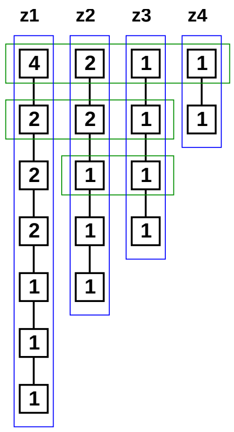
The neighbouring point sequences $(q_{j,k})_{0 \leq k}$, $1 \leq j \leq J$, truncated at K_j together with the equivalence relations $(\sim_k)_{0 \leq k \leq K}$ are called the *proper neighbouring point graph of* $\mathcal{C}$ *at* q.

Just to clarify notation we remark that the multiplicity of a neighbouring point $q_{j,k}$ w.r.t. to the curve-branch $\mathfrak{z}_j$ is $m_{j,k}$, whereas the multiplicity of $q_{j,k}$ w.r.t. to the curve $\mathcal{C}$ is $n_{j,k} = \sum_{j \sim_k j'} m_{j',k}$.

Example 5. Given a curve $\mathcal{C} \subset \mathbf{A}_k^2$ with four curve-branches $\mathfrak{z}_1$, $\mathfrak{z}_2$, $\mathfrak{z}_3$ and $\mathfrak{z}_4$ with origin $(0,0)$. The curve-branches have Puiseux-expansions

$$\begin{aligned} \mathfrak{z}_1 &: (t^4, t^6 + t^9), \\ \mathfrak{z}_2 &: (t^2, t^5), \\ \mathfrak{z}_3 &: (t, t^3 + t^5), \\ \mathfrak{z}_4 &: (t, t). \end{aligned}$$

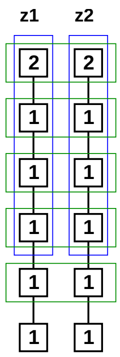
The proper multiplicity graph of $\mathcal{C}$ at $(0,0)$ has the following form.



Example 6. The curve $\mathcal{C} \subset \mathbf{A}_k^2$ defined by $f(x,y) = (x^3 + 2x^4 + x^5 - y^2)(x^3 + 4x^4 + 4x^5 - y^2)$ has at $(0,0)$ a singularity which decomposes into two curve-branches $\mathfrak{z}_1$ and $\mathfrak{z}_2$. The curve-branches have Puiseux-expansions

$$\begin{aligned} \mathfrak{z}_1 &: (t^2, t^3 + t^5 + \cdots), \\ \mathfrak{z}_2 &: (t^2, t^3 + 2t^5 + \cdots), \end{aligned}$$

and the proper multiplicity graph of $\mathcal{C}$ at $(0,0)$ is given by



The following lemma provides an easy way to determine the equivalence relations $(\sim_k)_{0 \le k}$ once the intersection multiplicities among the different curve-branches are given.

Lemma 6. *Let $\mathfrak{z}_1$, $\mathfrak{z}_2 \subset \mathbf{A}_k^2$ be two distinct curve-branches with origin q. Further, let their multiplicity graph be given by $(m_{1,k})_{0\leq k}$, $(m_{2,k})_{0\leq k}$ and $(\sim_k)_{0\leq k}$. Because $\mathfrak{z}_1$ and $\mathfrak{z}_2$ are distinct, there exists a natural number l, s.t. $1 \sim_k 2$, for $0 \leq k \leq l$, and $1 \not\sim_{l+1} 2$.*

The intersection multiplicity of $\mathfrak{z}_1$ and $\mathfrak{z}_2$ at q satisfies

$$\nu_q(\mathfrak{z}_1, \mathfrak{z}_2) = \sum_{k=0}^{l} m_{1,k} m_{2,k}.$$

Proof. A proof can be found e.g. in [Brieskorn and Knörrer, 1986]. □

Lemma 6 together with Lemma 1 and Theorem 3 provides an easy way to compute the equivalence relations $\sim_k$ in the case the curve-branches are given by their Puiseux-expansions.

Example 7. Let $\mathfrak{z}_1$ and $\mathfrak{z}_2$ be the two curve-branches with origin $q = (0,0)$ from Example 6. Their multiplicity sequences are

$$(m_{1,k})_{0\leq k} = (2, 1, 1, 1, 1, 1, \dots)$$

and

$$(m_{2,k})_{0\leq k} = (2, 1, 1, 1, 1, 1, \dots).$$

Using Lemma 1 and 6 we know that $1 \sim_k 2$, for $0 \leq k \leq 4$, and $1 \not\sim_k 2$, for $5 \leq k$.

Lemma 6 also motivates the following Definition of truncated intersection multiplicity which will be needed for the correctness proof of our algorithm.

Definition 19. Let $\mathfrak{z}_1, \mathfrak{z}_2 \subset \mathbf{A}_k^2$ be defined as in Lemma 6. The *intersection multiplicity $\nu_q^T(\mathfrak{z}_1, \mathfrak{z}_2)$ of $\mathfrak{z}_1$ and $\mathfrak{z}_2$ at q truncated at* T is defined as

$$\nu_q^T(\mathfrak{z}_1, \mathfrak{z}_2) = \sum_{k=0}^{\min(T,l)} m_{1,k} m_{2,k}.$$

For a curve-branch $\mathfrak{z} \subset \mathbf{A}_k^2$ and curve $\mathcal{C} \subset \mathbf{A}_k^2$ we define the *intersection multiplicity $\nu_q^T(\mathfrak{z}, \mathcal{C})$ of $\mathfrak{z}$ and $\mathcal{C}$ at q truncated at* T as

$$\nu_q^T(\mathfrak{z}, \mathcal{C}) = \sum_{\mathfrak{z}' \subset \mathcal{C} \text{ with origin } q} \nu_q^T(\mathfrak{z}, \mathfrak{z}').$$

It obviously holds that $\nu_q^{T_1}(\mathfrak{z}_1, \mathfrak{z}_2) \geq \nu_q^{T_2}(\mathfrak{z}_1, \mathfrak{z}_2)$ iff $T_1 \geq T_2$, and $\nu_q^K(\mathfrak{z}_1, \mathfrak{z}_2) = \nu_q(\mathfrak{z}_1, \mathfrak{z}_2)$, for $K \geq \max(K_1, K_2)$.

Definition 20. Let $\mathfrak{z} \subset \mathbf{A}_k^2$ be a curve-branch with Puiseux-expansion $(a_0 + t^m, B(t))$, where $B(t) = \sum_{0\leq k} b_k t^k$, and neighboring point sequence

$(q_k)_{0\leq k}$. By K we denote the proper resolution length of $\mathfrak{z}$. As shown in Lemma 4 for some $T \geq K$ the neighboring point q_T is free and there is a uniquely determined positive number S such that the coordinates of q_T depend on $a_0, b_0, \ldots, b_S$.

The *curve-branch* $\mathfrak{z}$ *truncated at* T *with parameter* h, written as $\mathfrak{z}_{T,h}$, is defined as

$$(a_0 + t^m, \sum_{k=0}^{S-1} b_k t^k + ht^S).$$

Lemma 7. *Let* $\mathcal{C} \subset \mathbf{A}_k^2$ *be curve with defining polynomial* $f \in k[x,y]$ *and let* $\mathfrak{z} \subset \mathbf{A}_k^2$ *be a curve-branch with proper resolution length* K. *Then it holds for all* $T \geq K$ *and some generically chosen* $h \in k$ *that*

$$\nu_q^T(\mathfrak{z}, \mathcal{C}) = \nu_q(\mathfrak{z}_{T,h}, \mathcal{C}) = \mathbf{o}_t\left(f(A(t), B_{T,h}(t))\right),$$

where $(A(t), B_{T,h}(t))$ *is the Puiseux-expansion of* $\mathfrak{z}_{T,h}$.

Proof. Let $\mathfrak{z}_i$, $1 \leq i \leq I$, be the curve-branches of $\mathcal{C}$ at q, $T \geq K$ and $\mathfrak{z}_{T,h}$ some generically chosen curve-branch of $\mathfrak{z}$ truncated at T. We will first show that $\nu_q^T(\mathfrak{z}, \mathfrak{z}_i) = \nu_q(\mathfrak{z}_{T,h}, \mathfrak{z}_i)$. Let the numbers l_i be defined as in Lemma 6 for $\mathfrak{z}_{T,h}$ and $\mathfrak{z}_i$. Obviously $l_i \leq T$ and hence we get

$$\begin{aligned} \nu_q^T(\mathfrak{z}, \mathfrak{z}_i) = \nu_q^T(\mathfrak{z}_{T,h}, \mathfrak{z}_i) &= \sum_{k=0}^{\min(l_i, T)} m_k m_{i,k} \\ &= \sum_{k=0}^{l_i} m_k m_{i,k} = \nu_q(\mathfrak{z}_{T,h}, \mathfrak{z}_i). \end{aligned}$$

Using Lemma 2 we finally have

$$\nu_q^T(\mathfrak{z}, \mathfrak{z}_i) = \mathbf{o}_t\left(f(A(t), B_{T,h}(t))\right).$$

$\square$

4. System of Adjoint Curves

In this section we will define adjoint curves using intersection multiplicities. That means, a curve $\mathcal{D}$ is adjoint to a curve $\mathcal{C}$ if the intersection multiplicities of $\mathcal{D}$ and all curve-branches of $\mathcal{C}$ are greater than certain numbers. We will further show, how these numbers can easily be computed using the polar curve of $\mathcal{C}$.

We will also give a first informal version of an algorithm to compute the linear system of adjoint curves of a given degree. A detailed description and a correctness proof of an improved version of this algorithm will be given in the next section.

We make the following assumptions: Let $\mathcal{C} \subset \mathbf{A}^2_k$ be an irreducible curve with defining polynomial $f(x,y) \in k[x,y]$ where the coordinate system is chosen in such a way that $\mathcal{C}$ has no singularities at infinity. Let $q_i \in \mathbf{A}^2_k$, for $1 \le i \le I$ be the singular points of $\mathcal{C}$.

For each singularity q_i let $\mathfrak{z}_{i,j}$, for $1 \le j \le J_i$, be the curve-branches of $\mathcal{C}$ with origin q_i. Further we define $m_{i,j,k}$ and $n_{i,j,k}$, for $1 \le k \le K_{i,j}$, as the multiplicities of the proper neighboring points $q_{i,j,k}$ of q_i w.r.t. to the curve-branch $\mathfrak{z}_{i,j}$ and the curve $\mathcal{C}$, respectively.

4.1. Adjoint curves.

The following definition is taken from [Keller, 1974].

Definition 21. For each curve-branch $\mathfrak{z}_{i,j} \subset \mathcal{C}$ we define a natural number $d_{i,j}$ as

$$d_{i,j} = \sum_{k=0}^{K_{i,j}} m_{i,j,k}(n_{i,j,k} - 1).$$

A curve $\mathcal{D} \subset \mathbf{A}^2_k$ is called an *adjoint curve of* $\mathcal{C}$, or *adjoint to* $\mathcal{C}$ if

$$\nu_{q_i}(\mathfrak{z}_{i,j}, \mathcal{D}) \ge d_{i,j},$$

for $1 \le i \le I$, $1 \le j \le J_i$.

Another way to define adjoint curves is based on prescribing certain multiplicities in the neighboring point graph, see [Brieskorn and Knörrer, 1986].

Definition 22. We fix some curve point q of $\mathcal{C}$. For each neighboring point $(q_k)_{0 \le k}$ of q let a natural number e_k be given. Let $\mathcal{D} \subset \mathbf{A}^2_k$ be a curve and let o_k be the multiplicities of q_k w.r.t. $\mathcal{D}$.

The curve $\mathcal{D}$ has *multiplicities at least* e_k *at the neighbouring points* q_k if

$$f_k = o_k - e_k + f_{k-1} \ge 0,$$

for all $0 \le k$, where $f_{-1} = 0$.

Proposition 1. *A curve $\mathcal{D} \subset \mathbf{A}_k^2$ is adjoint to $\mathcal{C}$ iff $\mathcal{D}$ has at all proper neighboring points $q_{i,j,k}$ of $\mathcal{C}$ multiplicity at least $n_{i,j,k}-1$, for $1 \leq i \leq I$, $1 \leq j \leq J_i$ and $1 \leq k \leq K_{i,j}$.*

Proof. Personal communication with Josef Schicho, based on [Schicho, 1995] or [Mňuk, 1995] and [Bliss, 1933]. □

The following Lemma shows that the intersection multiplicity in Definition 21 can be replaced by a truncated intersection multiplicity.

Lemma 8. *A curve $\mathcal{D} \subset \mathbf{A}_k^2$ is adjoint to $\mathcal{C}$ if and only if*

$$\nu_{q_i}^{K_{i,j}}(\mathfrak{z}_{i,j}, \mathcal{D}) \geq d_{i,j},$$

for $1 \leq i \leq I$, $1 \leq j \leq J_i$.

Proof. It is obvious that if $\nu_{q_i}^{K_{i,j}}(\mathfrak{z}_{i,j}, \mathcal{D}) \geq d_{i,j}$, for $1 \leq i \leq I$, $1 \leq j \leq J_i$ it follows that $\mathcal{D}$ is adjoint to $\mathcal{C}$.

So we assume that $\mathcal{D}$ is adjoint to $\mathcal{C}$ and we show that the intersection multiplicities fulfill the required inequalities. We fix some singular point q_i and some curve-branch $\mathfrak{z}_{i,j}$. Let $q = q_i$, $\mathfrak{z} = \mathfrak{z}_{i,j}$, $K = K_{i,j}$, $q_k = q_{i,j,k}$, $m_k = m_{i,j,k}$, $n_k = n_{i,j,k}$ and let o_k be the multiplicity of $\mathcal{D}$ in q_k. We then have to show

$$\nu_q^K(\mathfrak{z}, \mathcal{D}) \geq \sum_{k=0}^{K} m_k(n_k - 1) = d.$$

From Proposition 1 we know that $\mathcal{D}$ has at least multiplicity $n_k - 1$ in q_k. Hence one gets (f_k defined as in Definition 22)

$$\begin{aligned}
\nu_q^K(\mathfrak{z}, \mathcal{D}) &= \sum_{k=0}^{K} m_k o_k = \sum_{k=0}^{K} m_k(n_k - 1 + f_k - f_{k-1}) \\
&= d + \sum_{k=0}^{K-1} m_k f_k + m_K f_K - m_0 f_{-1} - \sum_{k=0}^{K-1} m_{k+1} f_k \\
&= d + \sum_{k=0}^{K-1} (m_k - m_{k+1}) f_k + m_K f_K \\
&\geq d.
\end{aligned}$$

□

4.2. **Polar Curve.** In order to be able to give a more convenient description of the adjoint curves we define the polar curve of a curve w.r.t. to some point.

Definition 23. Given a curve $\mathcal{C} \subset \mathbf{A}_k^2$ of degree n with defining polynomial $f(x, y) \in k[x, y]$ and a point $p = (p_0, p_1, p_2) \in \mathbf{P}_k^2$.

The *polar-curve* $P_{\mathcal{C},p}$ *of* $\mathcal{C}$ *with respect to* p is defined by the polynomial

$$p_0\left(nf(x,y) - x\frac{\partial}{\partial x}f(x,y) - y\frac{\partial}{\partial y}f(x,y)\right) + p_1\frac{\partial}{\partial x}f(x,y) + p_2\frac{\partial}{\partial y}f(x,y).$$

The next Lemma shows that the numbers $d_{i,j}$ can easily be computed from the intersection multiplicities of the curve-branches of $\mathcal{C}$ and its polar-curve.

Lemma 9. *Let $\mathcal{P} \subset \mathbf{A}^2_k$ be the polar-curve of $\mathcal{C}$ w.r.t. some generic point $(p_x, p_y, 0) \in \mathbf{P}^2_k$. The numbers $d_{i,j}$ from Definition 21 are equal to*

$$d_{i,j} = \nu_{q_i}(\mathfrak{z}_{i,j}, \mathcal{P}) - \nu_{q_i}(\mathfrak{z}_{i,j}) + 1.$$

Proof. [Stadelmeyer and Winkler, 1997] □

Again we can replace in the above Lemma the intersection multiplicity by the truncated intersection multiplicity.

Lemma 10. *Let $\mathcal{C} \subset \mathbf{A}^2_k$ be a curve and $\mathfrak{z} \subset \mathcal{C}$ a curve-branch of $\mathcal{C}$ with origin q. Further, let $a = (a_x, a_x)$, $b = (b_x, b_y) \in \mathbf{A}^2_k$ be two points not lying on $\mathcal{C}$ and let $\mathcal{L}_1$, $\mathcal{L}_2$ be the two lines connecting q and a, b. By $\mathcal{P}_1$ and $\mathcal{P}_2$ we denote the polar curves of $\mathcal{C}$ w.r.t. to the points $(a_x, a_y, 0)$ and $(b_x, b_y, 0)$.*

Then for all $T \geq K$, where K is the proper resolution length of $\mathfrak{z}$ within $\mathcal{C}$, it holds that

$$\nu^T_q(\mathfrak{z}, \mathcal{P}_1) - \nu^T_q(\mathfrak{z}, \mathcal{L}_1) = \nu^T_q(\mathfrak{z}, \mathcal{P}_2) - \nu^T_q(\mathfrak{z}, \mathcal{L}_2).$$

Proof. Let $f(x,y) \in k[x,y]$ be the defining polynomial of $\mathcal{C}$. W.l.o.g. we can assume that $q = (0,0)$ and we denote the sequence of neighboring points of q by $(q_k)_{0 \leq k}$. Let m_k and n_k be the multiplicities of q_k w.r.t. $\mathfrak{z}$ and $\mathcal{C}$. From Theorem 3 we know that the proper multiplicity sequence of $\mathfrak{z}$ can be written as

$$[m_r]^{\mu_r}[m_{r-1}]^{\mu_{r-1}} \cdots [m_2]^{\mu_2}[m_1]^{\mu_1}[m_1]^{\mu_1}[m_0]^{\mu_0}.$$

Furthermore, we can assume that $\mathcal{L}_1$ and $\mathcal{L}_2$ meet transversely at q. (Otherwise, we introduce a third point c, line $\mathcal{L}_3$ and polar curve $\mathcal{P}_3$ which meets both $\mathcal{L}_1$ and $\mathcal{L}_2$ transversely and compare them.)

So we can choose the coordinate system in such a way, that $\mathcal{L}_1$, $\mathcal{L}_2$, and $\mathcal{P}_1$, $\mathcal{P}_2$ are defined by $l_1(x,y) = y$, $l_2(x,y) = x$ and $p_1(x,y) = \frac{\partial}{\partial x}f(x,y)$, $p_2(x,y) = \frac{\partial}{\partial y}f(x,y)$, respectively.

Let $\mathfrak{z}_{T,h}$ be the curve-branch $\mathfrak{z}$ truncated at T with parameter $h \in k$. It is obvious that

$$\nu_q(\mathfrak{z}_{T,h}, \mathcal{C}) = \nu_q^T(\mathfrak{z}, \mathcal{C}) = \sum_{k=0}^{T} m_k n_k = M.$$

Using Lemma 7 we get

$$\mathbf{o}_t\left(\frac{d}{dt} f(A_{T,h}(t), B_{T,h}(t))\right) = M - 1$$

which is equal to

$$\mathbf{o}_t\left(G(t) + H(t)\right) = M - 1,$$

where

$$G(t) = \frac{\partial}{\partial x} f(A_{T,h}(t), B_{T,h}(t)) \frac{d}{dt} A_{T,h}(t),$$
$$H(t) = \frac{\partial}{\partial y} f(A_{T,h}(t), B_{T,h}(t)) \frac{d}{dt} B_{T,h}(t).$$

Now there are two possibilities

(i) $\mathbf{o}_t\left(G(t)\right) < M - 1$ or $\mathbf{o}_t\left(H(t)\right) < M - 1$, or
(ii) $\mathbf{o}_t\left(G(t)\right) \geq M - 1$ and $\mathbf{o}_t\left(H(t)\right) \geq M - 1$.

In the first case we then have

$$\mathbf{o}_t\left(G(t)\right) = \mathbf{o}_t\left(H(t)\right)$$

which implies that

$$\mathbf{o}_t\left(\frac{\partial}{\partial x} f(A_{T,h}(t), B_{T,h}(t))\right) - \mathbf{o}_t\left(\frac{d}{dt} B_{T,h}(t)\right) =$$
$$\mathbf{o}_t\left(\frac{\partial}{\partial y} f(A_{T,h}(t), B_{T,h}(t))\right) - \mathbf{o}_t\left(\frac{d}{dt} A_{T,h}(t)\right).$$

But this is equal to

$$\nu_q^T(\mathfrak{z}, \mathcal{P}_1) - \nu_q^T(\mathfrak{z}, \mathcal{L}_1) = \nu_q^T(\mathfrak{z}, \mathcal{P}_2) - \nu_q^T(\mathfrak{z}, \mathcal{L}_2).$$

In the second case we have

$$\nu_q^T(\mathfrak{z}, \mathcal{P}_1) + \nu_q^T(\mathfrak{z}, \mathcal{L}_2) \geq M$$
$$\nu_q^T(\mathfrak{z}, \mathcal{P}_2) + \nu_q^T(\mathfrak{z}, \mathcal{L}_1) \geq M.$$

As can easily be shown (since $T \geq K$) it holds that $\nu_q^T(\mathfrak{z}, \mathcal{L}_1) = \nu_q(\mathfrak{z}, \mathcal{L}_1)$ and $\nu_q^T(\mathfrak{z}, \mathcal{L}_2) = \nu_q(\mathfrak{z}, \mathcal{L}_2)$. So we get

$$\nu_q(\mathfrak{z}, \mathcal{P}_1) \geq \nu_q^T(\mathfrak{z}, \mathcal{P}_1) \geq M - \nu_q(\mathfrak{z}, \mathcal{L}_2)$$

Furthermore, we know from Lemma 9 that

$$\nu_q(\mathfrak{z}, \mathcal{P}_1) = \sum_{k=0}^{K-1} m_k(n_k - 1) + \nu_q(\mathfrak{z}, \mathcal{L}_1) - 1$$

and hence

$$\nu_q(\mathfrak{z}, \mathcal{L}_1) + \nu_q(\mathfrak{z}, \mathcal{L}_2) \geq \sum_{k=0}^{K} m_k + (T - K) = \sum_{j=0}^{r} \mu_j m_j + (T - K).$$

Because not both $\mathcal{L}_1$ and $\mathcal{L}_2$ are tangent to $\mathfrak{z}$ this implies that

$$\mu_r m_r + m_{r-1} + m_r \geq \sum_{j=0}^{r} \mu_j m_j + (T - K) + 1$$

which is equivalent to

$$m_{r-1} + m_r \geq \sum_{j=0}^{r-1} \mu_j m_j + (T - K) + 1.$$

But this contradicts Lemma 5 and therefore the second case can never occur. □

Lemma 11. *Let $\mathcal{P} \subset \mathbf{A}_k^2$ be the polar-curve of C w.r.t. some generic point p. The numbers $d_{i,j}$ from Definition 21 are equal to*

$$d_{i,j} = \nu_{q_i}^{K_{i,j}}(\mathfrak{z}_{i,j}, \mathcal{P}) - \nu_{q_i}(\mathfrak{z}_{i,j}) + 1.$$

Proof. Lemma 10 and [Stadelmeyer and Winkler, 1997]. □

4.3. **Linear System of Curves.** Since a curve $\mathcal{D} \subset \mathbf{A}_k^2$ of degree d is defined by a polynomial

$$g(x, y) = \sum_{j+k \leq d} a_{j,k} x^j y^k.$$

We can therefore set up a correspondence between the curves of degree d and the coefficients $a_{j,k}$ (which is an isomorphism up to a common factor of the $a_{j,k}$'s).

Hence the space of all curves of degree d can be identified with the projective space $\mathbf{P}_k^{N_d}$, where $N_d = \frac{d(d+3)}{2}$.

Definition 24. Let $d \in \mathbb{N}$. A linear subspace L of $\mathbf{P}^{N_d}$ is called *a linear system of curves of degree* d.

The linear system of curves of degree d which consists of all adjoint curves of C is called the *(linear) system of adjoint curves of degree* d *of* C.

Definition 21 can immediately be used for writing down a first version of an algorithm to compute the system of adjoint curves.

Let $g(x, y) \in k[x, y]$ be a polynomial of degree d with undetermined coefficients $a_{j,k}$ defining a curve $\mathcal{D} \subset \mathbf{A}_k^2$ and let $(A_{i,j}(t), B_{i,j}(t))$ be the Puiseux-expansions of the curve-branches $\mathfrak{z}_{i,j}$.

Requiring that

$$\mathbf{o}_t\left(g((q_i)_x + t^{m_{i,j}}, (q_i)_y + A_{i,j}(t))\right) \geq d_{i,j},$$

for $1 \leq i \leq r$, $1 \leq j \leq s_i$, we get linear conditions for the undetermined coefficients $a_{j,k}$.

Solving the resulting linear system (with indeterminates $a_{j,k}$) we get the system of adjoint curves of degree d of $\mathcal{C}$.

Example 8. Let $\mathcal{C} \subset \mathbf{A}_k^2$ be the curve defined by

$$y^5 - 2x^2y^2 + 3yx^4 - x^6.$$

The polar curve $\mathcal{P}$ of $\mathcal{C}$ w.r.t. $(0,1,0)$ is then of the form $5y^4 - 4x^2y + 3x^4$.

At the only singular point 0 the curve decomposes into 3 branches, namely

$$\begin{aligned}
\mathfrak{z}_1 : x_1(t) &= t\\
y_1(t) &= t^2 + t^6 + 3t^{10} + 13t^{14} + \cdots,\\
\mathfrak{z}_2 : x_2(t) &= t\\
y_2(t) &= \frac{1}{2}t^2 - \frac{1}{32}t^6 + \frac{3}{256}t^{10} - \frac{13}{2048}t^{13} + \cdots,\\
\mathfrak{z}_3 : x_3(t) &= 2t^3\\
y_3(t) &= 2t^2 - 2t^6 - \frac{8}{3}t^{10} - \frac{20}{3}t^{14} + \cdots.
\end{aligned}$$

We compute

$$\begin{aligned}
\mathfrak{z}_1 &: d_1 = \nu_0(\mathfrak{z}_1, \mathcal{P}) - \nu_0(\mathfrak{z}_1, \mathcal{L}) + 1 = 4,\\
\mathfrak{z}_2 &: d_2 = \nu_0(\mathfrak{z}_2, \mathcal{P}) - \nu_0(\mathfrak{z}_2, \mathcal{L}) + 1 = 4,\\
\mathfrak{z}_3 &: d_3 = \nu_0(\mathfrak{z}_3, \mathcal{P}) - \nu_0(\mathfrak{z}_3, \mathcal{L}) + 1 = 6.
\end{aligned}$$

A general curve $\mathcal{D}$ defined by

$$\begin{aligned}
g(x,y) = a_{0,0} + a_{1,0}x + a_{0,1}y + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2 +\\
a_{3,0}x^3 + a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3
\end{aligned}$$

of degree 3 is adjoint to $\mathcal{C}$ iff

$$\nu_0(\mathfrak{z}_1, \mathcal{D}) = \mathbf{o}_t\left(g(x_1(t), y_1(t))\right) =$$
$$\mathbf{o}_t\left(a_{0,0} + a_{1,0}t + (a_{0,1} + a_{2,0})t^2 + (a_{1,1} + a_{3,0})t^3 + \cdots\right) \geq 4,$$

$$\nu_0(\mathfrak{z}_2, \mathcal{D}) = \mathbf{o}_t\left(g(x_2(t), y_2(t))\right) =$$
$$\mathbf{o}_t\left(a_{0,0} + a_{1,0}t + (\frac{1}{2}a_{0,1} + a_{2,0})t^2 + (\frac{1}{2}a_{1,1} + a_{3,0})t^3 + \cdots\right) \geq 4,$$

$$\nu_0(\mathfrak{z}_3, \mathcal{D}) = \mathbf{o}_t\left(g(x_3(t), y_3(t))\right) =$$
$$\mathbf{o}_t\left(a_{0,0} + 2a_{0,1}t^2 + 2a_{1,0}t^3 + 4a_{0,2}t^4 + 4a_{1,1}t^5 + \cdots\right) \geq 6.$$

Hence, the system of adjoint curves is equal to

$$a_{2,1}x^2y + a_{1,2}xy^2 + a_{0,3}y^3.$$

5. Algorithm

In this section we describe an algorithm to compute the system of adjoint curves of a given degree. Whenever we speak of the curve f we mean the curve defined by f.

`singularPoints(f)`

In: $f \in k[x, y]$.

Out: List of singular points of f.

`puiseux(f, q, t, h)`

In: $f \in k[x, y]$, $q \in \mathbf{P}^2_k$

Out: List of all truncated Puiseux-expansions $\mathfrak{z}_{K,h}$ of the curve f with origin q, where K is the proper resolution length of $\mathfrak{z}_{K,h}$ within the curve f. I.e. this procedure computes the *singular part* of a Puiseux-series as described in [Duval, 1989].

`setCoeffZero(g, i, t)`

In: $g \in k'[t]$.

Out: Set of equations s.t. the coefficient of t^i in g is zero w.r.t. k'.

`substitute(ls, E)`

In: $ls \in k[x, y]$ with undetermined coefficients $a_{j,k}$, E a linear system of equations in the indeterminates $a_{j,k}$.

Out: ls, where the coefficients are substituted w.r.t. S, where S is a parametrization of the null-space of E.

```
adjoints(f, d)
   In:   f ∈ k[x, y], d ∈ ℕ.
  Out:   Linear system of adjoint curves of f of degree d.
begin
    sPointList := singularPoints(f)
    n := deg(f)
    p_x := ∂f/∂x
    p_y := ∂f/∂y
    ls := ∑_{j+k≤d} a_{j,k} x^j y^k
    E := {}
    for q in sPointList do
        peList := puiseux(f, q, t, h)
        for (a_0 + t^m, B_{T,h}(t)) in peList do
            mPoint := min(m, o_t (B_{T,h}(t) − b_0))
            mPolarX := o_t (p_x(a_0 + t^m, B_{T,h}(t)))
            mPolarY := o_t (p_y(a_0 + t^m, B_{T,h}(t)))
            mPolar := min(mPolarX, mPolarY)
            ls' := ls(a_0 + t^m, B_{T,h}(t))
            for i from 0 to mPoint − mPolar do
                E := E ∪ setCoeffZero(ls', i, t)
            od
        od
    od
    return substitute(ls, E)
end
```

6. Correctness

As in Section 4 let $\mathcal{C} \subset \mathbf{A}_k^2$ be an irreducible curve with defining polynomial $f(x_1, x_2) \in k[x_1, x_2]$ where the coordinate system is chosen in such a way that $\mathcal{C}$ has no singularities at infinity. Let $q_i \in \mathbf{A}_k^2$, for $1 \leq i \leq I$, be its singular points and let $\mathfrak{z}_{i,j}$, for $1 \leq j \leq J_i$, be the curve-branches of $\mathcal{C}$ with origin q_i. By $K_{i,j}$ we denote the proper resolution length of $\mathfrak{z}_{i,j}$.

Lemma 12. *The procedure puiseux(*f, q_i, t, h*) computes the Puiseux-expansions of the truncated curve-branches* ${\mathfrak{z}_{i,j}}_{K_{i,j},h}$.

Proof. Follows from [Duval, 1989]. □

Proposition 2. *The algorithm in Section 5 computes the system of adjoint curves of degree* d.

Proof. Follows from Lemmas 12, 8 and 11. □

6.1. **Complexity.** The algorithm described in Section 5 has a polynomial time complexity (counting ground field operations) due to the following facts:

- The singular points can be computed in polynomial time.
- According to [Duval, 1989] the computation of the singular part of a Puiseux-expansion can also be done in polynomial time.
- Substituting and solving the resulting linear system are again polynomial time operations.

A detailed complexity analysis will be given in "Computing the System of Adjoint Plane Curves by Puiseux Expansion, Part III: Complexity".

7. Examples

7.1. Example 1. Curve of degree 10 with genus 18.

$$f_1(x,y) = x^6y^4 - 23y^5x^4 + x^7y - 2y^3 + 3x^7 + 5x^{10}$$

The system of adjoint curves is given by

$$\begin{aligned}&a_{2,3,2}x^2y^3 + a_{7,0,0}x^7 + a_{6,0,1}x^6 + a_{5,0,2}x^5 + a_{4,0,3}x^4 +\\ &a_{6,1,0}x^6y + a_{5,1,1}x^5y + a_{4,1,2}x^4y + a_{3,1,3}x^3y + a_{2,1,4}x^2y +\\ &a_{5,2,0}x^5y^2 + a_{4,2,1}x^4y^2 + a_{3,2,2}x^3y^2 + a_{2,2,3}x^2y^2 +\\ &a_{1,2,4}xy^2 + a_{0,2,5}y^2 + a_{4,3,0}x^4y^3 + a_{3,3,1}x^3y^3.\end{aligned}$$

7.2. Example 2. Curve of degree 6 curve with genus 0.

$$\begin{aligned}f_2(x,y) = {}&2401yx^3 + 136857x^2y^2 + 1534239xy^3 - 614656y^4 -\\ &3176523y^6 - 146205 - 980x^2 + 302036yx - 755825y^2\end{aligned}$$

The system of adjoint curves is given by

$$\begin{aligned}&a_{2,0,2}x^2 + a_{1,0,3}x + a_{0,0,4} + a_{2,1,1}x^2y + a_{1,1,2}xy +\\ &\left(\frac{2051}{5}\,a_{1,0,3} + \frac{7524}{49}\,a_{2,1,1}\right)y +\\ &\left(\frac{147}{5}\,a_{1,0,3} + 46\,a_{2,1,1}\right)xy^2 +\\ &\left(\frac{2051}{5}\,a_{1,1,2} - \frac{4319461}{25}\,a_{2,0,2} + \frac{4296026}{4275}\,a_{0,0,4}\right)y^2 +\\ &\left(\frac{147}{5}\,a_{1,1,2} - \frac{318402}{25}\,a_{2,0,2} + \frac{105644}{1425}\,a_{0,0,4}\right)xy^3 +\\ &\left(397\,a_{2,1,1} + \frac{3381}{5}\,a_{1,0,3}\right)y^3 +\\ &\left(\frac{3381}{5}\,a_{1,1,2} - \frac{7226226}{25}\,a_{2,0,2} + \frac{2393797}{1425}\,a_{0,0,4}\right)y^4.\end{aligned}$$

Since this curve has genus 0, we can parameterize it. The parameterization is done as described in [Sendra and Winkler, 1991].

$$x(t) = -\frac{117649 + 6125t^4 + 9375t^6}{117649t},$$
$$y(t) = -\frac{5(196t + 25t^3)}{2401}.$$

7.3. Example 3. Curve of degree 8 with genus 0.

$$\begin{aligned}f_3(x,y) = {}&1144\,xy - 10341\,x^3 - 1351\,y^3 - 312\,x^2 + 416\,y^2 - 4421\,x^2y +\\ &8553\,xy^2 + 14852\,xy^3 - 24940\,yx^3 - 11660\,x^2y^2 - 346\,x^4 -\\ &3906\,y^4 + 7757\,y^5 - 38069\,xy^4 + 865\,x^3y^2 + 60447\,x^2y^3 -\\ &528\,x + 176\,y\end{aligned}$$

The system of adjoint curves is given by

$$
\begin{aligned}
&a_{3,0}x^3 + a_{0,0} + a_{2,1}x^2y + a_{0,2}y^2 + \\
&\left(\frac{833473}{8272}\,a_{0,0} - \frac{384025}{16262}\,a_{2,1} + \frac{2345948782}{1406663}\,a_{3,0} + \frac{1055}{94}\,a_{0,2}\right)x^2 + \\
&\left(\frac{1178220312}{1406663}\,a_{3,0} + \frac{13252}{517}\,a_{0,0} - \frac{97350}{8131}\,a_{2,1} + \frac{130}{47}\,a_{0,2}\right)x + \\
&\left(\frac{515463942}{1406663}\,a_{3,0} + \frac{34121}{8272}\,a_{0,0} - \frac{86237}{16262}\,a_{2,1} - \frac{127}{94}\,a_{0,2}\right)xy + \\
&\left(\frac{67474}{8131}\,a_{2,1} - \frac{815300616}{1406663}\,a_{3,0} - \frac{102}{47}\,a_{0,2} - \frac{19431}{1034}\,a_{0,0}\right)y + \\
&\left(-\frac{12773161109}{1406663}\,a_{3,0} - \frac{1072765}{4136}\,a_{0,0} + \frac{1055352}{8131}\,a_{2,1} - \frac{1362}{47}\,a_{0,2}\right)xy^2 + \\
&\left(-\frac{219623}{8131}\,a_{2,1} + \frac{2646619906}{1406663}\,a_{3,0} + \frac{209995}{4136}\,a_{0,0} + \frac{202}{47}\,a_{0,2}\right)y^3.
\end{aligned}
$$

7.4. Some more Examples.

$$f_4(x,y) = x^6y^4 - 23\,y^5x^4 + x^7y - 2\,y^3 + 3\,x^7 + 5\,x^{10}$$

$$f_5(x,y) = y^{10} - y^7x^2 - y^7x + 7\,y^6x^3 + 2\,y^5x^5 + x^{10} - x^5y^2 - y^3x^2$$

$$
\begin{aligned}
f_6(x,y) = {} & -52224\,y^6 - 7081701376\,y^2 + 934281216\,y^2x^2 - \\
& 358612992\,x^4y^2 + 19549126656\,x + 21535887360\,y + \\
& 4250880\,y^5 + 1187862528\,y^3 + 256\,y^7 + 9240576\,y^4x^2 - \\
& 268435456\,x^8 - 186384384\,y^3x^2 + 33554432\,x^5y^2 - \\
& 11961630720\,xy + 28311552\,x^4y^3 + 113246208\,x^3y^3 - \\
& 177209344\,xy^3 + 884736\,y^5x + 196608\,y^5x^2 - 5799936\,xy^4 - \\
& 1887436800\,y^2x^3 + 1279262720\,x^4y - 880803840\,x^4 + \\
& 10418651136\,x^3y - 4841275392\,x^2 + 1207959552\,x^5 - \\
& 26486267904 - 19025362944\,x^3 - 402653184\,x^5y - \\
& 104209920\,y^4 - 339738624\,x^2y + 2531524608\,y^2x
\end{aligned}
$$

$$
\begin{aligned}
f_7(x,y) = {} & -932960x^4y + 106624xy^2 + 2689120x^7 + 31360x^6y + \\
& 8704x + 13824y - 9216y^2 - 60928yx - 100352x^6 - \\
& 59904x^3 - 6912 + 2048y^3 + 66816x^3y - 17920x^3y^2 - \\
& 14000487x^8 - 268912x^7y - 186592x^4
\end{aligned}
$$

$$
\begin{aligned}
f_8(x,y) = {} & y^{10} - 2494\,y^8x^2 + 474\,y^8 + 84366\,y^6 + 2042158\,y^6x^4 - \\
& 660492\,y^6x^2 - 761328152\,x^6y^4 + 128361096\,y^4x^4 - \\
& 47970216\,y^4x^2 + 6697080\,y^4 + 134266087241\,y^2x^8 + \\
& 15052058268\,y^2x^6 - 12024807786\,x^4y^2 - 506101284\,y^2x^2 + \\
& 202172841\,y^2 - 9127158539954\,x^{10} - 3212722859346\,x^8 - \\
& 228715574724\,x^6 + 34263110700\,x^4 + 5431439286\,x^2 + 201803238
\end{aligned}
$$

7.5. **Timings.** Computations were done with Maple V release 5.1 under Linux (Debian 2.1, Intel PIII-450MHz). The time needed to compute the singular points together with all curve branches is denoted by "PE-time" (Puiseux-expansion time). The time needed to compute the required intersection multiplicities, building up the linear system and solving it can be seen from the row "solve-time". All times are given in seconds.

	degree	genus	PE-time	solve-time	total time
f_1	10	18	0.370	0.120	0.490
f_2	6	0	0.739	0.080	0.819
f_3	5	0	3.020	0.190	3.210
f_4	11	26	0.560	0.110	0.670
f_5	10	20	0.860	0.030	0.890
f_6	8	0	51.370	4.059	55.429
f_7	8	0	1137.280	52.881	1190.161
f_8	10	10	3.561	0.429	3.990

References

[Abhyankar, 1990] Abhyankar, S. S. (1990). *Algebraic Geometry for Scientists and Engineers*. American Mathematical Society.

[Bliss, 1933] Bliss, G. A. (1933). *Algebraic functions*. American Mathematical Society.

[Brieskorn and Knörrer, 1986] Brieskorn, E. and Knörrer, H. (1986). *Plane Algebraic Curves*. Birkhäuser.

[Duval, 1989] Duval, D. (1989). Rational puiseux expansion. *Compositio Math.*, 70:119–154.

[Keller, 1974] Keller, O.-H. (1974). *Vorlesung über algebraische Geometrie*. Akademische Verlagsgesellschaft Geest und Portig, Leipzig.

[Mňuk, 1995] Mňuk, M. (1995). *Algebraic and Geometric Approach to Parametrization of Rational Curves*. PhD thesis, Johannes Kepler Universität, A-4040 Linz, Europe.

[Newton, 1680] Newton, I. (1680). *Geometria Analytica*.

[Schicho, 1995] Schicho, J. (1995). *Rational Parametrization of Algebraic Surfaces*. PhD thesis, Johannes Kepler Universität, A-4040 Linz, Europe.

[Sendra and Winkler, 1991] Sendra, J. R. and Winkler, F. (1991). Symbolic parametrization of curves. *J. Symbolic Computation*, 12:607–631.

[Severi, 1921] Severi, F. (1921). *Vorlesungen über algebraische Geometrie*. B. G. Teubner Stuttgart.

[Stadelmeyer and Winkler, 1997] Stadelmeyer, P. and Winkler, F. (1997). Computing the system of adjoint plane curves by puiseux expansion. Technical Report 97-38, RISC – Linz, Johannes Kepler Universität, A-4040 Linz, Europe.

[van der Waerden, 1973] van der Waerden, B. (1973). *Einführung in die algebraische Geometrie*. Springer-Verlag.